GENERIC PROPERTIES FOR SEMIALGEBRAIC PROGRAMS

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ABSTRACT. In this paper we study genericity for the following parameterized class of nonlinear programs:

$$\text{minimize } f_u(x) := f(x) - \langle u, x \rangle \quad \text{subject to} \quad x \in S,$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial function and $S \subset \mathbb{R}^n$ is a closed semialgebraic set, which is not necessarily compact. Assume that the constraint set S is regular. It is shown that there exists an open and dense semialgebraic set $\mathscr{U} \subset \mathbb{R}^n$ such that for any $\bar{u} \in \mathscr{U}$, if the corresponding function $f_{\bar{u}}$ is bounded from below on S, then for all vectors $u \in \mathbb{R}^n$, sufficiently close to $\bar{u}$, the problem $\min_{x \in S} f_u(x)$ has the following properties: the objective function f_u is coercive on the constraint set S, there is a unique optimal solution, lying on a unique active manifold, and for which the strong second-order sufficient conditions, the quadratic growth condition, and the global sharp minima hold. Further, the active manifold is constant, and the optimal solution and the optimal value function vary analytically under local perturbations of the objective function. As a consequence, for almost all polynomial optimization problems, we can find a natural sequence of computationally feasible semidefinite programs, whose solutions give rise to a sequence of points in $\mathbb{R}^n$ converging to the optimal solution of the original problem.

1. INTRODUCTION

Genericity is the most desired property one expects when dealing with mathematical programming problems (see, for example, [5, 19, 31]).

The idea to study mathematical programming problems from the generic point of view goes back to the investigation of Saigal and Simon [36] for the complementarity problem. The studies of generic strict complementarity and primal and dual nondegeneracy for semidefinite programming by Alizadeh et al. [1] and Shapiro [39]. The study of generic properties of general conic convex programs was given by Pataki and Tunçel [35].

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Generic optimality conditions for parameterized classes of nonlinear programming problems are treated by Spingarn and Rockafellar [41] (see also [11, 20, 42]). The authors were demonstrated that, almost all problems in the class are such that the strong second-order sufficient conditions (the linear independence of gradients vectors of active constraints, the Karush–Kuhn–Tucker conditions with strict complementary slackness, and positive definiteness of the Hessian of the Lagrangian on the subspace perpendicular to the gradients of the active constraints) hold at every local minimizer.

In the paper [12], assuming compactness of the constraint set, Fujiwara showed generically that a regular program has a unique global solution and the global optimal value is twice continuously differentiable with respect to the function space of objective functions and constraints.

Note that "generic" in the above-mentionned results means that a given property holds for almost all problems, in the sense of Lebesgue measure.

It was shown in [17] by Hà and the second author that almost every linear objective function, which is bounded from below on a closed semialgebraic set, attains its infimum and has the same asymptotic growth at infinity.

Recently, Bolte et al. proved in [4] that any given fixed nonempty, compact and convex semialgebraic set, corresponding to a generic linear objective function is a unique optimal solution, lying on a unique active manifold, and for which partly smooth second-order sufficient optimality conditions hold. Further, the optimal solution varies smoothly on the manifold under local perturbations of the objective function.

In the very recent paper [9], Đoạt et. al. allow a more general perturbation to the semialgebraic program but prove a result instead about "well-posedness" (Dontchev and Zolezzi [8]).

The purpose of this paper is to study genericity in semialgebraic programs, where constraint sets are not necessarily compact. This investigation is a continuation of our recent work [24], where the case of compact constraint sets is treated. It is worth emphasizing that "generic" in the semialgebraic context means that a given property holds on a set that is dense and open rather than just full measure.

In order to formulate the results, let $f, g_1, \ldots, g_l, h_1, \ldots, h_m$ be real polynomials on $\mathbb{R}^n$, and define the set

$$S := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \ldots, g_l(x) = 0,\ h_1(x) \geq 0, \ldots, h_m(x) \geq 0\},$$

which is a closed semialgebraic set. For each parameter $u \in \mathbb{R}^n$ we define the corresponding polynomial f_u by

$$f_u(x) := f(x) - \langle u, x \rangle \quad \text{for} \quad x \in \mathbb{R}^n.$$

We will establish generic properties for the following parametrized class of nonlinear programs:

$$\phi(u) \quad := \quad \text{minimize } f_u(x) \quad \text{subject to} \quad x \in S. \tag{1}$$

Precisely, with the definitions in the next sections, the main result of this paper is as follows.

Theorem A. *Assume that the constraint set S is regular. There exists an open and dense semialgebraic set $\mathscr{U} \subset \mathbb{R}^n$ such that for any $\bar{u} \in \mathscr{U}$, if the corresponding objective function $f_{\bar{u}}$ is bounded from below on S, then there exist positive constants $\epsilon, R,$ and $c_i, i = 1, \ldots, 3,$ such that for any $u \in \mathbb{R}^n$, with $\|u - \bar{u}\| < \epsilon$, we have $u \in \mathscr{U}$ and the following statements satisfy:*

(i) [Uniform coercivity] *For all $x \in S$, with $\|x\| \geq R$, it holds that*

$$f_u(x) := f(x) - \langle u, x \rangle \quad \geq \quad c_1\|x\|.$$

In particular, f_u is coercive on S.

(ii) [Existence, uniqueness, and optimality conditions] *The problem $\min_{x \in S} f_u(x)$ has a unique minimizer $x(u) \in S$ for which the strong second-order sufficient conditions hold.*

(iii) [Analyticity of the optimal solution] *The corresponding*

$$\{u \in \mathbb{R}^n \mid \|u - \bar{u}\| < \epsilon\} \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m, \quad u \mapsto (x(u), \lambda(u), \nu(u)),$$

is an analytic map, where $\lambda(u) \in \mathbb{R}^l$ and $\nu(u) \in \mathbb{R}^m$ are (unique) Lagrange multipliers with respect to $x(u)$.

(iv) [Analyticity of the optimal value function] *The function*

$$\phi \colon \{u \in \mathbb{R}^n \mid \|u - \bar{u}\| < \epsilon\} \to \mathbb{R}, \quad u \mapsto \min_{x \in S} f_u(x),$$

is analytic and $\nabla\phi(u) = -x(u)$.

(v) [Local constancy of the set of active constraint indices] *The set of active constraint indices is locally constant:*

$$\{j \mid h_j(x(u)) = 0\} \quad = \quad \{j \mid h_j(x(\bar{u})) = 0\}.$$

(vi) [Uniform quadratic growth condition] *We have for any $x \in S$, with $\|x - x(u)\| \leq R$,*

$$f_u(x) - f_u(x(u)) \quad \geq \quad c_2\|x - x(u)\|^2.$$

(vii) [Uniform and global sharp minima] *For any $x \in S$, the following inequality holds*

$$[f_u(x) - f_u(x(u))] + [f_u(x) - f_u(x(u))]^{\frac{1}{2}} \quad \geq \quad c_3\|x - x(u)\|.$$

(viii) [Convergence of all minimizing sequences] *For any sequence $\{u^\ell\} \subset \mathbb{R}^n$ converging to $\bar{u}$, $\inf_{x \in S} f_{u^\ell}(x)$ is finite for large ℓ and any sequence $\{x^\ell\} \subset S$ such that $f_{u^\ell}(x^\ell) - \inf_{x \in S} f_{u^\ell}(x) \to 0$ converges to $x(\bar{u})$.*

Next let us consider the problem of computing numerically the optimal value of Problem (1). As is well-known, this is an NP-hard problem even when the degree of the polynomial f_u is fixed to be four [27]. For instance, Problem (1) contains *the partition problem* which is known to be NP-complete [13]. A standard approach for solving Problem (1) is the hierarchy of semidefinite program relaxations proposed by Lasserre [21] (see also [32, 33, 40]). It is based on results about moment sequences and (the dual theory of) representations of nonnegative polynomials as sums of squares. For details about these methods and their applications, see [6, 15, 16, 17, 18, 21, 22, 23, 25, 26, 28, 34, 38].

In the case the constraint set S is compact, Nie showed in [30] that Lasserre's hierarchy has finite convergence generically.

In the papers [6, 17, 26, 28, 29], the authors have proposed semidefinite program relaxations for finding the optimal value of Problem (1), under the assumption that the objective function f_u attains its optimal value. This assumption is non-trivial and the question of how to verify if a given polynomial has this property is important and difficult (see [28, Section 7]). As an application of Theorem A, it is shown that the semidefinite program relaxations for computing the optimal value can be applied to almost all polynomial optimization problems.

In practical, one is usually interested not only in finding the minimum value of f_u on S, but also in obtaining an optimal solution. The next result states that, for a generic polynomial f_u, we can find an appropriate sequence of computationally feasible SDP relaxations, whose optimal values converge *finitely* to the infimum value $\inf_{x\in S} f_u(x)$, and further, the (unique) optimal solution of the problem $\inf_{x\in S} f_u(x)$ can be approximated as closely as desired. These facts open up the possibility of solving previously intractable polynomial optimization problems.

