Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming

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Abstract

We describe simple and exact duals, and certificates of infeasibility and weak infeasibility in conic linear programming which do not rely on any constraint qualification, and retain most of the simplicity of the Lagrange dual. In particular, some of our infeasibility certificates generalize the row echelon form of a linear system of equations, and the "easy" proofs – as sufficiency of a certificate to prove infeasibility – will be trivial.

For many cones of interest, we provide an algorithm to generate *all* infeasible conic LP instances. For semidefinite programs we provide an algorithm to generate *all* weakly infeasible instances in a natural class.

As a byproduct, we obtain some fundamental geometric corollaries: an exact characterization of when the linear image of a closed convex cone is closed; an exact characterization of nice cones; and bounds on the number of constraints that can be dropped from, or added to a (weakly) infeasible conic LP while keeping it (weakly) infeasible.

We generate a public domain library of infeasible and weakly infeasible semidefinite programs. The status of our instances is easy to verify by inspection in exact arithmetic, but they turn out to be challenging for commercial and research codes.

Key words: conic linear programs; semidefinite programming; certificates of infeasibility and weak infeasibility

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1 Introduction and Main Results

Conic linear programs generalize linear programming by replacing the nonnegative orthant by a closed convex cone. They model a wide variety of practical optimization problems, and inherit some of the duality theory of linear programming: the Lagrange dual provides a bound on their optimal value and a simple generalization of Farkas' lemma yields a proof of infeasibility.

However, strong duality may fail (i.e., the Lagrange dual may yield a positive gap, or not attain its optimal value), and the simple Farkas' lemma may fail to prove infeasibility. In particular, these pathologies occur in semidefinite programs (SDPs) and second order conic programs (SOCPs), arguably the most useful classes of conic LPs.

To ground our discussion, we consider a conic linear program of the form

$$\begin{array}{rl} \sup & \langle c, x\rangle \\ s.t. & Ax \leq_K b, \end{array} \tag{P}$$

where $A : \mathbb{R}^m \rightarrow Y$ is a linear map, Y is a finite dimensional euclidean space, $K \subseteq Y$ is a closed convex cone, and $s \leq_K t$ stands for $t - s \in K$. Letting A^* be the adjoint of A, and K^* the dual cone of K, the Lagrange dual of (P) is

$$\begin{array}{rlcl} \inf & \langle b, y\rangle & & \\ s.t. & A^* y & = & c \\ & y & \geq_{K^*} & 0. \end{array} \tag{D}$$

Weak duality – the inequality $\langle c, x\rangle \leq \langle b, y\rangle$ between a pair of feasible solutions – is trivial. However, the optimal values of (P) and of (D) may differ, and/or may not be attained.

A suitable conic linear system can prove the infeasibility of (P) or of (D). Since (for convenience) we focus mostly on infeasibility of (D), we state its alternative system below:

$$\begin{array}{rcl} Ax & \geq_K & 0 \\ \langle c, x\rangle & = & -1. \end{array} \tag{$D_{\rm alt}$}$$

When $(D_{\rm alt})$ is feasible, (D) is trivially infeasible, and we call it *strongly infeasible.* However, $(D_{\rm alt})$ and (D) may both be infeasible, and in this case we call (D) *weakly infeasible.* Thus $(D_{\rm alt})$ is not an exact certificate of infeasibility.

While a suitable constraint qualification (CQ), (as assuming that an interior feasible solution in (P)) can make (D), and $(D_{\rm alt})$ exact, such CQs frequently fail to hold in practice. Three known approaches, which we review in detail below, provide exact duals, and certificates of infeasibility for conic LPs: facial reduction algorithms – see Borwein and Wolkowicz [7], Waki and Muramatsu [25], Pataki [16]; extended duals for SDPs and generalizations – see Ramana [19], Klep and Schweighofer [9] and [16]; and elementary reformulation for SDPs – see Pataki [14] and Liu and Pataki [10]. For the connection of these approaches, see Ramana, Tunçel and Wolkowicz [20], [16] and [10].

The nonexactness of the Lagrange dual and of Farkas' lemma is caused by the possible nonclosedness of the linear image of K or a related cone. For related studies, see Bauschke and Borwein [3]; Borwein and Moors [4, 5] and Pataki [13].

Here we unify, simplify and extend these approaches and develop a robust calculus of exact duals, and certificates of infeasibility in conic LPs with the following features:

(1) They do not rely on a CQ, and inherit most of the simplicity of the Lagrange dual: some of our infeasibility certificates generalize the row echelon form of a linear system of equations, and the "easy" proofs, as weak duality, and the proofs of infeasibility and weak infeasibility are nearly as simple as proofs in linear programming duality (see Section 2 and 3). Some of our duals generalize the exact SDP duals of Ramana [19] and Klep and Schweighofer [9] to the context of general conic linear programming.

(2) They yield some fundamental geometric results in convex analysis, as an exact characterization of when the linear image of a closed convex cone is closed; and bounds on how many constraints can be dropped or added in a conic LP while keeping it (weakly) infeasible (see Section 5);

(3) They provide algorithms to generate *all* infeasible conic LP instances over several important cones (see Section 2 and 3), and *all* infeasible SDPs in a natural class (Section 6);

(4) The above algorithms are easy to implement, and provide a challenging test set of infeasible and weakly infeasible SDPs: while we can verify the status of our instances by inspection in exact arithmetic, they are difficult for commercial and research codes (Section 7).

(5) Of possible independent interest is an elementary facial reduction algorithm (Section 2) with a much simplified proof of convergence; and the geometry of the *facial reduction cone*, a cone that we introduce and use to encode facial reduction algorithms (see Lemma 1).

As motivation, we now describe our main tools, and some of our main results with full proofs of the "easy" directions. We will often reformulate a conic LP in a suitable form from which its status (as infeasibility) is easy to read off. This process is akin to bringing a matrix to row echelon form, and most of the operations we use indeed come from Gaussian elimination. To begin, we represent A and A^* as

$$Ax \;=\; \sum_{i=1}^{m} x_i a_i,\; A^*y \;=\; (\langle a_1, y\rangle, \dots, \langle a_m, y\rangle)^T, \text{ where } a_i \in Y \text{ for } i = 1, \dots, m.$$

Definition 1. *We obtain an* elementary reformulation *or* reformulation *of (P)-(D) by a sequence of the operations:*

(1) Replace (a_i, c_i) by $(A\lambda, \langle c, \lambda\rangle)$, where $\lambda \in \mathbb{R}^m$, $\lambda_i \neq 0$.

(2) Switch (a_i, c_i) with (a_j, c_j), where $i \neq j$.

(3) Replace b by $b + A\mu$, where $\mu \in \mathbb{R}^m$.

If $K = K^*$ we also allow the operation:

(4) Replace a_i by $Ta_i (i = 1, \dots, m)$ and b by Tb, where T is an invertible linear map with $TK = K$.

We call operations (1)-(3) *elementary row operations (eros)*. Sometimes we reformulate only (P) or (D), or only the underlying systems, ignoring the objective function. Clearly, a conic linear system is infeasible, strongly infeasible, etc., exactly when its elementary reformulations are.

Facial reduction cones "encode" a facial reduction algorithm, in a sense that we make precise later, and will replace the usual dual cone to make our duals and certificates exact.

Definition 2. *The order k facial reduction cone of K is the set*

$$\mathrm{FR}_k(K) = \{\, (y_1, \dots, y_k) : k \geq 0,\, y_i \in (K \cap y_1^{\perp} \cap \dots \cap y_{i-1}^{\perp})^*, i = 1, \dots k \,\}.$$

We drop the index k when its value is clear from context. Clearly $K^* = \mathrm{FR}_1(K) \subseteq \mathrm{FR}_2(K) \subseteq \dots$ holds. Surprisingly, $\mathrm{FR}_k(K)$ is convex, which is only closed in trivial cases, but behaves as well as the usual dual cone K^* under the usual operations on convex sets: e.g. any invertible linear transformation that preserves K also preserves $\mathrm{FR}_k(K)$ – see Lemma 1.

We now state some of our main results with full proofs of the "easy" directions:

Theorem I If K is a general closed convex cone, then

(1) (D) is infeasible, if and only if it has a reformulation

$$\begin{array}{rcl} \langle a_i', y\rangle & = & 0\,(i = 1, \dots, k) \\ \langle a_{k+1}', y\rangle & = & -1 \\ \langle a_i', y\rangle & = & c_i'\,(i = k+2, \dots, m) \\ y & \geq_{K^*} & 0 \end{array} \qquad (D_{\mathrm{ref}})$$

where $k \geq 0$, $(a_1', \dots, a_{k+1}') \in \mathrm{FR}(K^*)$.

(2) (D) is not strongly infeasible, if and only if there is $(y_1, \dots, y_{\ell+1}) \in \mathrm{FR}(K)$, such that

$$\begin{array}{rcl} A^* y_i & = & 0 \, (i = 1, \dots, \ell) \\ A^* y_{\ell+1} & = & c. \end{array}$$

$\square$

To see how Theorem I extends known results, first assume that K^* is the whole space, hence (D) is a linear system of equations. Then $\mathrm{FR}_{k+1}(K^*) = \{0\}^{k+1}$, and the constraint $\langle 0, y \rangle = -1$ in (D_{ref}) proves infeasibility. Hence – in a sense – (D_{ref}) generalizes the row echelon form of a linear system of equations. Part (1) also generalizes the alternative system (D_{alt}) : if $k = 0$ then $a_1' \in K$, and $(a_1', -1) = (Ax, \langle c, x \rangle)$ for some x, so (D) is strongly infeasible. Part (2) generalizes feasibility of (D): if $\ell = 0$ then (D) is actually feasible.

Also, the "if" directions are trivial:

Proof of if in part (1) We prove that (D_{ref}) is infeasible, so suppose that y is feasible in it to obtain the contradiction

$$y \in K^* \cap a_1'^{\perp} \cap \dots \cap a_k'^{\perp} \Rightarrow \langle a_{k+1}', y \rangle \geq 0.$$

Proof of if in part (2) We prove that (D) is not strongly infeasible, so suppose it is. Let x be feasible in (D_{alt}), and $(y_1, \dots, y_{\ell+1})$ as stated. Then

$$Ax \in K \cap R(A) \subseteq K \cap y_1^{\perp} \cap \dots \cap y_\ell^{\perp} \Rightarrow \langle Ax, y_{\ell+1} \rangle \geq 0,$$

which yields the contradiction

$$\langle Ax, y_{\ell+1} \rangle = \langle x, A^* y_{\ell+1} \rangle = \langle c, x \rangle = -1.$$

$\square$

We illustrate Theorem I with a semidefinite program, with $Y = \mathcal{S}^n$ the set of order n symmetric matrices and $K = K^* = \mathcal{S}_+^n$ as the set of order n positive semidefinite matrices. The inner product of $a, b \in \mathcal{S}^n$ is $a \bullet b := \langle a, b \rangle := \mathrm{trace}(ab)$ and we write $\preceq$ in place of $\leq_K$. Note that we denote the elements of $\mathcal{S}^n$ by small letters, and reserve capital letters for operators.

