STABILITY AND GENERICITY FOR SEMI-ALGEBRAIC COMPACT PROGRAMS

GUE MYUNG LEE† AND TIẾN SƠN PHẠM‡

Abstract. In this paper we consider the class of polynomial optimization problems with inequality and equality constraints, in which every problem of the class is obtained by perturbations of the objective function, while the constraint functions are kept fixed. Under certain assumptions, we establish some stability properties (e.g., strong Hölder stability with explicitly determined exponents, semicontinuity, etc.) of the global solution map, the Karush-Kuhn-Tucker set-valued map, and of the optimal value function for all problems in the class. It is shown that for almost every problem in the class, there is a unique optimal solution for which the global quadratic growth condition and the strong second-order sufficient conditions hold. Further, under local perturbations to the objective function, the optimal solution and the optimal value function (resp., the Karush-Kuhn-Tucker set-valued map) vary smoothly (resp., continuously) and the set of active constraint indices is constant. As a nice consequence, for almost all polynomial optimization problems, we can find a natural sequence of computationally feasible semidefinite programs, whose solutions give rise to a sequence of points in $\mathbb{R}^n$ converging to the optimal solution of the original problem.

1. Introduction

Stability and genericity are the most desired properties one expects when dealing with optimization problems (see, for example, [7, 14]).

Recently, various continuity and differentiability properties of the global solution map, the local solution map, the Karush-Kuhn-Tucker set-valued map, and the optimal value function of the quadratic programs with linear constraints have been established (see [2, 8, 23, 28, 43, 53], and the references therein). The proofs of these properties have employed deeply the polyhedrality of the constraint sets. Very recently, stability results for the quadratic programs with unit ball constraints, known as the trust-region subproblem were given in

Date: July 1, 2015.

1991 *Mathematics Subject Classification.* 90C26 · 90C31 · 49J45 · 49J50 · 49K40.

Key words and phrases. Semi-algebraic programs · Global solution map · Optimal value function · Karush-Kuhn-Tucker set-valued map · Stability · Genericity · Strong second-order sufficient conditions · Quadratic growth.

†The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2013R1A1A2005378).

‡The second author was supported by the National Foundation for Science and Technology Development (NAFOSTED), Vietnam, grant 101.04-2013.07.

[29] by using the compactness of the constraint set and the simplicity of its representation in the Euclidean coordinates.

On the other hand, in the very recent paper [6], Bolte et al. consider linear optimization problems over a nonempty compact convex semi-algebraic set. Following the philosophy of Spingarn and Rockafellar [50] (see also [38, 51]), they show that for almost every linear objective function there is a unique optimal solution, lying on a unique "active" manifold, around which the constraint set is "partly smooth", and the second-order sufficient conditions hold.

Since the linear and quadratic programs can be regarded as very special forms of polynomial optimization problems, we may ask questions about stability and genericity properties of the global solution map, the Karush-Kuhn-Tucker set-valued map, and of the optimal value function for perturbed polynomial optimization problems with constraints.

Throughout this work we deal with the Euclidean space $\mathbb{R}^n$ equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. We suppose $1 \leq n \in \mathbb{N}$ and abbreviate $(x_1, x_2, \ldots, x_n)$ by x. For each integers d and n, let

$$n(d) \ := \ \{\alpha \in \mathbb{N}^n \mid |\alpha| \leq d\}.$$

By using the lexicographic ordering on the set of monomials x^α, $|\alpha| \leq d$, for each $x \in \mathbb{R}^n$ we define a corresponding vector

$$vec(x) := (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_1x_n, \ldots, x_1^d, \ldots, x_n^d)^T \in \mathbb{R}^{n(d)}.$$

Let $g_1, \ldots, g_l, h_1, \ldots, h_m$ be real polynomials on $\mathbb{R}^n$ with degree at most d, and let

$$S := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \ldots, g_l(x) = 0, \ h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}.$$

We consider a parameter $u := (u_\alpha)_{|\alpha| \leq d} \in \mathbb{R}^{n(d)}$ and the corresponding parameterized nonconvex optimization problem:

$$\text{minimize } f_u(x) \qquad \text{subject to} \qquad x \in S, \tag{1}$$

where $f_u(x) := vec(x)^T u = \sum_{|\alpha| \leq d} u_\alpha x^\alpha$ is a polynomial on $\mathbb{R}^n$ with degree at most d.

The sets of the Karush-Kuhn-Tucker points and of the global solutions of the problem (1) are abbreviated as $KKT(u)$ and $Sol(u)$, respectively. The optimal value of (1) is denoted by $\phi(u)$; i.e., $\phi(u) := \min_{x \in S} f_u(x)$ for $u \in \mathbb{R}^{n(d)}$.

In this paper, we are interested in studying stability and genericity properties of the set-valued maps $KKT(\cdot)$, $Sol(\cdot)$, and the function $\phi(\cdot)$. First, we establish the strong Hölder stability with exponent explicitly determined, the upper semicontinuity and the lower semicontinuity of global solution map $Sol(\cdot)$ and of Karush-Kuhn-Tucker set-valued map $KKT(\cdot)$ for polynomial optimization problems with perturbed objective functions. Furthermore, we give explicit formulas for computing the directional derivative and the Clarke subdifferential of the optimal value function $\phi(\cdot)$, and prove the semi-smoothness with explicit degree for the function. These generalize results obtained in the papers [29, 43, 53].

Second, we show in this work that, for almost every parameter $u \in \mathbb{R}^{n(d)}$, Problem (1) has a unique optimal solution for which the global quadratic growth condition and the strong second-order sufficient conditions hold. Further, under local perturbations to the objective function, the optimal solution and the optimal value function (resp., the Karush-Kuhn-Tucker set-valued map) vary smoothly (resp., continuously) and the set of active constraint indices is constant.

Finally, let us consider the problem of computing numerically the optimal value $\phi(u)$ of Problem (1). As is well-known, this is an NP-hard problem even when the degree of the polynomial f_u is fixed to be four [34]. For instance, Problem (1) contains *the partition problem* which is known to be NP-complete [16]. A standard approach for solving Problem (1) is the hierarchy of semidefinite program relaxations proposed by Lasserre [25] (see also [39, 40, 49]). It is based on results about moment sequences and (the dual theory of) representations of nonnegative polynomials as sums of squares. For details about these methods and their applications, see [12, 18, 19, 20, 21, 25, 26, 27, 31, 32, 35, 41, 48].

Assume that the constraint set S is compact. Very recently, Nie showed in [36] that Lasserre's hierarchy has finite convergence generically. On the other hand, in practical, one is usually interested not only in finding the minimum value $\phi(u)$ of f_u on S, but also in obtaining a minimizer $x(u) \in Sol(u)$. Section 6 deals with this problem. Speaking very roughly, we can find a natural sequence of computationally feasible semidefinite programs, whose solutions give rise to a sequence of points in $\mathbb{R}^n$ converging to the optimal solution of the original problem (see Theorem 6.2). This fact opens up the possibility of solving previously intractable polynomial optimization problems.

Main tools for our investigation come from semi-algebraic geometry (for example, Cell Decomposition Theorem, Sard's Theorem with parameter, Tarski-Seidenberg Theorem, etc.).

Remark 1.1. (i) It should be emphasized that we do not require the polynomials f_u, g_i, and h_j to be convex, and their degrees can be arbitrary.

(ii) The results in this paper can be established if we replace the perturbed objective function $f_u(x)$ by $f(x) + u^T x$, where f is a polynomial on $\mathbb{R}^n$. Furthermore, while all results are stated for polynomial functions, the authors believe analogous results hold for semi-algebraic C^2-functions or, more generally for sub-analytic C^2-functions and indeed for any "tame" class of C^2-functions (see Ioffe's survey [22]). However, quantities (e.g., the degree of Hölder stability, the degree of semismoothness, the number of connected components, etc.) cannot be explicitly calculated. Further, to lighten the exposition, we do not pursue this extension here.

The organization of the paper is as follows. Section 2 recalls some preliminary results from semi-algebraic geometry; some basic notions in set-valued analysis are also given there. Stability properties of the global solution map $Sol(\cdot)$, of the optimal value function $\phi(\cdot)$, and of the Karush-Kuhn-Tucker set-valued map $KKT(\cdot)$ are given in Sections 3, 4, and 5, respectively. Some generic properties for polynomial optimization problems are presented in Section 6.

2. Preliminaries

2.1. **Semi-algebraic geometry.** In this subsection, we recall some notions and results of semi-algebraic geometry, which can be found in [3, 4, 15].

Definition 2.1. (i) A subset of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_i(x) = 0, i = 1, \ldots, k; f_i(x) > 0, i = k+1, \ldots, p\}$$

where all f_i are polynomials.

(ii) Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be semi-algebraic sets. A map $F\colon A \to B$ is said to be *semi-algebraic* if its graph

$$\{(x, y) \in A \times B \mid y = F(x)\}$$

is a semi-algebraic subset in $\mathbb{R}^n \times \mathbb{R}^m$.

Semi-algebraic sets and functions enjoy a number of remarkable properties:

(i) The class of semi-algebraic sets is closed with respect to Boolean operators; a Cartesian product of semi-algebraic sets is a semi-algebraic set;
(ii) The closure and the interior of a semi-algebraic set is a semi-algebraic set;
(iii) A composition of semi-algebraic maps is a semi-algebraic map;
(iv) The image and inverse image of a semi-algebraic set under a semi-algebraic map are semi-algebraic sets;
(v) If A is a semi-algebraic set, then the distance function

$$d(\cdot, A)\colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto d(x, A) := \inf\{\|x - a\| \mid a \in A\},$$

is semi-algebraic.

A major fact concerning the class of semi-algebraic sets is its stability under linear projections (see, for example, [3, 4]).

Theorem 2.1 (Tarski-Seidenberg Theorem)**.** *The image of a semi-algebraic set by a semi-algebraic map is semi-algebraic.*

Remark 2.1. As an immediate consequence of Tarski-Seidenberg Theorem, we get semi-algebraicity of any set $\{x \in A \mid \exists y \in B, (x, y) \in C\}$, provided that A, B, and C are semi-algebraic sets in the corresponding spaces. It follows that also $\{x \in A \mid \forall y \in B, (x, y) \in C\}$ is a semi-algebraic set as its complement is the union of the complement of A and the set $\{x \in A \mid \exists y \in B, (x, y) \notin C\}$. Thus, if we have a finite collection of semi-algebraic sets, then any set obtained from them with the help of a finite chain of quantifiers is also semi-algebraic.

