Existence of Nash equilibrium for Chance-Constrained Games

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Abstract

We consider an n-player strategic game with finite action sets and random payoffs. We formulate this as a chance-constrained game by considering that the payoff of each player is defined using a chance-constraint. We consider that the components of the payoff vector of each player are independent normal/Cauchy random variables. We also consider the case where the payoff vector of each player follows a multivariate elliptically symmetric distribution. We show the existence of a Nash equilibrium in both cases.

Keywords: Chance-constrained game, Elliptically symmetric distribution, Normal/Cauchy distribution, Nash equilibrium.

1. Introduction

In 1928, John von Neumann [1] showed that there exists a mixed strategy saddle point equilibrium for a two player zero sum game with finite number of actions for each player. In 1950, John Nash [2] showed that there always exists a mixed strategy Nash equilibrium for an n-player general sum game with finite number of actions for each player. In both [1, 2], it is considered that the players' payoffs are deterministic. However, there can be practical cases where the players' payoffs are better modeled by random variables following certain distributions. The wholesale electricity markets are the good examples that capture

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this situation. A recent paper by Mazadi et al. [3] introduced wind integration on electricity markets due to which the payoffs become random variables. In some cases the consumers' demands are random that introduce randomness in the firms' payoffs [4, 5, 6]. One way to handle such type of games is to replace random variables by their expected values and solve the corresponding deterministic game [5, 6]. We hereafter mention few recent papers on the games with random payoffs using expected payoff criterion. Ravat and Shanbhag [7] studied stochastic Nash games where the payoff function of each player is random and each player solves expected value problem. They consider both smooth and non-smooth stochastic Nash games. In various cases they showed the existence of Nash equilibrium for stochastic Nash games. Xu and Zhang [8], Jadamba and Raciti [9], considered stochastic Nash equilibrium problems. To solve stochastic Nash equilibrium problems, a sample average approximation method is used in [8], and in [9] a variational inequality approach on probabilistic Lebesgue spaces is used. DeMiguel and Xu [10] proposed a two stage multiple-leader stochastic Stackelberg Nash-Cournot model where leaders use expected payoff criterion.

The expected payoff criterion does not take a proper account of stochasticity in the cases where the observed sample payoffs are large amounts with very small probabilities. To handle such cases the concept of satisficing has been considered in the literature, where a player is interested in a strategy which maximizes his total payoff that can be obtained with at least a given probability. Such payoff criterion is defined using chance-constrained programming [11, 12, 13], and due to which we call such games chance-constrained games. The papers [3, 4] mentioned above on electricity markets use chance-constrained game formulation to study the situation. In [3], the randomness in payoffs is due to the installation of wind generators on the electricity market. The authors consider the case where the random variables that represent the amount of wind are independent normal random variables, and they also consider the case where the random vector follows a multivariate normal distribution. Later, for better representation and ease in computation they discuss, in detail, the case where wind generator is present at only one bus station. In [4], the consumers' random demand is assumed to be normally distributed. In [3, 4], the action sets of the players are not finite. In [3], the game problem is formulated as an equivalent linear complementarity problem (LCP). Hence, the existence of Nash equilibrium depends on whether the corresponding LCP has a solution. The existence of Nash equilibrium for the games where the action sets are not finite is not easy to show even when the payoffs are deterministic. It depends on the nature of action sets and payoff functions [14, 15]. There is also a game theoretic situation in electricity market where the action sets are finite [16]. Although the players' payoffs are deterministic in [16], the counterpart of the model where the payoffs are random variables using chance-constrained game formulation can be considered. Only few theoretical results on zero sum chance-constrained games with finite action sets of the players are available in the literature so far [17, 18, 19, 20].

In this paper, we focus on the games where the payoffs of the players are random variables with known probability distributions. The case where probability distribution of the random payoffs are not known completely is considered in [21]. The authors use distributionally robust approach to handle these games. To the best of our knowledge there is no result on the existence of Nash equilibrium for chance-constrained games even when the action sets of all the players are finite. We consider an n-player game where the action set of each player is finite and the payoff vector of each player is a random vector. We formulate this as a chance-constrained game by considering that the payoff of each player is defined using a chance-constraint. We first consider the case where the components of the payoff vector of each player are independent random variables. Specifically, we discuss the case of normal and Cauchy random variables. We also consider the case where the components of the random payoff vector of each player are dependent random variables and we assume that the payoff vector of each player follows a multivariate elliptically symmetric distribution. For each case we show that there always exists a mixed strategy Nash equilibrium for the corresponding chance-constrained game.