Example 1. The semidefinite system

$$\begin{array}{rcl} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet y & = & 0 \\ \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix} \bullet y & = & -1 \\ y & \succeq & 0 \end{array} \tag{1.1}$$

is infeasible for any $\alpha \geq 0$, and weakly infeasible exactly when $\alpha = 0$.

Since the constraint matrices are in $\mathrm{FR}(\mathcal{S}_+^2)$, we see that (1.1) is in the form of (D_{ref}) (and itself is a proof of infeasibility).

Suppose $\alpha = 0$ and let

$$y_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \, y_2 = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}.$$

Then $(y_1, y_2) \in \mathrm{FR}(\mathcal{S}_+^2)$, $A^*(y_1) = (0, 0)^T$, $A^*(y_2) = (0, -1)^T$, so (y_1, y_2) proves that (1.1) is not strongly infeasible.

Classically, the set of right hand sides that make (D) weakly infeasible, is the *frontier* of A^*K^* defined as the difference between A^*K^* and its closure. In this case

$$\operatorname{front}(A^*K^*) \, = \, \operatorname{cl} A^*\mathcal{S}^2_+ \setminus A^*\mathcal{S}^2_+ \, = \, \{ \, (0, \lambda) : \, \lambda \neq 0 \, \}$$

For all such right hand sides a suitable (y_1, y_2) proves that (1.1) is not strongly infeasible.

We organize the rest of the paper as follows. In the rest of the introduction we review prior work, collect notation,and record basic properties of the facial reduction cone $\operatorname{FR}_k(K)$. In Section 2 we present our simple facial reduction algorithm, and our exact duals of (P) and (D). The exact dual of (P) is an explicit conic linear program over $\operatorname{FR}(K)$ which does not rely on a CQ, nor a computation, and is nearly as simple, as (D), with the proof of weak duality essentially a tautology. It extends the SDP duals of Ramana [19], Klep and Schweighofer [9] to conic LPs over general closed convex cones. The exact dual of (D) is obtained by reformulating the dual system using elementary row operations.

In Section 3 we describe our exact certificates of infeasibility and weak infeasibility of general conic LPs. In Section 4 we describe the corresponding certificates for SDPs. In Section 5 presents our geometric corollaries: an exact characterization of when the linear image of a closed convex cone is closed, an exact characterization of *nice cones* ([15], Roshchina [23]), and bounds on how many constraints can be dropped from, or added to a (weakly) infeasible conic LP, while keeping its feasibility status. Note that when K^* (and K) is polyhedral, and (D) is infeasible, a single equality constraint obtained using eros and membership in K^* proves infeasibility (by Farkas' lemma). In the general case, the number of necessary constraints is related to the length of the longest chain of faces in K.

In Section 6 we define a natural class of weakly infeasible SDPs, and provide a simple algorithm to generate *all* instances in this class. In Section 7 we present a library of infeasible, and weakly infeasible SDPs, and our computational results.

Prior work We first review the approaches, which do not rely on a CQ, and provide exact duals and exact infeasibility certificates for (P) and (D). Facial reduction algorithms – see Borwein and Wolkowicz [7, 6], Waki and Muramatsu [25], Pataki [16] – construct a suitable smaller cone, say F, to replace K in (P), and to replace K^* by F^* (a larger cone) in (D).

Extended duals for semidefinite programs and generalizations – see Ramana [19], Klep and Schweighofer [9], Pataki [16] – use polynomially many extra variables and constraints. We note that Ramana's dual relies on convex analysis, while Klep and Schweighofer's uses ideas from algebraic geometry. The paper [16] generalizes Ramana's dual to the context of conic LPs over *nice* cones.

The approaches of facial reduction and extended duals are related – see Ramana, Tunçel and Wolkowicz [20], and [16].

Elementary reformulation for SDPs – see Pataki [14] and Liu and Pataki [10] – use simple operations, as elementary row operations, to bring a semidefinite system into a form from which its status (as infeasibility) is trivial to read off.

We refer to Lourenco et al [11] for an error-bound based reduction procedure to simplify weakly infeasible SDPs, and a proof that weakly infeasbible SDPs contain another such system whose dimension is at most $n - 1$. We will generalize this result in Theorem 9.

For recent studies on the closedness of the linear image of a closed convex cone we refer to Bauschke and Borwein [3]; Pataki [13]; and Borwein and Moors [4, 5] for proofs that the set of linear maps under which the image is *not* closed is small both in terms of measure and category.

Notation and preliminaries We assume throughout that the operator A is surjective. For x and y in the same Euclidean space we sometimes write x^*y for $\langle x, y \rangle$. For a convex set C we denote its linear span, the orthogonal complement of its linear span, its closure, and relative interior by $\operatorname{lin} C$, $C^\perp$, $\operatorname{cl} C$, and $\operatorname{ri} C$, respectively.

We define the dual cone of K as

$$K^* = \{ y \,|\, \langle y, x\rangle \geq 0, \, \forall x \in K \},$$

and for convenience we set

$$K^{*\setminus\perp} := K^* \setminus K^\perp.$$

We say that strong duality holds between (P) and (D) if their values agree and the latter is attained when finite. This is true when (P) is strictly feasible, i.e., when there is $x \in \mathbb{R}^m$ with $b - Ax \in \operatorname{ri} K$.

For F, a convex subset of K we say that F is a *face* of K, if $y, z \in K$, and $1/2(y+z) \in F$ implies $y, z \in F$.

Definition 3. If H is an affine subspace with $H \cap K \neq \emptyset$, then we call the smallest face of K that contains $H \cap K$ the minimal cone of $H \cap K$.

If F is the minimal cone of $H \cap K$ then $H \cap \operatorname{ri} F \neq \emptyset$ (otherwise $H \cap K$ would be contained in a proper face of F). So if F is the minimal cone of $(\mathcal{R}(A) + b) \cap K$, then replacing K by F in (P) makes (P) strictly feasible, and keeps its feasible set the same. So if we also replace K^* by F^* in (D) then strong duality will hold between (P) and (D).

The traditional alternative system of (P) is

$$\begin{array}{rcl} A^* y & = & 0 \\ b^* y & = & -1 \\ y & \geq_{K^*} & 0; \end{array} \tag{P_{alt}}$$

if it is feasible, then (P) is infeasible and we say that it is strongly infeasible.

For a nonnegative integer r we denote by $\mathcal{S}_+^r \oplus \{0\}$ the subset of $\mathcal{S}_+^n$ (where n will be clear from the context) with psd upper left r by r block, and the rest zero, and write

$$\mathcal{S}_+^r \oplus \{0\} = \begin{pmatrix} \oplus & 0 \\ 0 & 0 \end{pmatrix}, \, (\mathcal{S}_+^r \oplus \{0\})^* = \begin{pmatrix} \oplus & \times \\ \times & \times \end{pmatrix} \tag{1.2}$$

where the $\times$ stand for matrix blocks with arbitrary elements. All faces of $\mathcal{S}_+^n$ are of the form $t^T(\mathcal{S}_+^r \oplus 0)t$ where t is an invertible matrix [2, 12]. We write

$$\operatorname{Aut}(K) = \{ T : Y \to Y \,|\, T \, \text{is linear and invertible}, T(K) = K \} \tag{1.3}$$

for the automorphism group of a closed convex cone K.

Definition 4. *We say that $F_1, \ldots, F_k$ faces of K form a* chain of faces, *if $F_1 \supsetneq F_2 \supsetneq \cdots \supsetneq F_k$ and we write ℓ_K for the length of the longest chain of faces in K.*

For instance, $\ell_{\mathcal{S}_+^n} = \ell_{\mathbb{R}_+^n} = n + 1$.

In Lemma 1 we record relevant properties of $\operatorname{FR}_k(K)$. Its proof is given in Appendix A.

Lemma 1. *For $k \geq 0$ the following hold:*

(1) $\operatorname{FR}_k(K)$ *is a convex cone.*

(2) $\operatorname{FR}_k(K)$ *is only closed if K is a subspace or $k = 1$.*

(3) If $T \in \operatorname{Aut}(K)$, and $(y_1, \ldots, y_k) \in \operatorname{FR}_k(K)$ then

$$(Ty_1, \ldots, Ty_k) \in \operatorname{FR}_k(K).$$

(4) If C is another closed convex cone, then

$$\operatorname{FR}_k(K \times C) = \operatorname{FR}_k(K) \times \operatorname{FR}_k(C).$$

2 Facial reduction and strong duality in conic linear programs

In this section we present exact duals of (P) and of (D). To start, we first describe a very simple facial reduction algorithm to find $F,$ the minimal cone of the system

$$H \cap K,$$

where H is an affine subspace with $H \cap K \neq \emptyset$. While simple facial reduction algorithms are available ([16, 25]) the convergence proof of Algorithm 1, with an upper bound on the number of steps, is particularly simple.

We rely on the following classic theorem of the alternative (recall $K^{*\setminus\perp} = K^* \setminus K^\perp$).

$$H \cap \operatorname{ri} K = \emptyset \Leftrightarrow H^\perp \cap K^{*\setminus\perp} \neq \emptyset, \tag{2.4}$$

and the definition

Definition 5. *For* $k \geq 1$ *we say that* $(y_1, \dots, y_k) \in \mathrm{FR}_k(K)$ *is* strict*, if*

$$y_i \in (K \cap y_1^\perp \cap \dots \cap y_{i-1}^\perp)^{*\setminus\perp} \text{ for } i = 1, \dots, k.$$

We say that it is pre-strict *if* $(y_1, \dots, y_{k-1})$ *is strict.*

If $(y_1, \dots, y_k)$ is strict, then these vectors are linearly independent. Assuming that they are not, for some $1 \leq i \leq k$ a contradiction follows:

$$y_i \in \operatorname{lin}\{ y_1, \dots, y_{i-1} \} \subseteq (K \cap y_1^\perp \cap \dots \cap y_{i-1}^\perp)^\perp.$$

Algorithm 1 repeatedly applies (2.4) to find F :

Algorithm 1 Facial Reduction

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Initialization: Let $y_0 = 0$, $F_0 = K$, $i = 1$.
  while $\exists y_i \in H^\perp \cap F_{i-1}^{*\setminus\perp}$ do
    Choose such a $y_i$.
    Let $F_i = F_{i-1} \cap y_i^\perp$.
    Let $i = i + 1$.
  end while
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If $y_1, \dots, y_k$ are found by Algorithm 1, then $(y_1, \dots, y_k) \in \mathrm{FR}_k(K)$ and

$$F \subseteq K \cap y_1^\perp \cap \dots \cap y_k^\perp, \tag{2.5}$$

since the set on the right hand side is a face of K that contains $H \cap K$ (by $y_i \in H^\perp$), and F is the smallest such face.

Recall that ℓ_K is the length of the longest chain of faces in K (Definition 4).