The next theorem (see [4, 15]) uses the concept of a cell whose definition we omit. We do not need the specific structure of cells described in the formal definition. For us it will be sufficient to think of a C^p-cell of dimension r as of an r-dimensional C^p-manifold which is the image of the cube $(0,1)^r$ under a semi-algebraic C^p-diffeomorphism. As follows from the definition, an n-dimensional cell in $\mathbb{R}^n$ is an open set.

Theorem 2.2 (Cell Decomposition Theorem)**.**

(i) *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set. Then for any integer p, A can be represented as a disjoint union of a finite number of cells of class C^p.*

(ii) *Let F be a semi-algebraic map from a semi-algebraic set $A \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Then there exists a partition of A into a finite number of cells of class C^p such that the restriction of F on each cell is a map of class C^p.*

By Cell Decomposition Theorem, for any $p \in \mathbb{N}$ and any semi-algebraic subset $A \subset \mathbb{R}^n$, we can write A as a disjoint union of finitely many semi-algebraic C^p-manifolds of different dimensions. The *dimension* $\dim A$ of a semi-algebraic set A can thus be defined as the dimension of the manifold of highest dimension of its decomposition. This dimension is well defined and independent of the decomposition of A.

Next we state a semi-algebraic version of Sard's Theorem with parameter, sufficient for the applications in the next section. Recall that, for a C^∞ map of manifolds $f : X \to Y$, a point $y \in Y$ is called a *regular value* for f if either $f^{-1}(y) = \emptyset$ or the derivative map $f'(x) : T_xX \to T_yY$ is surjective at every point x such that $f(x) = y$, where T_xX and T_yY denote the tangent spaces of X at x and of Y at y, respectively. A point $y \in Y$ that is not a regular value of f is called a *critical value.*

Theorem 2.3 (Sard's Theorem with parameter)**.** *Let $F : P \times X \to Y$ be a C^∞-semi-algebraic map between semi-algebraic manifolds. If $y \in Y$ is a regular value of F, then there exists a semi-algebraic set Σ in P of dimension at most $\dim P - 1$ such that for each $p \in P \setminus \Sigma$, y is a regular value of the map $F_p : X \to Y, x \mapsto F(p, x)$.*

Proof. The theorem is a direct consequence of the following two facts:

(a) For almost every $p \in P$, y is a regular value of the map F_p (see, for example, [17, The Transversality Theorem]).

(b) The set of points $p \in P$ such that y is not a regular value of the map F_p is a semi-algebraic subset of P.

In fact, from the Implicit Function Theorem it follows that the preimage $F^{-1}(y)$ is a submanifold in $P \times X$. Let $\pi \colon P \times X \to P, (p, x) \mapsto p$, be the natural projection map. By the semi-algebraic version of Sard's theorem (see, for example, [3, 4]), the set of critical values, denoted by Σ, of the restriction map $\pi \colon F^{-1}(y) \to P$ is a semi-algebraic subset of P, of dimension smaller than the dimension of P.

Now let p be an arbitrary point in $P \setminus \Sigma$. We shall prove that y is a regular value of F_p. In fact, let $x \in X$ be such that $F_p(x) = y$. Given any vector $a \in T_yY$, we want to exhibit a vector $b \in T_xX$ such that

$$F_p'(x)(b) = a.$$

Because y is a regular value of F, we know that

$$F'(p, x)\left(T_{(p,x)}(P \times X)\right) = T_yY.$$

Hence, there exists a vector $c \in T_{(p,x)}(P \times X)$ such that

$$F'(p, x)(c) = a.$$

Now

$$T_{(p,x)}(P \times X) = T_pP \times T_xX,$$

so $c = (d, e)$ for some vectors $d \in T_pP$ and $e \in T_xX$. If d were zero we would be done, for since the restriction of F to $\{p\} \times X$ is F_p, it follows that

$$a = F'(p, x)(0, e) = F_p'(x)(e).$$

Although d need not be zero, we may use the projection π to kill it off. As the derivative map

$$\pi'(p, x) \colon T_pP \times T_xX \to T_pP$$

is just projection onto the first factor, the regularity assumption that $\pi'(p, x)$ maps $T_{(p,x)}F^{-1}(y)$ onto T_pP tells us that there is some vector of the form (d, v) in $T_pF^{-1}(y)$. But the restriction of F on the manifold $F^{-1}(y)$ is constant y, so

$$F'(p, x)(d, v) = 0.$$

Consequently, the vector $b := e - v \in T_xX$ is our solution because we have that

$$\begin{aligned} F_p'(x)(b) &= F'(p, x)[(d, e) - (d, v)] \\ &= F'(p, x)(d, e) - F'(p, x)(d, v) \\ &= a - 0 \ = \ a. \end{aligned}$$

$\square$

In the sequel we will need the following result (see, for example, [3, 4]).

Theorem 2.4. *Let g_i as $i = 1, \dots, l$ and h_j as $j = 1, \dots, m$ be real polynomials on $\mathbb{R}^n$ with degree at most d, and let*

$$S := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \dots, g_l(x) = 0,\ h_1(x) \geq 0, \dots, h_m(x) \geq 0\}.$$

Then the number of connected components of the semi-algebraic set S is bounded above by $((m+1)d+1)(2(m+1)d+1)^n$.

For each integers d and n, let

$$\mathscr{R}(n,d) \;\; := \;\; \begin{cases} 1 & \text{if } d = 1, \\ d(3d-3)^{n-1} & \text{otherwise.} \end{cases}$$

We recall the following Hölder-type error bound with an explicit exponent (see [13, 24, 30, 42]).

Theorem 2.5. *Let g_i as $i = 1, \dots, l$ and h_j as $j = 1, \dots, m$ be real polynomials on $\mathbb{R}^n$ with degree at most d, and let*

$$S := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \dots, g_l(x) = 0,\ h_1(x) \geq 0, \dots, h_m(x) \geq 0\}.$$

Let R be a positive number such that S contains an element x with $\|x\| \leq R$. Then, there exists a constant $c > 0$ such that

$$d(x, S) \leq c\left(\sum_{i=1}^{l} |g_i(x)| + \sum_{j=1}^{m} [-h_j(x)]_+ \right)^{\frac{1}{\mathscr{R}(n+l+m,\, d+1)}} \qquad \textit{for all } x \textit{ with } \|x\| \leq R.$$

Here and in the following we let $[r]_+ := \max\{r, 0\}$.

2.2. **Basic notions in set-valued analysis.** In this subsection we recall the definitions of continuity (upper semi-continuity, lower semi-continuity) for set-valued maps, and other related notions from variational analysis. We refer to [1, 7, 33, 46] for more details.

We say that $\mathcal{S}$ is a *set-valued map* or a *multifunction* from X to Y, denoted by $\mathcal{S}\colon X \rightrightarrows Y$, if, for every $u \in X$, $\mathcal{S}(u)$ is a subset of Y. A set-valued map $\mathcal{S}\colon X \rightrightarrows Y$ is called *closed-valued* if $\mathcal{S}(x)$ is closed for all $u \in X$.

We now recall the definitions of upper and lower semi-continuity.

Definition 2.2. Let $\mathcal{S}\colon \mathbb{R}^N \rightrightarrows \mathbb{R}^n$ be a set-valued map. It is said that $\mathcal{S}$ is *upper semi-continuous* (usc) at $\bar{u} \in \mathbb{R}^N$ if for each open set $V \subset \mathbb{R}^n$ satisfying $\mathcal{S}(\bar{u}) \subset V$, there exists $\epsilon > 0$ such that $\mathcal{S}(u) \subset V$ whenever $\|u - \bar{u}\| < \epsilon$. If for each open set $V \subset \mathbb{R}^n$ satisfying $\mathcal{S}(\bar{u}) \cap V \neq \emptyset$ there exists $\epsilon > 0$ such that $\mathcal{S}(u) \cap V \neq \emptyset$ whenever $\|u - \bar{u}\| < \epsilon$, then $\mathcal{S}$ is said to be *lower semi-continuous* (lsc) at $\bar{u} \in \mathbb{R}^N$. If $\mathcal{S}$ is simultaneously usc and lsc at $\bar{u}$, we say that it is *continuous* at $\bar{u}$.

In the sequel we denote by $\mathbb{B}^N$ the closed unit ball in $\mathbb{R}^N$. For subsets $A_1, A_2 \subset \mathbb{R}^N$ and $r \in \mathbb{R}$ we set

$$A_1 + rA_2 := \{a_1 + ra_2 \mid a_1 \in A_1,\ a_2 \in A_2\}.$$

Definition 2.3. A set-valued map $\mathcal{S}\colon \mathbb{R}^N \rightrightarrows \mathbb{R}^n$ is said to be *strong Hölder stable* of degree $\mathcal{H}$ ($\mathcal{H} > 0$) if for any fixed $\bar{u} \in \mathbb{R}^N$ there exists a constant $c > 0$ such that

$$\mathcal{S}(u) \subset \mathcal{S}(\bar{u}) + c\|u - \bar{u}\|^{\mathcal{H}}\, \mathbb{B}^n \quad \text{for all} \quad u \in \mathbb{R}^N.$$

2.3. **Clarke subdifferential.** We also recall the notion of Clarke subdifferential-that is, an appropriate multivalued operator playing the role of the usual gradient map-which is crucial for our considerations.

Definition 2.4 ([9])**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function and $x \in \mathbb{R}^n$. The *Clarke subdifferential* of f at x, denoted by $\partial f(x)$, is defined by

$$\partial f(x) := \left\{ v \in \mathbb{R}^n \ : \ \limsup_{x' \to x,\ t \downarrow 0} \frac{f(x' + tu) - f(x')}{t} \geq \langle v, u\rangle, \forall u \in \mathbb{R}^n \right\}.$$

2.4. **Semi-smoothness.** Next we consider a locally Lipschitz function $F\colon \mathbb{R}^N \to \mathbb{R}^m$, and denote the set of points in $\mathbb{R}^N$ where F is differentiable by $\mathcal{D}$. (By Rademacher's theorem, the complement of $\mathcal{D}$ has Lebesgue measure zero.)

Definition 2.5. Following [44, 52], we call F *semismooth* of degree $\mathcal{H}$ ($\mathcal{H} > 0$) at a point $\bar{u} \in \mathbb{R}^N$ if its directional derivative

$$F'(\bar{u}; d) := \lim_{t \to 0^+} \frac{F(\bar{u} + td) - F(\bar{u})}{t}$$

exists for every vector $d \in \mathbb{R}^N$, and as $d \to 0$ with $\bar{u} + d \in \mathcal{D}$, we have

$$F(\bar{u} + d) - F(\bar{u}) - F'(\bar{u} + d)d = O(\|d\|^{1+\mathcal{H}}).$$

Remark 2.2. It was proved in [5] that any locally Lipschitz semi-algebraic map $F\colon \mathbb{R}^N \to \mathbb{R}^m$ is semismooth of degree $\mathcal{H}$ for some $\mathcal{H} > 0$. However, there is no estimation on the exponent $\mathcal{H}$.

