Solving SDP Completely with an Interior Point Oracle

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Abstract

We suppose the existence of an oracle which solves any semidefinite programming (SDP) problem satisfying strong feasibility (i.e., Slater's condition) simultaneously at its primal and dual sides. We note that such an oracle might not be able to directly solve general SDPs even after certain regularization schemes are applied. In this work we fill this gap and show how to use such an oracle to "completely solve" an arbitrary SDP. Completely solving entails, for example, distinguishing between weak/strong feasibility/infeasibility and detecting when the optimal value is attained or not. We will employ several tools, including a variant of facial reduction where all auxiliary problems are ensured to satisfy strong feasibility at all sides. Our main technical innovation, however, is an analysis of *double facial reduction*, which is the process of applying facial reduction twice: first to the original problem and then once more to the dual of the regularized problem obtained during the first run. Although our discussion is focused on semidefinite programming, the majority of the results are proved for general convex cones.

Keywords: double facial reduction, facial reduction, semidefinite programming, feasibility problem.

1 Introduction

Consider the following pair of primal and dual linear semidefinite programs (SDPs).

$$\begin{array}{lll} \inf\limits_{x} & \langle c, x\rangle & \text{(SDP-P)} \\ \text{subject to} & \mathcal{A}x = b & \\ & x \in \mathcal{S}^n_+ & \end{array} \qquad \begin{array}{lll} \sup\limits_{y} & \langle b, y\rangle & \text{(SDP-D)} \\ \text{subject to} & c - \mathcal{A}^*y \in \mathcal{S}^n_+, & \end{array}$$

where $\mathcal{S}^n_+$ denotes the cone of $n \times n$ symmetric positive semidefinite matrices, which is contained in $\mathcal{S}^n$ (the space of $n \times n$ real symmetric matrices). Here $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ is a linear map, $b \in \mathbb{R}^m$, $c \in \mathcal{S}^n$ and $\mathcal{A}^*$ denotes the adjoint map of $\mathcal{A}$. In addition, we assume that both $\mathbb{R}^m$ and $\mathcal{S}^n$ are equipped with the usual Euclidean product and the trace inner product, respectively. We will use the same symbol $\langle \cdot, \cdot \rangle$ to express both inner products.

We start with the following observation.

> *To the best of our knowledge, all (or almost all) methods for solving SDP require some kind of assumption on the problems* (SDP-P) *and* (SDP-D) *in order for its convergence theory to work. In addition, there seems to be no method that can solve arbitrary SDP instances and distinguish between all kinds of ill-behaviour that can happen in semidefinite programming.*

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On one hand, it might seem almost obvious that some condition must be imposed on the pair (SDP-P) and (SDP-D) in order to get meaningful convergence results, which is the pattern established in classical nonlinear programming since the early days, where many convergence results required some constraint qualification to hold. On the other hand, for linear programming we have the simplex method, which, at least in theory, is able to solve any linear program and detect all possible outcomes including infeasibility and unboundedness. Given that semidefinite programming is one of the most natural extensions of linear programming, it is somewhat disappointing that, as of this writing, we still cannot claim to be able to solve SDPs in the same thorough way.

However, there are indeed classes of SDPs that we can reasonably claim that are solvable by current methods. One of those classes consists of the SDPs for which strong feasibility (also called *Slater's condition*) holds at both (SDP-P) and (SDP-D). They are solvable, for instance, by using interior point methods [29, 2]. In this paper, we aim to show the following result.

> Suppose we have access to an oracle that can solve any SDP instance, provided that the instance is both primal and dual strongly feasible. Then, we can "completely solve" *any* SDP instance with polynomially (in n) many calls to this oracle.

Later in Section 1.2 we state our results more precisely including a suitable definition of "completely solving" but, for now, we give some background to our research and results.

1.1 Background and previous works

We now discuss briefly how strong feasibility is connected with some research trends in continuous optimization.