The structure of the rest of the paper is as follows: in Section 2 we give the definition of a chance-constrained game. Existence of mixed strategy Nash equilibrium is then given in Section 3.

2. The Model

We consider an n-player strategic game. Let $I = \{1, 2, \cdots, n\}$ be a set of all players. For each $i \in I$, let A_i be a finite action set of player i and its generic element is denoted by a_i. A vector $a = (a_1, a_2, \cdots, a_n)$ denotes an action profile of the game. Let $A = \times_{i=1}^{n} A_i$ be the set of all action profiles of the game. Denote, $A_{-i} = \times_{j=1;j\neq i}^{n} A_i$, and $a_{-i} \in A_{-i}$ is a vector of actions a_j, $j \neq i$. The action set A_i of player i is also called the set of pure strategies of player i. A mixed strategy of a player is represented by a probability distribution over his action set. For each $i \in I$, let X_i be the set of mixed strategies of player i, i.e., it is the set of all probability distributions over the action set A_i. A mixed strategy $\tau_i \in X_i$ is represented by $\tau_i = (\tau_i(a_i))_{a_i \in A_i}$, where $\tau_i(a_i) \geq 0$ is a probability with which player i chooses an action a_i and $\sum_{a_i \in A_i} \tau_i(a_i) = 1$. Let $X = \times_{i=1}^{n} X_i$ be the set of all mixed strategy profiles of the game and its element is denoted by $\tau = (\tau_i)_{i \in I}$. Denote, $X_{-i} = \times_{j=1;j\neq i}^{n} X_i$, and $\tau_{-i} \in X_{-i}$ is a vector of mixed strategies τ_j, $j \neq i$. We define (ν_i, τ_{-i}) to be a strategy profile where player i uses strategy ν_i and each player j, $j \neq i$, uses strategy τ_j. Let $r_i = (r_i(a))_{a \in A}$ be a payoff vector of player i whose components are real numbers. Specifically, player i gets payoff $r_i(a) \in \mathbb{R}$ at action profile a. For such games, Nash [2] showed that there always exists a Nash equilibrium in mixed strategies.

In real life applications the players' payoffs may be random due to uncertainty caused by various external factors. Therefore, we consider the case where the payoffs of each player are random variables and follow a certain distribution. We denote the random payoff vector by $r_i^w = (r_i^w(a))_{a \in A}$, where w is an uncertainty parameter. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then, for each $i \in I$, $r_i^w : \Omega \to \mathbb{R}^{|A|}$, where $|A|$ is the cardinality of set A. For a given strategy profile $\tau \in X$ the payoff of player i, $i \in I$, is defined by

$$r_i^w(\tau) = \sum_{a \in A} \prod_{j=1}^{n} \tau_j(a_j) r_i^w(a). \tag{2.1}$$

From (2.1), $r_i^w(\tau)$ is a random variable for all $\tau \in X$, and it is a linear combination of the components of the random payoff vector $r_i^w = (r_i^w(a))_{a\in A}$. We assume that each player uses satisficing payoff criterion, i.e., the payoff function of each player is defined using a chance-constraint. At strategy profile $\tau \in X$, the payoff of each player is the highest level of his payoff that he can attain with at least a specified level of confidence. The confidence level of each player is given a priori and it is known to other players. Let $\alpha_i \in [0,1]$ be the confidence level of player i and $\alpha = (\alpha_i)_{i\in I}$ be a confidence level vector. For a given strategy profile $\tau \in X$, and a given confidence level vector α the payoff function of player i, $i \in I$, is given by

$$u_i^{\alpha_i}(\tau) = \sup\{u | P(r_i^w(\tau) \geq u) \geq \alpha_i\}. \tag{2.2}$$

We assume that the probability distributions of the payoffs of each player are known to all the players. Then, for a given $\alpha \in [0,1]^n$, the payoff function of a player defined by (2.2) is known to all the players. That is, for a given $\alpha \in [0,1]^n$ the above chance constrained game is a non-cooperative game with complete information. The set of best response strategies of player i, $i \in I$, against a given strategy profile τ_{-i} of other players is given by

$$BR_i^{\alpha_i}(\tau_{-i}) = \left\{\bar{\tau}_i \in X_i | u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i}) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}),\ \forall\ \tau_i \in X_i\right\}.$$

Next, we give the definition of Nash equilibrium.