3. The global solution map

From now on, let g_i as $i = 1, \ldots, l$ and h_j as $j = 1, \ldots, m$ be real polynomials on $\mathbb{R}^n$ with degree at most d, and we assume that

$$S := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \ldots, g_l(x) = 0,\ h_1(x) \geq 0, \ldots, h_m(x) \geq 0\} \neq \emptyset.$$

We consider a parameter $u := (u_\alpha)_{|\alpha| \leq d} \in \mathbb{R}^{n(d)}$ and the corresponding parameterized non-convex optimization problem:

$$\phi(u) \ := \ \text{minimize } f_u(x) \quad \text{subject to} \quad x \in S,$$

where $f_u(x) := vec(x)^T u = \sum_{|\alpha|\le d} u_\alpha x^\alpha$ is a polynomial on $\mathbb{R}^n$ with degree at most d. Recall that, the set of all global optimal solutions of the problem is denoted by $Sol(u)$. The result of this section is as follows:

Theorem 3.1. *Assume that the constraint set S is nonempty compact. Then the following statements hold*

(i) *For each $u \in \mathbb{R}^{n(d)}$, $Sol(u)$ is a nonempty, compact, semi-algebraic set and it has at most $((m+1)d+1)(2(m+1)d+1)^n$ connected components.*
(ii) *The solution map $Sol\colon \mathbb{R}^{n(d)} \rightrightarrows \mathbb{R}^n$ is strong Hölder stable of degree $\mathcal{H} = \frac{1}{\mathscr{R}(n+l+m+1,d+1)}$.*
(iii) *$Sol(\cdot)$ is usc on $\mathbb{R}^{n(d)}$.*
(iv) *$Sol(\cdot)$ is lsc at $\bar{u} \in \mathbb{R}^{n(d)}$ if and only if $Sol(\bar{u})$ is a singleton.*
(v) *$Sol(\cdot)$ is continuous at $\bar{u} \in \mathbb{R}^{n(d)}$ if and only if $Sol(\bar{u})$ is a singleton.*

Proof. (i) Let u be arbitrary in $\mathbb{R}^{n(d)}$. Since the polynomial f_u is continuous and the constraint set S is compact, $Sol(u)$ is a nonempty, compact set. On the other hand, it is clear that

$$Sol(u) := \big\{x \in \mathbb{R}^n \mid g_i(x) = 0, i = 1, \dots, l,\ h_j(x) \ge 0, j = 1, \dots, m, \text{ and } \phi(u) - f_u(x) = 0\big\}.$$

By the assumptions, $g_i(\cdot), h_j(\cdot)$, and $\phi(u) - f_u(\cdot)$ are all polynomials on $\mathbb{R}^{n(d)}$ with degree at most d. Hence, $Sol(u)$ is a semi-algebraic set. Then, from Theorem 2.4, we get the desired conclusion.

(ii) Define

$$F(u,x) := \sum_{i=1}^{l} \big|g_i(x)\big| + \sum_{j=1}^{m} \big[-h_j(x)\big]_+ + \big|f_u(x) - \phi(u)\big|$$

and observe that $Sol(u) = \{x \in \mathbb{R}^n \mid F(u,x) = 0\}$.

Let $\bar{u}$ be arbitrary fixed in $\mathbb{R}^{n(d)}$. Since S is compact, it follows from Theorem 2.5 that there is a constant $c_0 > 0$ such that

$$d\big(x, Sol(\overline{u})\big) \le c_0\, F(\bar{u}, x)^{\mathcal{H}} \quad \text{for all} \quad x \in S,$$

where $\mathcal{H} := \frac{1}{\mathscr{R}(n+l+m+1,d+1)}$.

On the other hand, $L := \max_{x\in S} \|vec(x)\| < +\infty$ because the function $\mathbb{R}^n \to \mathbb{R}^{n(d)}, x \mapsto vec(x)$, is continuous on the compact set S. And so, we have, for all $x \in S$,

$$|f_u(x) - f_{\bar{u}}(x)| \;=\; |vec(x)^T(u-\bar{u})| \;\le\; \|vec(x)\|\|u-\bar{u}\| \;\le\; L\|u-\bar{u}\|.$$

For any $y \in Sol(u)$, since $Sol(\bar{u})$ is a closed set, we can select $z \in Sol(\overline{u})$ satisfying $\|y - z\| = d(y, Sol(\overline{u}))$. Note that

$$|f_{\overline{u}}(y) - \phi(\overline{u})| \;=\; F(\bar{u}, y) \;\ge\; c_1 d\big(y, Sol(\overline{u})\big)^{\frac{1}{\mathcal{H}}} \;=\; c_1\|y-z\|^{\frac{1}{\mathcal{H}}},$$

where $c_1 := c_0^{-\frac{1}{\mathcal{H}}} > 0$.

Since $z \in Sol(\overline{u})$, we have that $f_{\overline{u}}(z) = \phi(\overline{u}) \leq f_{\overline{u}}(y)$, and hence

$$\|y - z\|^{\frac{1}{\mathcal{H}}} \ \leq \ c_1^{-1}|f_{\overline{u}}(y) - \phi(\overline{u})| \ = \ c_1^{-1}\big(f_{\overline{u}}(y) - f_{\overline{u}}(z)\big). \tag{2}$$

Furthermore, it follows from $y \in Sol(u)$ that $f_u(z) \geq f_u(y)$, and therefore

$$\begin{aligned} f_{\overline{u}}(y) - f_{\overline{u}}(z) &= \big(f_u(y) - f_u(z)\big) + \big(f_u(z) - f_{\overline{u}}(z)\big) + \big(f_{\overline{u}}(y) - f_u(y)\big) \\ &\leq \big(f_u(z) - f_{\overline{u}}(z)\big) + \big(f_{\overline{u}}(y) - f_u(y)\big) \\ &\leq 2L\|u - \overline{u}\| \quad \text{as} \quad y, z \in S. \end{aligned}$$

It implies, together with Inequality (2), that

$$\|y - z\|^{\frac{1}{\mathcal{H}}} \ \leq \ c_1^{-1}\big(f_{\overline{u}}(y) - f_{\overline{u}}(z)\big) \ \leq \ 2c_1^{-1}L\|u - \overline{u}\|.$$

Thus

$$d\big(y, Sol(\overline{u})\big) \ = \ \|y - z\| \ \leq \ (2c_1^{-1}L)^{\mathcal{H}}\, \|u - \overline{u}\|^{\mathcal{H}},$$

which completes the proof of Item (ii).

(iii) It follows from Item (ii).

(iv) *Necessity.* On the contrary, suppose that $Sol(\cdot)$ is lsc at $\bar{u}$, but $Sol(\bar{u})$ is not a singleton. Since $Sol(\bar{u}) \neq \emptyset$, there are $\bar{x}, \bar{y} \in Sol(\bar{u})$ such that $\bar{x} \neq \bar{y}$. Choose $u \in \mathbb{R}^{n(d)}$ such that

$$\|u\| = 1 \quad \text{and} \quad f_u(\bar{x}) > f_u(\bar{y}).$$

Clearly, there exists an open neighborhood V of $\bar{x}$ such that

$$f_u(x) > f_u(\bar{y}) \quad \text{for all} \quad x \in V. \tag{3}$$

Given any $\epsilon > 0$, we fix a number $\delta \in (0, \epsilon)$ and put $u^\delta := \bar{u} + \delta u$. Then, $\|u^\delta - \bar{u}\| = \delta\|u\| = \delta < \epsilon$. Our next goal is to show that $Sol(u^\delta) \cap V = \emptyset$. In fact, for any $x \in S \cap V$, since $\bar{x}, \bar{y} \in Sol(\bar{u})$, by Inequality (3), we have

$$\begin{aligned} f_{u^\delta}(x) &= f_{\bar{u}}(x) + \delta f_u(x) \ \geq \ f_{\bar{u}}(\bar{x}) + \delta f_u(x) \\ &> f_{\bar{u}}(\bar{x}) + \delta f_u(\bar{y}) = f_{\bar{u}}(\bar{y}) + \delta f_u(\bar{y}) = f_{u^\delta}(\bar{y}). \end{aligned}$$

It follows that $x \notin Sol(u^\delta)$. Thus, for the chosen neighborhood V of $\bar{x} \in Sol(\bar{u})$ and for any $\epsilon > 0$, there exists a vector $u^\delta \in \mathbb{R}^{n(d)}$ satisfying $\|u^\delta - \bar{u}\| < \epsilon$ and $Sol(u^\delta) \cap V = \emptyset$. This contradicts the lower semi-continuity of the solution map $Sol(\cdot)$ and proves that $Sol(\bar{u})$ is a singleton.

Sufficiency. Suppose that $Sol(\bar{u}) = \{\bar{x}\} \subset S$. Let V be an open set containing $\bar{x}$. By Item (i), $Sol(u) \neq \emptyset$ for all $u \in \mathbb{R}^{n(d)}$. By Item (iii), the solution map $Sol(\cdot)$ is usc at $\bar{u}$. Hence, there exists a constant $\epsilon > 0$ such that for any $u \in \mathbb{R}^{n(d)}$ satisfying $\|u - \bar{u}\| < \epsilon$ we have $Sol(u) \subset V$. For such ϵ, the set $Sol(u) \cap V = Sol(u)$ is nonempty whenever $\|u - \bar{u}\| < \epsilon$. This proves that the solution map $u \mapsto Sol(u)$ is lsc at $\bar{u}$.

(v) It is a direct consequence of Items (iii) and (iv). $\square$

4. The optimal value function

Recall that $\phi(u) := \min_{x\in S} f_u(x)$ for $u \in \mathbb{R}^{n(d)}$. The result of this section is as follows:

Theorem 4.1. *Assume that the constraint set S is nonempty compact. Then the following statements hold*

(i) *The function $\phi\colon \mathbb{R}^{n(d)} \to \mathbb{R}, u \mapsto \phi(u)$, is concave and semi-algebraic.*

(ii) *The function $\phi(\cdot)$ is global Lipschitz:*

$$|\phi(u) - \phi(u')| \le L\|u - u'\|, \quad \text{for all} \quad u, u' \in \mathbb{R}^{n(d)},$$

where $L := \max_{x\in S} \|vec(x)\| < +\infty$.

