Solving the Probabilistic Traveling Salesman Problem by Linearising a Quadratic Approximation

J. Fabian Meier [a], Uwe Clausen [a]

[a] *Institute of Transport Logistics, TU Dortmund, Leonhard-Euler-Str. 2, 44227 Dortmund, Germany*

Abstract

The Probabilistic Traveling Salesman Problem, introduced in 1985 by Jaillet, is one of the fundamental stochastic versions of the Traveling Salesman Problem: After the tour is chosen, each vertex is deleted with given probability $1 - p$. The eliminated vertices are bypassed which leads to shorter tours. The aim is to minimize the expected tour length. The resulting Mixed Integer Program is difficult, because the objective function is a high-degree polynomial in the binary edge variables. Linearisations lead to a large number of variables.

We show that for large p, the model can be well approximated by a quadratic model. This quadratic model can be linearised to a small linear model which leads to good primal solutions and strong lower bounds up to 80 probabilistic nodes. We explain the approach and present numerical results.

Keywords: Quadratic Optimisation, Linearisation, Quadratic Traveling Salesman Problem

1. Introduction

The Probabilistic Traveling Salesman Problem was introduced by Jaillet [Jai85, Jai88]. It is a stochastic extension of the well-known Traveling Salesman Problem (TSP) which can be described as follows:

After the tour is constructed, some vertices are randomly removed from the problem. The resulting tour is then constructed by bypassing all removed vertices while keeping the non-removed vertices in the chosen order. It is assumed that every vertex is removed with given probability $q = 1 - p$ and that the removal of different vertices is not correlated.

Arbitrary values of p (down to 0.1) were discussed in the literature: We want to present a method that works well for values in a range about $90 - 98\%$ and leads to solutions with tight lower bounds. This might seem very restrictive, but for larger values of p the problem becomes indistinguishable from the TSP,

*Corresponding author
Email address: meier@itl.tu-dortmund.de (J. Fabian Meier)

and smaller values raise the question of applicability: A one-stage approach is justified if the disruptions are small—for larger cancellation probabilities, it becomes necessary to replan tours when disruptions happen instead of just bypassing eliminated nodes.

Jaillet and following authors distinguish *black* and *white* nodes, where black nodes are always part of the tour and the probability p only applies to the white nodes. In this paper, we will consider only white nodes so that every node is subject to possible deletion (which increases the difficulty of the problem drastically).

There are few exact approaches to the problem—notably [LLM94]: They introduced the stochastic components of the objective function by additional "optimality cuts" during the solution process. They solved some of the instances up to 50 nodes, but these larger instances contained at most 5 white nodes. This restricted the stochastic effect to a small set of nodes so that only few optimality cuts had to be applied.

Due to the limitations of the exact approaches, much of the PTSP literature focuses on heuristics. [BH93] introduce equations for efficiently evaluating the cost of local-search moves for the PTSP. Bianchi et al [BKB05, BG07] provide corrections for the equations in [BH93]. Work by Campbell [Cam06] and Tang/Miller-Hooks [TMH04] focuses on approximations for the PTSP. Many further metaheuristics were applied [MM10, Liu10, BGD02, BBSD10, BFB03], the newest example being the Ant Colony Optimization approach of Weyland et al. [WMG14]. The early works [Ber88, BH93] derived theoretical properties and asymptotic bounds.

Our aim is to derive a quadratic model that approximates the PTSP well if p is large. This model can be linearised to a model that adds only $O(n)$ variables and $O(n^2)$ to the TSP. We can therefore solve this approximation model with the usual MILP methods: This leads to a solution to the PTSP and a good lower bound.

In Sect. 2, we derive and describe a highly non-linear model for the considered situation. Section 3 constructs a quadratic model and shows that this model approximates the model of Sect. 2 well. Section 4 linearises the approximation model by using geometric methods similar to those of [MC15]. Section 5 gives numerical results while 6 contains the conclusion.

2. Description of the Model

Let us restate the Probabilistic Traveling Salesman Problem, following the definitions of [Jai85]. Let $V = \{1, \dots, n\}$ denote the set of vertices and let p the probability that a vertex is visited. For convenience, we define $q := 1 - p$ and use this notation from now on.