Theorem B. *Assume that the set S is regular. Then there exists an open and dense semi-algebraic set $\mathscr{U} \subset \mathbb{R}^n$ such that for each parameter $\bar{u} \in \mathscr{U}$, if the corresponding polynomial $f_{\bar{u}}$ is bounded from below on S then it attains its infimum on S at a unique point $\bar{x} \in S$ and we can find a sequence of semidefinite programmings, say* (SDP_ℓ) *for $\ell \in \mathbb{N}$, such that*

(i) *The sequence of the optimal values of* (SDP_ℓ) *converges finitely to the infimum value* $\inf_{x\in S} f_{\bar{u}}(x)$; *and*
(ii) *Every sequence of "nearly" optimal solutions of* (SDP_ℓ) *gives rise to a sequence of points in $\mathbb{R}^n$ converging to the unique global minimizer $\bar{x}$.*

Main tools for our investigation come from semialgebraic geometry (for example, Sard Theorem with parameter, Tarski–Seidenberg Theorem, etc.).

Remark 1.1. (i) It should be emphasized that we do not require the polynomials f_u, g_i, and h_j to be convex or the constraint set S is compact.

(ii) To be concrete, we state Theorem A for polynomial functions. Analogous results (replacing analyticity by differentiability), with essentially identical proofs, hold for C^2-functions definable in an "o-minimal structure" and, more generally, for "tame" C^2-functions. See [10] for more on the subject.

The organization of the paper is as follows. Section 2 recalls some preliminary results from semialgebraic geometry; some definitions in set-valued analysis are also given there. The proofs of Theorems A and B are given in Section 3.

2. Preliminaries

Throughout this work we deal with the Euclidean space $\mathbb{R}^n$ equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. We suppose $1 \le n \in \mathbb{N}$ and abbreviate $(x_1, x_2, \ldots, x_n)$ by x. We let $\mathbb{R}[x]$ denote the ring of real polynomials in n indeterminates.

2.1. **Semialgebraic geometry.** In this subsection, we recall some notions and results of semialgebraic geometry, which can be found in [2, 3, 10].

Definition 2.1. (i) A subset of $\mathbb{R}^n$ is called *semialgebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_i(x) = 0, i = 1, \ldots, k; f_i(x) > 0, i = k+1, \ldots, p\}$$

where all f_i are polynomials.

(ii) Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be semialgebraic sets. A map $F \colon A \to B$ is said to be *semialgebraic* if its graph

$$\{(x, y) \in A \times B \mid y = F(x)\}$$

is a semialgebraic subset in $\mathbb{R}^n \times \mathbb{R}^m$.

Semialgebraic sets and functions enjoy a number of remarkable properties:

(i) The class of semialgebraic sets is closed with respect to Boolean operators; a Cartesian product of semialgebraic sets is a semialgebraic set;
(ii) The closure and the interior of a semialgebraic set is a semialgebraic set;
(iii) A composition of semialgebraic maps is a semialgebraic map;
(iv) The image and inverse image of a semialgebraic set under a semialgebraic map are semialgebraic sets;
(v) If A is a semialgebraic set, then the distance function

$$d(\cdot, A) \colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto d(x, A) := \inf\{\|x - a\| \mid a \in A\},$$

is also semialgebraic.

A major fact concerning the class of semialgebraic sets is its stability under linear projections (see, for example, [2, 3]).

Theorem 2.1 (Tarski–Seidenberg Theorem). *The image of a semialgebraic set by a semialgebraic map is semialgebraic.*

Remark 2.1. As an immediate consequence of Tarski–Seidenberg Theorem, we get semialgebraicity of any set $\{x \in A \mid \exists y \in B, (x, y) \in C\}$, provided that A, B, and C are semialgebraic sets in the corresponding spaces. It follows that also $\{x \in A \mid \forall y \in B, (x, y) \in C\}$ is a semialgebraic set as its complement is the union of the complement of A and the set $\{x \in A \mid \exists y \in B, (x, y) \notin C\}$. Thus, if we have a finite collection of semialgebraic sets, then any set obtained from them with the help of a finite chain of quantifiers is also semialgebraic.

We also recall the Curve Selection Lemma at infinity which will be used in the paper (see, for example, [7]).

Lemma 2.1 (Curve Selection Lemma at infinity). *Let $A \subset \mathbb{R}^n$ be a semialgebraic set, and let $f := (f_1, \ldots, f_p) \colon \mathbb{R}^n \to \mathbb{R}^p$ be a semialgebraic map. Assume that there exists a sequence $\{x^\ell\}$ such that $x^\ell \in A$, $\lim_{l\to\infty} \|x^\ell\| = \infty$ and $\lim_{l\to\infty} f(x^\ell) = y \in (\overline{\mathbb{R}})^p$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Then there exists a smooth semialgebraic curve $\varphi \colon (0, \epsilon) \to \mathbb{R}^n$ such that $\varphi(t) \in A$ for all $t \in (0, \epsilon), \lim_{t\to 0} \|\varphi(t)\| = \infty,$ and $\lim_{t\to 0} f(\varphi(t)) = y$.*

In the sequel, we will need the following useful results (see, for example, [10]).

Lemma 2.2 (Growth Dichotomy Lemma). *Let $f \colon (0, \epsilon) \to \mathbb{R}$ be a semialgebraic function with $f(t) \neq 0$ for all $t \in (0, \epsilon)$. Then there exist constants $c \neq 0$ and $q \in \mathbb{Q}$ such that $f(t) = ct^q + o(t^q)$ as $t \to 0^+$.*

Lemma 2.3 (Monotonicity Lemma). *Let $a < b$ in $\mathbb{R}$. If $f \colon [a, b] \to \mathbb{R}^n$ is a semialgebraic function, then there is a partition $a =: t_1 < \cdots < t_N := b$ of $[a, b]$ such that $f|_{(t_l,t_{l+1})}$ is C^1, and either constant or strictly monotone, for $l \in \{1, \ldots, N - 1\}$.*

The next theorem (see [3, 10]) uses the concept of a cell whose definition we omit. We do not need the specific structure of cells described in the formal definition. For us it will be sufficient to think of a C^p-cell of dimension r as of an r-dimensional C^p-manifold which is the image of the cube $(0, 1)^r$ under a semialgebraic C^p-diffeomorphism. As follows from the definition, an n-dimensional cell in $\mathbb{R}^n$ is an open set.

Theorem 2.2 (Cell Decomposition Theorem). *Let $A \subset \mathbb{R}^n$ be a semialgebraic set. Then for any $p \in \mathbb{N}$, A can be represented as a disjoint union of a finite number of cells of class C^p.*

By Cell Decomposition Theorem, for any $p \in \mathbb{N}$ and any semialgebraic subset $A \subset \mathbb{R}^n$, we can write A as a disjoint union of finitely many semialgebraic C^p-manifolds of different dimensions. The *dimension* $\dim A$ of a semialgebraic set A can thus be defined as the dimension of the manifold of highest dimension of its decomposition. This dimension is well defined and independent of the decomposition of A. We will need the following result (see [3, 10]).

Proposition 2.1. (i) *Let* $A \subset \mathbb{R}^n$ *be a semialgebraic set and* $f\colon A \to \mathbb{R}^m$ *a semialgebraic map. Then* $\dim f(A) \leq \dim A$.

(ii) *Let* $A \subset \mathbb{R}^n$ *be a semialgebraic set. Then* $\dim(\overline{A} \setminus A) < \dim A$. *In particular,* $\dim \overline{A} = \dim A$.

Next we state a semialgebraic version of Sard Theorem with parameter, sufficient for the applications in the next section. Recall that, for a C^∞ map of manifolds $f\colon X \to Y$, a point $y \in Y$ is called a *regular value* for f if either $f^{-1}(y) = \emptyset$ or the derivative map $Df(x)\colon T_xX \to T_yY$ is surjective at every point x such that $f(x) = y$, where T_xX and T_yY denote the tangent spaces of X at x and of Y at y, respectively. A point $y \in Y$ that is not a regular value of f is called a *critical value.*

Theorem 2.3 (Sard Theorem with parameter)**.** *Let* $F\colon P \times X \to Y$ *be a* C^∞*-semialgebraic map between semialgebraic manifolds. If* $y \in Y$ *is a regular value of* F, *then there exists a semialgebraic set* Σ *in* P *of dimension at most* $\dim P - 1$ *such that, for each* $p \in P \setminus \Sigma$, y *is a regular value of the map* $F_p\colon X \to Y, x \mapsto F(p, x)$.

Proof. The theorem is a direct consequence of the following two facts:

(a) for almost every $p \in P$, y is a regular value of the map F_p (see, for example, [14, The Transversality Theorem]); and
(b) the set of points $p \in P$ such that y is not a regular value of the map F_p is a semialgebraic subset of P.

The details are left to the reader. □

In the sequel we will need the following result (see, for example, [2, 3]).

Theorem 2.4. *Let* g_i *as* $i = 1, \ldots, l$ *and* h_j *as* $j = 1, \ldots, m$ *be real polynomials on* $\mathbb{R}^n$ *with degree at most* d, *and let*

$$S := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \ldots, g_l(x) = 0,\ h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}.$$

Then the number of connected components of the semialgebraic set S *is bounded above by* $((m + 1)d + 1)(2(m + 1)d + 1)^n$.