(iii) *$\phi(\cdot)$ is directionally differentiable at $u \in \mathbb{R}^{n(d)}$ in direction $d \in \mathbb{R}^{n(d)}$ and*

$$\phi'(u; d) = \min\{vec(x)^T d \mid x \in Sol(u)\}.$$

(iv) *The Clarke subdifferential $\partial\phi(u)$ is the convex hull of the set $\{vec(x) \mid x \in Sol(u)\}$.*

(v) *$\phi(\cdot)$ is differentiable at $\bar{u}$ if and only if the solution set $Sol(\bar{u})$ is a singleton. In this case, if $Sol(\bar{u}) = \{\bar{x}\}$, then $\nabla\phi(\bar{u}) = vec(\bar{x})$, where $\nabla\phi(\bar{u})$ denotes the gradient vector of ϕ at $\bar{u}$.*

(vi) *$\phi(\cdot)$ is semi-smooth of degree $\mathcal{H} = \frac{1}{\mathscr{R}(n+l+m+1,d+1)}$.*

Proof. (i) The optimal value function $\phi(\cdot)$ is well-defined because the polynomial f_u is continuous and the constraint set S is nonempty compact. For any $u, u' \in \mathbb{R}^{n(d)}$ and $t \in [0, 1]$, we have

$$\begin{aligned}\phi(tu + (1-t)u') &= \min_{x\in S} f_{tu+(1-t)u'}(x) = \min_{x\in S}(tf_u(x) + (1-t)f_{u'}(x)) \\ &\ge t\min_{x\in S} f_u(x) + (1-t)\min_{x\in S} f_{u'}(x) = t\phi(u) + (1-t)\phi(u').\end{aligned}$$

Therefore, the function $\phi(\cdot)$ is concave. By definition, the graph of $\phi(\cdot)$ is

$$\begin{aligned}\{(u, v) \in \mathbb{R}^{n(d)} \times \mathbb{R} \quad | \quad & \forall x \in S\ v \le f_u(x) \text{ and} \\ & \forall\epsilon > 0 \ \Rightarrow\ \exists x \in S\ v > f_u(x) - \epsilon\},\end{aligned}$$

and hence is semi-algebraic by Tarski-Seidenberg Theorem (see Theorem 2.1 and Remark 2.1).

(ii) Take any $a \in Sol(u)$ and $a' \in Sol(u')$. Observe that

$$\begin{aligned}\phi(u) - \phi(u') &= f_u(a) - \min_{x\in S} f_{u'}(x) \\ &\ge f_u(a) - f_{u'}(a) = vec(a)^T(u - u') \\ &\ge -\|vec(a)\|\|u - u'\|\end{aligned}$$

and

$$\begin{array}{rcl}\phi(u)-\phi(u') & = & \min_{x\in S} f_u(x)-f_{u'}(a') \\ & \leq & f_u(a')-f_{u'}(a')=vec(a')^T(u-u') \\ & \leq & \|vec(a')\|\|u-u'\|.\end{array}$$

These inequalities imply that

$$\begin{array}{rcl}|\phi(u)-\phi(u')| & \leq & \max\{\|vec(a)\|\|u-u'\|, \|vec(a')\|\|u-u'\|\} \\ & \leq & \max_{x\in S} \|vec(x)\|\|u-u'\|,\end{array}$$

and so the optimal value function $\phi(\cdot)$ is global Lipschitz on $\mathbb{R}^{n(d)}$ with the Lipschitz constant $L := \max_{x\in S}\|vec(x)\| < +\infty$.

(iii) It follows from Items (i) and (ii) that $\phi(\cdot)$ is directionally differentiable at u in direction d. By Danskin's theorem [11], we have

$$\begin{array}{rcl}\phi'(u;d) & = & \min\{\nabla_u f_u(x)^T d \mid x\in Sol(u)\} \\ & = & \min\{vec(x)^T d \mid x \in Sol(u)\}.\end{array}$$

(iv) Thanks to [9, Theorem 2.1], it is easy to check that $\partial\phi(u)$ is the convex hull of the set $\{\nabla_u f_u(x) \mid x\in Sol(u)\} = \{vec(x) \mid x\in Sol(u)\}$.

(v) *Necessity.* If $\phi(\cdot)$ is differentiable at $\bar{u}$, then the set $\partial\phi(\bar{u})$ is singleton. From Item (iv) it follows that the set $Sol(\bar{u})$ is singleton.

Sufficiency. Suppose that $Sol(\bar{u})=\{\bar{x}\}$. By Item (iii), we have

$$\phi'(\bar{u};d)=vec(\bar{x})^T d \quad \text{for all} \quad d\in\mathbb{R}^{n(d)}.$$

Thus the function $\mathbb{R}^{n(d)}\to\mathbb{R}, d\mapsto\phi'(\bar{u};d)$, is linear. Since $\phi(\cdot)$ is concave, it follows from [45, Theorem 25.2] that the function $\phi(\cdot)$ is differentiable at $\bar{u}$, and so $\nabla\phi(\bar{u})=vec(\bar{x})$.

(vi) According to Theorem 3.1(ii), the solution map $Sol\colon \mathbb{R}^{n(d)} \rightrightarrows \mathbb{R}^n$ is strong Hölder stable of degree $\mathcal{H} := \frac{1}{\mathscr{R}(n+l+m+1,d+1)}$, or equivalently, that for any fixed $\bar{u}\in \mathrm{R}^{n(d)}$, there exists a constant $c>0$ such that

$$Sol(u)\subset Sol(\bar{u})+c\|u-\bar{u}\|^{\mathcal{H}}\,\mathbb{B}^n, \quad \text{for all} \quad u\in\mathbb{R}^{n(d)}. \tag{4}$$

Consider further a vector $\Delta u\in\mathbb{R}^{n(d)}$ such that $\phi(\cdot)$ is differentiable at $\bar{u}+\Delta u$; the existence of such a vector follows from the classical Rademacher's theorem due to the Lipschitz continuity of $\phi(\cdot)$; see, e.g., [46]. This implies that $\partial\phi(\bar{u}+\Delta u)$ is a singleton. Then we get from Item (iv) that

$$\partial\phi(\bar{u}+\Delta u)=\big\{\nabla\phi(\bar{u}+\Delta u)\big\}=\{vec(a)\}$$

for some $a\in Sol(\bar{u}+\Delta u)$.

To complete the proof of Item (v) it remains to show that

$$\phi(\bar{u}+\Delta u)-\phi(\bar{u})-vec(a)^T\Delta u = O(\|\Delta u\|^{1+\mathcal{H}}).$$

In fact, since the polynomial map $\mathbb{R}^n \to \mathbb{R}^{n(d)}, x \mapsto vec(x)$, is locally Lipschitz, there is a constant $\kappa > 0$ with

$$\|vec(x)-vec(y)\| \quad \leq \quad \kappa\|x-y\|$$

for all $x, y \in S$. (Note that S is compact.)

The set $Sol(\bar{u})$ is closed. So there exists a point $b \in Sol(\bar{u})$ such that $\|a-b\| = d(a, Sol(\bar{u}))$. Then Inclusion (4) implies that

$$\|a-b\| \leq c\|\Delta u\|^{\mathcal{H}}.$$

Note from Item (iv) that $vec(b) \in \partial\phi(\bar{u})$, which gives us by the concavity of $\phi(\cdot)$ the following estimate

$$\phi(\bar{u}+\Delta u)-\phi(\bar{u}) \leq vec(b)^T\Delta u.$$

Then we deduce successively

$$\begin{aligned}
\phi(\bar{u}+\Delta u)-\phi(\bar{u})-vec(a)^T\Delta u \quad &\leq \quad vec(b)^T\Delta u - vec(a)^T\Delta u \;=\; (vec(b)-vec(a))^T\Delta u \\
&\leq \quad \|vec(b)-vec(a)\|\|\Delta u\| \\
&\leq \quad \kappa\|b-a\|\|\Delta u\| \\
&\leq \quad c\kappa\|\Delta u\|^{1+\mathcal{H}}.
\end{aligned}$$

On the other hand, it follows from the equality $\nabla\phi(\bar{u}+\Delta u) = \{vec(a)\}$ and the concavity of the function $\phi(\cdot)$ that

$$-vec(a)^T\Delta u = vec(a)^T(\bar{u}-(\bar{u}+\Delta u)) \geq \phi(\bar{u})-\phi(\bar{u}+\Delta u).$$

Consequently, we obtain

$$\phi(\bar{u}+\Delta u)-\phi(\bar{u})-vec(a)^T\Delta u \geq 0.$$

Therefore,

$$-c\kappa\|\Delta u\|^{1+\mathcal{H}} \;\leq\; 0 \;\leq\; \phi(\bar{u}+\Delta u)-\phi(\bar{u})-vec(a)^T\Delta u \;\leq\; c\kappa\|\Delta u\|^{1+\mathcal{H}}.$$

This completes the proof of Item (vi). □

Corollary 4.1. *Assume that the constraint set S is nonempty compact. Let $\bar{u}$ be an arbitrary parameter in $\mathbb{R}^{n(d)}$. The following statements are equivalent:*

(i) *The optimal value function $\phi(\cdot)$ is continuously differentiable at $\bar{u}$.*
(ii) *The global solution map $Sol(\cdot)$ is continuous at $\bar{u}$.*
(iii) *The optimization problem $\min_{x\in S} f_{\bar{u}}(x)$ has a unique minimizer.*

Proof. It is a direct consequence of Theorems 3.1 and 4.1 and the fact that the map

$$\mathbb{R}^n \to \mathbb{R}^{n(d)}, \quad x \mapsto vec(x),$$

is continuous. □

5. The Karush-Kuhn-Tucker set-valued map

In this section we will establish the strong Hölder stability with exponent explicitly determined, the upper semicontinuity, the lower semicontinuity and the set-valued differentiability of the Karush-Kuhn-Tucker set-valued map $KKT(\cdot)$.

Recall that the set-valued map $KKT\colon \mathbb{R}^{n(d)} \rightrightarrows \mathbb{R}^n, u \mapsto KKT(u)$, corresponding to the optimization problem $\min_{x\in S} f_u(x)$, is defined by

$$\begin{aligned} KKT(u) := \{x \in S \quad | \quad & \text{there exist } \lambda := (\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l \text{ and } \nu := (\nu_1, \ldots, \nu_m) \in \mathbb{R}^m \text{ such that} \\ & \nabla f_u(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0, \\ & \nu_j h_j(x) = 0,\ \nu_j \geq 0, \quad \text{for } j = 1, \ldots, m\}. \end{aligned}$$

Lemma 5.1. *For each $u \in \mathbb{R}^{n(d)}$, the set $KKT(u)$ is closed semi-algebraic and has at most*

$$B_0(d, n, l, m) := ((2m + 1)(d + 1) + 1)(2(2m + 1)(d + 1) + 1)^{n+l+m}$$

connected components.