- *Interior point algorithms and software.* Most modern IPM softwares [44, 9, 47] including the commercial solver Mosek do not require explicit knowledge of an interior feasible point beforehand. SeDuMi [44], for instance, transforms a standard form problem into the so-called homogeneous self-dual formulation, which has a trivial starting point. SDPA [9] and SDPT3 [47] use an infeasible interior point method. The fact that these methods can work without explicit knowledge of an interior feasible point, does not mean that they do *not require the existence of an interior feasible point.* Quite the opposite, the absence of interior feasible points may introduce theoretical and numerical difficulties in recovering a solution for the original problem. Also, detection of infeasibility is a complicated task. Some interior point methods, such as the one discussed in [30] by Nesterov, Todd and Ye, are able to obtain a certificate of infeasibility if the problem is dual or primal strongly infeasible, but the situation is less clear in the presence of the so-called *weak infeasibility* [22]. These issues are also discussed by Karimi and Tunçel in [14], in the context of their software *DDS (Domain-Driven Solver)* [13, 15].

- *Ramana's extended dual.* Ramana [41, 42] developed an alternative duality theory for (SDP-D) beyond the usual Lagrangian duality. Remarkable features of Ramana's dual include the fact that whenever the optimal value of an SDP is finite, its Ramana's dual attains the same optimal value without any further assumptions. However, Ramana's dual is not necessarily suitable to be solved by IPMs due to the fact that it does not ensure the existence of interior feasible points at both sides.

- *Facial reduction.* Denote by $\mathcal{F}_{\mathrm{D}}^{S}$, the set of feasible slacks of (SDP-D), i.e.,

  $$\mathcal{F}_{\mathrm{D}}^{S} = \{s \in \mathcal{S}^n_+ \mid \exists y, s = c - \mathcal{A}^* y\}.$$

  Let $\mathcal{F}_{\min}^{\mathrm{D}}$ be the minimal face of $\mathcal{S}^n_+$ which contains $\mathcal{F}_{\mathrm{D}}^{S}$. If we replace $\mathcal{S}^n_+$ by $\mathcal{F}_{\min}^{\mathrm{D}}$ in (SDP-D), then the new (SDP-D) will be strongly feasible, because $\mathcal{F}_{\min}^{\mathrm{D}}$ is characterized as the unique face for which $\mathcal{F}_{\mathrm{D}}^{S}$ intersects the relative interior of $\mathcal{F}_{\min}^{\mathrm{D}}$. The process of finding $\mathcal{F}_{\min}^{\mathrm{D}}$ is called *facial reduction* [51, 32] and was developed originally by Borwein and Wolkowicz [5, 4] for convex optimization with conic constraints. Descriptions for the conic linear programming case have appeared, for instance, in Pataki [32] and in Waki and Muramatsu [51]. We will overview facial reduction in more detail in Section 3.

However, one important point is that facial reduction only guarantees that strong feasibility is satisfied at one side of the problem. So, again, even this regularized problem might fail to have interior solutions at both primal and dual sides.

We remark that strong feasibility at only one of the sides of the problem can also be a source of numerical difficulties. In Section 2 of [52], Waki, Nakata and Muramatsu shows an instance satisfying strong feasibility at the primal side, but not at the dual side. Its optimal value is zero but both SDPA [9] and SeDuMi [44] output 1 instead.

- *Algebraic approaches.* Henrion, Naldi and Din described an algebraic approach to the problem of obtaining a feasible solution to (SDP-D), see [10, 11]. Interesting features of their algorithm include, among others, the fact that their algorithm is implementable in exact arithmetic (as opposed to floating point arithmetic) and that, as long as (SDP-D) satisfies certain genericity assumptions, the algorithm can find solutions even in degenerate cases when strong feasibility is not satisfied. In addition, when a solution is found, a so-called rational parametrization is provided for it. A description of their package *Spectra* is given in [11]. Drawbacks, however, include that in most cases, only small instances can be solved, see Section 1 of [11]. Furthermore, optimization problems cannot be solved directly.

There is a growing body of research aimed at understanding SDPs and conic linear programs having pathological behaviours such as nonzero duality gaps and weak infeasibility. Here we will mention a few of them. A problem is called *weakly infeasible* if there is no feasible solution but the distance between the underlying affine space and the cone under consideration is zero. Weak infeasibility is known to be very hard to detect numerically, see for instance Pólik and Terlaky [38]. In [50], Waki showed that weakly infeasible problems sometimes arise from polynomial optimization. There is also a discussion on weak infeasibility semidefinite programming and second order cone programming in [22] and [24], respectively. Some of the results in [22] were generalized to arbitrary closed convex cones by Liu and Pataki, see [19] for more details. See also [23], where some results of [19] on weakly infeasible problems are sharpened when the polyhedral faces of the underlying cone are taken into account.

