

# Polyhedral studies of vertex coloring problems: The standard formulation

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## Abstract

Despite the fact that many vertex coloring problems are polynomially solvable on certain graph classes, most of these problems are not “under control” from a polyhedral point of view. The equivalence between optimization and separation suggests the existence of integer programming formulations for these problems whose associated polytopes admit elegant characterizations. In this work we address this issue. As a starting point, we focus our attention on the well-known *standard formulation* for the classical vertex coloring problem. We present some general results about this formulation and we give complete characterizations of the associated polytopes for trees and block graphs. Also, we show that the vertex coloring polytope associated to this formulation for a graph  $G$  and a set of colors  $C$  corresponds to a face of the *stable set polytope* of a particular graph  $\mathcal{S}_G^C$  and, based on this fact, we derive a new family of valid inequalities generalizing several known families from the literature. We conjecture that this new family of valid inequalities is sufficient to completely describe the vertex coloring polytope associated to cacti graphs. Finally, we study the perfectness of  $\mathcal{S}_G^C$ , in particular, we give a characterization of the graphs  $G$  for which  $\mathcal{S}_G^C$  is perfect and we show that those graphs are exactly block graphs, when  $|C| > 2$ .

KEYWORDS: vertex coloring, standard formulation, polyhedral characterization.

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## 1. Introduction

Given a graph  $G = (V, E)$ , a *coloring* of  $G$  is an assignment  $c : V \rightarrow \mathbb{N}$  of “colors” to vertices of  $G$ , such that  $c(v) \neq c(w)$  for each edge  $vw \in E$ . The *vertex coloring problem* consists in finding a coloring of  $G$  minimizing the number of used colors. This parameter is widely known as the *chromatic number* of  $G$ , and is denoted as  $\chi(G)$ . There are many variants of the graph coloring problem, motivated by real-life constraints; the following are some examples:

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*Precoloring extension* [2]: Given a graph  $G = (V, E)$  and a partial assignment  $\rho : V \rightarrow \mathbb{N}$ , this generalization of the classical vertex coloring problem asks for a coloring  $c$  with the smallest number of used colors such that  $c(v) = \rho(v)$  for every vertex  $v$  in the domain of  $\rho$ . In other words, a subset of vertices from  $G$  is already colored and the problem is to extend this coloring in a minimum fashion.

$\mu$ -coloring [3]: This generalization of the classical problem takes as additional input a function  $\mu : V \rightarrow \mathbb{N}$  defining an upper bound  $\mu(v)$  for the color assigned to each vertex  $v$ , i.e., the obtained coloring  $c : V \rightarrow \mathbb{N}$  must satisfy  $c(v) \leq \mu(v)$ , for every  $v \in V$ .

$(\gamma, \mu)$ -coloring [4]: Besides the upper-bounding function  $\mu : V \rightarrow \mathbb{N}$ , this generalization of the  $\mu$ -coloring problem considers also a function  $\gamma : V \rightarrow \mathbb{N}$  which establishes lower bounds on the assignments for the vertices of  $G$ . Now, a coloring  $c : V \rightarrow \mathbb{N}$  is asked for in such a way that  $\gamma(v) \leq c(v) \leq \mu(v)$  holds for every  $v \in V$ . Note that this problem is also a generalization of precoloring extension.

*List coloring* [27]: This problem considers a set  $L(v)$  of valid colors for each  $v \in V$  and asks for a coloring  $c$  such that  $c(v) \in L(v)$  for all  $v \in V$ . This version generalizes all the problems mentioned above.

There exist in the literature many other variants of the classical vertex coloring problem considering local constraints (see, for example, [27]). Although the classical vertex coloring problem is NP-hard [17], there are many graph classes for which this problem can be solved in polynomial time, one of the most important classes being perfect graphs [18]. A graph  $G$  is said to be *perfect* if  $\chi(H) = \omega(H)$  for every induced subgraph  $H$  of  $G$ , where  $\omega(H)$  represents the size of the maximum clique of  $H$ . However, the variants of the coloring problem mentioned above may not be polynomially solvable for perfect graphs, therefore it is interesting to study the computational complexities of these variants on subclasses of perfect graphs. In [4, 5], the complexity boundary between coloring and list-coloring is studied for several subclasses of perfect graphs. Table 1 shows a summary of known complexities for the graph classes studied in these previous works.

*Integer linear programming* (ILP) has proved to be a very suitable tool for solving combinatorial optimization problems [25], and in the last decade ILP has been successfully applied to graph coloring problems, by resorting to several formulations for the classical vertex coloring problem. The following are some examples of ILP formulations for this problem:

*Standard model* [11, 23, 24]: This model includes a binary variable  $x_{ic}$  for each vertex  $i \in V$  and each color  $c \in C$ , where  $C$  represents the set of available colors, asserting whether vertex  $i$  is assigned color  $c$  or not. This formulation may be extended with variables  $w_c$  for each color  $c \in C$

Class	Coloring	Precol	$\mu$ -col	$(\gamma, \mu)$ -col	List-col
Complete bipartite	P	P	P	P	NP-C
Bipartite	P	NP-C	NP-C	NP-C	NP-C
Cographs	P	P	P	?	NP-C
Distance-hereditary	P	NP-C	NP-C	NP-C	NP-C
Interval	P	NP-C	NP-C	NP-C	NP-C
Unit interval	P	NP-C	NP-C	NP-C	NP-C
Complete split	P	P	P	P	NP-C
Split	P	P	NP-C	NP-C	NP-C
Line of $K_{n,n}$	P	NP-C	NP-C	NP-C	NP-C
Line of $K_n$	P	NP-C	NP-C	NP-C	NP-C
Complements of bipartites	P	P	?	?	NP-C
Trees	P	P	P	P	P
Block	P	P	P	P	P
Cacti	P	P	P	P	P

“NP-C”: NP-complete problem      “P”: polynomial problem      “?”: open problem

Table 1: Known complexities on some graph classes [4, 5]

specifying whether this color is used or not; the minimum coloring is found by minimizing the sum of these variables.

*Orientation model* [6]: This model uses an integer variable  $x_i$  for each  $i \in V$  representing the color assigned to vertex  $i$ . In order to state the corresponding coloring constraints, the model introduces a binary orientation variable  $y_{ij}$  for each edge  $ij \in E$  such that  $y_{ij} = 1$  if and only if  $x_i < x_j$ .

*Distance model* [13]: This model uses an integer variable  $x_{ij}$  for each pair of vertices  $i, j \in V$  specifying the distance between the colors assigned to  $i$  and  $j$ , i.e.,  $x_{ij} = c(i) - c(j)$ , where  $c : V \rightarrow \mathbb{N}$  is the represented coloring. Orientation binary variables  $y_{ij}$  are necessary, as in the orientation model above.

*Representatives model* [8]: In this model, a coloring is determined by the color classes it induces, and each class is represented by one of its members. For each non-adjacent pair of vertices  $i, j \in V$ , the model uses a binary variable  $x_{ij}$  stating whether vertex  $i$  is the “representative” of the color class assigned to vertex  $j$  or not. Additionally, the model uses the variable  $x_{ii}$  for each  $i \in V$ , asserting whether  $i$  is the representative of its own color class or not.

*Stable sets model* [22]: Finding a proper vertex coloring is equivalent to finding a partition of  $V$  into stable sets. To this end, this model uses a binary variable  $x_S$  for each stable set  $S \subseteq V$  which indicates whether the set  $S$  is part of the solution or not. As the number of variables is potentially huge, a column generation approach is the natural computational approach to solve this model.

*Supernodal formulation* [7]: Given a graph  $G = (V, E)$ , another graph  $G' = (Q, E')$  is induced by contracting some particular vertices from  $G$ , such that every coloring for  $G$  corresponds to some (reversible) multicoloring for  $G'$ ; the supernodal formulation consists in finding a particular multicoloring among those. The model uses a binary variable  $x_{qc}$  for each “supernode”  $q \in Q$  and each color  $c \in C = \{1, \dots, |V|\}$  to determine if  $c$  is one of the colors assigned to  $q$  or not. Additionally, the model introduces a binary variable  $w_c$  for each color  $c \in C$  which specifies if color  $c$  is used by any supernode  $q \in Q$  or not. The objective function of this model asks to minimize the sum of these variables.

Although ILP is NP-hard, in many cases a complete description of the convex hull of its solutions is known and this description can be used to solve the *separation problem* associated to this polytope in polynomial time [26]. Based on the ellipsoid method, Grötschel, Lovász and Schrijver [19] proved that the separation problem and the optimization problem over a polytope are polynomially equivalent, i.e., if one problem is polynomially solvable, so is the other one (we properly define these two problems in the following section). From this equivalence stems the general belief that if a combinatorial optimization problem can be solved in polynomial time, then there should exist some ILP formulation of the problem for which the convex hull of its solutions admits an “elegant” characterization.

Despite the fact that many vertex coloring problems are polynomially solvable on certain graph classes, most of these problems are not “under control” from a polyhedral point of view. The mentioned equivalence between optimization and separation suggests that, for these problems, there must exist formulations with polynomially-solvable separation problems and, moreover, that these formulations may admit some elegant characterizations. The search for such characterizations is the main objective and motivation of our ongoing work.

From a theoretical point of view, our main objective is to complete the polyhedral counterpart of these combinatorially-solved graph coloring problems. On the other side, the study of these polytopes may lead us to a better understanding of their structures allowing us to (polyhedrally) find new classes of graphs colorable in polynomial time.