Proof. For each $u \in \mathbb{R}^{n(d)}$, the set

$$\begin{aligned} W(u) := \{(x, \lambda, \nu) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \quad | \quad & g_i(x) = 0,\ i = 1, \ldots, l,\ h_j(x) \geq 0, j = 1, \ldots, m, \\ & \nabla f_u(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0, \\ & \nu_j h_j(x) = 0,\ \nu_j \geq 0, \quad \text{for } j = 1, \ldots, m\} \end{aligned}$$

is semi-algebraic. It is clear that $KKT(u) = \pi(W(u))$, where the projection $\pi\colon \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^n$ is defined by $\pi(x, \lambda, \nu) := x$. Thanks to Tarski-Seidenberg Theorem (Theorem 2.1), the set $KKT(u)$ is semi-algebraic. Further, by Theorem 2.4, the number of connected components of the semi-algebraic set $W(u)$ is bounded above by $B_0(d, n, l, m)$. □

Sometimes the Karush-Kuhn-Tucker (KKT for short) system fails to hold at some minimizers. Hence, we usually make an assumption called a *constraint qualification* to ensure that the KKT system holds. Such a constraint qualification-probably the one most often used in the design of algorithms-is defined as follows:

Definition 5.1. (see [37, Definition 12.1]). For each $x \in S$, let $J(x)$ be the set of indices j for which h_j vanishes at x. The constraint set S is called *regular*, if for each $x \in S$, the gradient vectors $\nabla g_i(x),\ i = 1, \ldots, l$, and $\nabla h_j(x),\ j \in J(x)$, are linearly independent.

Remark 5.1. It is worth noting (see [19, 20]) that, if the constraint set S is regular, then for any fixed $u \in \mathbb{R}^{n(d)}$, the objective function f_u is constant on each connected component of $KKT(u)$; in particular, $f_u(KKT(u))$ is a finite set, and so, by Lemma 5.1, the set $f_u(KKT(u))$ has at most $B_0(d,n,l,m)$ points for all $u \in \mathbb{R}^{n(d)}$.

We also recall the following definition.

Definition 5.2. Let $u \in \mathbb{R}^{n(d)}$ and $\bar{x} \in S$. We say that the *strong second-order sufficient conditions* for the optimization problem $\min_{x \in S} f_u(x)$ hold at $\bar{x}$ if there exist Lagrange multipliers $\lambda_1, \ldots, \lambda_l$ and $\nu_1, \ldots, \nu_m$ such that the following conditions satisfy

$$
\begin{aligned}
&\nabla f_u(\bar{x}) - \sum_{i=1}^{l} \lambda_i \nabla g_i(\bar{x}) - \sum_{j=1}^{m} \nu_j \nabla h_j(\bar{x}) = 0,\\
&\nu_j h_j(\bar{x}) = 0, \ \nu_j \geq 0, \quad \text{for } j = 1, \ldots, m,\\
&\nu_j > 0 \ \text{ for all } \ j \in J(\bar{x}) = \{j \mid h_j(\bar{x}) = 0\},\\
&v^T \nabla^2 L(\bar{x}) v > 0 \quad \text{ for all } v \in \mathcal{M}(\bar{x})^{\perp}, v \neq 0.
\end{aligned}
$$

Here $\nabla^2 L(\bar{x})$ is the Hessian of the Lagrange function

$$
L(x) := f_u(x) - \sum_{i=1}^{l} \lambda_i g_i(x) - \sum_{j \in J(\bar{x})} \nu_j h_j(x),
$$

$\mathcal{M}(\bar{x})$ stands for the Jacobian of the active constraining polynomials

$$
\mathcal{M}(\bar{x}) := \left[\nabla g_i(\bar{x}), \ i = 1, \ldots, l, \ \nabla h_j(\bar{x}), \ j \in J(\bar{x})\right]^T,
$$

and $\mathcal{M}(\bar{x})^{\perp}$ denotes the null space of $\mathcal{M}(\bar{x})$.

Now we derive a generic property of the set-valued map $KKT(\cdot)$ that we shall need. Here the last statement is well known; for completeness, we provide a proof.

Lemma 5.2. *Assume that the constraint set S is regular. Then there exists an open and dense semi-algebraic set $\mathscr{U}$ in $\mathbb{R}^{n(d)}$ such that the following statements hold*

- (i) *The number of points of $KKT(u)$, denoted by $\#KKT(u)$, is at most $B_0(d,n,l,m)$ for any $u \in \mathscr{U}$.*
- (ii) *The set-valued map $u \mapsto KKT(u)$ is continuous on the set $\mathscr{U}$.*
- (iii) *The strong second-order sufficient conditions satisfy at every local (or global) minimizer of the polynomial f_u on S for any $u \in \mathscr{U}$.*

Proof. (i)-(ii) For each nonempty subset $J := \{j_1, \ldots, j_k\}$ of $\{1, \ldots, m\}$, we let $\tilde{\nu}_J := (\tilde{\nu}_j)_{j \in J} \in \mathbb{R}^{\#J}$ and

$$
V_J := \{(x, \lambda, \tilde{\nu}) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J} \ \mid \ h_j(x) > 0, \ \text{for } j \notin J\}.
$$

Clearly, V_J is an open semi-algebraic set in $\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}$. Assume that $V_J \neq \emptyset$. We define the semi-algebraic map $\Phi_J\colon \mathbb{R}^{n(d)} \times V_J \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}$ by

$$\Phi_J(u, x, \lambda, \tilde{\nu}_J) := \left(\nabla f_u(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j \in J} \tilde{\nu}_j^2 \nabla h_j(x), g_1(x), \ldots, g_l(x), h_{j_1}(x), \ldots, h_{j_k}(x) \right).$$

A direct computation shows that

$$\left(\frac{\partial \Phi_J}{\partial u_\alpha} \;\Big|\; \frac{\partial \Phi_J}{\partial x_k} \right)_{|\alpha|=1,\ k=1,\ldots,n} = \left(\begin{array}{c|c} I_n & \cdots \\ \hline 0 & \nabla g_1(x) \\ \vdots & \vdots \\ 0 & \nabla g_l(x) \\ \hline 0 & \nabla h_{j_1}(x) \\ \vdots & \vdots \\ 0 & \nabla h_{j_k}(x) \end{array} \right),$$

where I_n denotes the identity matrix of order n. Since the constraint set S is regular, it follows that $0 \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}$ is a regular value of Φ_J. By Sard's Theorem with parameter (Theorem 2.3), there exists an open and dense semi-algebraic set $\mathscr{U}_J$ in $\mathbb{R}^{n(d)}$ such that for each $u \in \mathscr{U}_J$, $0 \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}$ is a regular value of the map

$$\Phi_{J,u}\colon V_J \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{\#J}, \quad (x, \lambda, \tilde{\nu}_J) \mapsto \Phi_J(u, x, \lambda, \tilde{\nu}_J).$$

Let

$$W_J(u) := \{(x, \lambda, \tilde{\nu}_J) \in V_J \mid \Phi_{J,u}(x, \lambda, \tilde{\nu}_J) = 0\} \quad \text{for} \quad u \in \mathbb{R}^{n(d)}.$$

It follows from the Inverse Function Theorem that for each $u \in \mathscr{U}_J$, all points of $W_J(u)$ are isolated. Note that $W_J(u)$ is a semi-algebraic set; so in view of Theorem 2.4, it has finitely many connected components. Therefore, $W_J(u)$ is a finite (possibly empty) set for each $u \in \mathscr{U}_J$. Further, by the Implicit Function Theorem, all (local) solutions $(x, \lambda, \tilde{\nu}_J)$ of the system $\Phi_{J,u}(x, \lambda, \tilde{\nu}_J) = 0$ depend analytically on the parameter $u \in \mathscr{U}_J$, and hence the set-valued map $u \mapsto W_J(u)$ is continuous on the set $\mathscr{U}_J$.

Let $\mathscr{U} := \bigcap_J \mathscr{U}_J$, where the intersection is taken all subsets J of $\{1, \ldots, m\}$. Then $\mathscr{U}$ is an open and dense semi-algebraic set in $\mathbb{R}^{n(d)}$.

On the other hand, by the construction, it is not hard to see that $x \in KKT(u)$ if and only if $x \in \pi_J(W_J(u))$, where $\pi_J(x, \lambda, \tilde{\nu}_J) := x$ and $J = J(x)$. Therefore,

$$KKT(u) = \bigcup_J \pi_J(W_J(u)).$$

Consequently, for any $u \in \mathscr{U}$, the set $KKT(u)$ is a finite set, and so, by Lemma 5.1, it has at most $B_0(d, n, l, m)$ points. Moreover, since the set-valued map $u \mapsto W_J(u)$ is continuous on the set $\mathscr{U}_J$, the set-valued map $u \mapsto KKT(u)$ is continuous on the set $\mathscr{U}$.

(iii) Take any $u \in \mathscr{U}$ and let $\bar{x} \in S$ be a local (or global) minimizer of the polynomial f_u on S. Since S is regular, there exist (unique) Lagrange multipliers $\lambda_1, \ldots, \lambda_l$ and $\nu_1, \ldots, \nu_m$ such that

$$\begin{aligned} \nabla f_u(\bar{x}) - \sum_{i=1}^{l} \lambda_i \nabla g_i(\bar{x}) - \sum_{j=1}^{m} \nu_j \nabla h_j(\bar{x}) &= 0, \\ \nu_j h_j(\bar{x}) &= 0, \ \nu_j \geq 0, \quad \text{for } j = 1, \ldots, m. \end{aligned}$$

Let $L(x)$ be the associated Lagrange function

$$L(x) := f_u(x) - \sum_{i=1}^{l} \lambda_i g_i(x) - \sum_{j \in J(\bar{x})} \nu_j h_j(x).$$

Since the constraint set S is regular, the second order necessary condition holds at $\bar{x}$, i.e.,

$$v^T \nabla^2 L(\bar{x}) v \geq 0 \quad \text{ for all } v \in \mathcal{M}(\bar{x})^{\perp}, v \neq 0.$$

We will show that the above inequality is strict.