2.2. **Some definitions in set-valued analysis.** In this subsection we recall the definitions of continuity (upper semi-continuity, lower semi-continuity) for set-valued maps.

We say that $\mathcal{S}$ is a *set-valued map* or a *multifunction* from X to Y, denoted by $\mathcal{S}\colon X \rightrightarrows Y$, if, for every $u \in X$, $\mathcal{S}(u)$ is a subset of Y. A set-valued map $\mathcal{S}\colon X \rightrightarrows Y$ is called *closed-valued* if $\mathcal{S}(x)$ is closed for all $u \in X$.

Definition 2.2. Let $\mathcal{S}\colon \mathbb{R}^N \rightrightarrows \mathbb{R}^n$ be a set-valued map. It is said that $\mathcal{S}$ is *upper semi-continuous* (shortly, usc) at $\bar{u} \in \mathbb{R}^N$ if for each open set $V \subset \mathbb{R}^n$ satisfying $\mathcal{S}(\bar{u}) \subset V$, there exists $\delta > 0$ such that $\mathcal{S}(u) \subset V$ whenever $\|u-\bar{u}\| < \delta$. If for each open set $V \subset \mathbb{R}^n$ satisfying $\mathcal{S}(\bar{u}) \cap V \neq \emptyset$ there exists $\delta > 0$ such that $\mathcal{S}(u) \cap V \neq \emptyset$ whenever $\|u - \bar{u}\| < \delta$, then $\mathcal{S}$ is said to be *lower semi-continuous* (shortly, lsc) at $\bar{u} \in \mathbb{R}^N$. If $\mathcal{S}$ is simultaneously usc and lsc at $\bar{u}$, we say that it is *continuous* at $\bar{u}$.

3. Proofs of Theorems A and B

This section provides the proofs of Theorems A and B stated earlier in the introduction section. The proof of Theorem A will be divided into several steps, which, for convenience, will be called lemmas.

In what follows we let $f, g_1, \ldots, g_l, h_1, \ldots, h_m$ be real polynomials on $\mathbb{R}^n$ with degree at most d. We will assume that the closed semialgebraic set

$$S := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \ldots, g_l(x) = 0,\ h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}$$

is nonempty. For each parameter $u \in \mathbb{R}^n$ we define the corresponding polynomial f_u by

$$f_u(x) := f(x) - \langle u, x\rangle \quad \text{for} \quad x \in \mathbb{R}^n.$$

Recall that the Karush–Kuhn–Tucker set-valued map $KKT\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n, u \mapsto KKT(u)$, corresponding to the optimization problem $\min_{x\in S} f_u(x)$, is defined by

$$\begin{aligned} KKT(u) := \{x \in S \quad | \quad & \text{there exist } \lambda := (\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l \text{ and } \nu := (\nu_1, \ldots, \nu_m) \in \mathbb{R}^m \text{ such that} \\ & \nabla f_u(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0, \\ & \nu_j h_j(x) = 0,\ \nu_j \geq 0, \quad \text{for } j = 1, \ldots, m\}. \end{aligned}$$

Lemma 3.1. *For each $u \in \mathbb{R}^n$, the set $KKT(u)$ is closed, semialgebraic and has at most*

$$B_0(d, n, l, m) := ((2m+1)(d+1)+1)(2(2m+1)(d+1)+1)^{n+l+m}$$

connected components.

Proof. For each $u \in \mathbb{R}^n$, let

$$\begin{aligned} W(u) := \{(x, \lambda, \nu) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \quad | \quad & g_i(x) = 0,\ i = 1, \ldots, l,\ h_j(x) \geq 0, j = 1, \ldots, m, \\ & \nabla f_u(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0, \\ & \nu_j h_j(x) = 0,\ \nu_j \geq 0, \quad \text{for } j = 1, \ldots, m\}. \end{aligned}$$

It is clear that $KKT(u) = \pi(W(u))$, where the projection $\pi\colon \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^n$ is defined by $\pi(x, \lambda, \nu) := x$. So $KKT(u)$ is a closed set. Thanks to Tarski–Seidenberg Theorem

(Theorem 2.1), the set $KKT(u)$ is semialgebraic. Further, by Theorem 2.4, the number of connected components of the semialgebraic set $W(u)$ is bounded above by $B_0(d, n, l, m)$. $\square$

Sometimes the Karush–Kuhn–Tucker system fails to hold at some minimizers. Hence, we usually make an assumption called a *constraint qualification* to ensure that this system holds. Such a constraint qualification–probably the one most often used in the design of algorithms–is defined as follows.

Definition 3.1. (see [31, Definition 12.1]). For each $x \in S$, let $J(x)$ be the set of indices j for which h_j vanishes at x. The constraint set S is called *regular* if, for each $x \in S$, the gradient vectors $\nabla g_i(x),\ i = 1, \ldots, l$, and $\nabla h_j(x),\ j \in J(x)$, are linearly independent.

Remark 3.1. By Sard theorem, it is not hard to show that the regularity is a generic property (see [41, Theorem 1] for a proof).

From now on we assume that the constraint set S is regular.

Lemma 3.2. *If S is unbounded, then there exists a real number $R_0 > 0$ such that for all $R \geq R_0$, the set*

$$S_R := \{x \in S \mid \|x\|^2 = R^2\}$$

is a nonempty compact set, and it is is regular, i.e., for each $x \in S_R$, the vectors x, $\nabla g_i(x)$, $i = 1, \ldots, l$, and $\nabla h_j(x)$, $j \in J(x)$, are linearly independent.

Proof. See [16, Lemma 3.1]. $\square$

Let us recall the following definition.

Definition 3.2. Let $\bar{x} \in S$ and $u \in \mathbb{R}^n$. We say that the *strong second-order sufficient conditions* for the optimization problem $\min_{x \in S} f_u(x)$ hold at $\bar{x}$ if there exist Lagrange multipliers $\lambda_1, \ldots, \lambda_l$ and $\nu_1, \ldots, \nu_m$ such that the following conditions satisfy

$$\begin{aligned}
&\nabla f_u(\bar{x}) - \sum_{i=1}^{l} \lambda_i \nabla g_i(\bar{x}) - \sum_{j=1}^{m} \nu_j \nabla h_j(\bar{x}) = 0, \\
&\nu_j h_j(\bar{x}) = 0,\ \nu_j \geq 0, \quad \text{for } j = 1, \ldots, m, \\
&\nu_j > 0 \ \text{ for all } \ j \in J(\bar{x}) = \{j \mid h_j(\bar{x}) = 0\}, \\
&v^T \nabla^2 L(\bar{x}) v > 0 \quad \text{ for all } v \in \mathcal{M}(\bar{x})^{\perp}, v \neq 0.
\end{aligned}$$

Here $\nabla^2 L(\bar{x})$ is the Hessian of the Lagrange function

$$L(x) := f_u(x) - \sum_{i=1}^{l} \lambda_i g_i(x) - \sum_{j \in J(\bar{x})} \nu_j h_j(x),$$

$\mathcal{M}(\bar{x})$ stands for the Jacobian of the active constraint polynomials

$$\mathcal{M}(\bar{x}) := \left[\nabla g_i(\bar{x}),\ i = 1, \ldots, l,\ \nabla h_j(\bar{x}),\ j \in J(\bar{x})\right]^T,$$

and $\mathcal{M}(\bar{x})^{\perp}$ denotes the null space of $\mathcal{M}(\bar{x})$.

In the following lemma, the last statement is well known; for completeness, we provide a proof.

Lemma 3.3. *There exists an open and dense semialgebraic set $\mathscr{A}_f$ in $\mathbb{R}^n$ such that the following statements hold*

(i) *The number of points of $KKT(u)$, denoted by $\#KKT(u)$, is at most $B_0(d, n, l, m)$ for any $u \in \mathscr{A}_f$.*
(ii) *The set-valued map $u \mapsto KKT(u)$ is continuous on the set $\mathscr{A}_f$.*
(iii) *The strong second-order sufficient conditions hold at every local (or global) minimizer of the polynomial f_u on S for any $u \in \mathscr{A}_f$.*

Proof. (i)-(ii) For each subset $J := \{j_1, \ldots, j_k\}$ of $\{1, \ldots, m\}$, we let $\tilde{\nu}_J := (\tilde{\nu}_j)_{j \in J} \in \mathbb{R}^{\#J}$ and

$$V_J := \{(x, \lambda, \tilde{\nu}) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J} \quad | \quad h_j(x) > 0, \text{ for } j \notin J\}.$$

Clearly, V_J is an open semialgebraic set in $\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}$. Assume that $V_J \neq \emptyset$. We define the semialgebraic map $\Phi_J \colon \mathbb{R}^n \times V_J \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}$ by

$$\Phi_J(u, x, \lambda, \tilde{\nu}_J) := \left(\nabla f_u(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j \in J} \tilde{\nu}_j^2 \nabla h_j(x), g_1(x), \ldots, g_l(x), h_{j_1}(x), \ldots, h_{j_k}(x) \right).$$