As a starting point, in this work we focus our attention on the well-known *standard formulation* [11, 23, 24] and we give a characterization of the coloring polytope associated to this formulation for some graph classes. The remainder of this paper is organized as follows. In Section 2 we present some general results about the standard formulation and the studied variants of coloring problems. In Section 3 we give complete polyhedral characterizations for these problems, for trees and block graphs. In Section 4 we show that the vertex coloring polytope associated to the standard formulation for a graph  $G$  corresponds to a face of the

stable set polytope of a particular graph  $\mathcal{S}_G^C$ . Based on this fact, we give a new family of valid inequalities which generalizes several known families from the literature. We conjecture that this new family of valid inequalities is sufficient to completely describe the vertex coloring polytope associated to cacti graphs. We study the perfectness of  $\mathcal{S}_G^C$ , in particular, we give a characterization of the graphs  $G$  for which  $\mathcal{S}_G^C$  is perfect. Finally, we draw our conclusions and pose our ongoing and future work.

## 2. The standard formulation and general results

Let  $G = (V, E)$  be the input graph and let  $C$  be the set of available colors. The standard formulation for the classical vertex coloring problem uses a binary variable  $x_{ic}$  for each vertex  $i \in V$  and each color  $c \in C$  specifying whether the vertex  $i$  is assigned color  $c$  or not. With this encoding, all proper vertex colorings of  $G$  using colors from  $C$  are those satisfying the following constraints:

$$\sum_{c \in C} x_{vc} = 1 \quad \forall v \in V, \quad (1)$$

$$x_{vc} + x_{wc} \leq 1 \quad \forall vw \in E, \forall c \in C, \quad (2)$$

$$x_{vc} \in \{0, 1\} \quad \forall v \in V, \forall c \in C. \quad (3)$$

Constraints (1)-(3) characterize the colorings of  $G$  with colors from  $C$ , but no objective function is asked to be optimized. In order to find  $\chi(G)$ , this formulation may be extended by using a binary variable  $w_c$  for each color  $c \in C$  specifying whether color  $c$  is used or not, and a coloring with the smallest number of colors can be found by minimizing the sum of the  $w$ -variables. Nevertheless, in this work we will focus our attention on the non-extended formulation, i.e., we will not consider the  $w_c$  variables.

Although the formulation given by (1)-(3) corresponds to the classical vertex coloring problem, it can be easily extended to model the list-coloring problem (and thereby the precoloring extension,  $\mu$ -coloring and  $(\gamma, \mu)$ -coloring problems). To this end, for each  $i \in V$ , we may add the constraints

$$x_{ic} = 0 \quad \forall c \in C \setminus L(i) \quad (4)$$

where  $L(i)$  is the set of allowed colors for vertex  $i$ .

We define  $P_{\text{col}}(G, C)$  (resp.  $P_{\text{list}}(G, C, L)$ ) to be the convex hull of the points  $x \in \mathbb{R}^{|V||C|}$  satisfying constraints (1)-(3) (resp. (1)-(4)). If  $\mathcal{G}$  is a family of graphs, then  $P_{\text{col}}(\mathcal{G}, C)$  and  $P_{\text{list}}(\mathcal{G}, C, L)$  denote the corresponding families of polytopes. We may omit the set  $C$  in the previous definitions whenever it is clear from the context.

Given a family of polytopes  $\mathcal{P}$ , the associated *separation problem* takes a polytope  $P \in \mathcal{P}$  and a vector  $\hat{y}$  and asks to determine whether  $\hat{y}$  belongs to  $P$  or not, and if not, to find a hyperplane separating  $\hat{y}$  from  $P$ . In turn, the

*optimization problem* takes a polytope  $P \in \mathcal{P}$  and a vector  $c$  and asks for a vector  $\hat{x} \in P$  maximizing the objective function  $c^t \hat{x}$ , unless  $P = \emptyset$ . Based on the ellipsoid method, Grötschel, Lovász and Schrijver [19] proved that the separation problem and the optimization problem are polynomially equivalent, i.e., if one of these problems is polynomially solvable over a family of polytopes  $\mathcal{P}$ , so is the other one.

**Theorem 2.1.** *Given a graph family  $\mathcal{G}$  and a set of colors  $C$ , the separation problem over  $P_{\text{col}}(\mathcal{G}, C)$  can be solved in polynomial time if and only if the separation problem over  $P_{\text{list}}(\mathcal{G}, C, L)$  can be solved in polynomial time for any  $L : V \rightarrow 2^C$ .*

*Proof.* Let  $Q = \{x \in \mathbb{R}^{|V||C|} : x_{ic} = 0 \text{ for every } i \in V \text{ and every } c \in C \setminus L(i)\}$ . We claim that

$$P_{\text{list}}(G, C, L) = P_{\text{col}}(G, C) \cap Q. \quad (5)$$

For the reverse inclusion, take a point  $\hat{x} \in P_{\text{col}}(G, C) \cap Q$ . Since  $\hat{x} \in P_{\text{col}}(G, C)$ , then  $\hat{x}$  is a convex combination of colorings  $x^1, \dots, x^k$  of  $G$ . Since  $\hat{x} \in Q$ , then  $\hat{x}_{ic} = 0$  for every vertex  $i \in V$  and every color  $c \in C \setminus L(i)$ , and this implies  $\hat{x}_{ic}^j = 0$  for every  $j = 1, \dots, k$ . Therefore,  $x^j \in Q$  for  $j = 1, \dots, k$ . Then, all these colorings belong to  $P_{\text{list}}(G, C, L)$  and so does  $\hat{x}$ . This proves that  $P_{\text{col}}(G, C) \cap Q \subseteq P_{\text{list}}(G, C, L)$ .

For the forward inclusion, now take a point  $\hat{x} \in P_{\text{list}}(G, C, L)$ . This point is a convex combination of colorings of  $G$  which use only assignments allowed by  $L$ . Hence, all these colorings belong to  $Q$  and so does  $\hat{x}$ , implying  $P_{\text{list}}(G, C, L) \subseteq P_{\text{col}}(G, C) \cap Q$ . This shows that (5) holds.

Assume the separation problem over  $P_{\text{col}}(G, C)$  can be solved in polynomial time. Then, a point  $\hat{x} \notin P_{\text{list}}(G, C, L)$  either does not belong to  $P_{\text{col}}(G, C)$  or has  $\hat{x}_{ic} > 0$  for some  $i \in V$  and  $c \in C \setminus L(i)$ . Hence, to separate a point from  $P_{\text{list}}(G, C, L)$  we just need to test if  $x_{ic} = 0$ , for all  $i \in V$  and  $c \in C \setminus L(i)$  and, if these conditions hold, separate the point (in polynomial time) from  $P_{\text{col}}(G, C)$ . Hence, the separation problem over  $P_{\text{list}}(G, C, L)$  can be solved in polynomial time.

The converse is trivial, since  $P_{\text{col}}(G, C)$  equals  $P_{\text{list}}(G, C, L)$  with  $L(i) = C$  for all  $i \in V$ .  $\square$

The mentioned equivalence between polyhedral separation and optimization problems [19] yields the following corollary.

**Corollary 2.2.** *Given a graph family  $\mathcal{G}$  and a set of colors  $C$ , the optimization problem over  $P_{\text{col}}(\mathcal{G}, C)$  can be solved in polynomial time if and only if the optimization problem over  $P_{\text{list}}(\mathcal{G}, C, L)$  can be solved in polynomial time for any set  $L$ .*

Finally, the results above let us reach an important conclusion about the potential of the formulation (1)-(3).

**Theorem 2.3.** *Let  $\mathcal{G}$  be a family of graphs and let  $C$  be a set of colors. If the list-coloring problem on  $\mathcal{G}$  and  $C$  is NP-complete, then the optimization/separation problem over  $P_{\text{col}}(\mathcal{G}, C)$  is NP-complete.*

*Proof.* Suppose the optimization problem over  $P_{\text{col}}(\mathcal{G}, C)$  can be solved in polynomial time. Then, we can optimize over  $P_{\text{list}}(G, C, L)$  for any  $L$  in polynomial time by solving the list-coloring problem on  $G$  and  $C$ , thus contradicting the hypothesis.  $\square$

Theorem 2.3 implies that even when the classical vertex coloring problem is polynomially solvable over  $\mathcal{G}$ , the polytope associated with the standard formulation cannot be subject to an elegant characterization if list-coloring over  $\mathcal{G}$  is NP-complete. This severely limits the analysis scope for the standard formulation, since list-coloring is NP-complete for many classes of graphs (see, e.g., Table 1). Nevertheless, there are some few graph classes for which list-coloring is known to be polynomially solvable such as trees, block and cacti graphs. Hence, this formulation may be explored in order to find nice polyhedral descriptions for the corresponding polytopes.

The following general result allows to find a description for  $P_{\text{col}}(G)$  when  $G$  is obtained by a particular operation from other two graphs. For the subsequent proof, we shall resort to the following principle used by Edmonds in [15].

**Proposition 2.4.** [15] *Let  $S$  be a finite set of solutions*

$$S \subseteq P = \{x \in \mathbb{R}_+^{|I|} : \sum_{i \in I} a_{ji} x_i \leq b_j, j \in J\}.$$

*Then  $\text{conv}(S) = P$  if and only if for every vector  $\phi \in \mathbb{Z}^{|I|}$ , we have*

$$\max\{\phi^T x : x \in S\} = \min\left\{\sum_{j \in J} \lambda_j b_j : \sum_{j \in J} \lambda_j a_{ji} \geq \phi_i, i \in I, \lambda \in \mathbb{R}_+^{|J|}\right\}.$$

Given two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  such that  $V_1 \cap V_2 = \{v\}$ , the *one-vertex identification* of  $G_1$  and  $G_2$  is the graph  $G = (V_1 \cup V_2, E_1 \cup E_2)$ .