By contradiction, suppose that there exists a nonzero vector $v \in \mathcal{M}(\bar{x})^{\perp}$ such that $v^T \nabla^2 L(\bar{x}) v = 0$. It implies that v is a minimizer of the optimization problem

$$\min_{z \in \mathbb{R}^n} z^T \nabla^2 L(\bar{x}) z \quad \text{ such that } \quad \mathcal{M}(\bar{x}) z = 0.$$

By the first order optimality condition for the above problem, there exists a vector $w \in \mathbb{R}^n$ such that $\nabla^2 L(\bar{x}) v - \mathcal{M}(\bar{x})^T w = 0$, which then implies

$$\begin{pmatrix} \nabla^2 L(\bar{x}) & \mathcal{M}(\bar{x})^T \\ \mathcal{M}(\bar{x}) & 0 \end{pmatrix} \begin{pmatrix} v \\ -w \end{pmatrix} = 0.$$

Since $v \neq 0$, it follows that

$$\det \begin{pmatrix} \nabla^2 L(\bar{x}) & \mathcal{M}(\bar{x})^T \\ \mathcal{M}(\bar{x}) & 0 \end{pmatrix} = 0. \tag{5}$$

Now let $J := J(\bar{x})$, $\lambda := (\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l$, and $\widetilde{\nu}_J := (\sqrt{\nu_j})_{j \in J} \in \mathbb{R}^{\# J}$. Since $u \in \mathscr{U}$, we have $0 = \Phi_{J,u}(\bar{x}, \lambda, \widetilde{\nu}_J)$ is a regular value of the map $\Phi_{J,u}(\cdot, \cdot, \cdot)$. This contradicts Equality (5).

Finally, we show that $\nu_j > 0$ for all $j \in J(\bar{x})$. To see this, let us write $J(\bar{x}) := \{j_1, \ldots, j_k\}$ with $1 \leq j_1 < j_2 < \cdots < j_k \leq m$. It is easy to see that if for some $j_\ell \in J(\bar{x})$ we have $\lambda_{j_\ell} = 0$, then the $(n + \ell)$th column of the Jacobian of the map $\Phi_{J,u}(\cdot, \cdot, \cdot)$ at $(\bar{x}, \lambda, \widetilde{\nu}_J)$ will vanish, in contradiction to nonsingularity. □

The following result is useful for Theorem 5.1 below.

Lemma 5.3. *Let $a_i, b_j, c_k \colon \mathbb{R}^N \to \mathbb{R}$ be polynomial functions and $d_k \colon \mathbb{R}^N \to \mathbb{R}^{n(d)}$ be polynomial maps of degree at most d, where $i = 1, \ldots, \tilde{l},\ j = 1, \ldots, \tilde{m},$ and $k = 1, \ldots, n$. Let*

$$\tilde{S} := \{z \in \mathbb{R}^N \ \ | \ \ a_i(z) = 0, \ i = 1, \ldots, \tilde{l}, \ b_j(z) \geq 0, \ j = 1, \ldots, \tilde{m}\}$$

and define the set-valued map $\mathcal{S}\colon \mathbb{R}^{n(d)} \rightrightarrows \mathbb{R}^N, u \mapsto \mathcal{S}(u),$ *by*

$$\mathcal{S}(u) := \{z \in \tilde{S} \quad | \quad c_k(z) + d_k(z)^T u = 0, \ k = 1, \ldots, n\}.$$

If $\tilde{S}$ *is a compact set, then* $\mathcal{S}(\cdot)$ *is strong Hölder stable of degree* $\mathcal{H} = \frac{1}{\mathscr{R}(N+n+\tilde{l}+\tilde{m},d+1)}$.

Proof. Let us define the function $F\colon \mathbb{R}^{n(d)} \times \mathbb{R}^N \to \mathbb{R}, (u, z) \mapsto F(u, z)$, by

$$F(u, z) := \sum_{i=1}^{\tilde{l}} |a_i(z)| + \sum_{j=1}^{\tilde{m}} [-b_j(z)]_+ + \sum_{k=1}^{n} |c_k(z) + d_k(z)^T u|.$$

Then it is easy to see that F has the following properties:

(a) F is a continuous, semi-algebraic function.

(b) F is nonnegative on $\mathbb{R}^{n(d)} \times \mathbb{R}^N$ and, for each $u \in \mathbb{R}^{n(d)}$, it holds that

$$\mathcal{S}(u) = \{z \in \mathbb{R}^N \mid F(u, z) = 0\}.$$

(c) F is globally Lipschitz in u, uniformly for $z \in \tilde{S}$; in fact, for any $u, \bar{u} \in \mathbb{R}^{n(d)}$ and $z \in \tilde{S}$, we have

$$\begin{aligned} |F(u,z) - F(\bar{u},z)| &\leq \left| \sum_{k=1}^{n} |c_k(z) + d_k(z)^T u| - \sum_{k=1}^{n} |c_k(z) + d_k(z)^T \bar{u}| \right| \\ &\leq \sum_{k=1}^{n} \big| |c_k(z) + d_k(z)^T u| - |c_k(z) + d_k(z)^T \bar{u}| \big| \\ &\leq \sum_{k=1}^{n} \left| d_k(z)^T (u - \bar{u}) \right| \leq \sum_{k=1}^{n} \|d_k(z)\| \|u - \bar{u}\| \\ &\leq L\|u - \bar{u}\|, \end{aligned}$$

where $L := \max_{z \in \tilde{S}} \sum_{k=1}^{n} \|d_k(z)\| < +\infty$.

Let us fix $\bar{u}$ in $\mathbb{R}^{n(d)}$ arbitrarily. By Theorem 2.5, there exists a constant $c_0 > 0$ such that

$$d(z, \mathcal{S}(\bar{u})) \leq c_0 F(\bar{u}, z)^{\mathcal{H}}, \qquad \text{for all} \qquad z \in \tilde{S},$$

where $\mathcal{H} := \frac{1}{\mathscr{R}(N+n+\tilde{l}+\tilde{m},d+1)}$. Therefore, we have for any $u \in \mathbb{R}^{n(d)}$ and $z \in \mathcal{S}(u)$,

$$\begin{aligned} d(z, \mathcal{S}(\bar{u})) &\leq c_0 F(\bar{u}, z)^{\mathcal{H}} \\ &\leq c_0 \left(F(u, z) + L\|u - \bar{u}\| \right)^{\mathcal{H}} \\ &= c_0 L^{\mathcal{H}} \|u - \bar{u}\|^{\mathcal{H}}, \end{aligned}$$

which completes the proof of the lemma. □

The main result of this section is the next theorem.

Theorem 5.1. *Assume that the constraint set* S *is nonempty, compact and regular. Then the following statements hold*

(i) *For each* $u \in \mathbb{R}^{n(d)}$, $KKT(u)$ *is a nonempty, compact, and semi-algebraic set.*

(ii) *The KKT set-valued map* $KKT\colon \mathbb{R}^{n(d)} \rightrightarrows \mathbb{R}^n$ *is strong Hölder stable of degree* $\mathcal{H} = \frac{1}{\mathscr{R}(2n+2l+4m+2,d+1)}$.

(iii) $KKT(\cdot)$ *is usc on* $\mathbb{R}^{n(d)}$.

(iv) *If* $KKT(\cdot)$ *is lsc at* $\bar{u}$, *then* $KKT(\bar{u})$ *is a finite set.*

Proof. (i) By the assumptions, $KKT(u)$ is a nonempty compact set. Then the statement follows immediately from Lemma 5.1.

(ii) Define the set-valued map $\mathcal{S}\colon \mathbb{R}^{n(d)} \rightrightarrows \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^m, u \mapsto \mathcal{S}(u)$, by

$$\begin{aligned}\mathcal{S}(u) := \{(x,\kappa,\lambda,\nu) \in S \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^m \quad | \quad & \kappa \nabla f_u(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0, \\ & \nu_j h_j(x) = 0,\ \nu_j \geq 0, \quad \text{for } j = 1,\ldots,m, \\ & \|(\kappa,\lambda,\nu)\|^2 = 1\}.\end{aligned}$$

By assumptions, we have for each fixed $u \in \mathbb{R}^{n(d)}$, $\mathcal{S}(u)$ is a nonempty, compact, semi-algebraic set. Moreover, since S is regular, it is easy to see that $\pi(\mathcal{S}(u)) = KKT(u)$, where $\pi\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^n, (x,\kappa,\lambda,\nu) \mapsto x$, is the projection on the first component.

For simplicity, we put $N := n+1+l+m$ and $z := (x,\kappa,\lambda,\nu) \in \mathbb{R}^N$. For each $k = 1,\ldots,n$, let us consider the polynomial map

$$d_k\colon \mathbb{R}^N \to \mathbb{R}^{n(d)}, \quad z \mapsto \left(\kappa \frac{\partial x^\alpha}{\partial x_k}\right)_{|\alpha| \leq d}.$$

Since $f_u(x) = vec(x)^T u = \sum_{|\alpha| \leq d} u_\alpha x^\alpha$, it holds that

$$\kappa \frac{\partial f_u}{\partial x_k}(x) = d_k(z)^T u, \quad \text{for} \quad k = 1,\ldots,n.$$

We are in a position to apply Lemma 5.3; in fact, for any fixed $\bar{u} \in \mathbb{R}^{n(d)}$, there exists a constant $c > 0$ such that, for all $u \in \mathbb{R}^{n(d)}$,

$$\mathcal{S}(u) \subset \mathcal{S}(\bar{u}) + c\|u - \bar{u}\|^{\mathcal{H}} \mathbb{B}^N,$$

where $\mathcal{H} := \frac{1}{\mathscr{R}(2n+2l+4m+2,d+1)}$. So, for any $z = (x,\kappa,\lambda,\nu) \in \mathcal{S}(u)$, we have

$$d((x,\kappa,\lambda,\nu),\mathcal{S}(\bar{u})) \ \leq \ c\|u - \bar{u}\|^{\mathcal{H}}.$$

This, together with the fact that $x = \pi(z) \in KKT(u)$, yields that

$$d(x, KKT(\bar{u})) \ \leq \ d((x,\kappa,\lambda,\nu),\mathcal{S}(\bar{u})) \ \leq \ c\|u - \bar{u}\|^{\mathcal{H}},$$

which proves Item (ii).

(iii) It follows from Item (ii).

(iv) Assume that the set-valued map $KKT(\cdot)$ is lsc at $\bar{u} \in \mathbb{R}^{n(d)}$. According to Lemma 5.2, there exists an open and dense semi-algebraic set $\mathscr{U}$ in $\mathbb{R}^{n(d)}$ such that for each $u \in \mathscr{U}$, the set $KKT(u)$ is finite and has at most $B_0(d, n, l, m)$ points. In particular, there exists a sequence $\{u^\ell\}_{\ell\in\mathbb{N}} \subset \mathscr{U}$ such that $\lim_{\ell\to\infty} u^\ell = \bar{u}$ and $\#KKT(u^\ell) \leq B_0(d, n, l, m)$ for all ℓ. Thus, $\#KKT(\bar{u}) \leq B_0(d, n, l, m)$ due to the lower semicontinuity of the set-valued map $KKT(\cdot)$ at the point $\bar{u}$. □

Remark 5.2. The converse of Theorem 5.1(iv) is not true (see, for example, [29, Example 2.2]).