We consider 2^n possible scenarios: For each $J \subset V$, we have a scenario s_J, in which the vertices $i \in J$ are removed. This scenario occurs with probability $q^{|J|}p^{n-|J|}$. If σ is a permutation of V and $\sigma_1 \to \sigma_2 \to \dots \to \sigma_n \to \sigma_1$ is the chosen tour, then the tour in scenario s_J is formed by eliminating all $\sigma_i \in J$ from the tour.

The model we consider is a one-stage stochastic model: Once the removal of vertices is revealed, the tour changes in a predefined manner. Therefore, the constraints of the TSP remain unaltered, while the objective function changes. Let us first restate the constraints and fix the notation:

Following the standard TSP notation, we introduce binary variables x_{ij}, $i \neq j$, with cost factors c_{ij}, where $i \to j$ denotes the arc from i to j. For the edge cost c_{ij}, we assume the triangle inequality.

$$\sum_{i}^{i \neq j} x_{ij} = 1 \qquad \forall j \in V \tag{1}$$

$$\sum_{i}^{i \neq j} x_{ji} = 1 \qquad \forall j \in V \tag{2}$$

$$\sum_{i,j \in S}^{i \neq j} x_{ij} \leq |S| - 1 \qquad \forall S \subset V, 2 \leq |S| \leq n-2 \tag{3}$$

$$x_{ij} \in \{0,1\} \qquad \forall i \neq j, i,j \in V \tag{4}$$

All sums are understood to be over V if not stated otherwise. The constraint (1) and (2) make sure that each vertex is connected to exactly two others, while the subtour elimination constraints (3) forbid the forming of unconnected circles. Because the number of inequalities in (3) is exponential, they will (as usual) generated during the solution process.

We will now discuss the objective function. Before we elaborate on the MIP formulation, we will discuss it in a combinatorial way. Without loss of generality, we assume that the tour $1 \to 2 \to 3 \to \ldots \to n \to 1$ was chosen. Now, for any edge $i \to j$, $i,j \in V$, what is the probability that it is used, i.e. that the cost c_{ij} is realised? It is exactly the case when i and j are not removed, but all vertices in between. The number of those vertices is $|j-i|-1$, so that the appearance of edge $i \to j$ has probability $p^2 q^{|j-i|-1}$. The total cost is, therefore, $\sum_{i,j}^{i \neq j} p^2 q^{|j-i|-1} c_{ij}$.

Now the calculated cost function has to be transferred to a general tour, described by binary variables x_{ij}, $i,j \in V$. The resulting objective function is

$$f(x) = q^2 \sum_{i,j} c_{ij} x_{ij} + p^2 q \sum_{i,j,k} c_{ik} x_{ij} x_{jk} + p^2 q^2 \sum_{i,j,k,l} c_{il} x_{ij} x_{jk} x_{kl} + \ldots \tag{5}$$

We gain this formulation by ordering the c_{ij} by the number of arcs in between. The resulting objective function is polynomial of degree n in the binary variables x_{ij}. This formulation was already introduced by [Jai85, section 5]. Two linearisations were presented: The first adds $O(n^3)$ variables and $O(n^4)$ constraints to the TSP, the second one adds $O(n^3)$ variables and $O(n^3)$ constraints to the TSP. We use the next section to approximate the non-linear formulation by a quadratic function to eventually gain a smaller linearisation of the approximated model.

3. Approximating the Model

Theorem 1. *Let*

$$g(x) = p^2 \sum_{i,j} c_{ij} x_{ij} + p^2 q \sum_{i,j,k} c_{ik} x_{ij} x_{jk}$$

denote the first two terms of the objective function $f(x)$. Then we get

$$g(x) + p^2 q^2 T_n < f(x) \leq \Big(1 + \big(\tfrac{2q}{p}\big)^2\Big) g(x), \tag{6}$$

where T_n is the optimal solution of the TSP if $3 \not| n$ and of the following relaxed TSP if $3|n$: The subtour elimination constraints are only enforced for $|S| \leq n/3$.

In the case $3 \not| n$, we know that $T_n \geq g(\hat{x})$ for the optimal solution $\hat{x}$ of the approximated PTSP with objective function $g(x)$.

Before we prove the theorem, we want to stress that for large p, the theorem shows that—assuming $3 \not| n$—we get a good lower bound by solving the problem for $(1 + p^2 q^2) g(x)$ as objective function.