Theorem 1. *Equality holds in (2.5) for some strict* $(y_1, \dots, y_k) \in \mathrm{FR}_k(K)$ *with*

$$k \leq \min\{ \ell_K - 1, \dim H^\perp \}.$$

Proof If we choose $(y_1, y_2, \dots)$ in Algorithm 1 to be strict, then the algorithm eventually stops. Suppose it stops after finding $y_1, \dots, y_k$ and for brevity define F_k as the set on the right hand side of (2.5). Then $\operatorname{ri} F_k \cap H \neq \emptyset$, and

$$\operatorname{ri} F_k \cap H = \operatorname{ri} F_k \cap H \cap K = \operatorname{ri} F_k \cap H \cap F,$$

so $\operatorname{ri} F_k \cap F \neq \emptyset$. So by Theorem 18.1 in [22] we obtain $F_k \subseteq F$ with the reverse containment already given. The upper bound on k follows from strictness and the linear independence of $y_1, \dots, y_k$. ☐

Definition 6. *The singularity degree of the system $H \cap K$ which we denote by $d(H, K)$ is the minimum number of facial reduction steps needed to find its minimal cone.*

When (P) (resp. (D)) are feasible, we define the minimal cone (degree of singularity) of (P) and of (D) as the minimal cone (degree of singularity) of the systems

$$(\mathcal{R}(A) + b) \cap K, \text{ and } (\mathcal{N}(A^*) + y) \cap K^*,$$

where $A^* y = c$. We write $d(P)$ and $d(D)$ for the singularity degrees.

Example 2. Let F be the minimal cone of the semidefinite system

$$x_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

Since all slack matrices are contained in $\mathcal{S}^1_+ \oplus 0$ and there is a slack matrix whose $(1, 1)$ element is positive, we have $F = \mathcal{S}^1_+ \oplus 0$. Algorithm 1 may output the sequence

$$\begin{aligned} y_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ F_1 = \begin{pmatrix} \multicolumn{2}{c}{\oplus} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ y_2 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ F_2 = F \end{aligned} \tag{2.7}$$

Note that the singularity degree of (2.6) is two.

From Theorem 1 we immediately obtain an extended strong dual for (P), described below in (D_{ext}). Note that (D_{ext}) is an explicit conic linear program whose data is the same as the data of (P), thus it extends the exact SDP duals of Ramana [19] and Klep and Schweighofer [9] to the context of general conic LPs. The underlying cone in (D_{ext}) is the facial reduction cone: thus, somewhat counterintuitively, we find an exact dual of (P) over a convex cone, which is not closed.

Theorem 2. *Let $k \geq d(P)$. Then the problem*

$$\begin{array}{lrcl} \inf & b^* y_{k+1} & & \\ s.t. & A^* y_{k+1} & = & c \\ & A^* y_i & = & 0 \, (i = 1, \dots, k) \\ & b^* y_i & = & 0 \, (i = 1, \dots, k) \\ & (y_1, \dots, y_{k+1}) & \in & \mathrm{FR}_{k+1}(K) \end{array} \tag{D_{ext}}$$

is a strong dual of (P). If $k = d(P)$ and the value of (D_{ext}) is finite, then it is attained for some pre-strict $(y_1, \dots, y_{k+1})$.

Proof We first prove weak duality. Suppose that x is feasible in (P) and $(y_1, \dots, y_{k+1})$ in (D_{ext}) then

$$\begin{aligned} \langle b, y_{k+1} \rangle - \langle c, x \rangle &= \langle b, y_{k+1} \rangle - \langle A^* y_{k+1}, x \rangle \\ &= \langle b - Ax, y_{k+1} \rangle \geq 0 \end{aligned}$$

where the last inequality follows from $b - Ax \in K \cap y_1^\perp \cap \dots \cap y_k^\perp$.

To prove the remaining statements let F be the minimal cone of (P) and suppose that (P) has a finite value v. As we said in the Introduction, if we replace K^* by F^* in (D), then strong duality holds between (P) and (D), so we can choose $y \in F^*$ to satisfy the affine constraints of (D) with $b^* y = v$. We have that

$$F = K \cap y_1^\perp \cap \dots \cap y_k^\perp$$

for some $(y_1, \dots, y_k) \in \mathrm{FR}_k(K)$, with all y_i in $(\mathcal{R}(A)+b)^\perp$ and this sequence can be chosen strict, if $k = d(P)$.

Hence $(y_1, \dots, y_k, 0, \dots, 0, y)$ (where the number of zeros is $k - d(P)$) is feasible in (D_{ext}) with value v. $\square$

$\square$

Example 3. (Example 2 continued) The SDP

$$\begin{array}{ll} \sup & x_1 \\ s.t. & (x_1, x_2) \, \text{is feasible in} \, (2.6) \end{array} \tag{2.8}$$

has a zero optimal value. Its usual SDP dual, in which we denote the dual matrix by y and its components by y_{ij}, is equivalent to

$$\begin{array}{ll} \inf & y_{11} \\ s.t. & \begin{pmatrix} y_{11} & 1/2 & -y_{22}/2 \\ 1/2 & y_{22} & y_{23} \\ -y_{22}/2 & y_{23} & y_{33} \end{pmatrix} \succeq 0. \end{array} \tag{2.9}$$

Problem (2.9) has an unattained 0 infimum, since we can choose y_{11} close to zero, if we make y_{22} and y_{33} large, but we cannot set y_{11} to zero.

If in the dual program we replace $\mathcal{S}^3_+$ by F^* then the new dual attains with

$$y := \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in F^* = \begin{pmatrix} \oplus & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$$

an optimal solution, and (y_1, y_2, y) where y_1, y_2 are defined in (2.7) is an optimal solution to the extended dual of (2.8).

Theorem 3. *If (D) is feasible then it has a strictly feasible reformulation*

$$\begin{array}{lrcl} \inf & b^*y & & \\ s.t. & \langle a_i', y \rangle & = & 0 \, (i = 1, \dots, k) \\ & \langle a_i', y \rangle & = & c_i' \, (i = k+1, \dots, m) \\ & y & \in & K^* \cap a_1'^\perp \cap \dots \cap a_k'^\perp, \end{array} \tag{$D_{\text{ref,feas}}$}$$

with $k \geq 0$, $(a_1', \dots, a_k') \in \mathrm{FR}(K^*)$, *which can be chosen strict.* $\square$

Proof Let G be the minimal cone of (D). By Theorem 1 there is $k \geq 0$ and a strict $(a_1', \dots, a_k') \in \mathrm{FR}(K^*)$ such that

$$\begin{array}{rcl} a_i' & \in & \mathcal{R}(A) \cap y^\perp \, (i = 1, \dots, k), \\ G & = & K^* \cap a_1'^\perp \cap \dots \cap a_k'^\perp. \end{array}$$

Let us write $a_i' = Az_i$, with $z_i \in \mathbb{R}^m$ for $i = 1, \dots, k$. Since $a_1', \dots, a_k'$ are linearly independent, so are $z_1, \dots, z_k$, so we can expand $z_1, \dots, z_k$ to

$$Z = [z_1, \dots, z_m] \text{ a basismatrix of } \mathbb{R}^m.$$

Replacing A by AZ yields the required reformulation, since

$$\langle z_i, c \rangle = \langle z_i, A^*y \rangle = \langle Az_i, y \rangle = \langle a_i', y \rangle = 0 \, (i = 1, \dots, k),$$

and this completes the proof. $\square$

We now contrast Theorem 2 with Theorem 3. In the former the minimal cone of (P) is

$$K \cap y_1^\perp \cap \cdots \cap y_k^\perp,$$

where $(y_1, \dots, y_k, y_{k+1})$ is feasible in $(D_{\rm ext})$. In the latter the minimal cone of (D) is displayed by simply performing elementary row operations on the constraints. To illustrate Theorem 3, we continue Example 2:

Example 4. (Example 2 continued) We can rewrite the feasible set of this example in an equality constrained form, and choose

$$\begin{array}{rcl} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet y & = & 0 \\ \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \bullet y & = & 0 \end{array}$$

to be among the constraints: these matrices form a sequence in $\mathrm{FR}(\mathcal{S}_+^3)$.

To find the y_i in Algorithm 1 one needs to solve conic linear programs, and if K is the semidefinite cone, one needs to solve SDPs, which may not be easier to solve than the original problem (P). To overcome this practical difficulty, Permenter and Parrilo in [17] presented an implementation of a "partial" facial reduction algorithm, where they solve linear programming approximations of the SDP subproblems.

3 Certificates of infeasibility and weak infeasibility in conic LPs

We now describe a collection of certificates of infeasibility and weak infeasibility of (P) and of (D) below in Theorem 4, which contains Theorem I. The idea is simple: the exact dual of (P) provides an exact certificate of infeasibility of (P) by homogenization, and the remaining certificates are found by using duality and elementary linear algebra.

Theorem 4. *When K is a general closed, convex cone, the following hold:*

(1) (D) is infeasible, if and only if it has a reformulation

$$\begin{array}{rcl} \langle a_i', y \rangle & = & 0 \, (i = 1, \dots, k) \\ \langle a_{k+1}', y \rangle & = & -1 \\ \langle a_i', y \rangle & = & c_i' \, (i = k+2, \dots, m) \\ y & \geq_{K^*} & 0 \end{array} \qquad (D_{\rm ref})$$

where $(a_1', \dots, a_{k+1}') \in \mathrm{FR}(K^*)$.

(2) (D) is not strongly infeasible, if and only if there is $(y_1, \dots, y_{\ell+1}) \in \mathrm{FR}(K)$, such that

$$\begin{array}{rcl} A^* y_i & = & 0 \, (i = 1, \dots, \ell) \\ A^* y_{\ell+1} & = & c. \end{array}$$

(3) (P) is infeasible, if and only if there is $(y_1, \dots, y_{k+1}) \in \mathrm{FR}(K)$ such that

$$\begin{array}{rclrcl} A^* y_i & = & 0, & b^* y_i & = & 0 \, (i = 1, \dots, k) \\ A^* y_{k+1} & = & 0, & b^* y_{k+1} & = & -1 \end{array}$$

(4) (P) *is not strongly infeasible, if and only if it has a reformulation*

$$\sum_{i=1}^{m} x_i a_i' \leq_K b' \qquad (P_{\text{ref}})$$

where $(a_1', \ldots, a_\ell', b') \in \text{FR}(K^*)$ *for some* $\ell \geq 0$.

In all parts the facial reduction sequences can be chosen to be pre-strict.

□

We note that parts (1) through (4) in Theorem 4 should be read separately: the k integers in parts (1) and (3), the ℓ in parts (2) and (4), etc. may be different. We use the current notation for brevity. Also note that since K is a general closed, convex cone, in the reformulations we only use elementary row operations.

If $k = 0$ in part (3) then (P) is strongly infeasible; and if $\ell = 0$ in part (4) then (P) is actually feasible. The reader can check that the "if" directions are all trivial.

Part (3) in Theorem 4 essentially follows from [25], though their infeasbility certificate is not stated as a conic linear system.

Note that Part (1) Theorem 4 allows us to generate all infeasible conic LP instances over cones, whose facial structure (and hence their facial reduction cone) is well understood: to do so, we only need to generate systems of the form (D_{ref}) and reformulate them. By Part (4) we can systematically generate all systems that are not strongly infeasible, though this seems less interesting.