6. Some generic properties for semi-algebraic compact programs

In this section we establish genericity results for semi-algebraic compact programs. Namely, we have:

Theorem 6.1. *Assume that S is compact. Let $p \geq 2$ be an integer. Then there exists an open and dense semi-algebraic set $\mathscr{U} \subset \mathbb{R}^{n(d)}$ such that the following statements hold.*

(i) *The restriction of $\phi(\cdot)$ on $\mathscr{U}$ is differentiable of class C^p.*
(ii) *The restriction of the solution map $Sol(\cdot)$ on $\mathscr{U}$ is a single-valued map of class C^{p-1}.*
(iii) *The set-valued map $KKT(\cdot)$ is continuous on $\mathscr{U}$.*

Furthermore, for all parameters $\bar{u} \in \mathscr{U}$, the corresponding optimization problem $\min_{x\in S} f_{\bar{u}}(x)$ has the following properties.

(iv) *There is a unique minimizer $\bar{x} \in S$.*
(v) *The global quadratic growth condition holds at $\bar{x}$ in the sense that there exists a constant $c > 0$ such that*

$$f_{\bar{u}}(x) - f_{\bar{u}}(\bar{x}) \quad \geq \quad c\|x - \bar{x}\|^2 \quad \textit{for all} \quad x \in S.$$

(vi) *If S is regular, then the strong second-order sufficient conditions hold at $\bar{x}$ and the set of active constraint indices is locally constant:*

$$\{j \mid h_j(x(u)) = 0\} \quad = \quad \{j \mid h_j(\bar{x}) = 0\}, \qquad \textit{for all} \;\; u \;\; \textit{near} \;\; \bar{u},$$

where $x(u)$ is the unique minimizer of the problem $\min_{x\in S} f_u(x)$.

Proof. The desired set $\mathscr{U}$ will be the intersection of dense open semi-algebraic sets $\mathscr{U}_1, \mathscr{U}_2$, and $\mathscr{U}_3$, which are constructed as follows.

(i) Due to Theorem 4.1(i), the optimal value function $\phi\colon \mathbb{R}^{n(d)} \to \mathbb{R}$ is semi-algebraic. By Cell Decomposition Theorem (Theorem 2.2), there exist C^p-cells $A_1, \ldots, A_k$ such that

(a) $\mathbb{R}^{n(d)} = \cup_{i=1}^k A_i$;
(b) $A_i \cap A_j = \emptyset$ for $i \neq j$; and
(c) The restriction of $\phi(\cdot)$ on each cell A_i is a map of class C^p.

Let $I := \{i \mid \dim A_i = n(d)\}$. Then we have that $I \neq \emptyset$, A_i is an open semi-algebraic set for all $i \in I$, and that $\cup_{i\notin I} A_i$ is a semi-algebraic set of dimension at most $n(d) - 1$. Hence,

$\mathscr{U}_1 := \cup_{i \in I} A_i$ is an open and dense semi-algebraic set in $\mathbb{R}^{n(d)}$. Furthermore, the function $\phi(\cdot)$ is differentiable of class C^p on $\mathscr{U}_1$.

(ii) By Item (i), the optimal value function $\phi(\cdot)$ is differentiable of class C^p on $\mathscr{U}_1$. Thanks to Theorem 4.1(v), we have for all $u \in \mathscr{U}_1$ that the solution set $Sol(u)$ is a singleton, say $Sol(u) = \{x(u)\}$ for some $x(u) \in S$, and $\nabla \phi(u) = vec(x(u))$. It follows that $\left(\frac{\partial \phi}{\partial u_\alpha}(u)\right)_{|\alpha|=1} = x(u)$, and therefore, the restriction of the solution map $Sol(\cdot)$ on $\mathscr{U}_1$ is differentiable of class C^{p-1} on $\mathscr{U}_1$.

(iii) By Lemma 5.1, the set-valued map $KKT(\cdot)$ is semi-algebraic and closed-valued (the sets $KKT(u)$ are all closed). Applying [10, Theorem 28], we deduce that $KKT(\cdot)$ is continuous on a dense open semi-algebraic set $\mathscr{U}_2$ in $\mathbb{R}^{n(d)}$.

(iv) It is an immediate consequence of Theorem 3.1(v) and Item (ii) above.

(v) The function $\phi(\cdot)$ is differentiable of class C^p on $\mathscr{U}_1$. By Taylor expansion, we have for any fixed parameter $\bar{u} \in \mathscr{U}_1$,

$$\phi(u) = \phi(\bar{u}) + \nabla \phi(\bar{u})(u - \bar{u}) + \frac{1}{2}(u - \bar{u})^T \nabla^2 \phi(\bar{u})(u - \bar{u}) + o(\|u - \bar{u}\|^2)$$

for all $u \in \mathscr{U}$ near $\bar{u}$, where $\nabla^2 \phi(\bar{u})$ denotes the Hessian of the function ϕ at $\bar{u}$. Hence there exist constants $\epsilon > 0$ and $\rho > 0$ such that

$$\phi(u) \geq \phi(\bar{u}) + vec(x(\bar{u}))^T (u - \bar{u}) - \frac{\rho}{2}\|u - \bar{u}\|^2 \quad \text{for all} \quad \|u - \bar{u}\| < \epsilon.$$

Furthermore, since the constraint set S is compact, we can clearly assume that

$$\epsilon^{-1} \times \max_{x, y \in S} \|vec(x) - vec(y)\| \quad < \quad \rho. \tag{6}$$

Now consider any point $x \in S$. Since $\phi(u) \leq f_u(x)$ for all $u \in \mathbb{R}^{n(d)}$, we deduce successively

$$\begin{aligned}
0 \ &\leq \ \inf_{u \in \mathbb{R}^n} \{f_u(x) - \phi(u)\} \\
&\leq \ \inf_{\|u - \bar{u}\| < \epsilon} \{f_u(x) - \phi(u)\} \\
&\leq \ \inf_{\|u - \bar{u}\| < \epsilon} \{f_u(x) - \phi(\bar{u}) - vec(x(\bar{u}))^T (u - \bar{u}) + \frac{\rho}{2}\|u - \bar{u}\|^2\} \\
&= \ \inf_{\|u - \bar{u}\| < \epsilon} \{vec(x)^T u - vec(x(\bar{u}))^T u + \frac{\rho}{2}\|u - \bar{u}\|^2\} \\
&= \ \inf_{\|u - \bar{u}\| < \epsilon} \{(vec(x) - vec(x(\bar{u})))^T u + \frac{\rho}{2}\|u - \bar{u}\|^2\} \\
&= \ (vec(x) - vec(x(\bar{u})))^T \bar{u} + \inf_{\|u - \bar{u}\| < \epsilon} \{(vec(x) - vec(x(\bar{u})))^T (u - \bar{u}) + \frac{\rho}{2}\|u - \bar{u}\|^2\} \\
&= \ f_{\bar{u}}(x) - f_{\bar{u}}(x(\bar{u})) + \inf_{\|u - \bar{u}\| < \epsilon} \{(vec(x) - vec(x(\bar{u})))^T (u - \bar{u}) + \frac{\rho}{2}\|u - \bar{u}\|^2\}.
\end{aligned}$$

It follows easily from the inequality (6) that the above infimum is attained at the point $u = \bar{u} - \rho^{-1}(vec(x) - vec(x(\bar{u}))$ satisfying $\|u - \bar{u}\| < \epsilon$. Replacing this value in the above inequality, we deduce for all $x \in S$ that

$$\begin{aligned} 0 &\leq f_{\bar{u}}(x) - f_{\bar{u}}(x(\bar{u})) - \frac{1}{2\rho}\|vec(x) - vec(x(\bar{u}))\|^2 \\ &\leq f_{\bar{u}}(x) - f_{\bar{u}}(x(\bar{u})) - \frac{1}{2\rho}\|x - x(\bar{u})\|^2, \end{aligned}$$

which yields the desired conclusion with $c := (2\rho)^{-1}$.

(vi) By Lemma 5.2, there exists a dense open semi-algebraic set $\mathscr{U}_3 \subset \mathbb{R}^{n(d)}$ such that if $\bar{u} \in \mathscr{U}_3$, then the strong second-order sufficient conditions for the optimization problem $\min_{x\in S} f_{\bar{u}}(x)$ hold at every local (or global) minimum point of the polynomial $f_{\bar{u}}$ on S.

Take any $\bar{u} \in \mathscr{U} = \mathscr{U}_1 \cap \mathscr{U}_2 \cap \mathscr{U}_3$. Since $\mathscr{U}$ is open, for each u near $\bar{u}$, we have $u \in \mathscr{U}$ and hence the optimization problem $\min_{x\in S} f_u(x)$ has a unique minimizer $x(u) \in S$ for which the strong second-order sufficient conditions satisfy.