**Theorem 2.5.** *Let  $C$  be a set of colors and let  $G = (V, E)$  be the one-vertex identification of two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  with  $V_1 \cap V_2 = \{v\}$ . For  $k = 1, 2$ , let*

$$P_{\text{col}}(G_k) = \{x \in \mathbb{R}_+^{|V||C|} : \sum_{i \in V} \sum_{c \in C} a_{ic}^j x_{ic} \leq b_j, j \in J_k\}.$$

*Then,*

$$P_{\text{col}}(G) = \{x \in \mathbb{R}_+^{|V||C|} : \sum_{i \in V} \sum_{c \in C} a_{ic}^j x_{ic} \leq b_j, j \in (J_1 \cup J_2)\}.$$

The following proof is a straightforward adaptation of a proof by Fekete [16] for a particular vertex coloring polytope defined in [14]. The proof by Fekete is in turn an adaptation of a proof given by Chvátal for a similar result on the stable set polytope [10].

*Proof.* Let  $\phi = (\phi_{ic})_{i \in V, c \in C}$  be an integer-valued vector and call  $\phi^k = (\phi_{ic})_{i \in V_k, c \in C}$ , for  $k = 1, 2$ . Let  $m = \max\{\phi^T x : x \in P_{\text{col}}(G)\}$  and for each  $c' \in C$  and  $k = 1, 2$ , define  $m_k(c') = \max\{(\phi^k)^T x : x \in P_{\text{col}}(G_k), x_{vc'} = 1\}$ . Note that

$$\max\{(\phi^k)^T x - \sum_{c \in C} m_k(c) x_{vc} : x \in P_{\text{col}}(G_k)\} = 0.$$

Then, for  $k = 1, 2$ , by Proposition 2.4, there are non-negative reals  $\mu_j$ ,  $j \in J_k$ , such that  $\sum_{j \in J_k} \mu_j b_j = 0$  and

$$\sum_{j \in J_k} \mu_j a_{ic}^j \geq \begin{cases} \phi_{ic}, & \text{for } i \in V_k \setminus \{v\} \text{ and } c \in C, \\ \phi_{vc} - m_k(c), & \text{for } i = v \text{ and } c \in C. \end{cases}$$

On the other hand, it is clear that  $\max\{\sum_{c \in C} m x_{vc} : x \in P_{\text{col}}(G_1)\} = m$ . Hence, by Proposition 2.4, there are non-negative reals  $\mu_j^*$ ,  $j \in J_1$ , such that  $\sum_{j \in J_1} \mu_j^* b_j = m$  and

$$\sum_{j \in J_1} \mu_j^* a_{ic}^j \geq \begin{cases} 0, & \text{for } i \in V_k \setminus \{v\} \text{ and } c \in C, \\ m, & \text{for } i = v \text{ and } c \in C. \end{cases}$$

Define now

$$\lambda_j = \begin{cases} \mu_j + \mu_j^*, & \text{if } j \in J_1, \\ \mu_j, & \text{if } j \in J_2. \end{cases}$$

We shall prove that  $\sum_{j \in (J_1 \cup J_2)} \lambda_j b_j = m$ , and for every  $i \in V, c \in C$ , we have  $\sum_{j \in (J_1 \cup J_2)} \lambda_j a_{ic}^j \geq \phi_{ic}$ , hence Proposition 2.4 implies the desired conclusion.

For  $i \in V_1 \setminus \{v\}$  and  $c \in C$  we have

$$\sum_{j \in (J_1 \cup J_2)} \lambda_j a_{ic}^j = \sum_{j \in J_1} \lambda_j a_{ic}^j = \sum_{j \in J_1} \mu_j a_{ic}^j + \sum_{j \in J_1} \mu_j^* a_{ic}^j \geq \phi_{ic}.$$

For  $i \in V_2 \setminus \{v\}$  and  $c \in C$  we have

$$\sum_{j \in (J_1 \cup J_2)} \lambda_j a_{ic}^j = \sum_{j \in J_2} \lambda_j a_{ic}^j = \sum_{j \in J_1} \mu_j a_{ic}^j \geq \phi_{ic}.$$

For each  $c \in C$  we have

$$\begin{aligned} \sum_{j \in (J_1 \cup J_2)} \lambda_j a_{vc}^j &= \sum_{j \in J_1} \mu_j a_{vc}^j + \sum_{j \in J_1} \mu_j^* a_{vc}^j + \sum_{j \in J_2} \mu_j a_{vc}^j \\ &\geq (\phi_{vc} - m_1(c)) + m + (\phi_{vc} - m_2(c)) = \phi_{vc} \end{aligned}$$

Finally,

$$\sum_{j \in (J_1 \cup J_2)} \lambda_j b_j = \sum_{j \in J_1} \mu_j b_j + \sum_{j \in J_1} \mu_j^* b_j + \sum_{j \in J_2} \mu_j b_j = m.$$

□



### 3. Trees and block graphs

A tree is a connected and acyclic graph and the list-coloring problem over trees is polynomially solvable [21]. The following theorem gives a complete characterization for  $P_{\text{col}}(G)$  when  $G$  is a tree. Indeed, we prove that if  $G$  is a tree, then  $P_{\text{col}}(G, C)$  is completely described by (1) and (2), for every color set  $C$ .

**Theorem 3.1.** *If  $G = (V, E)$  is a tree and  $C$  is a set of colors, then  $P_{\text{col}}(G, C) = \{x \in \mathbb{R}_+^{|V||C|} : x \text{ satisfies (1) and (2)}\}$ .*

*Proof.* If  $G$  is just an edge, then inequalities (1) and (2) represent an assignment problem, hence the corresponding formulation yields an integer polytope. If  $G$  contains two or more edges, then it is a one-vertex identification of two smaller trees, and Theorem 2.5 implies the result.  $\square$

Theorem 2.1 and Theorem 3.1 show that the separation problem over  $P_{\text{list}}(G, L)$  can be solved in polynomial time when  $G$  is a tree. This result yields a polyhedral counterpart of the combinatorial algorithm presented in [21] for solving the list-coloring problem on trees in polynomial time.

A *block graph* is a graph in which every biconnected component is a clique (these graphs are sometimes called *clique trees* due to their structure). The list-coloring problem over block graphs is polynomially solvable [20]. We stated that constraints (1) and (2) are enough to describe  $P_{\text{col}}(G)$  when  $G$  is a tree. Despite the similarities with trees, block graphs admit bigger cliques than trees do, and it is known that some facets of  $P_{\text{col}}(G)$ , for a general graph  $G$ , are defined by the corresponding clique inequalities [11].

**Theorem 3.2.** [11] *Let  $G = (V, E)$  be a graph and  $C$  a set of colors. Given a clique  $K \subseteq V$  and a color  $c \in C$ , the clique inequality*

$$\sum_{v \in K} x_{vc} \leq 1 \tag{6}$$

*is valid for  $P_{\text{col}}(G, C)$ . If  $|C| > \chi(G)$  and  $K$  is a maximal clique, then (6) defines a facet of  $P_{\text{col}}(G, C)$ .*

We prove next that constraints (1) and (6) give a complete characterization for  $P_{\text{col}}(G)$  when  $G$  is a block graph. It is worth mentioning that the results from Section 4.3 have implications that give an alternative proof of this fact by resorting to well-established theorems by Chvátal [10]. However, the proof given here is self-contained and we think that the used techniques and structures may be useful in other contexts, i.e., they may be extended to other families of graphs for which the alternative proof from Section 4.3 does not hold.

Define  $Q(G) := \{x \in \mathbb{R}_+^{|V||C|} : x \text{ satisfies (1) and (6)}\}$ . We shall prove that  $Q(G)$  is an integer polytope, hence implying  $P_{\text{col}}(G) = Q(G)$ . To this end, we first define the *fractional graph*  $G_f(\hat{x})$ , associated to a vector  $\hat{x} \in [0, 1]^{|V||C|}$  as follows.

**Definition 3.1** (fractional graph). Given  $\hat{x} = (\hat{x}_{vc}) \in [0, 1]^{|V||C|}$ , the fractional graph associated to  $\hat{x}$  is  $G_f(\hat{x}) = (V \cup C, E(\hat{x}))$  where  $E(\hat{x}) = \{vc \in V \times C : 0 < \hat{x}_{vc} < 1\}$ . We call  $V$  (resp.  $C$ ) the set of  $V$ -nodes (resp.  $C$ -nodes) of  $G_f(\hat{x})$ .

Note that the sets  $V$  and  $C$  give a bipartition of  $G_f(\hat{x})$ . If  $\hat{x}$  satisfies (1), then

1. a  $V$ -node  $v$  is an isolated node if and only if  $x_{vc} = 1$  for some  $c \in C$ , and
2. each non-isolated  $V$ -node is connected to at least two  $C$ -nodes.

When  $G = K_n$ , i.e.,  $G$  is a complete graph, the fractional graph of a vector  $x \in Q(K_n)$  has many interesting properties, and we shall resort to some of them in order to prove that  $Q(K_n)$  is an integer polytope. For a vector  $x \in Q(K_n)$ , we say that a color  $c \in C$  is *saturated* if the corresponding constraint (6) is tight in  $x$ , otherwise we say that  $c$  is a *safe* color. Throughout this paper, we note  $x_{(U,D)} = \sum_{v \in U} \sum_{c \in D} x_{vc}$ . We also use  $x_{(v,D)}$  and  $x_{(U,c)}$  as shortcuts for  $x_{(\{v\},D)}$  and  $x_{(U,\{c\})}$ , respectively.