Proof. We obviously have

$$f(x) > g(x) + p^2 q^2 \sum_{i,j,k,l} c_{il} x_{ij} x_{jk} x_{kl}.$$

For $3 \not| n$, we know that $x_{il}^3 = \sum_{j,k} x_{ij} x_{jk} x_{kl}$ is a solution of the TSP, so that we have

$$\sum_{i,j,k,l} c_{il} x_{ij} x_{jk} x_{kl} \geq T_n$$

For $3|n$, we know that x_{il}^3 is a feasible solution of the relaxed TSP in which subtours of length $n/3$ are allowed.

To prove the other inequality, we make the following definition: Let

$$K_\alpha(x) := p^2 q^{\alpha-1} \sum_{i_1, i_2, \ldots, i_{\alpha+1}} c_{i_1 i_{\alpha+1}} x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_\alpha i_{\alpha+1}} \tag{7}$$

be the term of degree $1 \leq \alpha \leq n$ in $f(x)$. Obviously, we have $g(x) > K_1(x)$. So we want to establish the inequality

$$f(x) - g(x) \overset{?}{\leq} \left(\frac{2q}{1-q}\right)^2 K_1(x) \tag{8}$$

The proof relies on the appropriate use of the triangle inequality for c_{ij}. From it, we deduce

$$\frac{K_\alpha(x)}{p^2 q^{\alpha-1}} = \sum_{i_1, i_2, \ldots, i_{\alpha+1}} c_{i_1 i_{\alpha+1}} x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_\alpha i_{\alpha+1}}$$

Using the iterated triangle inequality:

$$
\begin{aligned}
&\leq \sum_{i_1,i_2,\ldots,i_{\alpha+1}} \left(c_{i_1i_2} + c_{i_2i_3} + \ldots + c_{i_\alpha i_{\alpha+1}}\right) x_{i_1i_2}x_{i_2i_3}\cdots x_{i_\alpha i_{\alpha+1}} \\
&= \sum_{i_1,i_2,\ldots,i_{\alpha+1}} c_{i_1i_2}\, x_{i_1i_2}x_{i_2i_3}\cdots x_{i_\alpha i_{\alpha+1}} \\
&\quad + \sum_{i_1,i_2,\ldots,i_{\alpha+1}} c_{i_2i_3}\, x_{i_1i_2}x_{i_2i_3}\cdots x_{i_\alpha i_{\alpha+1}} \\
&\quad + \ldots \\
&\quad + \sum_{i_1,i_2,\ldots,i_{\alpha+1}} c_{i_\alpha i_{\alpha+1}}\, x_{i_1i_2}x_{i_2i_3}\cdots x_{i_\alpha i_{\alpha+1}} \\
&= \sum_{i_1,i_2} c_{i_1i_2}\, x_{i_1i_2} \sum_{i_3,i_4,\ldots,i_{\alpha+1}} x_{i_2i_3}\cdots x_{i_\alpha i_{\alpha+1}} \\
&\quad + \sum_{i_2,i_3} c_{i_2i_3}\, x_{i_2i_3} \sum_{i_1,i_4,\ldots,i_{\alpha+1}} x_{i_1i_2}x_{i_3i_4}\cdots x_{i_\alpha i_{\alpha+1}} \\
&\quad + \ldots \\
&\quad + \sum_{i_\alpha,i_{\alpha+1}} c_{i_\alpha i_{\alpha+1}}\, x_{i_\alpha i_{\alpha+1}} \sum_{i_1,i_2,\ldots,i_{\alpha-1}} x_{i_1i_2}\cdots x_{i_{\alpha-1}i_\alpha}
\end{aligned}
$$

We split the sums into iterated sums over single indices:

$$
\begin{aligned}
&= \sum_{i_1,i_2} c_{i_1i_2}\, x_{i_1i_2} \sum_{i_3} x_{i_2i_3} \cdots \sum_{i_{\alpha+1}} x_{i_\alpha i_{\alpha+1}} \\
&\quad + \sum_{i_2,i_3} c_{i_2i_3}\, x_{i_2i_3} \sum_{i_1} x_{i_1i_2} \sum_{i_4} x_{i_3i_4} \cdots \sum_{i_{\alpha+1}} x_{i_\alpha i_{\alpha+1}} \\
&\quad + \ldots \\
&\quad + \sum_{i_\alpha,i_{\alpha+1}} c_{i_\alpha i_{\alpha+1}}\, x_{i_\alpha i_{\alpha+1}} \sum_{i_1} x_{i_1i_2} \cdots \sum_{i_{\alpha-1}} x_{i_{\alpha-1}i_\alpha}
\end{aligned}
$$