Example 1 already illustrates parts (1) and (2). A larger example, which also depicts the frontier of A^*K^* with $K = K^* = \mathcal{S}_+^3$ follows.

Example 5. Let

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ a_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then one easily checks

$$\begin{array}{rcl} \text{cl}(A^*\mathcal{S}_+^3) & = & \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \\ \text{front}(A^*\mathcal{S}_+^3) \ = \ \text{cl}(A^*\mathcal{S}_+^3) \setminus A^*\mathcal{S}_+^3 & = & \{(0, \lambda, \mu) \,|\, \lambda \neq \mu \geq 0\}. \end{array}$$

The set $A^*\mathcal{S}_+^3$ is shown on Figure 1 in blue, and its frontier in green. Note that the diagonal piece inside the green frontier actually belongs to $A^*\mathcal{S}_+^3$.

To see how parts (1) and (2) certify that elements of $\text{front}(A^*\mathcal{S}_+^3)$ are indeed in this set, for concreteness, consider the system

$$\begin{array}{rcl} A^*(y) & = & (0, 1, 2)^T \\ y & \succeq & 0. \end{array} \qquad (3.10)$$

The operations: 1) multiply the second equation by 3 and 2) subtract twice the third equation from it, bring (3.10) into the form of (D_{ref}) and show that it is infeasible. The y_1 and y_2 below prove that it is not strongly infeasible:

$$y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ y_2 = \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 2 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}. \qquad (3.11)$$

To illustrate parts (3) and (4) in Theorem 4, we modify Example 2 by simply exchanging two constraint matrices.

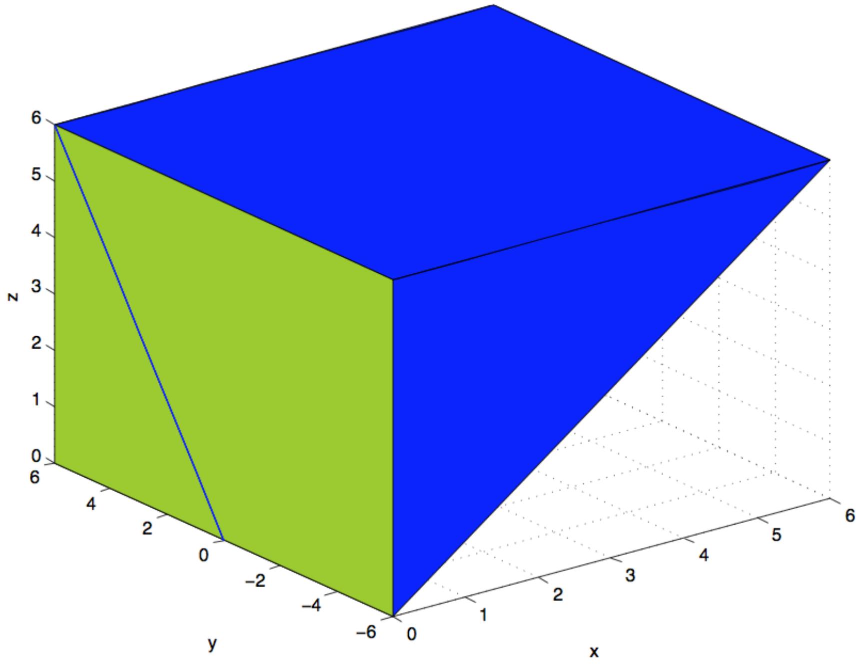

Figure 1: The set $A^*\mathcal{S}^3_+$ is in blue, and its frontier is in green

Example 6. (Example 2 continued) The semidefinite system below is weakly infeasible.

$$x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.12}$$

To prove it is infeasible, we use part (3) of Theorem 4 with $(y_1, y_2, y_3),$ where y_1, y_2 are given in (2.7) and

$$y_3 = \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix},$$

To prove it is not strongly infeasible, we use part (4). We write a_1, a_2, and b for the constraint matrices, and observe that $(a_1, b) \in \mathrm{FR}_2(\mathcal{S}^3_+)$, and is pre-strict, and $(a_1, a_2, b) \in \mathrm{FR}_3(\mathcal{S}^3_+)$.

Proof of (3) : Since the conic LP

$$\sup\{x_0 : Ax - bx_0 \leq_K 0\} \tag{3.13}$$

has value 0 iff (P) is infeasible, our claim follows from considering the strong dual of (3.13) from Theorem 2.

Proof of only if in (1) : Fix $y \in Y$ such that $A^*y = c$. By part (3) there is $k \geq 0$ and a pre-strict $(a_1', \dots, a_k', a_{k+1}') \in \mathrm{FR}(K^*)$ such that

$$\begin{aligned} a_i' &\in \mathcal{R}(A) \cap y^{\perp} \, (i = 1, \dots, k), \\ a_{k+1}' &\in \mathcal{R}(A), \, \langle a_{k+1}', y \rangle = -1. \end{aligned}$$

Since $(a_1', \dots, a_k', a_{k+1}')$ is pre-strict, $a_1', \dots, a_k'$ are linearly independent. Since $\langle a_{k+1}', y \rangle \neq 0$, also $a_1', \dots, a_k', a_{k+1}'$ are linearly independent. The proof now can be completed verbatim as the proof of Theorem 3.

Proof of only if in (2) Since (D) is not strongly infeasible, the alternative system $(D_{\rm alt})$ is infeasible. By Lemma 4 we deduce that

$$\mathrm{FR}_k(K \times \{0\}) = \mathrm{FR}_k(K) \times \mathbb{R}^{k+1} \,\text{holds for all}\, k \geq 0.$$

Combining this with part (3), there is a pre-strict $(y_1, \dots, y_{k+1}) \in \mathrm{FR}(K)$ and $(z_1, \dots, z_{k+1}) \in \mathbb{R}^{k+1}$ s.t.

$$\begin{array}{rclrcl} A^* y_i + c^* z_i & = & 0, & z_i & = & 0 \; (i = 1, \dots, k) \\ A^* y_{k+1} + c^* z_{k+1} & = & 0, & z_{k+1} & = & -1, \end{array}$$

so our claim follows.

Proof of (4) Since (P) is not strongly infeasible, the system $(P_{\rm alt})$ is infeasible, hence by part (1) it has a reformulation

$$\begin{array}{rcl} \langle a_i', y \rangle & = & 0 \, (i = 1, \dots, \ell) \\ \langle b', y \rangle & = & -1 \\ \langle a_i', y \rangle & = & c_i' \, (i = \ell + 1, \dots, m) \\ y & \geq_{K^*} & 0 \end{array} \tag{3.14}$$

with $(a_1', \dots, a_\ell', b') \in \mathrm{FR}(K^*)$ for some $\ell \geq 0$.

Since in $(P_{\rm alt})$ the only constraint with a nonzero right hand side is $\langle b, y \rangle = -1,$ we must have $b' = b + A\mu$ for some $\mu \in \mathbb{R}^m$. Since (3.14) is the alternative system of $(P_{\rm ref})$, the latter cannot be strongly infeasible. This completes the proof. ☐

4 Certificates of infeasibility and weak infeasibility in SDP

In this section we specialize the certificates of infeasibility and weak infeasibility of Section 3 to semidefinite programming. For this purpose we first introduce regularized facial reduction sequences in $\mathcal{S}^n_+$. These sequences have a certain staircase like structure and we will use them in Theorem 5, which is essentially obtained from Theorem 4 by replacing facial reduction sequences by regularized ones.

Definition 7. *The set of order k regularized facial reduction sequences for $\mathcal{S}^n_+$ is*

$$\mathrm{REGFR}_k(\mathcal{S}^n_+) \;=\; \Bigg\{ (y_1, \dots, y_k) : \, y_i \, = \begin{array}{c} \begin{array}{ccc} p_1 + \ldots + p_{i-1} & p_i & n - \sum_{j=1}^i p_j \end{array} \\ \left(\begin{array}{ccc} \times & \times & \times \\ \times & I & 0 \\ \times & 0 & 0 \end{array} \right) \end{array} \\ \text{where}\, p_i \geq 0, \; i = 1, \dots, k \Bigg\},$$

where the $\times$ symbols correspond to blocks with arbitrary elements. If the value of k is clear from the context, we simply write $\mathrm{REGFR}(\mathcal{S}^n_+)$.

Note that the constraint matrices in most examples actually form regularized facial reduction sequences. Clearly,

$$\mathrm{REGFR}(\mathcal{S}^n_+) \subseteq \mathrm{FR}(\mathcal{S}^n_+)$$

holds, and a sequence $(y_1, \dots, y_k) \in \mathrm{REGFR}(\mathcal{S}^n_+)$ with block sizes $p_1, \dots, p_k$ is strict, iff $p_1, \dots, p_k$ are positive.

The set $\mathrm{REGFR}_k(\mathcal{S}^n_+)$ is not convex. However, Lemma 2 below shows that any element of $\mathrm{FR}_k(\mathcal{S}^n_+)$ can be rotated to reside in $\mathrm{REGFR}_k(\mathcal{S}^n_+)$, i.e., $\mathrm{FR}_k(\mathcal{S}^n_+)$ is the *orbit* of $\mathrm{REGFR}_k(\mathcal{S}^n_+)$ under the automorphism group of $\mathcal{S}^n_+$. The proof of Lemma 2 is given in Appendix A.

Lemma 2. *Let* $(y_1, \dots, y_k) \in \operatorname{FR}(\mathcal{S}^n_+)$. *Then there is an invertible matrix* t *such that*

$$(t^T y_1 t, \dots, t^T y_k t) \in \operatorname{REGFR}(\mathcal{S}^n_+).$$

$\square$

The main result of this section follows.

Theorem 5. *When* $K = \mathcal{S}^n_+$, *the following hold:*

(1) (D) is infeasible, if and only if it has a reformulation

$$\begin{array}{rcl} \langle a'_i, y \rangle & = & 0 \, (i = 1, \dots, k) \\ \langle a'_{k+1}, y \rangle & = & -1 \\ \langle a'_i, y \rangle & = & c'_i \, (i = k+2, \dots, m) \\ y & \succeq & 0, \end{array} \tag{$D_{\text{ref,sdp}}$}$$

where $(a'_1, \dots, a'_{k+1}) \in \operatorname{REGFR}(\mathcal{S}^n_+)$.

(2) (D) is not strongly infeasible, if and only if it has a reformulation with data (A'', c'') *and* $(y_1, \dots, y_{\ell+1}) \in \operatorname{REGFR}(\mathcal{S}^n_+)$ *such that*

$$\begin{array}{rcl} A''^* y_i & = & 0 \, (i = 1, \dots, \ell) \\ A''^* y_{\ell+1} & = & c''. \end{array}$$

(3) (P) *is infeasible, if and only if it has a reformulation with data* (A', b') *and* $(y_1, \dots, y_{k+1}) \in \operatorname{REGFR}(\mathcal{S}^n_+)$ *such that*

$$\begin{array}{rclrcl} A'^* y_i & = & 0, & b'^* y_i & = & 0 \; (i = 1, \dots, k) \\ A'^* y_{k+1} & = & 0, & b'^* y_{k+1} & = & -1 \end{array}$$

(4) (P) *is not strongly infeasible, if and only if it has a reformulation*

$$\sum_{i=1}^m x_i a''_i \preceq b'' \tag{$P_{\text{ref,sdp}}$}$$

where $(a''_1, \dots, a''_\ell, b'') \in \operatorname{REGFR}(\mathcal{S}^n_+)$ *for some* $\ell \geq 0$.