**Proposition 3.3.** Let  $G_f(x) = (V \cup C, E(x))$  be the fractional graph associated to a solution  $x \in Q(K_n)$  and let  $H = (V_H \cup C_H, E_H)$  be an induced subgraph of  $G_f(x)$  with no isolated vertices. Call  $E_H^{\vec{vc}} = \{vc \in E(x) : v \in V_H \text{ and } c \notin C_H\}$  and  $E_H^{\overleftarrow{cv}} = \{cv \in E(x) : c \in C_H \text{ and } v \notin V_H\}$ . The following statements are true:

- i) If  $E_H^{\vec{vc}} = \emptyset$  then  $|V_H| \leq |C_H|$ .
- ii) If  $E_H^{\vec{vc}} = \emptyset$  then either  $E_H^{\overleftarrow{cv}} = \emptyset$  or  $|V_H| < |C_H|$ .
- iii) If  $E_H^{\vec{vc}} = \emptyset$  and there is a safe color in  $c \in C_H$  then  $|V_H| < |C_H|$ .
- iv) If  $E_H^{\overleftarrow{cv}} = \emptyset$  and there is no safe color in  $C_H$  then  $|C_H| \leq |V_H|$ .
- v) If  $E_H^{\overleftarrow{cv}} = \emptyset$  and there is no safe color in  $C_H$  then  $E_H^{\vec{vc}} = \emptyset$  or  $|C_H| < |V_H|$ .

*Proof.*

- i) Since  $H$  has no isolated  $V$ -nodes, then every  $v \in V_H$  with no neighbors in  $C \setminus C_H$  satisfies  $x_{(v,C_H)} = 1$ . Hence, if  $E_H^{\vec{vc}} = \emptyset$  then  $x_{(V_H,C_H)} = |V_H|$ . Also, by (6), every color  $c \in C$  satisfies  $x_{(V_H,c)} \leq 1$  and then  $|V_H| = x_{(V_H,C_H)} \leq |C_H|$ .
- ii) Since  $E_H^{\vec{vc}} = \emptyset$ , then  $x_{(V_H,C_H)} = |V_H|$ . For every  $c \in C_H$ , by (6) we have  $x_{(V,c)} \leq 1$ , hence

$$|C_H| \geq x_{(V,C_H)} = x_{(V_H,C_H)} + x_{(V \setminus V_H,C_H)} = |V_H| + x_{(V \setminus V_H,C_H)}.$$

If  $|V_H| = |C_H|$ , then  $x_{vc} = 0$  for every  $v \in V \setminus V_H$  and  $c \in C_H$ , implying  $E_H^{\overleftarrow{cv}} = \emptyset$ . Otherwise,  $|V_H| < |C_H|$  holds.

- iii) Since  $E_H^{\vec{vc}} = \emptyset$ , then  $x_{(V_H,C_H)} = |V_H|$  and  $c$  being a safe color implies  $x_{(V_H,c)} < 1$ . Then, by (6), it follows that  $|V_H| = x_{(V_H,C_H)} < |C_H|$ .
- iv) If there is no safe color in  $C_H$ , then  $x_{(V,C_H)} = |C_H|$  and if  $E_H^{\overleftarrow{cv}} = \emptyset$  then  $x_{(V,C_H)} = x_{(V_H,C_H)}$ . Hence,  $|C_H| = x_{(V,C_H)} = x_{(V_H,C_H)} \leq |V_H|$ .

v) As in the proof for item iv), we have  $|C_H| = X_{(V, C_H)} = X_{(V_H, C_H)}$ . If  $E_H^{\vec{v}\hat{c}} \neq \emptyset$  then  $X_{(V_H, C_H)} < |V_H|$ , and this implies  $|C_H| = X_{(V, C_H)} = X_{(V_H, C_H)} < |V_H|$ . Otherwise, we have  $E_H^{\vec{v}\hat{c}} = \emptyset$ .

□

**Lemma 3.4.** *Let  $G_f(x) = (V \cup C, E(x))$  be the fractional graph associated to the solution  $x \in Q(K_n)$ , and assume  $\hat{v}\hat{c}, \hat{v}\hat{c}' \in E(x)$  for some  $\hat{v} \in V$  and  $\hat{c}, \hat{c}' \in C$ . If the path  $P = \{\hat{c}, \hat{v}, \hat{c}'\}$  does not belong to any cycle of  $G_f(x)$ , then  $P$  belongs to a simple chordless path  $P' = \{c_1, \dots, \hat{c}, \hat{v}, \hat{c}', \dots, c_2\}$ , where  $c_1$  and  $c_2$  are safe colors in  $x$ .*

*Proof.* Since  $P$  does not belong to any cycle, then  $\hat{v}$  is a cut vertex, i.e., the removal of  $\hat{v}$  leaves  $\hat{c}$  and  $\hat{c}'$  in different connected components of the resulting graph. Let  $H = (V_H \cup C_H, E_H)$  be the connected component of  $G_f(x) \setminus \{\hat{v}\}$  including  $\hat{c}$ . Since  $H$  is a connected component, there are no C-nodes outside  $H$  connected to V-nodes in  $H$ . Given that  $\hat{v} \notin V_H$  and  $\hat{c} \in C_H$ , Proposition 3.3(ii) implies  $|V_H| < |C_H|$ , i.e.,  $|V_H| + 1 \leq |C_H|$ .

Let us now prove that there must be a safe C-node  $c_1 \in C_H$ , by showing that the sum of the edges incident to the C-nodes of  $H$  is strictly less than  $|C_H|$ . Since  $\hat{v}$  is the only V-node outside  $H$  connected to C-nodes in  $H$ , then

$$X_{(V \setminus V_H, C_H)} = X_{(\hat{v}, C_H)} < 1.$$

The strict inequality holds because  $x_{\hat{v}\hat{c}'} > 0$  and  $\hat{c}' \notin C_H$ . Therefore, the sum of the edges incident to C-nodes from  $H$  is

$$X_{(V, C_H)} = X_{(V_H, C_H)} + X_{(V \setminus V_H, C_H)} < |V_H| + 1 \leq |C_H|.$$

Thus, there exists a safe C-node  $c_1 \in C_H$  and also, as  $H$  is a connected graph, a minimum (chordless) path  $P_1$  between  $\hat{c}$  and  $c_1$ .

Similarly, we can deduce that in the connected component from  $G_f(x) \setminus \{\hat{v}\}$  including  $\hat{c}'$ , there exists a simple chordless path  $P_2$  between  $\hat{c}'$  and a safe color  $c_2$ . As these paths,  $P_1$  and  $P_2$ , lie on different connected components of  $G_f(x) \setminus \{\hat{v}\}$ , the simple chordless path  $P'$  may be obtained by concatenating  $P_1$ ,  $\hat{v}$ , and  $P_2$ . □

**Theorem 3.5.** *If  $G = (V, E)$  is a block graph, then  $P_{col}(G) = \{x \in \mathbb{R}_+^{|V||C|} : x \text{ satisfies (1) and (6)}\}$ .*

*Proof.* We shall prove that  $Q(G)$  is an integer polytope, hence implying  $P_{col}(G) = Q(G) = \{x \in \mathbb{R}_+^{|V||C|} : x \text{ satisfies (1) and (6)}\}$ . The proof proceeds by induction on the number of maximal cliques of  $G$ .

Assume first that  $G = K_n$  and let  $\hat{x} \in Q(G)$  have a fractional value  $\hat{x}_{vq}$  for some  $v \in V$  and  $c \in C$ . We will show that  $\hat{x}$  is a convex combination of two other solutions  $x^a, x^b \in Q(G)$ , thus implying that  $\hat{x}$  is not an extreme point.

As  $\hat{x}$  satisfies (1), there must be another color  $q' \in C$  with  $\hat{x}_{vq'}$  also fractional. Let  $\varepsilon \in \mathbb{R}_+$  such that  $\varepsilon < \min\{\hat{x}_{ic}, 1 - \hat{x}_{ic}\}$  for every  $(i, c) \in V \times C$  with a fractional value  $\hat{x}_{ic}$ , and  $x_{(V,c)} + \varepsilon < 1$  for each safe color  $c \in C$ . Let  $G_f(\hat{x})$  be the fractional graph associated to  $\hat{x}$ . As  $G_f(\hat{x})$  is bipartite, every cycle of  $G_f(\hat{x})$  is even. Assume that the edges  $vq$  and  $vq'$  belong to a cycle of  $G_f(\hat{x})$  and call  $H = \{q', v, q, v_2, \dots, c_t, v_t\}$  such a cycle. Note that  $H$  alternates between C-nodes and V-nodes due to the construction of  $G_f(\hat{x})$ . We may construct the solutions  $x^a$  and  $x^b$  as follows (considering cyclic precedences):

$$x_{ic}^a \text{ (resp. } x_{ic}^b) = \begin{cases} \hat{x}_{ic} + \varepsilon \text{ (resp. } \hat{x}_{ic} - \varepsilon) & \text{if } i \text{ precedes } c \text{ in the cycle} \\ \hat{x}_{ic} - \varepsilon \text{ (resp. } \hat{x}_{ic} + \varepsilon) & \text{if } c \text{ precedes } i \text{ in the cycle} \\ \hat{x}_{ic} \text{ (resp. } \hat{x}_{ic}) & \text{otherwise.} \end{cases}$$

It is easy to see that  $\hat{x} = \frac{1}{2}(x^a + x^b)$ . Let us verify that both solutions also belong to  $Q(K_n)$ . Firstly, every modified variable has fractional values in  $\hat{x}$  (since they correspond to edges of  $G_f(\hat{x})$ ), so the choice of  $\varepsilon$  implies that all new values remain between 0 and 1.