By (2) and (1), the sums are all equal to one:

$$\begin{aligned}
&= \sum_{i_1,i_2} c_{i_1 i_2}\, x_{i_1 i_2} \\
&\quad + \sum_{i_2,i_3} c_{i_2 i_3}\, x_{i_2 i_3} \\
&\quad + \dots \\
&\quad + \sum_{i_\alpha, i_{\alpha+1}} c_{i_\alpha i_{\alpha+1}}\, x_{i_\alpha i_{\alpha+1}} \\
&= \alpha \sum_{i,j} c_{ij}\, x_{ij} \\
&= \alpha \frac{K_1(x)}{p^2}
\end{aligned}$$

This implies $K_\alpha(x) \le \alpha q^{\alpha-1} K_1(x)$. From this, it follows that:

$$f(x) - g(x) \le K_1(x) \cdot \sum_{\alpha=3}^{n} \alpha q^{\alpha-1} \tag{9}$$

To establish (8), we need to show that

$$\sum_{\alpha=3}^{n} \alpha q^{\alpha-1} \overset{?}{\le} \left(\frac{2q}{1-q}\right)^2 \qquad 0 < q < 1 \tag{10}$$

Using the formulas $\sum_i q^i = \frac{1}{1-q}$ and $\sum_i i q^i = \frac{q}{(1-q)^2}$, this is an easy calculation. □

To ease the discussion, we will assume $3 \nmid n$ from now on. The problem for $3|n$ can be solved in a similar fashion, solving the relaxed TSP version described in the theorem first.

Following the good approximation properties, we will from now on consider the approximated model APTSP:

$$\text{Min} \quad g(x) = p^2 \sum_{i,j} c_{ij} x_{ij} + p^2 q \sum_{i,j,k} c_{ik} x_{ij} x_{jk}$$

$$\begin{aligned}
&(1), (2), (3) \\
&x_{ij} \in \{0,1\} \qquad \forall i,j \in V, i \ne j
\end{aligned} \tag{11}$$

We know that optimal solutions to this model are lower bounds to the PTSP and that the optimal solution of the PTSP lies within a range of $[(1+p^2q^2)g(x), (1+4\frac{q^2}{p^2})g(x)]$. The model is a special case of the Quadratic Traveling Salesman Problem (see e.g. [Fis13] for a recent, thorough discussion), in which the objective function contains a quadratic term involving consecutive arcs.

4. Linearising the Approximated Model

The quadratic term $\sum_{i,j,k} c_{ik}x_{ij}x_{jk}$ in $g(x)$ has a special structure because c_{ik} fulfils the triangle inequality. We can therefore develop a similar reasoning as in [MC15].

Let us introduce variables $y_j \geq 0$, $j \in V$, for the "bypassing distance" of j. We can write the model as:

$$\text{Min} \quad g(x) = p^2 \sum_{i,j} c_{ij}x_{ij} + p^2 q \sum_j y_j$$

$$(1), (2), (3)$$

$$y_j \geq \sum_{i,k} c_{ik}x_{ij}x_{jk} \tag{12}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in V, i \neq j \tag{13}$$

We will now prove

Theorem 2. *The set of constraints* (1)*,* (2)*,* (12) *is equivalent to the set of constraints* (1)*,* (2)*,* (14)*:*

$$y_j \geq \sum_i c_{il}x_{ij} - \sum_k c_{kl}x_{jk} \qquad \forall j, l \in V \tag{14}$$

Proof. Let us assume (1), (2), (12). For every $l \in V$, we have:

$$\begin{aligned} y_j \geq \sum_{i,k} c_{ik}x_{ij}x_{jk} &\geq \sum_{i,k} (c_{il} - c_{kl})x_{ij}x_{jk} \\ &= \sum_{i,k} c_{il}x_{ij}x_{jk} - \sum_{i,k} c_{kl}x_{ij}x_{jk} \\ &= \sum_i c_{il}x_{ij} \sum_k x_{jk} - \sum_k c_{kl}x_{jk} \sum_i x_{ij} \\ &= \sum_i c_{il}x_{ij} - \sum_k c_{kl}x_{jk} \end{aligned}$$

This implies (14), even for non-integral values of x_{ij}.