Definition 2.1 (Nash equilibrium). *A strategy profile $\tau^* \in X$ is said to be a Nash equilibrium for a given $\alpha \in [0,1]^n$, if for all $i \in I$ the following inequality holds,*

$$u_i^{\alpha_i}(\tau_i^*, \tau_{-i}^*) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}^*),\ \forall\ \tau_i \in X_i.$$

That is, τ^* is a Nash equilibrium if and only if $\tau_i^* \in BR_i^{\alpha_i}(\tau_{-i}^*)$ for all $i \in I$.

3. Existence of Nash equilibrium

We assume that the payoffs of each player are random variables following a certain distribution. We consider various cases and show the existence of a mixed strategy Nash equilibrium of chance-constrained game for different values of α.

3.1. Payoffs following normal/Cauchy distribution

For a given strategy profile $\tau \in X$, the probability distribution of random payoff $r_i^w(\tau)$, $i \in I$, plays an important role in defining the payoff function of player i given by (2.2). For each $i \in I$, $r_i^w(\tau)$ is a linear combination of the components of the random payoff vector $r_i^w = (r_i^w(a))_{a\in A}$. We consider the probability distributions that are closed under the linear combination. It is well known that the independent random variables following normal or Cauchy distributions possess this property [22]. We first consider the case where for each $i \in I$, $\{r_i^w(a)\}_{a\in A}$ are independent normal random variables, where the mean and variance of $r_i^w(a)$ are $\mu_i(a)$ and $\sigma_i^2(a)$ respectively. Then, for $\tau \in X$, $r_i^w(\tau)$ follows a normal distribution with mean $\mu_i(\tau) = \sum_{a\in A}\prod_{j=1}^n \tau_j(a_j)\mu_i(a)$ and variance $\sigma_i^2(\tau) = \sum_{a\in A}\prod_{j=1}^n \tau_j^2(a_j)\sigma_i^2(a)$, and $Z_i^N = \frac{r_i^w(\tau)-\mu_i(\tau)}{\sigma_i(\tau)}$ follows a standard normal distribution. Let $F^{-1}_{Z_i^N}(\cdot)$, $i \in I$, be a quantile function of a standard normal distribution. From (2.2), for a given $\tau \in X$ and α, we have

$$\begin{aligned} u_i^{\alpha_i}(\tau) &= \sup\{u | P(r_i^w(\tau) \geq u) \geq \alpha_i\} \\ &= \sup\left\{u | P\left(Z_i^N \leq \frac{u-\mu_i(\tau)}{\sigma_i(\tau)}\right) \leq 1-\alpha_i\right\} \\ &= \mu_i(\tau) + \sigma_i(\tau)F^{-1}_{Z_i^N}(1-\alpha_i). \end{aligned}$$

That is,

$$u_i^{\alpha_i}(\tau) = \sum_{a\in A}\prod_{j=1}^n \tau_j(a_j)\mu_i(a) + \left(\sum_{a\in A}\prod_{j=1}^n \tau_j^2(a_j)\sigma_i^2(a)\right)^{\frac{1}{2}} F^{-1}_{Z_i^N}(1-\alpha_i),\ i \in I. \tag{3.1}$$

Lemma 3.1. *$u_i^{\alpha_i}(\cdot, \tau_{-i})$, $i \in I$, defined by* (3.1) *is a concave function of τ_i for all $\alpha_i \in$* [0.5, 1].