A direct computation shows that

$$\left(D_u\Phi_J \,\middle|\, D_x\Phi_J \right) = \left(\begin{array}{c|c} -I_n & \cdots \\ \hline 0 & \nabla g_1(x) \\ \vdots & \vdots \\ 0 & \nabla g_l(x) \\ \hline 0 & \nabla h_{j_1}(x) \\ \vdots & \vdots \\ 0 & \nabla h_{j_k}(x) \end{array} \right),$$

where $D_u\Phi_J$ (resp., $D_x\Phi_J$) denotes the derivative of Φ_J with respect to u (resp., x), and I_n denotes the identity matrix of order n. Since the constraint set S is regular, it follows that $0 \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}$ is a regular value of Φ_J. By Sard Theorem with parameter (Theorem 2.3), there exists an open and dense semialgebraic set $\mathscr{U}_J$ in $\mathbb{R}^n$ such that for each $u \in \mathscr{U}_J$, $0 \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}$ is a regular value of the map

$$\Phi_{J,u} \colon V_J \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}, \quad (x, \lambda, \tilde{\nu}_J) \mapsto \Phi_J(u, x, \lambda, \tilde{\nu}_J).$$

Let

$$W_J(u) := \{(x, \lambda, \tilde{\nu}_J) \in V_J \mid \Phi_{J,u}(x, \lambda, \tilde{\nu}_J) = 0\} \quad \text{for} \quad u \in \mathbb{R}^n.$$

It follows from the Inverse Function Theorem that for each $u \in \mathscr{U}_J$, all points of $W_J(u)$ are isolated. Note that $W_J(u)$ is a semialgebraic set; so in view of Theorem 2.4, it has finitely many connected components. Therefore, $W_J(u)$ is a finite (possibly empty) set for each $u \in \mathscr{U}_J$. Further, by the Implicit Function Theorem, all (local) solutions $(x, \lambda, \tilde{\nu}_J)$ of the system $\Phi_{J,u}(x, \lambda, \tilde{\nu}_J) = 0$ depend analytically on the parameter $u \in \mathscr{U}_J$, and hence the set-valued map $u \mapsto W_J(u)$ is continuous on the set $\mathscr{U}_J$.

Let $\mathscr{A}_f := \bigcap_J \mathscr{U}_J$, where the intersection is taken all subsets J of $\{1, \ldots, m\}$. Then $\mathscr{A}_f$ is an open and dense semialgebraic set in $\mathbb{R}^n$.

On the other hand, by construction, it is not hard to see that $x \in KKT(u)$ if and only if $x \in \pi_J(W_J(u))$, where $\pi_J(x, \lambda, \tilde{\nu}_J) := x$ and $J := J(x)$. Therefore,

$$KKT(u) \ = \ \bigcup_J \pi_J(W_J(u)).$$

Consequently, for any $u \in \mathscr{A}_f$, the set $KKT(u)$ is a finite set, and so, by Lemma 3.1, it has at most $B_0(d, n, l, m)$ points. Moreover, since the set-valued map $u \mapsto W_J(u)$ is continuous on the set $\mathscr{U}_J$, the set-valued map $u \mapsto KKT(u)$ is continuous on the set $\mathscr{A}_f$.

(iii) Take any $u \in \mathscr{A}_f$ and let $\bar{x} \in S$ be a local (or global) minimizer of the polynomial f_u on S. Since S is regular, there exist (unique) Lagrange multipliers $\lambda_1, \ldots, \lambda_l$ and $\nu_1, \ldots, \nu_m$ such that

$$\begin{aligned} \nabla f_u(\bar{x}) - \sum_{i=1}^{l} \lambda_i \nabla g_i(\bar{x}) - \sum_{j=1}^{m} \nu_j \nabla h_j(\bar{x}) \ &= \ 0, \\ \nu_j h_j(\bar{x}) \ &= \ 0, \ \nu_j \geq 0, \quad \text{for } j = 1, \ldots, m. \end{aligned}$$

Let $L(x)$ be the associated Lagrange function

$$L(x) := f_u(x) - \sum_{i=1}^{l} \lambda_i g_i(x) - \sum_{j \in J(\bar{x})} \nu_j h_j(x).$$

Since the constraint set S is regular, the second order necessary condition holds at $\bar{x}$, i.e.,

$$v^T \nabla^2 L(\bar{x}) v \geq 0 \quad \text{ for all } v \in \mathcal{M}(\bar{x})^\perp, v \neq 0.$$

We will show that the above inequality is strict.

By contradiction, suppose that there exists a nonzero vector $v \in \mathcal{M}(\bar{x})^\perp$ such that $v^T \nabla^2 L(\bar{x}) v = 0$. It implies that v is a minimizer of the optimization problem

$$\min_{z \in \mathbb{R}^n} z^T \nabla^2 L(\bar{x}) z \quad \text{ such that } \quad \mathcal{M}(\bar{x}) z = 0.$$

By the first order optimality condition for the above problem, there exists a vector $w \in \mathbb{R}^n$ such that $\nabla^2 L(\bar{x}) v - \mathcal{M}(\bar{x})^T w = 0$, which then implies

$$\begin{pmatrix} \nabla^2 L(\bar{x}) & \mathcal{M}(\bar{x})^T \\ \mathcal{M}(\bar{x}) & 0 \end{pmatrix} \begin{pmatrix} v \\ -w \end{pmatrix} = 0.$$

Since $v \neq 0$, it follows that

$$\det \begin{pmatrix} \nabla^2 L(\bar{x}) & \mathcal{M}(\bar{x})^T \\ \mathcal{M}(\bar{x}) & 0 \end{pmatrix} = 0. \tag{2}$$

Now let $J := J(\bar{x})$, $\lambda := (\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l$, and $\widetilde{\nu}_J := (\sqrt{\nu_j})_{j \in J} \in \mathbb{R}^{\#J}$. Since $u \in \mathscr{A}_f$, we have $0 = \Phi_{J,u}(\bar{x}, \lambda, \widetilde{\nu}_J)$ is a regular value of the map $\Phi_{J,\bar{u}}(\cdot,\cdot,\cdot)$. This contradicts Equality (2).

Finally, we show that $\nu_j > 0$ for all $j \in J(\bar{x})$. To see this, let us write $J(\bar{x}) := \{j_1, \ldots, j_k\}$ with $1 \le j_1 < j_2 < \cdots < j_k \le m$. It is easy to see that if for some $j_\ell \in J(\bar{x})$ we have $\lambda_{j_\ell} = 0$, then the $(n+\ell)$th column of the Jacobian of the map $\Phi_{J,u}(\cdot,\cdot,\cdot)$ at $(\bar{x}, \lambda, \widetilde{\nu}_J)$ will vanish, in contradiction to nonsingularity. □

Remark 3.2. It is not hard to see that the proof of Lemma 3.3 implies that for each $\bar{u} \in \mathscr{A}_f$, there exist $\epsilon > 0$ and $N := \#KKT(\bar{u})$ analytic maps

$$x^i \colon \{u \in \mathbb{R}^n \mid \|u - \bar{u}\| < \epsilon\} \to \mathbb{R}^n, \quad u \mapsto x^i(u),$$

for $i = 1, \ldots, N$, such that $x^i(u) \neq x^j(u)$, with $i \neq j$, and $KKT(u) = \{x^1(u), \ldots, x^N(u)\}$.

On the other hand, connected semialgebraic sets are path connected (see, for example, [3, Theorem 2.4.5]). Hence, the number of points of $KKT(u)$ is constant on each connected component of $\mathscr{A}_f$.

Corollary 3.1. *If the set-valued map $KKT(\cdot)$ is lower semi-continuous at $u \in \mathbb{R}^n$, then $KKT(u)$ is a finite set and has at most $B_0(d,n,l,m)$ points.*

Proof. In fact, according to Lemma 3.3, there exists an open and dense semialgebraic set $\mathscr{A}_f$ in $\mathbb{R}^n$ such that for each $u \in \mathscr{A}_f$, the set $KKT(u)$ is finite and has at most $B_0(d,n,l,m)$ points. In particular, there exists a sequence $\{u^\ell\}_{\ell \in \mathbb{N}} \subset \mathscr{A}_f$ such that $\lim_{\ell \to +\infty} u^\ell = u$ and $\#KKT(u^\ell) \le B_0(d,n,l,m)$ for all ℓ. Thus, $\#KKT(u) \le B_0(d,n,l,m)$ due to the lower semicontinuity of the set-valued map $KKT(\cdot)$ at the point u. □

Lemma 3.4. *There exists an open and dense semialgebraic set $\mathscr{B}_f \subset \mathbb{R}^n$ such that for each parameter $u \in \mathscr{B}_f$, the corresponding polynomial f_u has distinct values on the set*

$$\begin{aligned}\{x \in \mathbb{R}^n \quad | \quad & \textit{there exist } \lambda := (\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l \textit{ and } \nu := (\nu_1, \ldots, \nu_m) \in \mathbb{R}^m, \textit{ such that} \\ & \nabla f_u(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0, \\ & \nu_j h_j(x) = 0,\ \nu_j \ge 0,\ \textit{and } \nu_j + h_j(x) > 0, \quad \textit{for } j = 1, \ldots, m\}.\end{aligned}$$