In all parts the facial reduction sequences can be chosen as pre-strict. $\square$

We first note that parts (1) and (2) above should be read together, but separately from parts (3) and (4), and vice versa. (So the k integers in parts (1) and in part (3) may be different, and so on.) We use the double primes to emphasize that the reformulations in the first two and the last two parts are different.

Example 1 illustrates part (1) in Theorem 5, since the constraint matrices are in $\operatorname{REGFR}(\mathcal{S}^2_+)$. It also illustrates part (2), after we apply a trivial rotation on the constraint matrices:

Example 7. (Example 1 continued) After exchanging the first row and column in this example, and assuming $\alpha = 0$ we obtain the system

$$\begin{array}{rcl} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bullet y & = & 0 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bullet y & = & -1 \\ y & \succeq & 0. \end{array} \tag{4.15}$$

The fact that (4.15) is not strongly infeasible is proved by

$$y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ y_2 = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix},$$

and $(y_1, y_2) \in \text{REGFR}(\mathcal{S}^2_+)$.

Example 5 also illustrates part (2) in Theorem 5, since after a trivial rotation of the y_j in (3.11) they are in $\text{REGFR}(\mathcal{S}^3_+)$. Example 6 illustrates parts (3) and (4).

Proof of Theorem 5 To see part (1) we consider the reformulation given in part (1) of Theorem 4, and a t invertible matrix such that $(t^T a'_1 t, \dots, t^T a'_{k+1} t) \in \text{REGFR}(\mathcal{S}^n_+)$. We replace a'_i by $t^T a'_i t$ for all i and obtain $(D_{\text{ref,sdp}})$.

To see (2) we consider the sequence $(y_1, \dots, y_{\ell+1}) \in \text{FR}(\mathcal{S}^n_+)$ given by part (2) of Theorem 4, and a t invertible matrix such that

$$(t^T y_1 t, \dots, t^T y_{\ell+1} t) \in \text{REGFR}(\mathcal{S}^n_+).$$

For $i = 1, \dots, m$ and $j = 1, \dots, \ell + 1$ we have

$$\langle t^{-1} a_i t^{-T}, t^T y_j t \rangle \quad = \quad \langle a_i, y_j \rangle. \tag{4.16}$$

We set $a''_i := t^{-1} a_i t^{-T}$ and replace y_j by $t^T y_j t$ for all i and j and this completes the proof.

The proof of (3) is analogous to the proof of (2); and the proof of (4) to the proof of (1), hence we omit these. $\square$

Note that part (1) in Theorem 5 recovers Theorem 1 in [10]. Parts (2) and (4) are related to the recent paper of Lourenco et al [11]. The authors there show that if a semidefinite system, say of the form (P) is weakly infeasible, then a sequence $(a'_1, \dots, a'_\ell) \in \text{REGFR}_\ell(\mathcal{S}^n_+)$ with all a'_i can be found by taking linear combinations of the a_i and applying rotations.

In contrast, we exactly characterize systems that are *infeasble* and systems that are *not strongly infeasible.* Putting these parts together yields our geometric corollaries (in Section 5) and our algorithm to generate weakly infeasible SDPs (in Section 6).

5 Geometric corollaries

In this section we use the preceding results to address several fundamental questions in convex analysis. We begin by asking the question:

- Under what conditions is the linear image of a closed convex cone closed?

This question is fundamental, due to its role in constraint qualifications in convex programming. Due to its importance, Chapter 9 in Rockafellar's classic text [22] is entirely devoted to it: see e.g. Theorem 9.1 therein. Surprisingly, the literature on the subject (beyond [22] and other textbooks) appears to be scant. Bauschke and Borwein [3] gave a necessary and sufficient condition for the continuous image of a closed convex cone to be closed. Their condition (due to its greater generality) is more involved than Theorem 9.1 in [22]. See also [1], and the references in [13]. We refer to Borwein and Moors [4, 5] for proofs that the set of linear maps under which the image is not closed is small both in terms of measure and category.

For convenience we restate our question in an equivalent form:

- Given A and K, when is $A^* K^*$ closed?

In [13] we gave the very simple *necessary* condition

$$\mathcal{R}(A) \cap (\operatorname{cl} \operatorname{dir}(z, K) \setminus \operatorname{dir}(z, K)) = \emptyset, \tag{5.17}$$

for A^*K^* to be closed: here z is in the relative interior of $\mathcal{R}(A) \cap K,$ and $\operatorname{dir}(z, K)$ is the set of feasible directions at z in K. Note that (5.17) subsumes two seemingly unrelated classical sufficient conditions for the closedness of A^*K^*, as it trivially holds when K is polyhedral, or when $z \in \operatorname{ri} K$. It is also sufficient, when the set

$$K^* + F^{\perp}$$

is closed, where F is the minimal cone of $\mathcal{R}(A) \cap K$. Thus (5.17) becomes an exact characterization when $K^* + F^{\perp}$ is closed for *all* F faces of K. Such cones are called *nice*, and reassuringly, most cones that occur in optimization (such as polyhedral, semidefinite, and p-order cones) are nice.

Thus it is also of interest to characterize nice cones. To review previous results on nice cones we recall that $y \in K^*$ is said to *expose* the face $K \cap y^{\perp}$; a face G of K is said to be *exposed*, if it equals $K \cap y^{\perp}$ for some $y \in K^*$; and it is not exposed iff

$$K^* \cap G^{\perp} = K^* \cap F^{\perp}$$

for some F face of K that strictly contains G (i.e. all vectors that expose G actually expose a larger face). The cone K is said to be facially exposed if all of its faces are exposed.

For characterizations of nice cones, and a proof that they must be facially exposed, we refer to [15]; for an example of a facially exposed, but not nice cone, see [23]; and [8] for a proof that the linear pre-image of a nice cone is also nice.

As a byproduct of the preceding results, here we obtain an *exact* and simple characterization of when A^*K^* is closed when K is an arbitrary closed convex cone, and an exact characterization of nice cones.

We build on the following basic fact:

$$A^*K^* \text{ is not closed } \Leftrightarrow \ (D) \text{ is weakly infeasible for some } c. \tag{5.18}$$

Theorem 6. *The set* A^*K^* *is not closed, if and only if there is* $(a_1, \dots, a_{k+1}) \in \mathrm{FR}_{k+1}(K^*)$ *with* $k \geq 1$, *and* $(y_1, \dots, y_{\ell+1}) \in \mathrm{FR}_{\ell+1}(K)$ *with* $\ell \geq 1$ *such that*

$$\begin{array}{rcl} a_i & \in & \mathcal{R}(A)\,(i = 1, \dots, k+1), \\ y_j & \in & \mathcal{N}(A^*)\,(j = 1, \dots, \ell) \end{array} \tag{5.19}$$

and

$$\langle a_i, y_{\ell+1} \rangle \ = \left\{ \begin{array}{rl} 0 & \text{if } i \leq k \\ -1 & \text{if } i = k+1. \end{array} \right. \tag{5.20}$$

Proof Starting with the forward implication, we choose c such that (D) is weakly infeasible. We take $(a_1, 0), \dots, (a_k, 0), (a_{k+1}, -1)$ as constraints in a reformulation that proves infeasibility of (D), and $(y_1, \dots, y_{\ell+1})$ that proves that it is not strongly infeasible: cf. parts (1) and (2) in Theorem 4.

For the backward implication, fix $a := (a_1, \dots, a_{k+1})$ and $y := (y_1, \dots, y_{\ell+1})$ as stated. First we prove that they can be assumed to be pre-strict, so suppose that, say, a is not. Then

$$a_{i+1} \in (K \cap a_1^{\perp} \dots a_i^{\perp})^{\perp} \text{ for some } i < k.$$

Then $a_{i+1}^{\perp} \supseteq K \cap a_1^{\perp} \cdots \cap a_i^{\perp},$ so

$$K \cap a_1^{\perp} \cdots \cap a_i^{\perp} \ = \ K \cap a_1^{\perp} \cdots \cap a_i^{\perp} \cap a_{i+1}^{\perp},$$

so we can drop a_{i+1} from a while keeping all required properties of a and y. Continuing like this we arrive at both a and y being pre-strict, and to ease notation, we still assume $a \in \mathrm{FR}_{k+1}(K)$ and $y \in \mathrm{FR}_{\ell+1}(K^*)$.

Now $a_1, \ldots, a_k$ are linearly independent. Since $\langle a_{k+1}, y_{\ell+1} \rangle \neq 0$, so are $a_1, \ldots, a_k, a_{k+1}$.

Thus we can expand

$$A' = [a_1, \ldots, a_k, a_{k+1}, a_{k+2}, \ldots, a_m] \text{ a basis of } \mathcal{R}(A),$$

and let

$$c' = (0, \ldots, 0, -1, \langle a_{k+1}, y_{\ell+1} \rangle, \ldots, \langle a_m, y_{\ell+1} \rangle)^T.$$

Write $A' = TA$, with T an m by m invertible matrix, and let $c = T^{-1}c'$. Then (D) with this c is weakly infeasible (since it has a reformulation with data (A', c') proving infeasibility; and $(y_1, \ldots, y_{\ell+1})$ proving that it is not strongly infeasible: cf. Theorem 4). $\square$

Theorem 6 also characterizes when a cone is (not) nice:

Theorem 7. *Let F be a face of K. Then $K^* + F^\perp$ is not closed, if and only if there is $(a_1, \ldots, a_{k+1}) \in \mathrm{FR}_{k+1}(K^*)$ with $k \geq 1$, and $(y_1, \ldots, y_{\ell+1}) \in \mathrm{FR}_{\ell+1}(K)$ with $\ell \geq 1$ such that*

$$\begin{array}{rcl} a_i & \in & \operatorname{lin} F \, (i = 1, \ldots, k+1), \\ y_j & \in & F^\perp \, (j = 1, \ldots, \ell) \end{array} \tag{5.21}$$

and

$$\langle a_i, y_{\ell+1} \rangle = \begin{cases} 0 & \text{if } i \leq k \\ -1 & \text{if } i = k+1. \end{cases} \tag{5.22}$$

Proof The result follows from Theorem 6 by considering a linear operator A with $\mathcal{R}(A) = \operatorname{lin} F$, $\mathcal{N}(A^*) = F^\perp$ and noting that A^*K^* is not closed, iff $\mathcal{N}(A^*) + K^*$ is. $\square$

Theorems 6 and 7 provide a hierarchy of conditions, and it is natural to ask, how these relate to the simpler, but less general known conditions on closedness, and niceness. To address this question, we need a definition:

Definition 8. *We say that the nonclosedness of A^*K^* (of $K^* + F^\perp$) has an $(k+1, \ell+1)$-proof, if there is $(a_1, \ldots, a_{k+1})$ and $(y_1, \ldots, y_{\ell+1})$ as in Theorem 6 (Theorem 7).*

Theorem 8. *The following hold:*

(1) Suppose that condition (5.17) is violated, and let ℓ be the degree of singularity of $\mathcal{R}(A) \cap K$. Then there is an $(2, \ell+1)$-proof of the nonclosedness of A^*K^*.