Proof. For fixed τ_{-i}, the first term of $u_i^{\alpha_i}(\cdot, \tau_{-i})$ defined by (3.1) is a linear function of τ_i. If $\alpha_i \in [0.5, 1]$, then the second term of $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of τ_i because $\left(\sum_{a\in A}\prod_{j=1}^n \tau_j^2(a_j)\sigma_i^2(a)\right)^{\frac{1}{2}}$ is a convex function of τ_i, and $F^{-1}_{Z_i^N}(1-\alpha_i) \leq 0$ for all $\alpha_i \in [0.5, 1]$. Hence, we can say that for each $i \in I$, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of τ_i for all $\alpha_i \in [0.5, 1]$. □

Theorem 3.2. *Consider an n-player game where each player has finite number of actions. If for each player i, $i \in I$, $\{r_i^w(a)\}_{a \in A}$ are independent random variables, where $r_i^w(a)$ follows a normal distribution with mean $\mu_i(a)$ and variance $\sigma_i^2(a)$, there exists a mixed strategy Nash equilibrium for all $\alpha \in [0.5, 1]^n$.*

Proof. Let $\mathcal{P}(X)$ be a power set of X. Define a set valued map $G : X \to \mathcal{P}(X)$ such that $G(\tau) = \prod_{i=1}^{n} BR_i^{\alpha_i}(\tau_{-i})$. A strategy profile $\tau \in X$ is said to be a fixed point of the set valued map G if $\tau \in G(\tau)$. It is easy to see that a fixed point of G is a Nash equilibrium. Then, it is sufficient to show that G has a fixed point. In order to show that G has a fixed point, we show that G satisfies all the following conditions of Kakutani fixed point theorem [23]:

1. X is a non-empty, convex, and compact subset of a finite dimensional Euclidean space.
2. $G(\tau)$ is non-empty and convex for all $\tau \in X$.
3. $G(\cdot)$ has closed graph: If $(\tau_n, \bar{\tau}_n) \to (\tau, \bar{\tau})$ with $\bar{\tau}_n \in G(\tau_n)$ for all n then $\bar{\tau} \in G(\tau)$.

Condition 1 holds from the definition of X. Fix $\alpha \in [0.5, 1]^n$. For fixed τ_{-i}, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a continuous function of τ_i from (3.1). For each $i \in I$, $BR_i^{\alpha_i}(\tau_{-i})$, is non-empty because a continuous function $u_i^{\alpha_i}(\cdot, \tau_{-i})$ over a compact set X_i always attains maxima. Hence, $G(\tau)$ is non-empty for all $\tau \in X$. For each $i \in I$, $BR_i^{\alpha_i}(\tau_{-i})$ is a convex set because $u_i^{\alpha_i}(\cdot, \tau_{-i})$ given by (3.1) is a concave function of τ_i from Lemma 3.1. Hence, $G(\tau)$ is a convex set for all $\tau \in X$. Now, we prove that $G(\cdot)$ is a closed graph. Assume that $G(\cdot)$ is not a closed graph, i.e., there is a sequence $(\tau^n, \bar{\tau}^n) \to (\tau, \bar{\tau})$ with $\bar{\tau}^n \in G(\tau^n)$ for all n, but $\bar{\tau} \notin G(\tau)$. In this case $\bar{\tau}_i \notin BR_i(\tau_{-i})$ for some $i \in I$. Then, there is an $\epsilon > 0$ and a $\tilde{\tau}_i$ such that

$$u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}) > u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i}) + 3\epsilon. \quad (3.2)$$

Since, $u_i^{\alpha_i}(\cdot)$ is a continuous function of τ from (3.1), $u_i^{\alpha_i}(\bar{\tau}_i^n, \tau_{-i}^n) \rightarrow u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i})$. Then, there exists an integer N_1 such that

$$u_i^{\alpha_i}(\bar{\tau}_i^n, \tau_{-i}^n) < u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i}) + \epsilon, \ \forall\, n \geq N_1. \tag{3.3}$$

From (3.2) and (3.3), we have

$$u_i^{\alpha_i}(\bar{\tau}_i^n, \tau_{-i}^n) < u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}) - 2\epsilon, \ \forall\, n \geq N_1. \tag{3.4}$$

Similarly, $u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}^n) \rightarrow u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i})$. Then, there exists an integer N_2 such that

$$u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}) < u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}^n) + \epsilon, \ \forall\, n \geq N_2. \tag{3.5}$$