For the other direction, integrality is required. From (1) it follows that we can choose a $k_j \in V$ for every $j \in V$ so that $x_{jk_j} = 1$ and $x_{jk} = 0$ if $k \neq k_j$. If we consider (14) for $l = k_j$, we get:

$$\begin{aligned} y_j &\geq \sum_i c_{ik_j}x_{ij} - \sum_k c_{kk_j}x_{jk} \\ &= \sum_i c_{ik_j}x_{ij}x_{jk_j} - c_{k_jk_j} \\ &= \sum_{i,k} c_{ik}x_{ij}x_{jk} \end{aligned}$$

which proves the proposition. □

Table 1: Numerical results the LAPTSP. We set $g'(x) = (1 + p^2q^2)g(x)$, which forms the generated lower bound. Time limit is one hour.

n	p	$g'(x)$	LB $g'(x)$	$f(x)$	gap $f(x)$ (%)	time (s)
10	0.98	1585.59	opt	1586.29	0.04	0.5
10	0.95	1565.69	opt	1570.09	0.28	0.2
10	0.90	1523.10	opt	1540.88	1.15	0.2
19	0.98	1911.65	opt	1912.68	0.05	0.7
19	0.95	1891.96	opt	1898.46	0.34	1.0
19	0.90	1846.66	opt	1873.04	1.41	1.0
28	0.98	2155.85	opt	2156.90	0.05	1.0
28	0.95	2127.90	opt	2134.53	0.31	1.1
28	0.90	2067.13	opt	2093.73	1.27	1.6
37	0.98	2395.57	opt	2396.63	0.04	5.4
37	0.95	2362.32	opt	2369.10	0.29	9.7
37	0.90	2293.50	opt	2321.43	1.20	39.4
46	0.98	2783.27	opt	2784.58	0.05	15.5
46	0.95	2746.35	opt	2754.65	0.30	43.3
46	0.90	2668.78	opt	2702.77	1.26	1178.4
55	0.98	3002.46	opt	3003.91	0.05	32.9
55	0.95	2964.71	opt	2973.87	0.31	94.3
55	0.90	2883.95	opt	2921.40	1.28	1999.9
64	0.98	3160.44	opt	3161.98	0.05	61.5
64	0.95	3119.93	opt	3129.70	0.31	311.8
64	0.90	3033.85	2971.08	3073.71	3.34	3603.9
73	0.98	3261.18	opt	3262.74	0.05	48.2
73	0.95	3216.54	opt	3226.45	0.31	213.7
73	0.90	3123.71	3048.68	3164.33	3.65	3604.4
82	0.98	3578.27	opt	3579.90	0.05	307.8
82	0.95	3527.57	opt	3537.97	0.30	3474.1
82	0.90	3421.63	3215.75	3462.96	7.14	3606.5
91	0.98	3682.04	opt	3683.83	0.05	279.5
91	0.95	3634.39	opt	3645.67	0.31	2200.8
91	0.90	3530.76	3296.66	3575.83	7.81	3606.4

Following the theorem, we consider the linear model LAPTSP:

$$\text{Min} \quad g(x) = q^2 \sum_{i,j} c_{ij} x_{ij} + q^2 p \sum_{j} y_j$$

$$(1), (2), (3), (14)$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in V, i \neq j \qquad (15)$$

5. Computational Results

Our test instances are sampled by randomly choosing points from $[0, 500] \times [0, 500]$. We consider the values $p = 0.98$, $p = 0.95$ and $p = 0.9$ for every node—meaning that no black nodes are involved. The instances are solved by the Gurobi 6.0 solver, using callbacks to generate the subtour elimination constraints. For our tests, we used a 3.4 GHz computer with 16GB RAM, coding our procedure in C$^\sharp$. The results are gathered in table 1.

We see that the model LAPTSP can be solved to (near) optimal values up to approximately 80 nodes. The runtime depends extremely on the parameter p. For $p = 0.95$, 91 nodes can be solved to optimality. The solution values x_{ij} of the LAPTSP can be directly use to compute the value $f(x)$. This value can subsequently be compared with the lower bound $(1 + p^2q^2)g(x)$. We see that for $p = 0.98$ we get the negligible gap of about 0.04%. For $p = 0.95$, the gap is about 0.30% which is a very good value. Those instances for $p = 0.9$ which solve to optimality lead to a gap of about 1.3% for $f(x)$.