(2) Suppose that K has a nonexposed face, say G, and F is the smallest exposed face of K that contains it. Then there is a $(2,2)$-proof that $K^* + F^\perp$ is not closed.

Since the proof of this result is somewhat technical, we defer it to Appendix B. t is also natural to ask, as to what values of k and ℓ are actually necessary to prove nonclosedness of A^*K^* (or of $K^* + F^\perp$). We will explore this in a followup paper.

Also, in recent work, Roshchina and Tunçel gave a condition to strengthen the facial exposedness condition of [15]: it would be interesting to see how their condition fits into our hierarchy.

We next turn to another basic question in the theory of conic LPs: given a (weakly) infeasible conic linear system

$$H \cap K, \tag{5.23}$$

where H is an affine subspace, what is the maximal/minimal dimension of an affine subspace H' with $H' \supseteq H$ (or $H' \subseteq H$) such that $H' \cap K$ has the same feasibility status as (5.23)?

Note that by (weak) infeasibility of (5.23) we mean (weak) infeasibility of a representation in either the primal (P) or the dual (D) form.

For instance, if K is polyhedral, and (5.23) is infeasible, then by Farkas' lemma we can take H' as an affine subspace defined by a single equality constraint. To further illustrate this question, consider the SDP

$$\begin{array}{rcl} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet y & = & 0 \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet y & = & -1, \end{array}$$

which is weakly infeasible. Dropping the first constraint keeps it weakly infeasible, and so does adding a constraint that fixes y_{12} to zero.

To state our main result, we recall that ℓ_K denotes the length of the longest chain of faces in K (see Definition 4).

Theorem 9. *The following hold.*

(1) If (5.23) is infeasible, then there is $H' \supseteq H$ such that

$$\operatorname{codim} H' \leq \ell_K \text{ and } H' \cap K \text{ is infeasible.}$$

(2) If (5.23) is not strongly infeasible, then there is $H'' \subseteq H$ such that

$$\dim H'' \leq \ell_K - 1 \text{ and } H'' \cap K \text{ is not strongly infeasible.}$$

(3) If (5.23) is weakly infeasible, then there is $H'' \subseteq H \subseteq H'$ as in parts (1) and (2) such that

$$H' \cap K \text{ and } H'' \cap K \text{ are both weakly infeasible.}$$

(4) If $K = K^*$ is the cone of psd matrices, then the bounds above can be tightened to

$$\operatorname{codim} H' \leq \ell_{\mathcal{S}^n_+} - 1 = n, \text{ and } \dim H'' \leq \ell_{\mathcal{S}^n_+} - 2 = n - 1.$$

Proof For part (1) we first represent (5.23) as a dual type problem (D) (with K in place of K^*), and apply part (1) of Theorem 4. We let H' be the affine subspace defined by the first $k+1$ constraints in (D_{ref}), and by pre-strictness of $a'_1, \ldots, a'_k$ we deduce

$$\dim H' = k + 1 \leq \ell_K - 1 + 1 = \ell_K,$$

as required. For part (2) we represent (5.23) as a primal type problem (P) and apply part in part (4) of Theorem 4. We let H'' be spanned by the first ℓ generators and the right hand side in (P_{ref}). By pre-strictness, we find

$$\ell \leq \ell_K - 1,$$

and this completes the proof.

For part (3) we choose H' as in part (1). Since (5.23) is not strongly infeasible, and $H' \supseteq H$, the system $H' \cap K$ is also not strongly infeasible. We construct H'' as in part (2) with an analogous justification.

For part (4) suppose that (5.23) is infeasible, represent it as a dual problem (with $\mathcal{S}^n_+$ in place of K^*) and apply part (1) of Theorem 5, by which in the reformulated system (D_{ref}) we can drop all but $k+1$ constraints while keeping it infeasible. We have

$$k \leq \ell_{\mathcal{S}^n_+} - 1 = n.$$

If $k < n,$ then there is nothing to prove. If $k = n,$ then a linear combination of $a_1', \dots, a_k'$ is positive definite, hence in this case a subsystem with only two constraints is infeasible. $\square$ $\square$

We note that the upper bound on the dimension of H'' in part (4) follows from [11].

6 Generating infeasible, and weakly infeasible SDPs

We now turn to a practical aspect of our work, generating infeasible, and weakly infeasible SDP instances. Having a library of such instances is important, since detecting infeasibility is a weak point of commercial and research codes: when they report this status, they also return a feasible solution to the alternative system (D_{alt}). When the instance is weakly infeasible, the returned certificate is necessarily inaccurate.

We first state an elementary algorithm, based on part (1) of Theorem 5, to generate infeasible SDPs. By Theorem 5 all infeasible SDPs are a reformulation of a a possible output of Algorithm 2. This algorithm may generate a strongly or a weakly infeasible SDP, and the latter outcome is likelier if k is small with respect to $m,$ but weak infeasibility is not guaranteed.

Algorithm 2 Infeasible SDP

```
Choose integers $m, n, k, p_1, \dots, p_k > 0$ and $p_{k+1} \geq 0$ s.t. $k + 1 \leq m, \sum_{i=1}^{k+1} p_i \leq n$.
Let $(a_1, \dots, a_{k+1}) \in \text{REGFR}_{k+1}(\mathcal{S}_+^n)$ with block sizes $p_1, \dots, p_{k+1}$ and $c_1 = \dots = c_k = 0, c_{k+1} = -1$.
Let $a_{k+2}, \dots, a_m \in \mathcal{S}^n$ and $c_{k+2}, \dots, c_m \in \mathbb{R}$ be arbitrary.
```

Next we turn to generating weakly infeasible SDP instances with a proof of weak infeasibility. We first note that Waki in [24] described a method to generate such SDPs from Lasserre's relaxation of polynomial optimization problems. In contrast, we will generate our instances by solving simple systems of equations. In fact, we will define a natural class of weakly infeasible SDPs, and show that a simple algorithm generates *all* instances in this class.

Although our framework is different – since we generate objects in a countably infinite set – our algorithms to generate *all* SDP instances in a certain class fit into the framework of *listing* combinatorial objects, as cycles, paths, spanning trees and cuts: see e.g., [21, 18].

We will use part (1) of Theorem 4 to find an infeasible instance, and part (2) to find a (y_j) sequence to prove that it is not strongly infeasible, so we will solve a *bilinear* system of equations over the (a_i) and (y_j). While this may be difficult in general, it is easy if we impose a structure: we will require that the (a_i) be regularized (cf. Definition 7), and that the (y_j) have the same structure, but "reversed" in the sense defined below:

Definition 9. *The set of reversed regularized facial reduction sequences in* $\mathcal{S}_+^n$ *is*

$$\text{REVREGFR}_\ell(\mathcal{S}_+^n) \quad = \quad \Bigg\{ (y_1, \dots, y_\ell) : \, y_i \; = \; \begin{array}{c} \begin{array}{ccc} n - \sum_{j=1}^{i} q_j & q_i & \sum_{j=1}^{i-1} q_j \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & \times \\ 0 & I & \times \\ \times & \times & \times \end{array} \right) \end{array} \\ \text{where}\, q_i \geq 0, \, i = 1, \dots, \ell \Bigg\},$$

where the $\times$ *symbols correspond to blocks with arbitrary elements. We drop the subscript, if its value is clear from the context.*

For instance, the y_i matrices in Example 1 are in $\text{REVREGFR}(\mathcal{S}_+^2)$.

Definition 10. *An SDP instance*

$$\begin{array}{rcl} A^*y & = & c \\ y & \succeq & 0 \end{array} \tag{6.24}$$

is nonoverlapping weakly infeasible*, if*

1. *(1) it is in the form (D_{ref}) as in part (1) of Theorem 4 with $(a_1, \dots, a_{k+1}) \in \mathrm{REGFR}(\mathcal{S}^n_+)$.*
2. *(2) There is $(y_1, \dots, y_{\ell+1}) \in \mathrm{REVREGFR}(\mathcal{S}^n_+)$ as in part (2) of Theorem 4 which proves it is not strongly infeasible;*
3. *(3) The block sizes p_i of $(a_1, \dots, a_{k+1})$ and the block sizes q_j of $(y_1, \dots, y_{\ell+1})$ satisfy*

$$\sum_{i=1}^{k+1} p_i + \sum_{j=1}^{\ell+1} q_j \leq n. \tag{6.25}$$

Note that condition (6.25) means that the identity blocks in the (a_i) and (y_j) sequences do not overlap. Example 1 is such an instance with $p_1 = q_1 = 1$ and $p_2 = q_2 = 0$.

A larger example follows:

Example 8. Letting

$$k = \ell = 2,\ p_1 = p_2 = p_3 = 1, q_1 = q_2 = 1,\ q_3 = 0,$$

the (a_i) and (y_j) below define a nonoverlapping weakly infeasible SDP. Some matrix entries are underlined, since we will return to this instance to explain our algorithm.

$$a_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ a_2 = \begin{pmatrix} 5 & 1 & 2 & \underline{2} & \underline{0} \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ \underline{2} & 0 & 0 & 0 & 0 \\ \underline{0} & 0 & 0 & 0 & 0 \end{pmatrix},\ a_3 = \begin{pmatrix} 3 & 2 & 1 & 3 & -2 \\ 2 & 0 & 0 & \underline{1} & \underline{1} \\ 1 & 0 & 1 & 0 & 0 \\ 3 & \underline{1} & 0 & 0 & 0 \\ -2 & \underline{1} & 0 & 0 & 0 \end{pmatrix}$$

$$y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & \underline{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \underline{1} & \underline{2} & 1 & 0 & 0 \end{pmatrix},\ y_3 = \begin{pmatrix} 0 & 0 & 0 & \underline{0} & 3 \\ 0 & 0 & 0 & \underline{1} & 5 \\ 0 & 0 & 0 & 4 & 1 \\ \underline{0} & \underline{1} & 4 & 1 & 2 \\ 3 & 5 & 1 & 2 & 3 \end{pmatrix}.$$

To proceed, for $(a_1, \dots, a_{k+1}) \in \mathrm{REGFR}(\mathcal{S}^n_+)$ with block sizes $p_1, \dots, p_{k+1}$ we denote the ith block containing p_i integers by P_i, i.e.,

$$P_1 = \{1, \dots, p_1\},\ p_2 = \{p_1 + 1, \dots, p_1 + p_2\}, \dots$$

For $(y_1, \dots, y_{\ell+1}) \in \mathrm{REVREGFR}(\mathcal{S}^n_+)$ with block sizes $q_1, \dots, q_{\ell+1}$ we similarly denote the jth block containing q_j integers by

$$Q_1 = \{n - q_1 + 1, \dots, n\},\ Q_2 = \{n - q_1 - q_2 + 1, \dots, n - q_1\}, \dots$$

For instance, in Example 8

$$P_1 = \{1\},\ P_2 = \{2\},\ P_3 = \{3\},\ Q_1 = \{5\},\ Q_2 = \{4\},\ Q_3 = \emptyset. \tag{6.26}$$

For $a \in \mathcal{S}^n$ and $P, Q \subseteq \{1, \dots, n\}$ we denote by $a(P, Q)$ the union of the block of a indexed by rows corresponding to P and columns corresponding to Q; and the block symmetric with it (i.e., rows indexed by Q and columns indexed by P.).