It is hard to obtain finite certificates of infeasibility for SDPs, because there is no straightforward extension of Farkas' Lemma for non-polyhedral cones. Another issue is that, as shown by Porkolab and Khachiyan [39], even a reasonably sized SDP may only have exponentially small feasible solutions, which makes it hard to detect feasibility/infeasibility numerically.

Nevertheless, the first finite infeasibility certificate was obtained by Ramana in [41] using his extended duality theory. Since then, Sturm mentioned the possibility of obtaining a finite certificate for infeasibility by using the directions produced in his regularization procedure, see page 1243 of [45]. More recently, Liu and Pataki have also obtained finite certificates through elementary reformulations [18]. Interestingly, Klep and Schweighofer [16] also obtained certificates through a completely different approach using tools from real algebraic geometry. As we move from SDPs to conic linear programs over arbitrary cones, facial reduction seems to one of the few approaches that can provide finite certificates of infeasibility see, for example, [19].

In [52], Waki, Nakata and Muramatsu discussed SDP instances for which known solvers failed to obtain the correct answer and in one case, this happened even though the problem had an interior feasible point at the primal side. In [33, 35], Pataki gave a definition of "bad behaviour" and showed that all SDPs in that class can be put in the same form, after performing an elementary reformulation. A discussion on duality gaps and many interesting examples of pathological SDPs are given by Tunçel and Wolkowicz in [48]. Pataki has recently provided an extensive study of duality gaps in semidefinite programming in [34]. He showed, for instance, that all SDPs with positive duality gap and $m = 2$ (i.e., the dual problem has two variables) have a common reformulation, see Theorem 1 therein.

1.2 Summary and contributions of this work

We consider the following oracle, which we will denote by $\mathcal{O}_{\text{int}}$.

The interior point oracle $\mathcal{O}_{\text{int}}$ for SDPs

Input : The problem data: $\mathcal{A}, b, c$. Both (SDP-P) and (SDP-D) must be strongly feasible.
Output: A zero duality gap optimal solution pair x^*, y^*. That is, x^* and y^* satisfy

$$\begin{aligned} \langle c, x^* \rangle &= \langle b, y^* \rangle \\ c - \mathcal{A}^* y^* &\in \mathcal{S}^n_+ \\ \mathcal{A} x^* &= b \\ x^* &\in \mathcal{S}^n_+. \end{aligned}$$

We can regard $\mathcal{O}_{\text{int}}$ as a machine running an idealized version of either the homogeneous self-dual embedding method [40, 7, 26], an infeasible interior point method [30], the ellipsoid method or even an augmented Lagrangian method. An important point is that no assumption is made on the inner workings of the oracle. Now we are ready to define the meaning of *completely solving an SDP*.

Definition 1 (Completely solving (SDP-D))**.** *An algorithm, procedure or a scheme is said to* completely solve (SDP-D)*, if it receives as input $\mathcal{A}, b, c$ and $\epsilon > 0$ and achieves the following goals.*

(a) *It decides whether the* (SDP-D) *is feasible or not.*

(b) *When* (SDP-D) *is feasible, it computes the optimal value. If the optimal value is attained, it computes an optimal solution. If the optimal value is finite but not attained, it computes an ϵ-optimal solution. If* (SDP-D) *is unbounded (i.e., $\theta_{\text{D}} = +\infty$) this must be detected.*

(c) *When* (SDP-D) *is infeasible, it correctly distinguishes between strong infeasibility and weak infeasibility. If* (SDP-D) *is weakly infeasible, then it finds a matrix that is arbitrarily close to feasibility (this will be made precise later).*

Although we focus on semidefinite programming, the majority of our results will be proved for general conic linear programs (CLPs). Keeping this remark in mind, we now state our contributions in this paper.