Proof. Let $\mathscr{L} \colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ be a polynomial function defined by

$$\mathscr{L}(u, x, \lambda, \tilde{\nu}) := f_u(x) - \sum_{i=1}^{l} \lambda_i g_i(x) - \sum_{j=1}^{m} \tilde{\nu}_j^2 h_j(x).$$

Consider the semialgebraic map

$$\Phi \colon \mathbb{R}^n \times V \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$$

where

$$\Phi(u,x,\lambda,\tilde{\nu}) := \left(\nabla_x \mathscr{L}(u,x,\lambda,\tilde{\nu}), \nabla_\lambda \mathscr{L}(u,x,\lambda,\tilde{\nu}), \nabla_{\tilde{\nu}} \mathscr{L}(u,x,\lambda,\tilde{\nu})\right),$$

and

$$V := \{(x,\lambda,\tilde{\nu}) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \mid \tilde{\nu}_j^2 + [h_j(x)]^2 > 0 \quad \text{for} \quad j = 1,\ldots,m\}.$$

A direct computation shows that

$$\left(D_u\Phi \,\middle|\, D_x\Phi \,\middle|\, D_{\tilde{\nu}}\Phi\right) = (-1) \times \left(\begin{array}{c|c|ccc} I_n & \cdots & \cdot & \cdot & \cdot \\ \hline 0 & \nabla g_1(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \nabla g_l(x) & 0 & \cdots & 0 \\ \hline 0 & 2\tilde{\nu}_1 \nabla h_1(x) & 2h_1(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 2\tilde{\nu}_m \nabla h_m(x) & 0 & \cdots & 2h_m(x) \end{array}\right).$$

Since the constraint set S is regular, it follows that $0 \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ is a regular value of Φ. By Sard Theorem with parameter (Theorem 2.3), there exists an open and dense semialgebraic set $\mathscr{V}$ in $\mathbb{R}^n$ such that for each $u \in \mathscr{V}$, $0 \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ is a regular value of the map

$$\Phi_u \colon V \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m, \quad (x,\lambda,\tilde{\nu}) \mapsto \Phi(u,x,\lambda,\tilde{\nu}).$$

Next we define the semialgebraic map

$$\Psi \colon \mathscr{V} \times ((V \times V) \setminus \Delta) \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$$

by

$$\Psi(u,x,\lambda,\tilde{\nu},x',\lambda',\tilde{\nu}') := (\mathscr{L}(u,x,\lambda,\tilde{\nu}) - \mathscr{L}(u,x',\lambda',\tilde{\nu}'), \Phi(u,x,\lambda,\tilde{\nu}), \Phi(u,x',\lambda',\tilde{\nu}')),$$

where we put

$$\Delta := \{(x,\lambda,\tilde{\nu},x',\lambda',\tilde{\nu}') \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \mid x = x'\}.$$

A direct computation shows that

$$\left(D_u\Psi \,\middle|\, D_{(x,\lambda,\tilde{\nu})}\Psi \,\middle|\, D_{(x',\lambda',\tilde{\nu}')}\Psi\right) = \begin{pmatrix} x - x' & \Phi(u,x,\lambda,\tilde{\nu}) & -\Phi(u,x',\lambda',\tilde{\nu}') \\ \bullet & D_{(x,\lambda,\tilde{\nu})}\Phi(u,x,\lambda,\tilde{\nu}) & 0 \\ \bullet & 0 & D_{(x,\lambda,\tilde{\nu})}\Phi(u,x',\lambda',\tilde{\nu}') \end{pmatrix}.$$

Let $(x,\lambda,\tilde{\nu},x',\lambda',\tilde{\nu}') \in (V \times V) \setminus \Delta$ and $u \in \mathscr{V}$ be such that

$$\Phi(u,x,\lambda,\tilde{\nu}) = \Phi(u,x',\lambda',\tilde{\nu}') = 0.$$

Note that $0 \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ is a regular value of $\Phi_u(\cdot,\cdot,\cdot)$. Therefore,

$$\begin{aligned} & x - x' \neq 0, \quad \text{and} \\ & \mathrm{rank} D_{(x,\lambda,\tilde{\nu})}\Phi(u,x,\lambda,\tilde{\nu}) = \mathrm{rank} D_{(x,\lambda,\tilde{\nu})}\Phi(u,x',\lambda',\tilde{\nu}') = n + l + m. \end{aligned}$$

Consequently, $0 \in \mathbb{R} \times \mathbb{R}^{2n+2l+2m}$ is a regular value of Ψ. By Sard theorem with parameter (Theorem 2.3), there exists an open and dense semialgebraic set $\mathscr{B}_f \subset \mathscr{V}$ in $\mathbb{R}^n$ such that for each $u \in \mathscr{B}_f$, 0 is a regular value of the map

$$\begin{array}{rccl} \Psi_u\colon & (V \times V) \setminus \Delta & \longrightarrow & \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \\ & (x, \lambda, \tilde{\nu}, x', \lambda', \tilde{\nu}') & \longmapsto & \Psi(u, x, \lambda, \tilde{\nu}, x', \lambda', \tilde{\nu}'). \end{array}$$

On the other hand, it is clear that

$$\begin{aligned} \dim\left((V \times V) \setminus \Delta\right) &= 2n + 2l + 2m \\ &< 1 + 2n + 2l + 2m \ = \ \dim(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m). \end{aligned}$$

Therefore, $\Psi_u^{-1}(0) = \emptyset$; in other words, the following system has no solution in $V \times V$:

$$x \neq x', \quad \mathscr{L}(u, x, \lambda, \tilde{\nu}) = \mathscr{L}(u, x', \lambda', \tilde{\nu}'), \quad \Phi(u, x, \lambda, \tilde{\nu}) = \Phi(u, x', \lambda', \tilde{\nu}') = 0;$$

or equivalently, the following system has no solution in $V \times V$:

$$x \neq x', \quad f_u(x) = f_u(x'), \quad \Phi(u, x, \lambda, \tilde{\nu}) = \Phi(u, x', \lambda', \tilde{\nu}') = 0.$$

Consequently, the polynomial f_u has distinct values on the set

$$\{x \in \mathbb{R}^n \mid \text{there exist } \lambda \in \mathbb{R}^l \text{ and } \tilde{\nu} \in \mathbb{R}^m, \text{ s.t. } (x, \lambda, \tilde{\nu}) \in V, \ \Phi(u, x, \lambda, \tilde{\nu}) = 0\},$$

which completes the proof. □

For each subset J of $\{1, \ldots, m\}$, we let

$$S_J \ := \ \{x \in \mathbb{R}^n \quad | \quad g_i(x) = 0 \text{ for } i = 1, \ldots, l, \ h_j(x) = 0, \text{ for } j \in J, \ h_j(x) > 0 \text{ for } j \notin J\}.$$

By definition, S_J is a semialgebraic set in $\mathbb{R}^n$ and $S = \cup_J S_J$.

Let J be a subset of $\{1, \ldots, m\}$ such that $S_J \neq \emptyset$. We define the semialgebraic map $\mathscr{F}_J \colon S_J \times \mathbb{R}^l \times \mathbb{R}^{\#J} \to \mathbb{R}^n$ by

$$\mathscr{F}_J(x, \lambda, \nu_J) := \left(\nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j \in J} \nu_j \nabla h_j(x)\right),$$

where $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^l$ and $\nu_J := (\nu_j)_{j \in J} \in \mathbb{R}^{\#J}$.

Lemma 3.5. *With the above notations, let*

$$\begin{aligned} \Sigma_J := \{u \in \mathbb{R}^n \quad | \quad & \exists \{(x^\ell, \lambda^\ell, \nu_J^\ell)\}_{\ell \in \mathbb{N}} \subset S_J \times \mathbb{R}^l \times \mathbb{R}^{\#J} \textit{ such that} \\ & \lim_{\ell \to \infty} \|(x^\ell, \lambda^\ell, \nu_J^\ell)\| = +\infty \textit{ and } \lim_{\ell \to \infty} \mathscr{F}_J(x^\ell, \lambda^\ell, \nu_J^\ell) = u\}. \end{aligned}$$

Then Σ_J is a semialgebraic set of dimension at most $n - 1$.