6. Conclusion

The numerical results show that one can use the model LAPTSP to produce good primal solutions and strong lower bounds if p is large. Future research will address two issues: One the one hand, we aim to improve the lower bound to include better approximations of the term $f(x) - g(x)$. On the other hand, we aim to improve the solvability of the LAPTSP: Although the model is small in terms of variables and constraints, the LP relaxation is not very strong and has to be improved.

7. Acknowledgements

This research has been funded by the German Research Foundation (DFG) within the project “Lenkung des Güterflusses in durch Gateways gekoppelten Logistik-Service-Netzwerken mittels quadratischer Optimierung” (CL 318/14 and BU 2313/2)

References

[BBSD10] Prasanna Balaprakash, Mauro Birattari, Thomas Stützle, and Marco Dorigo. Estimation-based metaheuristics for the probabilistic traveling salesman problem. *Computers & operations research*, 37(11):1939–1951, 2010.

[Ber88] Dimitris Bertsimas. *Probabilistic combinatorial optimization problems*. PhD thesis, Massachusetts Institute of Technology, 1988.

[BFB03] Neill E Bowler, Thomas MA Fink, and Robin C Ball. Characterization of the probabilistic traveling salesman problem. *Physical Review E*, 68(3):036703, 2003.

[BG07] Leonora Bianchi and Luca Maria Gambardella. Ant colony optimization and local search based on exact and estimated objective values for the probabilistic traveling salesman problem. *Manno: IDSIA*, 2007.

[BGD02] Leonora Bianchi, Luca Maria Gambardella, and Marco Dorigo. An ant colony optimization approach to the probabilistic traveling salesman problem. In *Parallel Problem Solving from NaturePPSN VII*, pages 883–892. Springer, 2002.

[BH93] Dimitris Bertsimas and Louis H Howell. Further results on the probabilistic traveling salesman problem. *European Journal of Operational Research*, 65(1):68–95, 1993.

[BKB05] Leonora Bianchi, Joshua Knowles, and Neill Bowler. Local search for the probabilistic traveling salesman problem: correction to the 2-p-opt and 1-shift algorithms. *European Journal of Operational Research*, 162(1):206–219, 2005.

[Cam06] Ann Melissa Campbell. Aggregation for the probabilistic traveling salesman problem. *Computers & Operations Research*, 33(9):2703–2724, 2006.

[Fis13] Anja Fischer. *A Polyhedral Study of Quadratic Traveling Salesman Problems*. PhD thesis, TU Chemnitz, 2013. `http://nbn-resolving.de/urn:nbn:de:bsz:ch1-qucosa-118345`.

[Jai85] P Jaillet. *Probabilistic traveling salesman problem*. PhD thesis, Operations Research Center, Massachusetts Institute of Technology, Cambridge, MA, 1985. `http://web.mit.edu/jaillet/www/general/jaillet-phd-mit-orc-85.pdf`.

[Jai88] Patrick Jaillet. A priori solution of a traveling salesman problem in which a random subset of the customers are visited. *Operations research*, 36(6):929–936, 1988.

[Liu10] Yu-Hsin Liu. Different initial solution generators in genetic algorithms for solving the probabilistic traveling salesman problem. *Applied mathematics and computation*, 216(1):125–137, 2010.

[LLM94] Gilbert Laporte, Francois V Louveaux, and Hélene Mercure. A priori optimization of the probabilistic traveling salesman problem. *Operations Research*, 42(3):543–549, 1994.

[MC15] J. Fabian Meier and Uwe Clausen. Solving classical and new single allocation hub location problems on euclidean data. Technical report, Optimization Online, 2015. `http://www.optimization-online.org/DB_HTML/2015/03/4816.html`.

[MM10] Yannis Marinakis and Magdalene Marinaki. A hybrid multi-swarm particle swarm optimization algorithm for the probabilistic traveling salesman problem. *Computers & Operations Research*, 37(3):432–442, 2010.

[TMH04] Hao Tang and Elise Miller-Hooks. Approximate procedures for probabilistic traveling salesperson problem. *Transportation Research Record: Journal of the Transportation Research Board*, 1882(1):27–36, 2004.

[WMG14] Dennis Weyland, Roberto Montemanni, and Luca Maria Gambardella. An enhanced ant colony system for the probabilistic traveling salesman problem. In *Bio-Inspired Models of Network, Information, and Computing Systems*, pages 237–249. Springer, 2014.