1. We present an algorithm for completely solving general CLPs, provided that we can solve certain auxiliary problems that are strongly feasible, see Section 4 and Algorithm 4. In particular, we will show that an *arbitrary* SDP can be completely solved by $O(n)$ calls to $\mathcal{O}_{\text{int}}$. This implies that even though an arbitrary SDP may have unfavourable properties, we can always completely solve it in the sense of Definition 1 if we assume that we are capable of solving instances that are both primal and dual strongly feasible. An important feature of our approach is that it is *method agnostic* and does not rely in any way on the inner working of $\mathcal{O}_{\text{int}}$. See Appendix B for an example of applying Algorithm 4 to a particularly ill-behaved instance.

2. We present a detailed discussion of *double facial reduction* for general conic linear programs, which is the process of applying facial reduction twice: first to an CLP and then, to the dual of the regularized CLP obtained at the first step.

   Through double facial reduction, whenever the optimal value of (SDP-D) is finite, we are ensured to obtain a new pair of primal and dual strongly feasible problems and whose common optimal value coincides with the optimal value of (SDP-D).

   Although we cannot always recover optimal solutions for (SDP-D) from this new pair of problems (after all, (SDP-D) might not even have optimal solutions in the first place), we will show how it is possible to obtain feasible solutions that are arbitrarily close to optimality, for any desired accuracy, by using the directions that appear when applying facial reduction. See Section 4.2 and Algorithm 2 for more details. The discussion on obtaining almost optimal solution leads naturally to an approach for obtaining almost feasible solution for weakly infeasible problems and this is discussed in Section 4.3.

3. We present several technical results about facial reduction that we believe might be of independent interest. For example, we show how to perform facial reduction by solving auxiliary problems that are ensured to be both primal and dual strongly feasible, see Lemma 10 and Algorithm 1.

   We also provide a technical result on how the feasibility properties of a problem might change when facial reduction is applied to its dual, see Proposition 14 and Theorem 15.

We remark that this paper is a thorough extension and reformulation of an earlier technical report [21], where the results were only proved for semidefinite programming by different techniques.

1.3 Limitations of this work

A limitation of this work is that the algorithm for completely solving conic linear programs (Algorithm 4) is somewhat hypothetical. This is because, except in very special cases [49, 53, 28], we cannot solve exactly an SDP even if it is primal and dual strongly feasible. Usually, the best we can do is to compute solutions that are approximately feasible and approximately optimal to some specified tolerance $\epsilon > 0$ or, under special circumstances, provide a rational parametrization to the solution set as in [10, 11]. So, strictly speaking, only an approximate version of the oracle $\mathcal{O}_{\rm int}$ might be practically implementable.

Missing from our analysis is how to deal with the case where there is some imprecision in the answer returned by $\mathcal{O}_{\rm int}$. This is a very complex issue because since regularity conditions might fail, small perturbations in the input data might lead to problems whose optimal values are vastly different. Furthermore, impreciseness whilst doing facial reduction might lead to a wrong face being computed and feasible solutions could be inadvertently removed.

We believe however, that the analysis of the exact case is an important stepping stone and we see a similar pattern in many subareas of optimization. For example, for augmented Lagrangian methods, understanding the behavior of the algorithm when subproblems are solved exactly seems to be quite important for getting the larger picture of the algorithm and its convergence analysis, even though, in practice, the subproblems are only approximately solved.

We remark that related approaches by de Klerk, Roos and Terlaky [7] and Permenter, Friberg and Andersen [36] also assume that exact solutions are obtainable. However, numerical experiments are provided in [36] to check how their approach fare under inexactness. We provide a detailed comparison between [7, 36] and our approach in Section 6.

1.4 Structure of this paper

This paper is organized as follows. Section 2 discusses the notation used throughout the paper and contains a review of the necessary notions from convex analysis. Some technical aspects related to the faces of $\mathcal{S}^n_+$ and interior point oracle $\mathcal{O}_{\rm int}$ are discussed in Section 2.2. Section 3 presents a facial reduction algorithm that is suitable to be used in conjunction with $\mathcal{O}_{\rm int}$. Section 4 discusses double facial reduction and how it can be used to obtain almost optimal solutions and analyze weak infeasibility. Section 5 contains the description of an algorithm for completely solving a general conic linear program which can be adapted to use $\mathcal{O}_{\rm int}$ when the underlying cone is $\mathcal{S}^n_+$. Section 6 contains a discussion on related approaches. Section 7 concludes this work.