Proof. Let $\overline{G}$ be the closure of the set

$$G := \{(u, x, \lambda, \nu_J) \in \mathbb{R}^n \times S_J \times \mathbb{R}^l \times \mathbb{R}^{\#J} \mid u = \mathscr{F}_J(x, \lambda, \nu_J)\} \subset \mathbb{R}^n \times \mathbb{R}^N$$

in $\mathbb{R}^n \times \mathbb{P}^N$, where $N := n + l + \#J$ and $\mathbb{P}^N$ is the real projective space. Then, the sets G and $\overline{G}$ are semialgebraic. Hence, the set $\overline{G} \setminus G$ is semialgebraic. Moreover, if $\pi \colon \mathbb{R}^n \times \mathbb{P}^N \to \mathbb{R}^n$ is the projection on the first factor, then $\Sigma_J = \pi(\overline{G} \setminus G)$. In view of Tarski–Seidenberg Theorem (Theorem 2.1), the set Σ_J is semialgebraic. It follows from Proposition 2.1 that

$$\dim \Sigma_J = \dim \pi(\overline{G} \setminus G) \leq \dim(\overline{G} \setminus G) < \dim G = \dim(S_J \times \mathbb{R}^l \times \mathbb{R}^{\#J}).$$

On the other hand, since the constraint set S is regular, it follows easily that

$$\dim S_J \quad = \quad n - l - \#J.$$

Therefore, $\dim \Sigma_J \leq n - 1$, which completes the proof of the lemma. $\square$

Lemma 3.6. *There exists an open and dense semialgebraic set $\mathscr{C}_f \subset \mathbb{R}^n$ such that for each parameter $u \in \mathscr{C}_f$, if the corresponding polynomial f_u is bounded from below on S, then there exist constants $c > 0$ and $R > 0$ such that*

$$f_u(x) \quad \geq \quad c\|x\| \quad \text{ for all } x \in S \quad \text{and} \quad \|x\| \geq R.$$

In particular, f_u is coercive on S.

Proof. For each subset J of $\{1, \ldots, m\}$, let Σ_J be the semialgebraic set given in Lemma 3.5. Put $\mathscr{C}_f := \mathbb{R}^n \setminus \overline{\cup_J \Sigma_J}$. Then $\mathscr{C}_f$ is an open and dense semialgebraic set in $\mathbb{R}^n$.

Take any $u \in \mathscr{C}_f$ and assume that the polynomial $f_u \colon \mathbb{R}^n \to \mathbb{R}, x \mapsto f_u(x) := f(x) - \langle u, x \rangle$, is bounded from below on S. We will show that there exist constants $c > 0$ and $R > 0$ such that

$$f_u(x) \quad \geq \quad c\|x\| \quad \text{ for all } x \in S \quad \text{and} \quad \|x\| \geq R.$$

Suppose, by contradiction, that there exist sequences $\{x^\ell\}_{\ell \in \mathbb{N}} \subset S$, with $\lim_{\ell \to \infty} \|x^\ell\| = +\infty$, and $\{c^\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{R}$, with $c_\ell > 0$ and $\lim_{\ell \to \infty} c^\ell = 0$, such that

$$f_u(x^\ell) \quad < \quad c^\ell \|x^\ell\| \quad \text{ for all } \quad \ell.$$

For each ℓ, let $y^\ell \in S$ be a minimizer of the following problem

$$\min_{x \in S, \ \|x\|^2 = \|x^\ell\|^2} f_u(x).$$

(The existence of y^ℓ follows direct from the fact that the objective function f_u is continuous on the compact set $\{x \in S, \ \|x\|^2 = \|x^\ell\|^2\}$.) Then we have for all ℓ,

$$-\infty \ < \ \inf_{x \in S} f_u(x) \ \leq \ f_u(y^\ell) \ \leq \ f_u(x^\ell) \ < \ c^\ell \|x^\ell\| \ = \ c^\ell \|y^\ell\|,$$

which yields that

$$\lim_{\ell \to +\infty} \frac{f_u(y^\ell)}{\|y^\ell\|} = 0.$$

By Lemma 3.2, for all $\ell \gg 1$, the vectors $\nabla g_i(y^\ell), i = 1, \ldots, l$, $\nabla h_j(y^\ell)$, $j \in J(y^\ell)$, and the vector y^ℓ are linearly independent. We therefore deduce from Lagrange's multipliers theorem that: there exist sequences $\{\lambda^\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{R}^l$, $\{\nu^\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{R}^m$, and $\{\mu^\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{R}$ such that

- $\nabla f_u(y^\ell) - \sum_{i=1}^l \lambda_i^\ell \nabla g_i(y^\ell) - \sum_{j=1}^m \nu_j^\ell \nabla h_j(y^\ell) - \mu^\ell y^\ell = 0$;
- $\nu_j^\ell h_j(y^\ell) = 0$, and $\nu_j^\ell \geq 0$ for $j = 1, \ldots, m$.

By the Curve Selection Lemma (Lemma 2.1), there are a smooth semialgebraic curve $\varphi(t)$ and semialgebraic functions $\lambda_i(t), \nu_j(t), \mu(t), t \in (0, \epsilon]$, such that

(a) $\lim_{t \to 0^+} \|\varphi(t)\| = +\infty$;
(b) $\lim_{t \to 0^+} \frac{f_u(\varphi(t))}{\|\varphi(t)\|} = 0$;
(c) $\nabla f_u(\varphi(t)) - \sum_{i=1}^l \lambda_i(t) \nabla g_i(\varphi(t)) - \sum_{j=1}^m \nu_j(t) \nabla h_j(\varphi(t)) - \mu(t)\varphi(t) \equiv 0$; and
(d) For all $t \in (0, \epsilon]$, we have $\varphi(t) \in S$ and $\nu_j(t) h_j(\varphi(t)) = 0$, for $j = 1, \ldots, m$.

Thanks to Monotonicity Lemma (Lemma 2.3), for $\epsilon > 0$ small enough, the functions ν_j and $h_j \circ \varphi$ are either constant or strictly monotone. Then, by (d), we can see that either $\nu_j(t) \equiv 0$ or $h_j \circ \varphi(t) \equiv 0$; in particular,

$$\nu_j(t) \frac{d}{dt}(h_j \circ \varphi)(t) \equiv 0, \quad j = 1, \ldots, m.$$

Hence, it follows from (c) that

$$\begin{aligned}
\frac{d}{dt}(f_u \circ \varphi)(t) &= \langle \nabla f_u(\varphi(t)), \frac{d\varphi}{dt} \rangle \\
&= \sum_{i=1}^l \lambda_i(t) \langle \nabla g_i(\varphi(t)), \frac{d\varphi}{dt} \rangle + \sum_{j=1}^m \nu_j(t) \langle \nabla h_j(\varphi(t)), \frac{d\varphi}{dt} \rangle + \mu(t) \langle \varphi(t), \frac{d\varphi}{dt} \rangle \\
&= \sum_{i=1}^l \lambda_i(t) \frac{d}{dt}(g_i \circ \varphi)(t) + \sum_{j=1}^m \nu_j(t) \frac{d}{dt}(h_j \circ \varphi)(t) + \frac{\mu(t)}{2} \frac{d\|\varphi(t)\|^2}{dt} \\
&= \frac{\mu(t)}{2} \frac{d\|\varphi(t)\|^2}{dt}.
\end{aligned}$$

This, together with Condition (c) again, implies that

$$\begin{aligned}
\left| \frac{d}{dt}(f_u \circ \varphi)(t) \right| &= \left| \frac{\mu(t)}{2} \frac{d\|\varphi(t)\|^2}{dt} \right| \\
&= \frac{\|\nabla f_u(\varphi(t)) - \sum_{i=1}^l \lambda_i(t) \nabla g_i(\varphi(t)) - \sum_{j=1}^m \nu_j(t) \nabla h_j(\varphi(t))\|}{2\|\varphi(t)\|} \left| \frac{d\|\varphi(t)\|^2}{dt} \right|.
\end{aligned}$$

Since the function $f_u \circ \varphi \colon (0, \epsilon] \mapsto \mathbb{R}, t \mapsto f_u(\varphi(t))$, is semialgebraic, it follows from Monotonicity Lemma (Lemma 2.3) that for $\epsilon > 0$ sufficiently small, this functions is either constant or strictly monotone. If the function $f_u \circ \varphi$ is constant, then $\mu(t) \equiv 0$, and hence

$$\nabla f_u(\varphi(t)) - \sum_{i=1}^l \lambda_i(t) \nabla g_i(\varphi(t)) - \sum_{j=1}^m \nu_j(t) \nabla h_j(\varphi(t)) \equiv 0,$$

Or equivalently,

$$\nabla f(\varphi(t)) - \sum_{i=1}^{l} \lambda_i(t) \nabla g_i(\varphi(t)) - \sum_{j=1}^{m} \nu_j(t) \nabla h_j(\varphi(t)) \equiv u.$$

Combining this equality with Condition (a) gives a contradiction to the assumption that $u \in \mathscr{C}_f$.

Hence, we may assume that the function $(0, \epsilon] \mapsto \mathbb{R}, t \mapsto f_u(\varphi(t))$, is not constant. Thanks to Growth Dichotomy Lemma (Lemma 2.2), we may write

$$\begin{aligned} \|\varphi(t)\| &= c_\alpha t^\alpha + \text{higher order terms in } t, \\ f_u(\varphi(t)) &= c_\beta t^\beta + \text{higher order terms in } t, \end{aligned}$$

here $c_\alpha \neq 0, c_\beta \neq 0$, $\alpha < 0$, and $\alpha < \beta$ (because of Conditions (a)-(b)). By a direct computation, therefore

$$\left\| \nabla f_u(\varphi(t)) - \sum_{i=1}^{l} \lambda_i(t) \nabla g_i(\varphi(t)) - \sum_{j=1}^{m} \nu_j(t) \nabla h_j(\varphi(t)) \right\| = ct^{\beta-\alpha} + \text{higher order terms in } t,$$

for some constant $c \neq 0$, which now yields

$$\lim_{t \to 0^+} \|\nabla f_u(\varphi(t)) - \sum_{i=1}^{l} \lambda_i(t) \nabla g_i(\varphi(t)) - \sum_{j=1}^{m} \nu_j(t) \nabla h_j(\varphi(t))\| = 0.$$

Consequently, we get

$$\lim_{t \to 0^+} \left[\nabla f(\varphi(t)) - \sum_{i=1}^{l} \lambda_i(t) \nabla g_i(\varphi(t)) - \sum_{j=1}^{m} \nu_j(t) \nabla h_j(\varphi(t)) \right] = u.$$

This equality and Condition (a) give us a contradiction to the fact that $u \in \mathscr{C}_f$. □

Now we are in a position to finish the proof of Theorem A.