Let $N = \max\{N_1, N_2\}$. Then, from (3.4) and (3.5), we have

$$u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}^n) > u_i^{\alpha_i}(\bar{\tau}_i^n, \tau_{-i}^n) + \epsilon, \ \forall\, n \geq N.$$

That is, $\tilde{\tau}_i$ performs better than $\bar{\tau}_i^n$ against τ_{-i}^n for all $n \geq N$ which contradicts $\bar{\tau}_i^n \in BR_i^{\alpha_i}(\tau_{-i}^n)$ for all n. Hence, $G(\cdot)$ is a closed graph. That is, the set valued map $G(\cdot)$ satisfies all the conditions of Kakutani fixed point theorem. Hence, $G(\cdot)$ has a fixed point τ^*. Such τ^* is a Nash equilibrium of the game. The confidence level vector $\alpha \in [0.5, 1]^n$ is arbitrary, therefore, there always exists a mixed strategy Nash equilibrium for all $\alpha \in [0.5, 1]^n$. $\square$

Now, we consider the case where for each $i \in I$, $\{r_i^w(a)\}_{a \in A}$ are independent Cauchy random variables, where the location and scale parameters of $r_i^w(a)$ are $\mu_i(a)$ and $\sigma_i(a)$ respectively. Then, for a given $\tau \in X$, $r_i^w(\tau)$, $i \in I$, follows a Cauchy distribution with location parameter $\mu_i(\tau) = \sum_{a \in A} \prod_{j=1}^n \tau_j(a_j)\mu_i(a)$, and scale parameter $\sigma_i(\tau) = \sum_{a \in A} \prod_{j=1}^n \tau_j(a_j)\sigma_i(a)$ because the components of τ are non-negative [22]. Hence, $Z_i^C = \frac{r_i^w(\tau) - \mu_i(\tau)}{\sigma_i(\tau)}$ follows a standard Cauchy distribution. Let $F_{Z_i^C}^{-1}(\cdot)$ be a quantile function of a standard Cauchy distribution. Similar to the case of normal distribution, for a given $\tau \in X$ and α we have from (2.2),

$$u_i^{\alpha_i}(\tau) = \sum_{a \in A} \prod_{j=1}^n \tau_j(a_j)\mu_i(a) + \sum_{a \in A} \prod_{j=1}^n \tau_j(a_j)\sigma_i(a) F_{Z_i^C}^{-1}(1 - \alpha_i), \ i \in I. \tag{3.6}$$

The quantile function $F^{-1}_{Z_i^C}(\cdot)$ of Cauchy distribution is not finite at 0 and 1. Therefore, we consider the case of $\alpha_i \in (0,1)$, $i \in I$, so that the payoff function defined by (3.6) is finite. For fixed $\tau_{-i} \in X_{-i}$ the function $u_i^{\alpha_i}(\cdot, \tau_{-i})$, $i \in I$, defined by (3.6) is a linear function of τ_i.

Theorem 3.3. *Consider an n-player game where each player has finite number of actions. If for each player i, $i \in I$, $\{r_i^w(a)\}_{a\in A}$ are independent random variables, where $r_i^w(a)$ follows a Cauchy distribution with location parameter $\mu_i(a)$ and scale parameter $\sigma_i(a)$, there exists a mixed strategy Nash equilibrium for all $\alpha \in (0,1)^n$.*

Proof. For each $i \in I$, $BR_i^{\alpha_i}(\tau_{-i})$ is a convex set for all $\alpha_i \in (0,1)$ because $u_i^{\alpha_i}(\cdot, \tau_{-i})$ defined by (3.6) is a linear function of τ_i. From (3.6), $u_i^{\alpha_i}(\cdot)$, $i \in I$, is a continuous function of τ. Then, the proof follows from the similar arguments used in Theorem 3.2. □

3.1.1. Special cases

In previous section, we show that a Nash equilibrium of chance-constrained game exists, for all $\alpha \in [0.5,1]^n$, for the case where all the components of random payoff vector r_i^w are independent normal random variables. We now consider a special case where the components of the payoff vector of each player corresponding to only one of his action are independent normal random variables and rest of the components are deterministic. We show that there exists a Nash equilibrium for all $\alpha \in [0,1]^n$. Further, if only one component of each player's payoff vector is a random variable and all other components are deterministic, the Nash equilibrium existence for all $\alpha \in [0,1]^n$ can be extended to all the continuous probability distributions whose quantile function exist.