2 Preliminary discussion and review of relevant notions

Let $C \subseteq \mathcal{E}$ be a closed convex set contained in a real finite dimensional space $\mathcal{E}$. Its relative interior, closure, linear span and dimension are denoted by $\operatorname{ri} C$, $\operatorname{cl} C$, $\operatorname{span} C$ and $\dim C$, respectively. We assume that $\mathcal{E}$ is equipped with some inner product $\langle \cdot, \cdot \rangle$ and we will denote by $C^\perp$ the subspace of $\mathcal{E}$ which contains the elements orthogonal to C with respect to $\langle \cdot, \cdot \rangle$. We will denote by $\|\cdot\|$ the norm induced by $\langle \cdot, \cdot \rangle$. For a pair of sets $C, D \subseteq \mathcal{E}$ we define the distance between C and D as

$$\operatorname{dist}(C, D) \coloneqq \inf\{\|x - s\| \mid x \in C, s \in D\}.$$

If $x \in \mathcal{E}$, we will use $\mathrm{dist}\,(x, C)$ as a shorthand for $\mathrm{dist}\,(\{x\}, C)$.

If $\mathcal{A}$ is a linear map, we will denote its image, kernel and adjoint by $\mathrm{range}\,\mathcal{A}, \ker \mathcal{A}$ and $\mathcal{A}^*$, respectively.

For $\mathcal{K} \subseteq \mathcal{E}$ a closed convex cone, we denote by $\mathrm{lin}\,\mathcal{K}$ the *lineality space* of $\mathcal{K}$, i.e.,

$$\mathrm{lin}\,\mathcal{K} \coloneqq \mathcal{K} \cap -\mathcal{K}.$$

We denote by $\mathcal{K}^*$ the dual cone of $\mathcal{K}$:

$$\mathcal{K}^* \coloneqq \{x \in \mathcal{E} \mid \langle s, x \rangle \geq 0, \forall s \in \mathcal{K}\}.$$

A closed convex subset $\mathcal{F}$ contained in $\mathcal{K}$ is said to be a *face* of $\mathcal{K}$ if

$$s, \hat{s} \in \mathcal{K}, \frac{s + \hat{s}}{2} \in \mathcal{F} \Rightarrow s, \hat{s} \in \mathcal{F}.$$

The *conjugate face* of $\mathcal{F}$ is defined as

$$\mathcal{F}^{\Delta} \coloneqq \mathcal{K}^* \cap \mathcal{F}^{\perp}.$$

Given $x \in \mathcal{K}$, we write $\mathcal{F}(x, \mathcal{K})$ for the intersection of all faces of $\mathcal{K}$ containing x. $\mathcal{F}(x, \mathcal{K})$ is the minimal (with respect to inclusion) face of $\mathcal{K}$ containing x.

For a given $x \in \mathcal{K}$, we write $\mathrm{dir}\,(x, \mathcal{K})$ for the *cone of feasible directions* of $\mathcal{K}$ at x. This is the set

$$\mathrm{dir}\,(x, \mathcal{K}) \coloneqq \{z \in \mathcal{E} \mid \exists t > 0, x + tz \in \mathcal{K}\}.$$

The closure of $\mathrm{dir}\,(x, \mathcal{K})$ is the *tangent cone* of $\mathcal{K}$ at x and is denoted by $\tan\,(x, \mathcal{K})$. The *tangent space* of $\mathcal{K}$ at x is the lineality space of $\tan\,(x, \mathcal{K})$ and is denote by $T_x\mathcal{K}$. In summary, we have

$$\begin{aligned} \tan\,(x, \mathcal{K}) &\coloneqq \mathrm{cl}\,\mathrm{dir}\,(x, \mathcal{K}), \\ T_x\mathcal{K} &\coloneqq \tan\,(x, \mathcal{K}) \cap -\tan\,(x, \mathcal{K}). \end{aligned}$$

Some of the relationships between the sets defined so far will be summarized at Lemma 3.