We are now ready to state our algorithm. The input of Algorithm 3 is $(a_1, \ldots, a_{k+1}) \in \mathrm{REGFR}(\mathcal{S}^n_+)$ and $(y_1, \ldots, y_{\ell+1}) \in \mathrm{REVREGFR}(\mathcal{S}^n_+)$ with block sizes p_i and q_j which satisfy inequality (6.25). We fix all entries of all a_i and of all y_j in advance, except we leave free the entries in

$$a_i(P_{i-1}, Q_1 \cup \cdots \cup Q_\ell) \,\text{and}\, y_j(Q_{j-1}, P_1 \cup \cdots \cup P_k). \tag{6.27}$$

The algorithm then sets the entries in these free blocks to satisfy the equations

$$a_i \bullet y_j \;=\; \begin{cases} 0 & \text{if } (i,j) \neq (k+1, \ell+1), \\ -1 & \text{if } (i,j) = (k+1, \ell+1). \end{cases} \tag{6.28}$$

This way we find the first $k+1$ equations in (D) and the last part of the algorithm generates the remaining $m-k-1$.

Algorithm 3 Nonoverlapping weakly infeasible SDP

```
for $j = 2 : (\ell + 1)$ do
    for $i = 2 : (k + 1)$ do
        (*) Set $a_i(P_{i-1}, Q_{j-1})$ and $y_j(P_{i-1}, Q_{j-1})$ to satisfy the equation for $a_i \bullet y_j$.
    end for
end for
Find $a_{k+2}, \ldots, a_m$ orthogonal to $y_1, \ldots, y_\ell$.
Set $c = (0, \ldots, 0, -1, a_{k+2} \bullet y_{\ell+1}, \ldots, a_m \bullet y_{\ell+1})^T$.
```

Algorithm 3 can generate Example 1 by starting with only the offdiagonal element of a_2 and y_2 free, then setting these to satisfy the equation $a_2 \bullet y_2 = -1$.

Algorithm 3 can also generate Example 8. It starts with the underlined entries free, and successively sets the entries in (note the definition of P_i and Q_j in (6.26))

(1) $a_2(P_1, Q_1)$ and $y_2(P_1, Q_1)$

(2) $a_3(P_2, Q_1)$ and $y_2(P_2, Q_1)$

(3) $a_2(P_1, Q_2)$ and $y_3(P_1, Q_2)$

(4) $a_3(P_2, Q_2)$ and $y_3(P_2, Q_2)$

Theorem 10. *Algorithm 3 always succeeds, and every nonoverlapping weakly infeasible instance is among its possible outputs.*

Proof To show that the algorithm always succeeds assume that at some point we execute Step (*). All previously satisfied equations which involve a_i have left hand side

$$a_i \bullet y_t \,\text{with}\, t \leq j-1.$$

Since for all such t we have

$$y_t(P_{i-1}, Q_{j-1}) = 0, \;\text{since}\, P_{i-1} \subseteq \{1, \ldots, n\} \setminus (Q_1 \cup \cdots \cup Q_{\ell+1}),$$

all these equations remain satisfied. Similarly, all previously satisfied equations that involve y_j remain true.

It is trivial to prove that all nonoverlapping instances are among the outputs: suppose that such an instance is identified by $(\bar{a}_1, \ldots, \bar{a}_m)$ and $(\bar{y}_1, \ldots, \bar{y}_{\ell+1})$ with $(\bar{a}_1, \ldots, \bar{a}_{k+1})$ having block sizes $p_1, \ldots, p_{k+1}$, and the $\bar{y}_j$ having block sizes q_j. Suppose that before we start Algorithm 3 we set all entries in $(a_1, \ldots, a_{k+1})$ and $(y_1, \ldots, y_{\ell+1})$ other than the ones in (6.27) to the corresponding values in the $(\bar{a}_i)$ and $(\bar{y}_j)$. Then there is a possible run of the algorithm which completes the a_i and y_j to be equal to the $\bar{a}_i$ and $\bar{y}_j$. $\square$

7 Computational experiments

To generate a test suite of challenging infeasible and weakly infeasible SDPs (in the dual form (D)) we implemented Algorithms 2 and 3 We ran Algorithm 2 with parameters

$$n = 10,\, k = 2,\, p_1 = 2, p_2 = 3, p_3 = 2,\; m = 10\,\text{or}\, m = 20, \tag{7.29}$$

and we call its outputs *infeasible* instances (these may be strongly or weakly infeasible). All entries in the generated instances are integers.

We ran Algorithm 2 with parameters

$$n = 10,\, k = 2,\, \ell = 1,\, r = (2,3,2),\, s = (2,1),\, m = 10\,\text{or}\, m = 20, \tag{7.30}$$

and we call the instances it generates *weakly infeasible.* (These are guaranteed to be weakly infeasible.) By choosing the components of the $a_1, \dots, a_{k+1}$ in the support of the y_j as integers in $[-2, 2]$ so the entries of $a_{k+2}, \dots, a_m$ and of y_1, y_2 turn out to be "near" integers with components in $\{0, \pm 1, \pm 1/2, \dots, \pm 1/7\}$.

Hence one can easily verify the status of our instances in exact arithmetic.

To generate instances, in which the structure proving (weak) infeasibility is less readily apparent, we add the optional

Messing step: Choose $t = (t_{ij}) \in \mathbb{Z}^{m \times m}$ and $v = (v_{ij}) \in \mathbb{Z}^{n \times n}$ random invertible matrices with entries in $[-2, 2]$ and let

$$a_i \;=\; v^T\Big(\sum_{j=1}^{m} t_{ij} a_j\Big) v \text{ for } i = 1, \dots, m.$$

The t matrix encodes elementary row operations performed on (D), and v encodes a rotation.

We call the instances output by Algorithms 2 and 3 *clean*, and the instances we find after the Messing step *messy.*

The choices: "clean/messy, infeasible/weakly infeasible, $m = 10/m = 20$" provide eight categories and we generated 100 instances in each. We set the objective function as I to ensure that the primal problem (P) is feasible.

We tested four solvers: we first ran the solvers Sedumi, SDPT3 and MOSEK from the YALMIP environment, and the preprocessing algorithm of Permenter and Parrilo [17] interfaced with Sedumi.

As the solvers consider our dual problem to be the primal, the only correct solution status ‘is ‘primal infeasible." We report the results in Tables 1 and 2. In these tables "PP+Sedumi" stands for the preprocessing algorithm of [17] interfaced with Sedumi.

	Infeasible		Weakly Infeasible	
	Clean	Messy	Clean	Messy
SEDUMI	87	27	0	0
SDPT3	10	5	0	0
MOSEK	63	17	0	0
PP+SEDUMI	100	27	100	0

Table 1: Result for instances with $n = 10, m = 10$

We can see that

	Infeasible		Weakly Infeasible	
	Clean	Messy	Clean	Messy
SEDUMI	100	100	1	0
SDPT3	100	96	0	0
MOSEK	100	100	11	0
PP+SEDUMI	0	100	0	0

Table 2: Result for instances with $n = 10, m = 20$

(1) The standalone solvers do better when m goes from 10 to 20 as for larger m the portion of strongly infeasible instances is likely to be higher.

(2) The standalone solvers mostly fail on the weakly infeasible problems, though MOSEK detects infeasibility of some. These are "almost" strongly infeasible, i.e., the alternative system (D_{alt}) is almost feasible. (Of course, in exact arithmetic (D_{alt}) is infeasible.)

(3) The preprocessing of Permenter and Parrilo considerably helps Sedumi when $m = 10$; and it is the only method to work consistently well on the weakly infeasible instances with $m = 10$. Somewhat surprisingly, it does not work, however, on the infeasible instances with $m = 20$.

Clearly, a preprocessing algorithm like [17] could easily scan for entire facial reduction sequences in the input, and it is likely that some of the instances coming from applications also contain such sequences.

The SDP instances are available from

`www.unc.edu/~pataki/SDP.zip`

8 Discussion and conclusion

Here we briefly discuss how some of our results can be further extended. First we note by Theorem 4 and part 4 in Theorem 1 we can write exact duals, and exact certificates of infeasibility for more involved conic linear systems. For instance, the system

$$\begin{array}{rcl} A_1 x & \leq_{K_1} & b_1 \\ A_2 x & \leq_{K_2} & b_2 \end{array} \tag{8.31}$$

(where K_1 and K_2 are closed convex cones) is infeasible iff there is $k \geq 0$ and $(y_1, \ldots, y_{k+1}) \in \mathrm{FR}_{k+1}(K_1^*)$ and $(z_1, \ldots, z_{k+1}) \in \mathrm{FR}_{k+1}(K_2^*)$ with

$$\begin{array}{rclrcl} A_1^* y_i + A_2^* z_i & = & 0, & b_1^* y_i + b_2^* z_i & = & 0 \ (i = 1, \ldots, k) \\ A_1^* y_{k+1} + A_2^* z_{k+1} & = & 0, & b_1^* y_{k+1} + b_2^* z_{k+1} & = & -1, \end{array}$$

and at least one of the y_j and z_j sequences can be chosen pre-strict.

Further, if the cone K is well-described – which is the case for all the cones over which one can efficiently optimize, as $\mathcal{S}_+^n$, polyhedral and p-order cones – then so is $\mathrm{FR}_k(K)$. We will say that K is a *smooth cone* if it is pointed, full-dimensional, and all faces distinct from $\{0\}$ and K itself are one-dimensional (i.e., extreme rays). For instance, the p-order cone

$$\{ (x_0, x) \mid x_0 \geq \| x \|_p \}$$

is a smooth cone, when p is not equal to 1 or ∞. The facial reduction cone $\mathrm{FR}(K)$ is trivial for such cones, and for their direct products, using part 4 of Theorem 1.

Therefore, we can easily generate all infeasible instances over direct products of smooth cones.