It remains to show that

$$J(x(u)) = J(x(\bar{u}))$$

for all u near $\bar{u}$. Indeed, if $j \notin J(x(\bar{u}))$ then $h_j(x(\bar{u})) > 0$, and by continuity, we have for all u near $\bar{u}$, $h_j(x(u)) > 0$ and hence $J(x(u)) \subseteq J(x(\bar{u}))$. Further, the equality $J(x(u)) = J(x(\bar{u}))$ holds for all u near $\bar{u}$. Indeed, if it is not the case, then there exist a sequence $\{u^\ell\}_{\ell\in\mathbb{N}} \subset \mathscr{U}$, with $\lim_{\ell\to+\infty} u^\ell = \bar{u}$, and an index $j \in J(x(\bar{u})) \setminus J(x(u^\ell))$. Then $h_j(x(u^\ell)) > 0$. The strict complementarity condition implies that $\tilde{\nu}_j(u^\ell) = 0$ for all ℓ. By continuity, we get $\tilde{\nu}_j(\bar{u}) = 0$, which contradicts the facts that $\tilde{\nu}_j(\bar{u}) + h_j(x(\bar{u})) > 0$ and $h_j(x(\bar{u})) = 0$ (because $j \in J(x(\bar{u}))$). The proof of Theorem 6.1 is completed now. □

As a consequence of the above theorem, we obtain the following result:

Theorem 6.2. *Assume that the constraint set S is nonempty, compact and regular. Then there exists an open and dense semi-algebraic set $\mathscr{U} \subset \mathbb{R}^{n(d)}$ such that for each parameter $u \in \mathscr{U}$, the corresponding optimization problem $\min_{x\in S} f_u(x)$ has a unique optimal solution $\bar{x} \in S$ and we can find a sequence $\{x^\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{R}^n$ satisfying the following properties*

(i) *The problem of computing x^ℓ is a semidefinite program;*
(ii) *The sequence x^ℓ converges to the minimizer $\bar{x}$.*

Proof. Let $\mathscr{U} \subset \mathbb{R}^{n(d)}$ be an open and dense semi-algebraic set satisfying the conclusions of Theorem 6.1. Then, by [36, Theorem 1.1], for each parameter $u \in \mathscr{U}$, we can construct a finite sequence of semidefinite programs, say (SDP$_\ell$), $\ell \in \mathbb{N}$, whose optimal values converge monotonically, increasing to the optimal value $\phi(u)$; furthermore, since the problem $\min_{x\in S} f_u(x)$ has a unique minimizer $\bar{x} \in S$, it follows from [47, Corollary 13] (see also [26, Theorem 5.6]) that that every sequence of "nearly" optimal solutions of (SDP$_\ell$) gives rise to a sequence of points in $\mathbb{R}^n$ converging to the unique global minimizer $\bar{x}$. □

Acknowledgments. A part of this work was performed during a research visit of the second author at Department of Applied Mathematics, Pukyong National University, Busan, Korea. He wishes to thank the first author for his kind hospitality and support.

References

[1] J.-P. Aubin, and H. Frankowska, *Set-valued analysis,* Birkhauser, 1990.

[2] B. Bank, J. Guddat, D. Klatte, B. Kummer, and K. Tammer, *Nonlinear parametric optimization,* Akademie-Verlag, Berlin, 1982.

[3] R. Benedetti, and J. Risler, *Real algebraic and semi-algebraic sets*, Hermann, 1991.

[4] J. Bochnak, M. Coste, and M.-F. Roy, *Real algebraic geometry*, Vol. 36, Springer, 1998.

[5] J. Bolte, A. Daniilidis, and A. S. Lewis, *Tame functions are semismooth,* Math. Program., **117** (2009), 5–19.

[6] J. Bolte, A. Daniilidis, and A. S Lewis, *Generic optimality conditions for semialgebraic convex programs,* Math. Oper. Res., **36**, no. 1 (2011), 55–70.

[7] J. Bonnans, and A. Shapiro, *Perturbation analysis of optimization problems,* Springer Series in Operations Research. Springer-Verlag, New York, 2000.

[8] R.W. Cottle, J.-S. Pang, and R.E. Stone, *The linear complementarity problem,* Academic Press, New York, 1992.

[9] F. H. Clarke, *Generelized gradients and applications*, Trans. Amer. Math. Soc., **205** (1975), 247–262.

[10] A. Daniilidis, and J. Pang, *Continuity and differentiability of set-valued maps revisited in the light of tame geometry,* J. Lond. Math. Soc., (2) **83** (2011), No. 3, 637–658.

[11] J. Danskin, *The theory of max-min, with applications,* SIAM J. Appl. Math., **14** (1966), 641–664.

[12] J. Demmel, J.W. Nie, and V. Powers, *Representations of positive polynomials on noncompact semi-algebraic sets via KKT ideals,* J. Pure Appl. Algebra. 209(1), 189–200 (2007).

[13] S. T. Dinh, H. V. Hà, and T. S. Phạm, *Hölder-type global error bounds for non-degenerate polynomial systems,* Available online from http://arxiv.org/pdf/1411.0859v1.pdf

[14] A. L. Dontchev and T. Zolezzi, *Well-posed optimization problems,* Springer-Verlag, Berlin, 1993.

[15] L. van den Dries, and C. Miller, *Geometric categories and o-minimal structures,* Duke Math. J., **84** (1996), pp. 497–540.

[16] M. R. Garey and D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness,* San Francisco, W.H. Freeman & Company, Publishers, (1979).

[17] V. Guillemin, and A. Pollack, *Differential topology,* Reprint of the 1974 original. AMS Chelsea Publishing, Providence, RI, 2010. xviii+224 pp. ISBN: 978-0-8218-5193-7

[18] H. V. Hà, and T. S. Phạm, *Global optimization of polynomials using the truncated tangency variety and sums of squares,* SIAM J. Optim., 19(2) (2008), 941–951.

[19] H. V. Hà, and T. S. Phạm, *Solving polynomial optimization problems via the truncated tangency variety and sums of squares,* J. Pure Appl. Algebra, **213** (2009), 2167–2176.

[20] H. V. Hà, and T. S. Phạm, *Representations of positive polynomials and optimization on noncompact semi-algebraic sets,* SIAM J. Optim., **20** (2010), 3082–3103.

[21] D. Jibetean and M. Laurent, *Semidefinite approximations for global unconstrained polynomial optimization,* SIAM J. Optim., 16, 490–514 (2005).

[22] A. D. Ioffe, *An invitation to tame optimization,* SIAM J. Optim., **19** (2009), 1894–1917.

[23] D. Klatte, *On the Lipschitz behavior of optimal solutions in parametric problems of quadratic optimization and linear complementarity,* Optimization, **16** (1985), 819–831.

[24] K. Kurdyka and S. Spodzieja, *Separation of real algebraic sets and the Łojasiewicz exponent,* Proc. Amer. Math. Soc., **142** (2014), 3089–3102.

[25] J. B. Lasserre, *Global optimization with polynomials and the problem of moments,* SIAM J. Optim., **11** (3) (2001), 796–817.

[26] J. B. Lasserre, *Moments, positive polynomials and their applications,* Imperial College Press, 2009.

[27] M. Laurent, Sums of squares, moment matrices and optimization over polynomials. *Emerging Applications of Algebraic Geometry*, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds.), Springer, 157–270 (2009).

[28] G. M. Lee, N. N. Tam, and N. D. Yen, *Quadratic programming and affine variational inequalities: a qualitative study,* Series: "Nonconvex Optimization and its Applications", Vol. **78**. Springer Verlag, New York, (2005).

[29] G. M. Lee, N. N. Tam, and N. D. Yen, *Stability of linear-quadratic minimization over Euclidean balls,* SIAM J. Optim., **22** (2012), No. 3, 936–952.

[30] C. Li, B. S. Mordukhovich, and T. S. Phạm, *New fractional error bounds for polynomial systems with applications to Holderian stability in optimization and spectral theory of tensors*, Math. Program. Ser. A, (1014), DOI: 10.1007/s10107-014-0806-9.

[31] M. Marshall, *Positive polynomials and sums of squares,* Mathematical Surveys and Monographs 146. Providence, RI: American Mathematical Society, (2008).

[32] M. Marshall, *Representations of non-negative polynomials, degree bounds and applications to optimization,* Canad. J. Math., **61**(1) (2009), 205–221.

[33] B. S. Mordukhovich, *Variational analysis and generalized differentiation, I: Basic Theory, II: Applications*, Springer, Berlin, 2006.

[34] Y. Nesterov, *Squared functional systems and optimization problems,* High Performance Optimization, H. Frenk et al., eds., Kluwer Academic Publishers, Norwell, MA, (2000), pp. 405–440.

[35] J. Nie, J. Demmel, and B. Sturmfels, *Minimizing polynomials via sum of squares over the gradient ideal,* Math. Program. Ser. A 106, no. 3, 587–606 (2006).

[36] J. Nie, *Optimality conditions and finite convergence of Lasserre's hierarchy,* Math. Program. Ser. A, **146** (1-2) (2014), 97–121.

[37] J. Nocedal, and S. J. Wright, *Numerical optimization,* Springer Series in Operations Research, Springer-Verlag, NewYork, 1999.

[38] O. Fujiwara, *Morse programs: a topological approach to smooth constrained optimization. I.,* Math. Oper. Res., **7** (1982), no. 4, 602–616.

[39] P. A. Parrilo, *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization,* Ph.D. thesis, California Institute of Technology, May 2000.

[40] P. A. Parrilo and B. Sturmfels, *Minimizing polynomial functions,* DIMACS Series in Discrete Mathematics and Theoretical Computer Science, AMS, 60 (2003), 83–99.

[41] P. A. Parrilo, *Semidefinite programming relaxations for semialgebraic problems,* Math. Program., Ser. B, 96 (2) (2003), 293–320.

[42] T. S. Phạm, *An explicit bound for the Łojasiewicz exponent of real polynomials*, Kodai Mathematical Journal, **35** (2012), No. 2, 311–319.

[43] H. X. Phu, N. D. Yen, *On the stability of solutions to quadratic programming problems,* Math. Program., **89** (2001), 385–394.

[44] L. Qi, and J. Sun, *A nonsmooth version of Newton's method,* Math. Prog., **58** (1993), 353–367.

[45] R. T. Rockafellar, *Convex analysis,* Princeton University Press, Princeton (1970).

[46] R. T. Rockafellar, and R. J-B. Wets, *Variational analysis*, Springer, Berlin, 1998.

[47] M. Schweighofer, *Optimization of polynomials on compact semialgebraic sets,* SIAM J. Optim., 15 (3) (2005) 805–825.

[48] M. Schweighofer, *Global optimization of polynomials using gradient tentacles and sums of squares,* SIAM J. Optim. 17, no. 3, 920–942 (2006).

[49] N. Z. Shor, *An approach to obtaining global extremums in polynomial mathematical programming problems,* Kibernetika, **5** (1987), 102–106.

[50] J. E. Spingarn, and R. T. Rockafellar, *The generic nature of optimality conditions in nonlinear programming,* Math. Oper. Res., **4** (1979), 425–430.

[51] J. E. Spingarn, *On optimality conditions for structured families of nonlinear programming problems,* Math. Program., **22** (1) (1982), 82–92.

[52] D. Sun, and J. Sun, *Semismooth matrix-valued functions,* Math. Oper. Res., **27** (2002), 150–169.

[53] N. N. Tam, and N. D. Yen, *Continuity properties of the Karush-Kuhn-Tucker point set in quadratic programming problems,* Math. Program., **85** (1999), 193–206.

Department of Applied Mathematics, Pukyong National University, Busan 608-737, Korea
E-mail address: `gmlee@pknu.ac.kr`

Institute of Research and Development, Duy Tan University, K7/25, Quang Trung, Danang, Vietnam, and Department of Mathematics, University of Dalat, 1 Phu Dong Thien Vuong, Dalat, Vietnam
E-mail address: `sonpt@dlu.edu.vn`