Although our focus is on semidefinite programming, most of the results will be proved for the following primal and dual pair of general conic linear programs:

$$\begin{aligned} \inf_{x} \quad & \langle c, x \rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{K}^* \end{aligned} \qquad \text{(Conic-P)}$$

$$\begin{aligned} \sup_{y} \quad & \langle b, y \rangle \\ \text{subject to} \quad & c - \mathcal{A}^*y \in \mathcal{K}, \end{aligned} \qquad \text{(Conic-D)}$$

where $\mathcal{A} : \mathcal{E} \to \mathbb{R}^m$ is a linear map, $b \in \mathbb{R}^m$, $c \in \mathcal{E}$. Semidefinite programming corresponds to the specific case where $\mathcal{E} = \mathcal{S}^n$ and $\mathcal{K} = \mathcal{S}^n_+$.

We will denote by θ_{P} and θ_{D}, the optimal values of (Conic-P) and (Conic-D) respectively. It is understood that $\theta_{\mathrm{P}} = +\infty$ if (Conic-P) is infeasible and $\theta_{\mathrm{D}} = -\infty$ if (Conic-D) is infeasible. The primal and dual feasible regions are defined as follows:

$$\begin{aligned} \mathcal{F}_{\mathrm{P}} &\coloneqq \{x \in \mathcal{K}^* \mid \mathcal{A}x = b\}, \\ \mathcal{F}_{\mathrm{D}} &\coloneqq \{y \in \mathbb{R}^m \mid c - \mathcal{A}^*y \in \mathcal{K}\}, \\ \mathcal{F}^S_{\mathrm{D}} &\coloneqq \{s \in \mathcal{K} \mid \exists y \in \mathbb{R}^m, s = c - \mathcal{A}^*y\} = (c + \mathrm{range}\,\mathcal{A}^*) \cap \mathcal{K}. \end{aligned}$$

If $s \in \mathcal{E}$ can be written as $s = c - \mathcal{A}^*y$ for some y, then s is said to be a *dual slack*. Furthermore, if $s \in \mathcal{F}^S_{\mathrm{D}}$ then s is called a *dual feasible slack*. The dual optimal value θ_{D} is said to be *attained* if there is $y \in \mathcal{F}_{\mathrm{D}}$ such that $\langle b, y \rangle = \theta_{\mathrm{D}}$. The notion of primal attainment is analogous. We recall the following basic constraint qualification.

Proposition 2 (Slater)**.** *Consider the pair* (Conic-P) *and* (Conic-D)*.*

(i) If there exists $x \in (\operatorname{ri}\mathcal{K}^*) \cap \mathcal{F}_P$ *then* $\theta_{\mathrm{P}} = \theta_{\mathrm{D}}$*. If, in addition,* θ_{P} *is finite then* θ_{D} *is attained.*

(ii) If there exists $s \in (\operatorname{ri}\mathcal{K}) \cap \mathcal{F}_D^S$ *then* $\theta_{\mathrm{P}} = \theta_{\mathrm{D}}$*. If, in addition,* θ_{D} *is finite then* θ_{P} *is attained.*

For the reader's convenience, before we proceed we recall a few basic facts from convex analysis. We provide references for the items and/or short proofs.

Lemma 3. *Let* $\mathcal{K} \subset \mathcal{E}$ *be a closed convex cone,* $e \in \operatorname{ri}\mathcal{K}$*,* $x \in \mathcal{K}$ *and* $z \in \mathcal{K}^*$*.*

(i) $\mathcal{K}^\perp = \operatorname{lin}(\mathcal{K}^*)$.

(ii) $x + e \in \operatorname{ri}\mathcal{K}$.

(iii) There exists $\alpha > 1$ *such that* $\alpha e + (1-\alpha)x \in \mathcal{K}$.

(iv) $z \in \mathcal{K}^\perp$ *if and only if* $\langle e, z\rangle = 0$.

(v) $\mathcal{F}(x,\mathcal{K})^\Delta = \mathcal{K}^* \cap \{x\}^\perp$.

(vi) $(\tan(x,\mathcal{K}))^* = \mathcal{F}(x,\mathcal{K})^\Delta$.

(vii) $T_x\mathcal{K} = \mathcal{F}(x,\mathcal{K})^{\Delta\perp}$.