(i) Normal random payoffs of each player corresponding to only one of his action

We assume that for each player i, $i \in I$, there exists an action $\bar{a}_i \in A_i$ such that $\{r_i^w(\bar{a}_i, a_{-i})\}_{a_{-i}\in A_{-i}}$ are independent normal random variables and all other components of r_i^w are deterministic. That is, for each $i \in I$, $r_i^w(a_i, a_{-i}) = r_i(a_i, a_{-i})$ for all $a_i \in A_i$, $a_i \neq \bar{a}_i$, and $a_{-i} \in A_{-i}$. Then, for each $i \in I$,

$\sigma_i^2(a_i, a_{-i}) = 0$, and $\mu_i(a_i, a_{-i}) = r_i(a_i, a_{-i})$, for all $a_i \in A_i$, $a_i \neq \bar{a}_i$, and $a_{-i} \in A_{-i}$. By putting these values in (3.1), we have,

$$
\begin{aligned}
u_i^{\alpha_i}(\tau) = & \sum_{a_i \in A_i; a_i \neq \bar{a}_i} \tau_i(a_i) \left(\sum_{a_{-i} \in A_{-i}} \prod_{j=1; j\neq i}^{n} \tau_j(a_j) r_i(a_i, a_{-i}) \right) \\
& + \tau_i(\bar{a}_i) \Bigg[\sum_{a_{-i} \in A_{-i}} \prod_{j=1; j\neq i}^{n} \tau_j(a_j) \mu_i(\bar{a}_i, a_{-i}) \\
& + \left(\sum_{a_{-i} \in A_{-i}} \prod_{j=1; j\neq i}^{n} \tau_j^2(a_j) \sigma_i^2(\bar{a}_i, a_{-i}) \right)^{\frac{1}{2}} F_{Z_i^N}^{-1}(1-\alpha_i) \Bigg], \; i \in I. \quad (3.7)
\end{aligned}
$$

Theorem 3.4. *Consider an n-player game where each player has finite number of actions. If for each player i, $i \in I$, there exists an action $\bar{a}_i \in A_i$ such that $\{r_i^w(\bar{a}_i, a_{-i})\}_{a_{-i} \in A_{-i}}$ are independent random variables, where $r_i^w(\bar{a}_i, a_{-i})$ follows a normal distribution with mean $\mu_i(\bar{a}_i, a_{-i})$ and variance $\sigma_i^2(\bar{a}_i, a_{-i})$, and all other components of r_i^w are deterministic, there exists a mixed strategy Nash equilibrium for all $\alpha \in [0,1]^n$.*

Proof. For each $i \in I$, $BR_i^{\alpha_i}(\tau_{-i})$ is a convex set for all $\alpha_i \in [0,1]$ because $u_i^{\alpha_i}(\cdot, \tau_{-i})$ defined by (3.7) is a linear function of τ_i. From (3.7), $u_i^{\alpha_i}(\cdot)$ is a continuous function of τ. Then, the proof follows from the similar arguments used in Theorem 3.2. □

(ii) Payoff with one random variable component

We assume that for each player i, $i \in I$, there exists an action profile $a^i \in A$ such that $r_i^w(a^i)$ is a normal random variable and all other components of r_i^w are deterministic, i.e., $r_i^w(a) = r_i(a)$ for all $a \in A$, $a \neq a^i$. Then $\mu_i(a) = r_i(a)$ and $\sigma_i^2(a) = 0$ for all $a \in A$, $a \neq a^i$. By putting these values in (3.1), we have,

$$
u_i^{\alpha_i}(\tau) = \sum_{a \in A; a \neq a^i} \prod_{j=1}^{n} \tau_j(a_j) r_i(a) + \prod_{j=1}^{n} \tau_j(a_j^i) \left(\mu_i(a^i) + \sigma_i(a^i) F_{Z_i^N}^{-1}(1-\alpha_i) \right). \quad (3.8)
$$