Appendix A: Proof of Lemmas 1 and 2

Proof of Lemma 1

Proof of (1) It is trivial that $\mathrm{FR}_k(K)$ contains all nonnegative multiples of its elements, so we only need to show that it is convex. To this end, we use the following Claim, whose proof is an easy exercise:

Claim If C is a closed, convex cone and $y, z \in C^*$, then

$$C \cap (y+z)^\perp = C \cap y^\perp \cap z^\perp.$$

We let $(y_1, \dots, y_k), (z_1, \dots, z_k) \in \mathrm{FR}_k(K)$, and for brevity, for $i = 1, \dots, k$ we set

$$\begin{array}{rcl} K_{y,i} & = & K \cap y_1^\perp \cap \cdots \cap y_i^\perp, \\ K_{z,i} & = & K \cap z_1^\perp \cap \cdots \cap z_i^\perp, \\ K_{y+z,i} & = & K \cap (y_1+z_1)^\perp \cap \cdots \cap (y_i+z_i)^\perp. \end{array}$$

We first prove that for $i = 1, \dots, k$ the relation

$$K_{y+z,i} \quad = \quad K_{y,i} \cap K_{z,i} \text{ holds.} \tag{A.32}$$

For $i = 1$ this follows from the Claim. Suppose now that (A.32) is true with $i-1$ in place of i. Then

$$y_i \in K_{y,i-1}^* \subseteq (K_{y,i-1} \cap K_{z,i-1})^* = K_{y+z,i-1}^*, \tag{A.33}$$

where the first containment is by definition, the inclusion is trivial, and the equality is by using the induction hypothesis. Analogously,

$$z_i \quad \in \quad K_{y+z,i-1}^*. \tag{A.34}$$

Hence

$$\begin{array}{rcl} K_{y+z,i} & = & K_{y+z,i-1} \cap (y_i+z_i)^\perp \\ & = & K_{y+z,i-1} \cap y_i^\perp \cap z_i^\perp \\ & = & K_{y,i-1} \cap K_{z,i-1} \cap y_i^\perp \cap z_i^\perp \\ & = & K_{y,i} \cap K_{z,i}, \end{array}$$

where the first equation is trivial. The second follows since by (A.33) and (A.34) we can use the Claim with $C = K_{y+z,i-1}$, $y = y_i$, $z = z_i$. The third is by the inductive hypothesis, and the last is by definition. This completes the proof of (A.32).

Now we use (A.33), (A.34) and the convexity of $K_{y+z,i-1}^*$ to deduce that

$$y_i + z_i \in K_{y+z,i-1}^* \text{ holds for } i = 1, \dots, k.$$

This completes the proof of (1).

Proof of (2) Let $L = K \cap -K$, assume $K \neq L$, and $k \geq 2$. Let $\{y_{1i}\} \subseteq \mathrm{ri}\, K^*$, s.t. $y_{1i} \to 0$. Then

$$\begin{array}{rcccrcl} K \cap y_{1i}^\perp & = & L, & \Rightarrow & (K \cap y_{1i}^\perp)^* & = & L^\perp \\ K \cap 0^\perp & = & K & \Rightarrow & (K \cap 0^\perp)^* & = & K^*. \end{array}$$

Let $y_2 \in L^\perp \setminus K^*$. (Such a y_2 exists, since $K^* \neq L^\perp$.) Then $(y_{i1}, y_2, 0, \dots, 0) \in \mathrm{FR}_k(K)$, and it converges to $(0, y_2, 0, \dots, 0) \notin \mathrm{FR}_k(K)$.

Proof of (3) Let us fix $T \in \mathrm{Aut}(K)$ and let S be an arbitary set. Then we claim that

$$
\begin{aligned}
(TS)^* &= T^{-1}S^*, && \text{(A.35)}\\
(TS)^{\perp} &= T^{-1}S^{\perp}, && \text{(A.36)}\\
(K \cap (TS)^{\perp})^* &= T(K \cap S^{\perp})^* && \text{(A.37)}
\end{aligned}
$$

hold. The first two statements are an easy calculation, and the third follows by

$$
\begin{aligned}
(K \cap (TS)^{\perp})^* &= (K \cap T^{-1}S^{\perp})^* \\
(A.36)) &= (T^{-1}(K \cap S^{\perp}))^* \\
&= T(K \cap S^{\perp})^*,
\end{aligned}
$$

where in the first equation we used (A.36), in the second equation we used $T^{-1}K = K$ and in the last we used (A.35).

Now let $(y_1, \dots, y_k) \in \mathrm{FR}_k(K)$, and $S_i = \{y_1, \dots, y_{i-1}\}$ for $i = 1, \dots, k$. Then by definition we have $y_i \in (K \cap S_i^{\perp})^*$ and (A.37) implies

$$
Ty_i \in (K \cap (TS_i)^{\perp})^*,
$$

which completes the proof.

Proof of (4) We prove that

$$
\big((y_1, z_1), \dots, (y_{k+1}, z_{k+1})\big) \in FR_k(K \times C) \tag{A.38}
$$

if and only if

$$
\big(y_1, \dots, y_{k+1}\big) \in \mathrm{FR}_k(K)\,\text{and}\,\big(z_1, \dots, z_{k+1}\big) \in \mathrm{FR}_k(C). \tag{A.39}
$$

The equivalence is trivial for $k = 0$ so let us assume that $k \geq 1$ and we proved it for $0, \dots, k-1$. Statement (A.38) is equivalent to

$$
\big((y_1, z_1), \dots, (y_k, z_k)\big) \in FR_{k-1}(K \times C)
$$

and

$$
(y_{k+1}, z_{k+1}) \in \big((K \times C) \cap (y_1, z_1)^{\perp} \cap \dots \cap (y_k, z_k)^{\perp}\big)^*. \tag{A.40}
$$

By the inductive hypothesis the set on the right hand side of (A.40) is

$$
(K \cap y_1^{\perp} \cap \dots \cap y_k^{\perp})^* \times (C \cap z_1^{\perp} \cap \dots \cap z_k^{\perp})^*,
$$

and this completes the proof.

Proof of Lemma 2 First let us note that $T \in \mathrm{Aut}(\mathcal{S}_+^n)$ (cf. equation (1.3)) iff $T(x) = t^T x t$ for some invertible matrix t.

Suppose that $\ell \geq 0$ is an integer, and we computed a t invertible matrix such that

$$
\begin{aligned}
(t^T y_1 t, \dots, t^T y_k t) &\in \mathrm{FR}(\mathcal{S}_+^n), && \text{(A.41)}\\
(t^T y_1 t, \dots, t^T y_\ell t) &\in \mathrm{REGFR}(\mathcal{S}_+^n), && \text{(A.42)}
\end{aligned}
$$

and the block sizes in the latter sequence are $p_1, \dots, p_\ell$, respectively. If $\ell = k$, we stop.

Otherwise, define $p := p_1 + \dots + p_\ell$ and $y_i' := t^T y_i t$ for $i = 1, \dots, k$. Let

$$
K = \mathcal{S}_+^n \cap y_1'^{\perp} \cap \dots \cap y_\ell'^{\perp}.
$$

Then K and K^* are of the form

$$K = \begin{array}{c} \begin{array}{cc} p & n-p \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & \oplus \end{pmatrix} \end{array}, \; K^* = \begin{array}{c} \begin{array}{cc} p & n-p \end{array} \\ \begin{pmatrix} \times & \times \\ \times & \oplus \end{pmatrix} \end{array}$$

Let z be the lower $n-p$ by $n-p$ block of $y'_{\ell+1}$. Since z is psd, there is a q invertible matrix such that

$$q^T zq = \begin{pmatrix} I_{p_{\ell+1}} & 0 \\ 0 & 0 \end{pmatrix},$$

where $p_{\ell+1}$ is the rank of z.

Let $v := I_p \oplus q$ and replace t by tv. Then by part (3) in Theorem 1 statement (A.41) still holds, and by the choice of v equation (A.42) now holds with $\ell+1$ in place of ℓ. ☐

Appendix B: Proof of Theorem 8

Proof of (1) We assume that condition (5.17) is violated, let

$$\begin{array}{rcl} a_1 & \in & \mathrm{ri}(\mathcal{R}(A) \cap K), \\ a_2 & \in & \mathcal{R}(A) \cap (\mathrm{cl}\,\mathrm{dir}(a_1, K) \setminus \mathrm{dir}(a_1, K)), \end{array}$$

and let F the minimal cone of $\mathcal{R}(A) \cap K$ (i.e., the smallest face of K that contains a_1). Then

$$(K^* \cap a_1^\perp)^* = (K^* \cap F^\perp)^* = \mathrm{cl}\,\mathrm{dir}(a_1, K),$$

where the first equality comes from $a_1 \in \mathrm{ri}\, F$ and the second can be found e.g., in [12]. Hence

$$\begin{array}{rcl} a_1, a_2 & \in & \mathcal{R}(A) \\ (a_1, a_2) & \in & \mathrm{FR}(K^*) \end{array}$$

holds. To construct the y_j we recall that ℓ is the degree of singularity of $\mathcal{R}(A) \cap K$, so

$$F = K \cap y_1^\perp \cap \cdots \cap y_\ell^\perp$$

for some $(y_1, \dots, y_\ell) \in \mathrm{FR}(K)$ with all y_j in $\mathcal{N}(A^*)$. Since $a_2 \notin \mathrm{lin}\, F$ (otherwise a_2 would be in $\mathrm{dir}(a_1, K)$) we can choose $y_{\ell+1} \in F^* \perp$ such that

$$\begin{array}{rcl} \langle a_1, y_{\ell+1} \rangle & = & 0 \\ \langle a_2, y_{\ell+1} \rangle & = & -1 \end{array}$$

hold, and this completes the proof. ☐

Proof of (2) For brevity we define $F^\triangle = K^* \cap F^\perp$ and for a face H of K^* we define $H^\triangle = K \cap H^\perp$. Our assumption means:

$$F^\triangle = G^\triangle \,\mathrm{and}\, F^{\triangle\triangle} = F.$$

We first choose y_1 and y_2 with the properties

$$y_1 \in F^\perp, \tag{B.43}$$

$$(y_1, y_2) \in \mathrm{FR}(K). \tag{B.44}$$

To do this, we choose $y_1 \in \operatorname{ri} F^{\triangle},$ and $y_2 \in (F^* \cap G^{\perp}) \setminus F^{\perp}$ (this is doable, since $G \subsetneq F$). Then

$$K \cap y_1^{\perp} = K \cap (F^{\triangle})^{\perp} = F^{\triangle\triangle} = F,$$

where the first equation comes from $y_1 \in \operatorname{ri} F^{\triangle},$ and the last from our assumption. Hence property (B.44) follows.

To complete the construction we will define a_1 and a_2 to satisfy the conditions

$$\begin{aligned} a_1, a_2 &\in \operatorname{lin} F, && \text{(B.45)} \\ (a_1, a_2) &\in \operatorname{FR}(K^*), && \text{(B.46)} \\ \langle a_2, y_2 \rangle &= -1. && \text{(B.47)} \end{aligned}$$

To do so, we choose $a_1 \in \operatorname{ri} G,$ and $a_2 \in \operatorname{lin} F$ to satisfy (B.47) (this can be done since $y_2 \notin F^{\perp}$) and claim that they then satisfy (B.46). To see this, we observe

$$K^* \cap a_1^{\perp} \;=\; K^* \cap G^{\perp} \;=\; K^* \cap F^{\perp},$$

where the first equation follows from $a_1 \in \operatorname{ri} G,$ and the second from $F^{\triangle} = G^{\triangle}$. Hence

$$(K^* \cap a_1^{\perp})^* \;=\; (K^* \cap F^{\perp})^* \;\supseteq\; \operatorname{lin} F,$$

so (B.46) follows, and this completes the proof. ☐

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