Constraints (1) for the V-nodes not in  $H$  are not modified, and the same happens for constraints (6) for the C-nodes not in  $H$ . For any V-node in  $H$ , the corresponding constraint (1) remains satisfied, both in  $x^a$  and  $x^b$ , since  $\varepsilon$  is added in one color and subtracted in other (i.e., the colors before and after such V-node in the cycle). Finally, for any C-node in  $H$ , the corresponding constraint (6) remains satisfied, both in  $x^a$  and  $x^b$ , since  $\varepsilon$  is added in one V-node and subtracted in other (i.e., the V-nodes before and after such C-node in the cycle). Therefore, both  $x^a$  and  $x^b$  belong to  $Q(K_n)$ .

Assume now that edges  $vq$  and  $vq'$  do not belong to a cycle of the fractional graph. In this case, by Lemma 3.4, these edges belong to a simple chordless path  $P = \{c_1, \dots, q', v, q, \dots, c_2\}$ , where  $c_1$  and  $c_2$  are safe colors in  $\hat{x}$ . Note that  $P$  alternates between C-nodes and V-nodes due to the construction of  $G_f(\hat{x})$ . Let us define  $x^a$  and  $x^b$  using  $P$  in the same way as before (regardless cyclic precedence in this case). As before,  $\hat{x} = \frac{1}{2}(x^a + x^b)$  and it is easy to see that both solutions remain in  $[0, 1]^{|V||C|}$  and that constraints (1) are satisfied. As for constraints (6), for every C-node outside  $P$  these remain unmodified and for every C-node in  $P$ , except  $c_1$  and  $c_2$ , the argument is the same as before. Finally, as  $c_1$  and  $c_2$  are safe colors, constraints (6) associated to these colors are satisfied, both in  $x^a$  and  $x^b$ , for the given  $\varepsilon$ . This ends the proof for the base case  $G = K_n$ .

Assume now that  $G$  has two or more maximal cliques. As  $G$  is a block graph, it contains a maximal clique  $K$  with a vertex  $v \in K$  such that  $K \cap K' \subseteq \{v\}$ , for every other maximal clique  $K'$  of  $G$ . Hence,  $G$  is the one-vertex identification of  $G[K]$  and  $G[(V \setminus K) \cup \{v\}]$ . Since both are block graphs with less maximal cliques than  $G$ , the inductive hypothesis and Theorem 2.5 imply the desired conclusion.  $\square$

It is worth noting that Theorem 3.5 generalizes Theorem 3.1. Nevertheless,

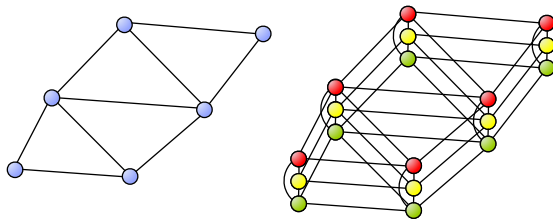


Figure 1: A graph  $G$  and the auxiliary graph  $\mathcal{S}_G^C$  for  $|C| = 3$ .

the particular case of trees is interesting by itself as the characterization uses no further inequalities than the ones from the classical formulation.

From Theorem 2.1 and Theorem 3.5 we have that the separation problem over  $P_{\text{list}}(G, L)$  can be solved in polynomial time when  $G$  is a block graph. This result yields a polyhedral method for solving the list-coloring problem over block graphs in polynomial time, as a counterpart of the result presented in [20].

#### 4. A relation with the stable set polytope

A stable set on a graph is a set of pairwise non-adjacent vertices and the *stable set polytope*  $STAB(H)$  of a graph  $H$  is the convex hull of the characteristic vectors of the stable sets of  $H$ . There is a well-known reduction from the vertex coloring problem to the maximum cardinality stable set problem [12]. Inspired by this reduction, we show in this section the polyhedral relation between  $P_{\text{col}}(G)$  and the stable set polytope of an auxiliary graph  $\mathcal{S}_G^C$ , for which every coloring of  $G$  defines a stable set of  $\mathcal{S}_G^C$ .

Given a graph  $G = (V, E)$  and a set of colors  $C$ , construct the graph  $\mathcal{S}_G^C = (V \times C, E_{\mathcal{S}_G^C})$ , where

$$E_{\mathcal{S}_G^C} = \{(v, c)(v, c') : v \in V \wedge c, c' \in C \wedge c \neq c'\} \cup \{(v, c)(w, c) : vw \in E \wedge c \in C\}.$$

For each vertex in  $G$ ,  $\mathcal{S}_G^C$  contains a clique of size  $|C|$ . Additionally, for each pair of adjacent vertices  $vw \in E$ , there are  $|C|$  edges between the corresponding cliques in  $\mathcal{S}_G^C$ , joining the vertices corresponding to the same color. Figure 1 shows an example for  $|C| = 3$ . Note that every coloring  $c : V \rightarrow C$  of  $G$  defines the stable set  $\{(v, c(v)) : v \in V\}$  in  $\mathcal{S}_G^C$ .

The characteristic vectors of stable sets of  $\mathcal{S}_G^C$  can be described by the constraints

$$\sum_{c \in C} x_{(v, c)} \leq 1 \quad \forall v \in V, \quad (7)$$

$$x_{(v, c)} + x_{(w, c)} \leq 1 \quad \forall vw \in E, \forall c \in C, \quad (8)$$

$$x_{(v, c)} \in \{0, 1\} \quad \forall v \in V, \forall c \in C. \quad (9)$$

Constraints (7) ensure that at most one vertex is selected from each of the cliques representing vertices from  $G$ . Additionally, there is a constraint of type (8) for each of the remaining edges of  $\mathcal{S}_G^C$ , to avoid having both end-points in the stable set. Hence,  $STAB(\mathcal{S}_G^C) = \text{conv}(\{x \in \mathbb{R}_+^{|V||C|} : x \text{ satisfies (7)-(9)}\})$ .

Recall that, for any graph  $G$ ,  $P_{\text{col}}(G) = \text{conv}(\{x \in \mathbb{R}_+^{|V||C|} : x \text{ satisfies (1)-(3)}\})$  and note that the only difference between (1)-(3) and (7)-(9) is the equality on (1), hence  $P_{\text{col}}(G) \subseteq STAB(\mathcal{S}_G^C)$  and so every valid inequality for  $STAB(\mathcal{S}_G^C)$  is valid for  $P_{\text{col}}(G)$ . Furthermore,  $P_{\text{col}}(G)$  is the face of  $STAB(\mathcal{S}_G^C)$  defined by the facet-inducing inequalities (7). Therefore, many facets of  $P_{\text{col}}(G)$  can be potentially described by known facet-inducing inequalities of  $STAB(\mathcal{S}_G^C)$ .

#### 4.1. A new family of valid inequalities for $P_{\text{col}}(G)$

A *hole* in a graph is a chordless cycle and it is known that some facets of  $STAB(G)$ , for an arbitrary graph  $G$ , are defined by the following *odd hole* inequalities:

$$\sum_{v \in H} x_v \leq \frac{|H| - 1}{2} \quad \forall \text{ odd hole } H \subseteq V(G). \quad (10)$$

Graphs for which these inequalities suffice to describe  $STAB(G)$  are called *t-perfect graphs* [19]. The inequality (10) is valid for  $STAB(G)$  even when  $H$  has chords. If  $H$  is even, then the inequality obtained from (10) by replacing the right-hand side by  $|H|/2$  is also valid. However, in these cases the resulting *circuit inequality* does not define a facet of  $STAB(G)$ . Based on these results, we derive a new family of valid inequalities for  $P_{\text{col}}(G)$  for any graph  $G$ , as follows.

**Theorem 4.1.** *Let  $G = (V, E)$  be a graph,  $C$  a set of colors and  $J = \{v_1, \dots, v_{|J|}\} \subseteq V$  any cycle of  $G$ , i.e., with  $v_i v_{i+1} \in E$  for  $i = 1, \dots, |J| - 1$  and  $v_1 v_{|J|} \in E$ . Given  $|J|$  lists of colors  $l_1, \dots, l_{|J|} \subseteq C$  such that the last color of  $l_i$  equals the first color of  $l_{i+1}$ , for  $i = 1, \dots, |J|$  (where  $l_{|J|+1} = l_1$ ), then the stable cycle inequality*

$$\sum_{i=1}^{|J|} \sum_{c \in l_i} x_{v_i c} \leq \left\lfloor \frac{L}{2} \right\rfloor \quad (11)$$

with  $L = \sum_{i=1}^{|J|} |l_i|$ , is valid for  $P_{\text{col}}(G, C)$ .

*Proof.* The variables on the left hand side of (11) can be ordered to form a circuit of  $\mathcal{S}_G^C$  of size  $L$ . Hence, (11) is a valid circuit inequality for  $STAB(\mathcal{S}_G^C)$ . Since  $P_{\text{col}}(G) \subseteq STAB(\mathcal{S}_G^C)$ , the result follows.  $\square$

Every stable cycle inequality (11) for  $P_{\text{col}}(G)$  can be matched with a unique circuit inequality (10) for  $STAB(\mathcal{S}_G^C)$ . Hence the separation problems for these families are equivalent, and the results in [19] show that both can be solved in polynomial time. This implies the following result.