Let $F_{r_i^w(a^i)}^{-1}(\cdot)$ denote the quantile function of normal random variable $r_i^w(a^i)$. Then, $F_{r_i^w(a^i)}^{-1}(1-\alpha_i) = \left(\mu_i(a^i) + \sigma_i(a^i) F_{Z_i^N}^{-1}(1-\alpha_i) \right)$. Hence, we can write

(3.8) as

$$u_i^{\alpha_i}(\tau) = \sum_{a\in A}\prod_{j=1}^{n} \tau_j(a_j)\tilde{r}_i(a),\ i \in I,$$

where,

$$\tilde{r}_i(a) = \begin{cases} F^{-1}_{r_i^w(a^i)}(1-\alpha_i), \text{ if } a = a^i, \\ r_i(a), \text{ if } a \neq a^i. \end{cases} \tag{3.9}$$

Hence, the game is equivalent to a deterministic strategic game with payoff vector $\tilde{r}_i = (\tilde{r}_i(a))_{a\in A}$, $i \in I$, where $\tilde{r}_i(a) \in \mathbb{R}$ is defined by (3.9). Therefore, the existence of Nash equilibrium for all $\alpha \in [0,1]^n$ follows from [2]. Since, the deterministic payoff vector $\tilde{r}_i$ depends only on the quantile function of normal random variable $r_i^w(a^i)$, then the result on the existence of Nash equilibrium in this case can be extended for all the continuous probability distributions whose quantile function exists.

3.2. Payoffs following multivariate elliptical distributions

In Section 3.1, we consider the case where the components of the random payoff vector of each player are independent random variables. Here, we consider the case where the components of the random payoff vector of each player are dependent random variables. We assume that the payoff vector of each player follows a multivariate elliptically symmetric distribution. The distributions belonging to the class of elliptically symmetric distributions generalize the multivariate normal distribution. Some famous multivariate distributions like normal, Cauchy, t, Laplace, and logistic distributions belong to the family of elliptically symmetric distributions. For more details about elliptically symmetric distributions see [24].

We assume that the payoff vector $(r_i^w(a))_{a\in A}$ of player i, $i \in I$, follows a multivariate elliptically symmetric distribution with parameters μ_i and Σ_i. The vector $\mu_i = (\mu_i(a))_{a\in A}$ represents a location parameter and Σ_i is a scale matrix. We assume Σ_i to be a positive definite matrix. Then, all linear combinations of the components of the payoff vector follow a univariate elliptically symmetric distribution [24]. For a given $\tau \in X$, let $\eta^\tau = (\eta^\tau(a))_{a\in A}$ be a vector, where

$\eta^\tau(a) = \prod_{j=1}^n \tau_j(a_j)$. Then for $\tau \in X$, $r_i^w(\tau)$, $i \in I$, follows a univariate elliptically symmetric distribution with parameters $\mu_i^T\eta^\tau$ and $(\eta^\tau)^T\Sigma_i\eta^\tau$; T denotes transposition. Since, Σ_i is a positive definite matrix, then, $\sqrt{(\eta^\tau)^T\Sigma_i\eta^\tau}$ will be a norm and it is denoted by $||\eta^\tau||_{\Sigma_i}$. For each $i \in I$, $Z_i^S = \frac{r_i^w(\tau)-\mu_i^T\eta^\tau}{||\eta^\tau||_{\Sigma_i}}$ follows a univariate spherically symmetric distribution with parameters 0 and 1 [24]. Let $F^{-1}_{Z_i^S}(\cdot)$ be a quantile function of a univariate spherically symmetric distribution. From (2.2), for a given $\tau \in X$, we obtain

$$\begin{aligned}
u_i^{\alpha_i}(\tau) &= \sup\{u|P(r_i^w(\tau) \geq u) \geq \alpha_i\} \\
&= \sup\left\{u|P\left(\frac{r_i^w(\tau)-\mu_i^T\eta^\tau}{||\eta^\tau||_{\Sigma_i}} \leq \frac{u-\mu_i^T\eta^\tau}{||\eta^\tau||_{\Sigma_i}}\right) \leq 1-\alpha_i\right\} \\
&= \mu_i^T\eta^\tau + ||\eta^\tau||_{\Sigma_i} F^{-1}_{Z_i^S}(1-\alpha_i).
\end{aligned}$$