Proof of Theorem A. By Lemmas 3.3, 3.4 and 3.6, $\mathscr{U} := \mathscr{A}_f \cap \mathscr{B}_f \cap \mathscr{C}_f$ is an open and dense semialgebraic set in $\mathbb{R}^n$.

Take any $\bar{u} \in \mathscr{U}$ and assume that the function $x \mapsto f_{\bar{u}}(x) = f(x) - \langle \bar{u}, x \rangle$ is bounded from below on the constraint set S.

(i) By Lemma 3.6, there exist positive constants c' and R such that

$$f_{\bar{u}}(x) \geq c'\|x\|, \quad \text{for all} \quad x \in S, \ \|x\| \geq R.$$

Since the set $\mathscr{U}$ is open, there exists $\epsilon \in (0, \frac{c'}{2})$ such that $\{u \in \mathbb{R}^n \mid \|u - \bar{u}\| < \epsilon\} \subset \mathscr{U}$. Then we have for all $x \in S$ and $\|x\| \geq R$,

$$\begin{aligned} f_u(x) &= f_{\bar{u}}(x) - \langle u - \bar{u}, x \rangle \ \geq \ f_{\bar{u}}(x) - \|u - \bar{u}\| \|x\| \\ &\geq (c' - \|u - \bar{u}\|)\|x\| \ > \ (c' - \epsilon)\|x\| \ \geq \ \frac{c'}{2}\|x\|. \end{aligned}$$

Clearly, the positive constant $c_1 := \frac{c'}{2}$ satisfies the required inequality. Consequently, the function f_u is coercive on S.

(ii) Take any $u \in \mathbb{R}^n$, with $\|u - \bar{u}\| < \epsilon$. Then $u \in \mathscr{U}$. By Lemma 3.3, the strong second-order sufficient conditions for the optimization problem $\min_{x\in S} f_u(x)$ hold at every local (or global) minimum point of the polynomial f_u on S. Since the polynomial f_u is coercive on S, it has a global minimizer on S. Then it follows from Lemma 3.4 that the optimization problem $\min_{x\in S} f_u(x)$ has a unique minimizer $x(u) \in S$ for which the strong second-order sufficient conditions satisfy.

(iii) We first show that $\lim_{u\to\bar{u}} x(u) = x(\bar{u})$. Indeed, we see that if $\|x(u)\| \geq R$, $\|u-\bar{u}\| < \epsilon$, then

$$\begin{aligned} c_1\|x(u)\| \ \leq \ f_u(x(u)) \ \leq \ f_u(x(\bar{u})) \ &= \ f(x(\bar{u})) - \langle u, x(\bar{u})\rangle \\ &= \ f(x(\bar{u})) - \langle \bar{u}, x(\bar{u})\rangle - \langle u - \bar{u}, x(\bar{u})\rangle \\ &= \ f_{\bar{u}}(x(\bar{u})) - \langle u - \bar{u}, x(\bar{u})\rangle \\ &\leq \ f_{\bar{u}}(x(\bar{u})) + \|u - \bar{u}\|\|x(\bar{u})\| \\ &\leq \ f_{\bar{u}}(x(\bar{u})) + \epsilon\|x(\bar{u})\|. \end{aligned}$$

This implies that the set $\{x(u) \mid \|u - \bar{u}\| < \epsilon\}$ is bounded. Let $\bar{y} \in \{\lim_{u\to\bar{u}} x(u)\}$. Then $\bar{y} \in S$. Note that $f_u(x(u)) \leq f_u(x(\bar{u}))$. Consequently,

$$f_{\bar{u}}(\bar{y}) \ \leq \ \lim_{u\to\bar{u}} f_u(x(\bar{u})) \ = \ \lim_{u\to\bar{u}} [f(x(\bar{u})) - \langle u, x(\bar{u})\rangle] \ = \ f(x(\bar{u})) - \langle \bar{u}, x(\bar{u})\rangle \ = \ f_{\bar{u}}(x(\bar{u})),$$

which yields $\bar{y} = x(\bar{u})$ because $x(\bar{u})$ is the unique minimizer of $f_{\bar{u}}$ on S. Therefore, there exists the limit $\lim_{u\to\bar{u}} x(u) = x(\bar{u})$.

Since the constraint set S is regular, for each $u \in \mathbb{R}^n$, with $\|u - \bar{u}\| < \epsilon$, there exist the unique Lagrange multipliers $\lambda(u) \in \mathbb{R}^l$ and $\nu(u) \in \mathbb{R}^m$ corresponding to the minimizer $x(u)$. It is easy to see that $\lim_{u\to\bar{u}} \lambda(u) = \lambda(\bar{u})$ and $\lim_{u\to\bar{u}} \nu(u) = \nu(\bar{u})$.

Keeping the notations as in the proof of Lemma 3.4. Let $\tilde{\nu}(u) := \left(\sqrt{\nu_1(u)}, \ldots, \sqrt{\nu_m(u)}\right) \in \mathbb{R}^m$. We have $(x(u), \lambda(u), \tilde{\nu}(u)) \in V$ and $\Phi_u(x(u), \lambda(u), \tilde{\nu}(u)) = 0$. Since $0 \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ is a regular value of the map $\Phi_{\bar{u}}$, the Jacobian $D_{(x,\lambda,\tilde{\nu})}\Phi_{\bar{u}}(x(\bar{u}), \lambda(\bar{u}), \tilde{\nu}(\bar{u}))$ is nonsingular. By the Implicit Function Theorem, the system $\Phi_u(x, \lambda, \tilde{\nu}) = 0$ has a unique solution, which depends analytically on u in a some neighborhood of the point $(x(\bar{u}), \lambda(\bar{u}), \tilde{\nu}(\bar{u}))$. This, together with the fact that $\lim_{u\to\bar{u}} x(u) = x(\bar{u})$, implies that if $\epsilon > 0$ small enough, then the map $u \mapsto (x(u), \lambda(u), \tilde{\nu}(u))$ is analytic.

(iv) By definition, we have for all $u \in \mathbb{R}^n$, with $\|u - \bar{u}\| < \epsilon$,

$$\phi(u) = \min_{x\in S} f_u(x) = f_u(x(u)) = ((f \circ x) - \langle u, x\rangle)(u).$$

The map $u \mapsto x(u)$ is analytic, so is $u \mapsto \phi(u)$.

We have shown that for all $u \in \mathbb{R}^n$, with $\|u - \bar{u}\| < \epsilon$,

$$\begin{aligned} 0 &= \nabla f(x(u)) - u - \sum_{i=1}^{l} \lambda_i(u) \nabla g_i(x(u)) - \sum_{j \in J(x(\bar{u}))} \nu_j(u) \nabla h_j(x(u)); \\ & \quad g_i(x(u)) = 0, \; i = 1, \ldots, l, \quad \text{and} \quad h_j(x(u)) = 0, \; j \in J(x(\bar{u})). \end{aligned}$$

It follows successively that

$$\begin{aligned} 0 &= \nabla f(x(u))Dx(u) - uDx(u) - \sum_{i=1}^{l} \lambda_i(u) \nabla g_i(x(u))Dx(u) - \sum_{j \in J(x(\bar{u}))} \nu_j(u) \nabla h_j(x(u))Dx(u) \\ &= \nabla (f \circ x)(u) - uDx(u) - \sum_{i=1}^{l} \lambda_i(u) \nabla (g_i \circ x)(u) - \sum_{j \in J(x(\bar{u}))} \nu_j(u) \nabla (h_j \circ x)(u) \\ &= \nabla (f \circ x)(u) - uDx(u). \end{aligned}$$

Hence

$$\begin{aligned} \nabla \phi(u) &= \nabla (f_u \circ x)(u) = \nabla \left((f \circ x) - \langle u, x \rangle \right)(u) \\ &= \nabla (f \circ x)(u) - uDx(u) - x(u) = -x(u). \end{aligned}$$

This equality proves Item (iv).