**Proposition 4.2.** *For any graph  $G$  and set of colors  $C$ , there exists a polynomial time algorithm that takes a point  $x \in \mathbb{R}_+^{|V||C|}$  and determines if  $x$  violates any stable cycle inequality.*

As can be seen, inequalities (11) are strongly related with the odd hole inequalities (10), however, the circuit of  $\mathcal{S}_G^C$  associated with (11) is an odd hole if and only if  $l_1, \dots, l_{|J|}$  are such that

1.  $L = \sum_{i=1}^{|J|} |l_i|$  is an odd number,
2.  $1 \leq |l_i| \leq 2$ , for  $i = 1, \dots, |J|$ , and
3. If  $v_i v_j \in E$  for some  $v_i, v_j \in J$ , then  $|l_i \cap l_j| \leq 1$  and if  $v_i$  and  $v_j$  are non-consecutive vertices of  $J$ , then  $l_i \cap l_j = \emptyset$ .

Items 2 and 3 force the circuit to be chordless. Therefore, (11) may define a facet of  $STAB(\mathcal{S}_G^C)$  only if the above properties hold. This suggests that these properties would also be necessary for (11) to define facets of  $P_{\text{col}}(G)$ , however this is not trivially true since the face of  $P_{\text{col}}(G)$  defined by (11) may be a facet of  $P_{\text{col}}(G)$  even when the corresponding face of  $STAB(\mathcal{S}_G^C)$  is not a facet of  $STAB(\mathcal{S}_G^C)$ .

Based on computational experiments on small graphs, we conjecture that constraints (1), (2) and (11) give a complete characterization for  $P_{\text{col}}(G)$  when  $G$  is a cycle. Furthermore, if this conjecture is true, then Proposition 2.4, can be used to prove that constraints (1), (6) and (11) give a complete characterization of  $P_{\text{col}}(G)$  when  $G$  is a cacti graph (as these graphs can be obtained as one-vertex identifications of cycles, edges and triangles).

It is trivial to see that this new family of valid inequalities for  $P_{\text{col}}$  generalizes the (vertex coloring) odd hole inequalities presented in [11]. Furthermore, if we assign a distinct color  $c_i$  for each vertex  $v_i \in J \setminus \{v_{|J|}\}$  and take  $l_1 = [c_1]$  and  $l_i = [c_{i-1}, c_i]$ , for  $i = 2, \dots, |J|$  (where  $c_{|J|} := c_1$ ), then (11) yields the chain colors inequalities presented in [11]. Therefore, (11) is also a generalization of this family. Both of the above families of valid inequalities define facets of  $P_{\text{col}}$  under particular hypotheses.

#### 4.2. Other known inequalities for $STAB$

Many families of facet-inducing inequalities for  $STAB(G)$  are known as, e.g., inequalities based on cliques, odd-antiholes, odd-wheels, and webs. One may try to derive new valid inequalities for  $P_{\text{col}}$  from them in the same fashion we derived (11) from the odd hole inequalities. Unfortunately, in these cases the obtained valid inequalities for  $P_{\text{col}}$  are just the same (stable set) inequalities but restricted to a fixed color  $c \in C$ . For instance, take a complete subgraph  $K$  of  $\mathcal{S}_G^C$  (not included in some clique  $\{(v, c) / c \in C\}$ ). The corresponding stable set version of the clique inequality would be

$$\sum_{(v,c) \in K} x_{(v,c)} \leq 1. \quad (12)$$

Inequality (12) is valid for  $P_{\text{col}}$ , however, by the construction of  $\mathcal{S}_G^C$ , it is easy to see that  $K$  spans over vertices  $(v, c) \in V(\mathcal{S}_G^C)$  for some fixed  $c \in C$ . Then, the set  $\{v \in V \mid (v, c) \in K\}$  defines a clique of  $G$  for which the vertex coloring clique inequality (6) using  $c$  gives also (12).

Indeed, the construction of  $\mathcal{S}_G^C$  implies that if  $\{(v_1, c_1), (v_2, c_2), (v_3, c_3)\} \subseteq V(\mathcal{S}_G^C)$  is a complete subgraph of  $\mathcal{S}_G^C$ , then either  $v_1 = v_2 = v_3$  or  $c_1 = c_2 = c_3$ . Hence, we can prove that if a structure  $H$  of  $\mathcal{S}_G^C$  is a clique, an odd-antihole, an odd-wheel or a web and is not contained in a clique  $\{(v, c) \mid c \in C\}$  for a fixed  $v \in V$ , then the corresponding valid inequality for  $STAB(\mathcal{S}_G^C)$  uses variables  $x_{(v,c)}$  with some fixed  $c \in C$ . Hence the obtained valid inequalities for  $P_{\text{col}}$  are just the vertex coloring versions of the clique, odd-antihole, odd-wheel and web inequalities using color  $c$ . We give a formal proof for the case of odd-antiholes, as this implies some further properties of  $\mathcal{S}_G^C$ , which we discuss later.

**Proposition 4.3.**  $H = \{(v_1, c_1), \dots, (v_k, c_k)\} \subseteq V(\mathcal{S}_G^C)$  induces an odd-antihole of  $\mathcal{S}_G^C$  if and only if  $c_i = c_j$ , for all  $i, j = 1, \dots, k$  and  $H' = \{v_1, \dots, v_k\}$  induces an odd-antihole of  $G$ .

*Proof.* By the construction of  $\mathcal{S}_G^C$ , it is easy to see that every odd-antihole  $H' = \{v_1, \dots, v_k\}$  of  $G$  yields an odd-antihole  $H = \{(v_1, c), \dots, (v_k, c)\}$ , for each  $c \in C$ , and viceversa. Hence, we just need to prove the forward implication. In fact, it suffices to prove that if  $H = \{(v_1, c_1), \dots, (v_k, c_k)\} \subseteq V(\mathcal{S}_G^C)$  induces an odd-antihole of  $\mathcal{S}_G^C$  then  $c_i = c_j$ , for all  $i, j = 1, \dots, k$ .

Let  $H = \{(v_1, c_1), \dots, (v_k, c_k)\} \subseteq V(\mathcal{S}_G^C)$  induce an odd-antihole of  $\mathcal{S}_G^C$ . Call  $E_H$  the set of edges of the induced antihole and  $w_i = (v_i, c_i)$ . Assume first that  $k = 5$ , i.e.,  $E_H = \{w_1w_3, w_3w_5, w_5w_2, w_2w_4, w_4w_1\}$ , and suppose that  $c_1 \neq c_2$ . Since  $w_1w_2 \notin E_H$ , we know that  $v_1 \neq v_2$  and as  $w_1w_4, w_2w_4 \in E_H$ , then either  $w_4 = (v_1, c_2)$  or  $w_4 = (v_2, c_1)$ . Assume, w.l.o.g., that  $w_4 = (v_1, c_2)$  and so the edge  $v_1v_2$  exists in  $G$ , as  $w_2w_4 \in E_H$ . Since  $w_3w_4 \notin E_H$ , we have that  $v_3 \neq v_1$  and this implies that  $c_3 = c_1$  and that the edge  $v_1v_3$  exists in  $G$  (as  $w_1w_3 \in E_H$ ). Also,  $v_3 \neq v_2$ , otherwise we would have  $w_2w_3 \in E_H$ . Lastly, as  $w_2w_5, w_3w_5 \in E_H$ , then either  $w_5 = (v_2, c_1)$  or  $w_5 = (v_3, c_2)$ . If  $w_5 = (v_2, c_1)$ , then  $w_1w_5 \in E_H$  (as  $v_1v_2$  is an edge of  $G$ ), which is false, and if  $w_5 = (v_3, c_2)$ , then  $w_4w_5 \in E_H$  (as  $v_1v_3$  is an edge of  $G$ ), which is also false. In any case we have a contradiction which came from the assumption that  $c_1 \neq c_2$ . Since  $w_1$  and  $w_2$  may be any two consecutive vertices from  $H$ , this proves that  $c_i = c_j$ , for every  $i, j = 1, \dots, 5$  and so  $H'$  induces an odd-antihole of  $G$ .

Assume now that  $k \geq 7$ . Since  $\{w_1, w_3, w_6\}$  is a clique, then either  $v_1 = v_3 = v_6$  or  $c_1 = c_3 = c_6$ . If  $v_1 = v_3 = v_6 := v^*$ , then  $v_4 = v^*$  also, since  $\{w_1, w_4, w_6\}$  is also a clique and  $v_1 = v_6$ . But this cannot happen since  $w_3w_4 \notin E_H$ . Hence  $c_1 = c_3 = c_6 = c_4$ . Finally, as  $\{w_2, w_4, w_6\}$  is a clique and  $c_4 = c_6$ , then  $c_2 = c_4 = c_6$ . This proves that  $c_1 = c_2$  and as  $w_1$  and  $w_2$  may be any two consecutive vertices from  $H$ , we have that  $c_i = c_j$ , for every  $i, j = 1, \dots, k$  and so  $H'$  induces an odd-antihole of  $G$ .  $\square$



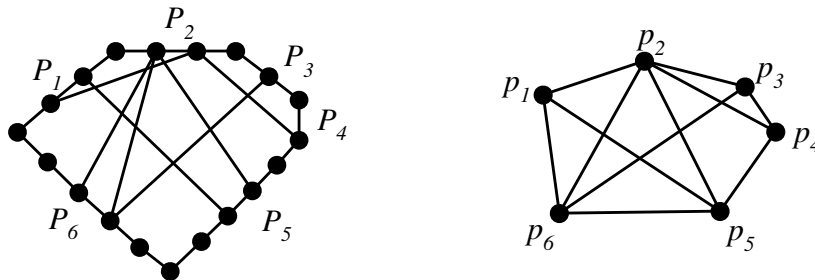


Figure 2: An odd-path-cycle  $\mathcal{C}$  and the corresponding chord-graph  $\mathcal{H}(\mathcal{C})$ .