That is,

$$u_i^{\alpha_i}(\tau) = \mu_i^T\eta^\tau + ||\eta^\tau||_{\Sigma_i} F^{-1}_{Z_i^S}(1-\alpha_i),\ i \in I. \tag{3.10}$$

We know that the quantile function $F^{-1}_{Z_i^S}(1-\alpha_i) \leq 0$ for all $\alpha_i \in (0.5, 1]$. If Z_i^S has strictly positive density then $F^{-1}_{Z_i^S}(1-\alpha_i) \leq 0$ for all $\alpha_i \in [0.5, 1]$ (see [25]).

Lemma 3.5. *$u_i^{\alpha_i}(\cdot, \tau_{-i})$, $i \in I$, defined by* (3.10) *is a concave function of τ_i for all $\alpha_i \in$* $(0.5, 1]$.

Proof. Fix $i \in I$, $\alpha_i \in (0.5, 1]$, and $\tau_{-i} \in X_{-i}$. Let τ_i^1, $\tau_i^2 \in X_i$. Take $\lambda \in [0, 1]$. Then, for a strategy profile $(\lambda\tau_i^1 + (1-\lambda)\tau_i^2, \tau_{-i})$ we have

$$\eta^{(\lambda\tau_i^1+(1-\lambda)\tau_i^2,\tau_{-i})}(a) = \left(\lambda\tau_i^1(a_i) + (1-\lambda)\tau_i^2(a_i)\right) \prod_{j\in I; j\neq i} \tau_j(a_j),$$

for all $a \in A$. That is, $\eta^{(\lambda\tau_i^1+(1-\lambda)\tau_i^2,\tau_{-i})} = \lambda\eta^{(\tau_i^1,\tau_{-i})} + (1-\lambda)\eta^{(\tau_i^2,\tau_{-i})}$. From (3.10), we have

$$\begin{aligned}
u_i^{\alpha_i}(\lambda\tau_i^1 + (1-\lambda)\tau_i^2, \tau_{-i}) = \mu_i^T &\left(\lambda\eta^{(\tau_i^1,\tau_{-i})} + (1-\lambda)\eta^{(\tau_i^2,\tau_{-i})}\right) \\
&+ ||\lambda\eta^{(\tau_i^1,\tau_{-i})} + (1-\lambda)\eta^{(\tau_i^2,\tau_{-i})}||_{\Sigma_i} F^{-1}_{Z_i^S}(1-\alpha_i) \\
&\geq \lambda u_i^{\alpha_i}(\tau_i^1, \tau_{-i}) + (1-\lambda)u_i^{\alpha_i}(\tau_i^2, \tau_{-i}).
\end{aligned}$$

That is, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of τ_i for all $\alpha_i \in (0.5, 1]$. □

Remark 3.6. *If the random payoff vector* $(r_i^w(a))_{a\in A}$*,* $i \in I$*, has strictly positive density then Lemma 3.5 holds for all* $\alpha_i \in [0.5, 1]$.

Theorem 3.7. *Consider an* n*-player game where each player has finite number of actions. If the random payoff vector* $(r_i^w(a))_{a\in A}$ *of player* i*,* $i \in I$*, follows a multivariate elliptically symmetric distribution with location parameter* $\mu_i = (\mu_i(a))_{a\in A}$ *and scale matrix* Σ_i *which is positive definite, there exists a mixed strategy Nash equilibrium for all* $\alpha \in (0.5, 1]^n$.

PROOF. For each $i \in I$, $BR_i^{\alpha_i}(\tau_{-i})$ is a convex set for all $\alpha_i \in (0.5, 1]$ because the payoff function $u_i^{\alpha_i}(\cdot, \tau_{-i})$ defined by (3.10) is a concave function of τ_i from Lemma 3.5. From (3.10), $u_i^{\alpha_i}(\cdot)$ is a continuous function of τ. Then, the proof follows from the similar arguments used in Theorem 3.2. □

Remark 3.8. *For each* $i \in I$*, if the random payoff vector* $(r_i^w(a))_{a\in A}$ *has strictly positive density then Theorem 3.7 holds for all* $\alpha \in [0.5, 1]^n$.

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