(v) We next show that

$$J(x(u)) = J(x(\bar{u})) \quad \text{for all } u \text{ near } \bar{u}.$$

Indeed, if $j \notin J(x(\bar{u}))$ then $h_j(x(\bar{u})) > 0$, and by continuity, we have for all u near $\bar{u}$, $h_j(x(u)) > 0$ and hence $J(x(u)) \subseteq J(x(\bar{u}))$. Further, the equality $J(x(u)) = J(x(\bar{u}))$ holds for all u near $\bar{u}$. Indeed, if it is not the case, then there exist a sequence $\{u^\ell\}_{\ell \in \mathbb{N}} \subset \mathscr{U}$, with $\lim_{\ell \to +\infty} u^\ell = \bar{u}$, and an index $j \in J(x(\bar{u})) \setminus J(x(u^\ell))$. Then $h_j(x(u^\ell)) > 0$. The strict complementarity condition implies that $\tilde{\nu}_j(u^\ell) = 0$ for all ℓ. By continuity, we get $\tilde{\nu}_j(\bar{u}) = 0$, which contradicts the facts that $\tilde{\nu}_j(\bar{u}) + h_j(x(\bar{u})) > 0$ and $h_j(x(\bar{u})) = 0$ (because $j \in J(x(\bar{u}))$). Therefore, Item (iv) holds.

(vi) We will follow an argument analogous to one given in [4, Proposition 2.2] to show that the uniform quadratic growth condition for f_u holds.

The function $\phi(\cdot)$ is differentiable of class C^2 on $\{u \in \mathbb{R}^n \mid \|u - \bar{u}\| < \epsilon\}$. By Taylor expansion, we have for any fixed parameter $u \in \mathbb{R}^n$, with $\|u - \bar{u}\| < \epsilon$,

$$\begin{aligned} \phi(v) &= \phi(u) + \langle v - u, \nabla \phi(u) \rangle + \frac{1}{2} \langle v - u, \nabla^2 \phi(u)(v - u) \rangle + o(\|v - u\|^2) \\ &= \phi(u) - \langle v - u, x(u) \rangle + \frac{1}{2} \langle v - u, \nabla^2 \phi(u)(v - u) \rangle + o(\|v - u\|^2) \end{aligned}$$

for all $v \in \mathscr{U}$ near u, where $\nabla^2 \phi(u)$ denotes the Hessian of the function ϕ at u. Since the Hessian $\nabla^2 \phi(\cdot)$ is continuous on $\{u \in \mathbb{R}^n \mid \|u - \bar{u}\| < \epsilon\}$, shrinking $\epsilon > 0$ if necessary we may assume that $\nabla^2\phi(v)$ is bounded for all v satisfying $\|v-\bar{u}\| < \epsilon$. Hence there exist positive constants $\delta < \epsilon - \|u-\bar{u}\|$ and ρ such that

$$\phi(v) \geq \phi(u) - \langle v-u, x(u)\rangle - \frac{\rho}{2}\|v-u\|^2 \quad \text{for all} \quad \|v-u\| < \delta.$$

Furthermore, since the set $S_R := \{x \in S \mid \|x - x(u)\| \leq R\}$ is compact, we can clearly assume that

$$\delta^{-1} \times \max_{x,y\in S_R} \|x-y\| \;<\; \rho. \tag{3}$$

Now consider any point $x \in S_R$. Since $\phi(v) \leq f_v(x)$ for all $v \in \mathbb{R}^n$, we deduce successively

$$\begin{aligned}
0 &\leq \inf_{v\in\mathbb{R}^n}\{f_v(x) - \phi(v)\} \\
&\leq \inf_{\|v-u\|<\delta}\{f_v(x) - \phi(v)\} \\
&\leq \inf_{\|v-u\|<\delta}\{f_v(x) - \phi(u) + \langle v-u, x(u)\rangle + \frac{\rho}{2}\|v-u\|^2\} \\
&= \inf_{\|v-u\|<\delta}\{f_v(x) - f_u(x(u)) + \langle v-u, x(u)\rangle + \frac{\rho}{2}\|v-u\|^2\} \\
&= \inf_{\|v-u\|<\delta}\{f(x) - \langle v, x\rangle - f_u(x(u)) + \langle v-u, x(u)\rangle + \frac{\rho}{2}\|v-u\|^2\} \\
&= \inf_{\|v-u\|<\delta}\{f_u(x) - f_u(x(u)) - \langle v-u, x-x(u)\rangle + \frac{\rho}{2}\|v-u\|^2\} \\
&= f_u(x) - f_u(x(u)) + \inf_{\|v-u\|<\delta}\{-\langle v-u, x-x(u)\rangle + \frac{\rho}{2}\|v-u\|^2\}.
\end{aligned}$$

It follows easily from the inequality (3) that the above infimum is attained at the point $v = u + \rho^{-1}(x - x(u))$ satisfying $\|v-u\| < \delta$. Replacing this value in the above inequality, we deduce for all $x \in S_R$ that

$$0 \;\leq\; f_u(x) - f_u(x(u)) - \frac{1}{2\rho}\|x - x(u)\|^2,$$

which yields the desired conclusion with $c_2 := (2\rho)^{-1}$.

(vii) By Item (i), we can find positive constants c_1' and $R' \geq R$ such that

$$[f_u(x) - f_u(x(u))] \;\geq\; c_1'\|x - x(u)\| \quad \text{for} \quad x \in S,\; \|x - x(u)\| \geq R'.$$

By applying Item (vi) we obtain the inequality

$$[f_u(x) - f_u(x(u))]^{\frac{1}{2}} \;\geq\; c_2'\|x - x(u)\| \quad \text{for} \quad x \in S,\; \|x - x(u)\| \leq R$$

where $c_2' := \sqrt{c_2} > 0$.

On the other hand, it is not hard to show that

$$[f_u(x) - f_u(x(u))] \;\geq\; c_3'\|x - x(u)\| \quad \text{for} \quad x \in S,\; R \leq \|x - x(u)\| \leq R'.$$

for some $c_3' > 0$.

From the above inequalities we obtain

$$[f_u(x) - f_u(x(u))] + [f_u(x) - f_u(x(u))]^{\frac{1}{2}} \quad \geq \quad c\|x - x(u)\| \quad \text{ for } \quad x \in S,$$

where $c_3 := \min\{c_1', c_2', c_3'\}$. This implies the required inequality.

(viii) Let $\{u^\ell\} \subset \mathbb{R}^n$ be a sequence converging to $\bar{u}$. Then, for all sufficiently large ℓ, $\|u^\ell - \bar{u}\| < \epsilon$, and hence the optimal value $\inf_{x \in S} f_{u^\ell}(x) = f_{u^\ell}(x(u^\ell))$ is finite.

Take any sequence $\{x^\ell\} \subset S$ with $f_{u^\ell}(x^\ell) - \inf_{x \in S} f_{u^\ell}(x) \to 0$. By Item (vii), then for all large ℓ,

$$\left[f_{u^\ell}(x^\ell) - f_{u^\ell}(x(u^\ell))\right] + \left[f_{u^\ell}(x^\ell) - f_{u^\ell}(x(u^\ell))\right]^{\frac{1}{2}} \quad \geq \quad c_3\|x^\ell - x(u^\ell)\|.$$

This implies clearly that $\lim_{\ell \to \infty} x^\ell = \lim_{\ell \to \infty} x(u^\ell) = x(\bar{u})$.

The proof of Theorem A is now complete. □

Proof of Theorem B. Let $\mathscr{U} \subset \mathbb{R}^n$ be an open and dense semialgebraic set satisfying the conclusions of Theorem A. Take any $\bar{u} \in \mathscr{U}$ and assume that the corresponding polynomial $f_{\bar{u}}$ is bounded from below on S. We have

(a) the restriction of $f_{\bar{u}}$ on S is coercive;
(b) the problem $\min_{x \in S} f_{\bar{u}}(x)$ has a unique minimizer $\bar{x} \in S$; and
(c) the strong second-order sufficient conditions hold at $\bar{x}$.

Let M be any real number such that $M > f_{\bar{u}}(x^0)$ for some $x^0 \in S$. By condition (a), the basic semi-algebraic set

$$\begin{aligned} S_M \quad &:= \quad \{x \in S \mid M - f_{\bar{u}}(x) \geq 0\} \\ &= \quad \{x \in \mathbb{R}^n \mid M - f_{\bar{u}}(x) \geq 0, g_1(x) = 0, \ldots, g_l(x) = 0, \ h_1(x) \geq 0, \ldots, h_m(x) \geq 0\} \end{aligned}$$

is nonempty compact. Moreover, it is clear that $\bar{x} \in S_M$ and

$$\phi(\bar{u}) := \inf_{x \in S} f_{\bar{u}}(x) = \inf_{x \in S_M} f_{\bar{u}}(x).$$

Therefore, we can construct a sequence of semidefinite programs, say $(\mathrm{SDP}_\ell), \ell \in \mathbb{N}$, whose optimal values converge monotonically, increasing to the optimal value $\phi(\bar{u})$. Indeed it suffices to replace S by S_M and apply the so-called Lasserre's hierarchy of semidefinite relaxations defined for the compact case [21]. Further, it follows from [30, Theorem 1.1] that the sequence of semidefinite programs (SDP_ℓ) stops after a finite number of steps. This, together with the uniqueness of the optimal solution $\bar{x} \in S_M$ and [37, Corollary 13] (see also [22, Theorem 5.6]), yields the existence of a sequence $\{x^\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{R}^n$ satisfying the required properties. □

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