#### 4.3. On the perfectness of $\mathcal{S}_G^{\mathcal{C}}$

Recall that a graph  $H$  is said to be *perfect* if  $\chi(H') = \omega(H')$  for every induced subgraph  $H'$  of  $H$ , where  $\omega(H')$  represents the size of the maximum clique of  $H'$ . Chvátal proved in [10] that the stable set polytope of a perfect graph is completely described by the non-negativity constraints and (the stable set version of) the clique inequalities (as in (12)). Since  $P_{\text{col}}(G)$  is a face of  $STAB(\mathcal{S}_G^{\mathcal{C}})$ , it is interesting to study the perfectness of  $\mathcal{S}_G^{\mathcal{C}}$ .

In 2006, Chudnovsky, Robertson, Seymour and Thomas [9] published the *strong perfect graph theorem* proving that a graph is perfect if and only if it does not contain any odd-hole nor odd-antihole as an induced subgraph. Proposition 4.3 states that  $\mathcal{S}_G^{\mathcal{C}}$  does not contain odd-antiholes, besides from those coming strictly from odd-antiholes from  $G$ , however, it is easy to see that  $\mathcal{S}_G^{\mathcal{C}}$  may contain odd-holes, even when  $G$  does not. In order to analyze the perfectness of  $\mathcal{S}_G^{\mathcal{C}}$ , it is imperative to characterize the structures of  $G$  that yield odd-holes on  $\mathcal{S}_G^{\mathcal{C}}$ .

For a graph  $G = (V, E)$ , we define a *path-cycle* of  $G$  to be a collection of induced paths  $\mathcal{C} = \{P_1, \dots, P_k\}$ , with  $P_i = \{v_1^i, \dots, v_{|P_i|}^i\}$  and  $|P_i| > 1$ , such that  $v_{|P_i|}^i = v_1^{i+1}$ , for  $i = 1, \dots, k$  (where  $v_1^{k+1} = v_1^1$ ), and the concatenation of  $P_1, \dots, P_k$  forms a simple cycle of  $G$  (we consider the concatenation of two consecutive paths  $P_i$  and  $P_{i+1}$  by removing one copy of repeated vertex  $v_1^{i+1}$ ), i.e., the paths are disjoint except at their extreme vertices. Note that the cycle can have edges between vertices from two paths  $P_i$  and  $P_j$  as long as  $i \neq j$  (as the paths are asked to be induced paths). The *multilength* of  $\mathcal{C}$  is defined as  $\sum_{i=1}^k |P_i|$ , where  $|P_i|$  is the number of vertices in  $P_i$ . An odd-path-cycle is a path-cycle with odd multilength. The *chord-graph*  $\mathcal{H}(\mathcal{C})$  of a path-cycle  $\mathcal{C} = \{P_1, \dots, P_k\}$ , has one vertex  $p_i$  associated with each path  $P_i$  in  $\mathcal{C}$  and there is an edge joining two vertices  $p_i$  and  $p_j$  if and only if  $P_i$  and  $P_j$  are consecutive paths in  $\mathcal{C}$  or there exists an edge joining a vertex from  $P_i$  with a vertex from  $P_j$ . Figure 2 gives an example of an odd-path-cycle (with multilength 25) and the corresponding chord-graph.

**Theorem 4.4.** *Given a graph  $G$  and a set of colors  $C$ , we have that  $\mathcal{S}_G^C$  has an odd-hole with length  $L$  if and only if  $G$  has an odd-hole with length  $L$  or it has an odd-path-cycle  $\mathcal{C}$  with multilength  $L$  and  $\chi(\mathcal{H}(\mathcal{C})) \leq |C|$ .*

*Proof.* For the forward implication, let  $H = \{(v_1, c_1), \dots, (v_L, c_L)\} \subseteq V(\mathcal{S}_G^C)$  induce an odd-hole of  $\mathcal{S}_G^C$ . Call  $w_i = (v_i, c_i)$  and let  $E_H = \{w_1w_2, w_2w_3, \dots, w_{L-1}w_L, w_Lw_1\}$  the set of edges of the induced hole. Note that a vertex of  $G$  can only appear in two vertices  $w_i$  and  $w_j$  from  $H$  if these are consecutive vertices in  $H$ , as this implies that  $w_iw_j \in E_H$  and  $H$  is a hole. Furthermore, it can only appear in at most two vertices from  $H$ . Note also that if  $v_i \neq v_{i+1}$ , then  $c_i = c_{i+1}$  (with  $L+1 := 1$ ). Let  $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, L\}$  be the set of indices  $i$  such that  $v_i = v_{i-1}$  (considering index 0 as  $L$ ). If  $I = \emptyset$  then  $\{v_1, \dots, v_L\}$  induces an odd-hole of  $G$  and there is nothing else to prove. Assume then that  $I \neq \emptyset$ , hence  $|I| \geq 2$ . Assume also, w.l.o.g., that  $i_1 < \dots < i_k$  and, for each  $j = 1 \dots, k$ , define  $P_j = \{v_{i_j}, \dots, v_{i_{j+1}-1}\}$  (where  $k+1 := 1$ ). Each  $P_j$  induces a chordless path in  $G$ , as  $c_{i_j} = \dots = c_{i_{j+1}-1}$  and  $H$  is a hole. Then, it is easy to see that  $\mathcal{C} = \{P_1, \dots, P_k\}$  is an odd-path-cycle of  $G$  with multilength  $L$ . Given the chord-graph  $\mathcal{H}(\mathcal{C})$ , assign color  $c_{i_j}$  to the vertex  $p_j$ , for  $j = 1, \dots, k$ . Note that  $c_{i_j} \neq c_{i_{j+1}}$ , as  $i_{j+1} \in I$ . Also, if  $p_j$  and  $p_l$  are adjacent in  $\mathcal{H}(\mathcal{C})$  (and related to non-consecutive paths  $P_j$  and  $P_l$ ), then there exists an edge joining a vertex from  $P_j$  with a vertex from  $P_l$ . Hence,  $c_{i_j} \neq c_{i_l}$ , otherwise  $H$  is not a hole. Therefore, the given assignment gives a proper coloring of  $\mathcal{H}(\mathcal{C})$  using colors from  $C$  and so  $\chi(\mathcal{H}(\mathcal{C})) \leq |C|$ .

For the backwards implication, it is easy to see that if  $G$  has an odd-hole  $\{v_1, \dots, v_L\}$ , then  $\{(v_1, c), \dots, (v_L, c)\}$  induces an odd-hole of  $\mathcal{S}_G^C$ , for every  $c \in C$ . Assume then that  $G$  has no odd-antihole and let  $\mathcal{C} = \{P_1, \dots, P_k\}$  be an odd-path-cycle of  $G$  with multilength  $L$  such that  $\chi(\mathcal{H}(\mathcal{C})) \leq |C|$ . Take a coloring of  $\mathcal{H}(\mathcal{C})$  using colors from  $C$  and call  $c_j$  the color assigned to  $p_j$ . We claim that  $H = \{(v, c_j) : j = 1, \dots, k \text{ and } v \in P_j\}$  induces an odd-hole of  $\mathcal{S}_G^C$  with length  $L$ . As the paths from  $\mathcal{C}$  are disjoint except at their joining extreme vertices, and the colors assigned to two consecutive paths are different, then  $H$  has a unique vertex for each vertex in each path of  $\mathcal{C}$ . Hence, the length of  $H$  is clearly  $L$ , and so is odd. Each chordless path  $P_j$  corresponds to the chordless path  $P'_j = \{(v, c_j) : v \in P_j\}$  in  $\mathcal{S}_G^C$  and there is an edge between vertices  $(v, c_j)$  and  $(v, c_{j+1})$  where  $v$  is the last (resp. the first) vertex of path  $P_j$  (resp.  $P_{j+1}$ ). Hence,  $H$  is a cycle of  $\mathcal{S}_G^C$ . Suppose  $H$  does not induce a hole and so there is an edge joining a vertex  $w_1 = (v_1, c_j) \in P'_j$  and a vertex  $w_2 = (v_2, c_i) \in P'_i$ , with  $i \neq j$ , such that  $w_1w_2$  is not part of the cycle. Hence,  $v_1 \neq v_2$  since  $P_j$  and  $P_i$  are disjoint sets of vertices (except maybe at the joining extremes but we are in the case that  $w_1w_2$  is not part of the cycle). Then  $c_i = c_j$  and there exists an edge  $v_1v_2$  in  $G$ . So, the edge  $p_jp_i$  exists in  $\mathcal{H}(\mathcal{C})$ , contradicting the fact that the assignment is a proper coloring. Therefore,  $H$  induces a hole of  $\mathcal{S}_G^C$ .  $\square$

**Corollary 4.5.** *Given a graph  $G$  and a set of colors  $C$ , the graph  $\mathcal{S}_G^C$  is perfect if and only if  $G$  is perfect and has no odd-path-cycle  $\mathcal{C}$  with  $\chi(\mathcal{H}(\mathcal{C})) \leq |C|$ .*

A *diamond* is the graph obtained by removing one edge from  $K_4$  and a *chordal* graph is a graph that does not contain cycles of four or more vertices as induced subgraphs. A graph is a block graph if and only if it is chordal and it has no induced diamonds [1]. By resorting to this fact, the following theorem implies that the family of graphs satisfying the properties on Corollary 4.5 are exactly block graphs, when  $|C| > 2$ .

**Theorem 4.6.** *Block graphs have no odd-path-cycles. Moreover, a perfect graph with no odd-path-cycle  $\mathcal{C}$  with  $\chi(\mathcal{H}(\mathcal{C})) = 3$  is a block graph, hence it has no odd-path-cycles at all.*

*Proof.* If  $G$  is a block graph then it is easy to see that  $\mathcal{S}_G^C$  has no odd-holes, regardless of the size of  $C$ . Hence, Theorem 4.4 implies the result.

Assume now that  $G = (V, E)$  is perfect and has no odd-path-cycle  $\mathcal{C}$  with  $\chi(\mathcal{H}(\mathcal{C})) = 3$ . Suppose there is a diamond in  $G$  with vertices  $a, b, c, d \in V$ , where  $ac \notin E$ . Take  $P_1 = \{a, b, c\}$ ,  $P_2 = \{c, d\}$ , and  $P_3 = \{d, a\}$ . Clearly,  $\mathcal{C} = \{P_1, P_2, P_3\}$  is an odd-path-cycle and  $\mathcal{H}(\mathcal{C})$  is a triangle, hence  $\chi(\mathcal{H}(\mathcal{C})) = 3$ . Suppose now that  $G$  contains a hole  $\{c_1, \dots, c_k\}$ . As  $G$  is perfect, then  $k$  is even. Take  $P_1 = \{c_1, c_2\}$ ,  $P_2 = \{c_2, c_3\}$ , and  $P_3 = \{c_3, \dots, c_k, c_1\}$ . Again,  $\mathcal{C} = \{P_1, P_2, P_3\}$  is an odd-path-cycle and  $\mathcal{H}(\mathcal{C})$  is a triangle, hence  $\chi(\mathcal{H}(\mathcal{C})) = 3$ . Therefore,  $G$  does not contain a diamond nor a hole as an induced subgraph, hence  $G$  is a block graph.  $\square$

**Corollary 4.7.** *Given a graph  $G$  and a set of colors  $C$  with  $|C| > 2$ , the graph  $\mathcal{S}_G^C$  is perfect if and only if  $G$  is a block graph.*

When  $|C| = 2$ , there are graphs  $G$  not being block graphs for which  $\mathcal{S}_G^C$  is a perfect graph. A simple example is the diamond and a more general case is any complete graph without an edge. When  $|C| = 1$  it is clear that  $\mathcal{S}_G^C$  is perfect if and only if  $G$  is perfect, as both graphs coincide.

Recall that a clique from  $\mathcal{S}_G^C$  is either one of the cliques  $\{(v, c) : c \in C\}$  or it comes from a clique of  $G$ . Therefore, Corollary 4.5 and Theorem 4.6 imply that non-negativity constraints and constraints (1) and (6) give a complete characterization for  $P_{\text{col}}(G)$  when  $G$  is a block graph, thus giving an alternative proof of Theorem 3.5.

## 5. Final remarks

The main goal of our ongoing work is to study the polytopes associated to different integer programming formulations for vertex coloring problems on particular families of graphs. For those cases where these problems can be polynomially solved, we pretend to find complete characterizations for the polytopes associated to at least one formulation. As an additional goal, we pretend to extend our results in order to find such characterizations for polytopes associated to open problems, proving by this means that these variantes of the vertex coloring problem can be solved in polynomial time. It is also expected that from

this kind of studies several intermediate results arise, such as new families of valid inequalities for known formulations.

In this work we presented a polyhedral study in search of complete descriptions for vertex coloring polytopes arising from the formulation presented in [11, 23, 24], which we call the *standard formulation*. Theorem 2.3 implies that polytopes arising from this formulation will not yield complete characterizations for those families of graphs for which list-coloring is NP-complete, unless P=NP. Hence, we studied in this work graph classes for which list-coloring is known to be polynomially solvable, in order to find elegant polyhedral descriptions for the corresponding polytopes. Within our study, we show that the vertex coloring polytope associated to this formulation for a graph  $G$  corresponds to a face of the *stable set polytope* of a particular graph  $\mathcal{S}_G^C$  and, based on this fact, we derive a new family of valid inequalities generalizing several known families from the literature, namely the stable cycle inequalities. As a major open problem, we conjecture the stable cycle inequalities to be sufficient to describe  $P_{\text{col}}(G)$  when  $G$  is a cycle. Moreover, if this conjecture is true, these inequalities would also suffice to describe  $P_{\text{col}}(G)$  for cacti graphs.

The stable set polytope has been widely studied and many facet-inducing inequalities are known for this polytope. Furthermore, complete characterizations are known for some graph families, most notably perfect graphs. Corollary 4.5 characterizes graphs  $G$  for which  $\mathcal{S}_G^C$  is perfect and Corollary 4.7 gives a more precise characterization of these graphs when  $|C| > 2$ . As a future work, it would be also interesting to find other characterizations for graphs  $G$  such that  $\mathcal{S}_G^C$  belongs to a family (besides perfect graphs) for which  $STAB(\mathcal{S}_G^C)$  is also known, as this would give a description of  $P_{\text{col}}(G)$  for such graphs.

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## References

- [1] H. Bandelt, and H. Mulder, *Distance-hereditary graphs*, Journal of Combinatorial Theory, Series B **41**-2 (1986) 182–208.
- [2] M. Biro, M. Hujter, and Z. Tuza, *Precoloring extension. I. Interval graphs*, Discrete Mathematics **100** (1–3) (1992) 267–279.

- [3] F. Bonomo and M. Cecowski, *Between coloring and list-coloring:  $\mu$ -coloring*, Electronic Notes in Discrete Mathematics **19** (2005) 117–123.
- [4] F. Bonomo, G. Durán, and J. Marenco, *Exploring the complexity boundary between coloring and list-coloring*, Annals of Operations Research **169-1** (2009) 3–16.
- [5] F. Bonomo, Y. Faenza, and G. Oriolo, *On coloring problems with local constraints*, Discrete Mathematics **312-12–13** (2012) 2027–2039.
- [6] R. Borndörfer, A. Eisenblätter, M. Grötschel, and A. Martin, *The Orientation Model for Frequency Assignment Problems*, ZIB-Berlin TR 98-01, 1998.
- [7] E. Burke, J. Marecek, A. Parkes, and H. Rudová, *A supernodal formulation of vertex colouring with applications in course timetabling*, Annals of Operations Research **179-1** (2010) 105–130.
- [8] M. Campêlo, R. Corrêa, and Y. Frota, *Cliques, holes and the vertex coloring polytope*, Information Processing Letters **89-4** (2004) 159–164.
- [9] M. Chudnovsky, N. Robertson, P.D. Seymour, and R.Thomas, *The strong perfect graph theorem*, Annals of Mathematics, **164** (2006), 51–229.
- [10] V. Chvátal, *On Certain Polytopes Associated with Graphs*, Journal of Combinatorial Theory (B) **18** (1975) 138–154.
- [11] P. Coll, J. Marenco, I. Méndez-Díaz, and P. Zabala, *Facets of the Graph Coloring Polytope*, Annals of Operations Research **116-12** (2002) 79–90.
- [12] D. Cornaz, and V. Jost, *A one-to-one correspondence between colorings and stable sets*, Operations Research Letters **36** (2008) 673–676.
- [13] D. Delle Donne, *Un algoritmo Branch & Cut para un problema de asignación de frecuencias en redes de telefonía celular*. Undergraduate thesis, University of Buenos Aires, 2009.
- [14] D. Delle Donne, and J. Marenco, *A branch & cut algorithm for the minimum-adjacency vertex coloring problem*. Discrete Optimization **8-4** (2011) 540–554.
- [15] J. Edmonds, *Maximum matching and a polyhedron with 0,1-vertices*. Journal of Research of the National Bureau of Standards **69** (1965) 125–130.
- [16] Fekete, P., and Marenco, J. *A complete description of the minimum-adjacency vertex coloring polytope for trees*. Unpublished manuscript (2015).
- [17] M. Garey and D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman, 1979.

- [18] M. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, North Holland, 2004.
- [19] M. Gröschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, 1988.
- [20] K. Jansen, *The optimum cost chromatic partition problem*, Lecture Notes in Computer Science **1203** (1997), 25–36.
- [21] K. Jansen and P. Scheffler. *Generalized coloring for tree-like graphs*. Discrete Applied Mathematics **75-2** (1997) 135–155.
- [22] A. Mehrotra and M. Trick, *A Column Generation Approach for Graph Coloring*, INFORMS Journal On Computing **8-4** (1996) 344–354.
- [23] I. Méndez-Díaz and P. Zabala, *A branch-and-cut algorithm for graph coloring*. Discrete Applied Mathematics **154-5** (2006) 826–847.
- [24] I. Méndez-Díaz and P. Zabala, *A cutting plane algorithm for graph coloring*. Discrete Applied Mathematics **156-2** (2008) 159–179.
- [25] G. Nemhauser and L. Wolsey, *Integer and Combinatorial Optimization*, John Wiley & Sons, 1988.
- [26] A. Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, Springer-Verlag, 2003.
- [27] Z. Tuza, *Graph colorings with local constraints - A survey*, *Discussiones Mathematicae – Graph Theory* **17** (1997) 161–228.