Proof. *(i)* See item (a) of Proposition 2.1 in [46].

(ii) Since $e \in \operatorname{ri}\mathcal{K}$, for any $z \in \mathcal{K}$ we have that all points in the relative interior of the line segment connecting z and e also belong to the relative interior of $\mathcal{K}$, see Theorem 6.1 of [43]. Since

$$x + e = \frac{1}{2}e + \frac{1}{2}(2x + e),$$

we have $x + e \in \operatorname{ri}\mathcal{K}$.

(iii) See Theorem 6.4 in [43].

(iv) If $z \in \mathcal{K}^\perp$, then $\langle e, z\rangle$ is zero. Conversely, suppose that $\langle e, z\rangle$ is zero. By item *(iii)*, there is $\alpha > 1$ such that

$$u \coloneqq \alpha e + (1-\alpha)x \in \mathcal{K}.$$

On one hand, since $z \in \mathcal{K}^*$, we have $\langle u, z\rangle \geq 0$. On the other, $\langle u, z\rangle = (1-\alpha)\langle x, z\rangle \leq 0$. So, we must have $\langle x, z\rangle = 0$. As x is an arbitrary element, it holds that $z \in \mathcal{K}^\perp$.

(v) and *(vi)* First, we observe that $\operatorname{dir}(x,\mathcal{K})$ coincides $\{\alpha(w - x) \mid w \in \mathcal{K}, \alpha \geq 0\}$. The latter is called the *cone of* $\mathcal{K}$ *at* x in the terminology of [46] and its dual is given by $\mathcal{K}^* \cap \{x\}^\perp$. With this in mind, both items follow from Proposition 3.1 and Corollary 3.2 in [46].

(vii) Follows from *(i)* and *(vi)*.

□

2.1 Types of feasibility, almost optimality, almost feasibility

Here, we review the fact that a conic linear program can be in four different mutually exclusive feasibility statuses. We say that (Conic-D) is

(i) strongly feasible if $(\operatorname{ri}\mathcal{K}) \cap (c + \operatorname{range}\mathcal{A}^*) \neq \emptyset$ (i.e., Slater's condition hold),

(ii) weakly feasible if it is feasible but not strongly feasible,

(iii) weakly infeasible if it is infeasible but $\operatorname{dist}(c + \operatorname{range}\mathcal{A}^*, \mathcal{K}) = 0$,

(*iv*) strongly infeasible if $\mathrm{dist}(c + \mathrm{range}\,\mathcal{A}^*, \mathcal{K}) > 0$.

Strong/weak feasibility/infeasibility of (Conic-P) is defined analogously by replacing $(c + \mathrm{range}\,\mathcal{A}^*)$ by $\mathcal{V} := \{x \mid \mathcal{A}x = b\}$. As a matter of convention, if $\mathcal{V} = \emptyset$, we will say that (Conic-P) is strongly infeasible. If a problem is either weakly infeasible or weakly feasible we will say that it is in *weak status*. In view of these definitions, the usual assumption underlying interior point methods amounts to requiring both primal and dual strong feasibility.

We have the following characterization of strong infeasibility, see Lemma 5 in [25].

Proposition 4 (Characterization of strong infeasibility)**.** *The following hold.*

(*i*) (Conic-P) *is strongly infeasible if and only if there exists y such that*

$$\langle b, y\rangle = 1 \quad \text{and} \quad -\mathcal{A}^* y \in \mathcal{K}. \tag{1}$$

(*ii*) (Conic-D) *is strongly infeasible if and only if there exists x such that*

$$\langle c, x\rangle = -1 \quad \text{and} \quad x \in \mathcal{K}^* \cap \ker \mathcal{A} \tag{2}$$

Moving on, let $y \in \mathcal{E}$ and $\epsilon > 0$. We say that y is an ϵ*-feasible solution to* (Conic-D) if $\mathrm{dist}\,(c - \mathcal{A}^* y, \mathcal{K}) \le \epsilon$. In addition, we say that y is an ϵ*-optimal solution to* (Conic-D) if y is feasible for (SDP-D) and $\langle b, y\rangle \ge \theta_{\mathrm{D}} - \epsilon$. These notions will be used in Sections 4.2 and 4.3.

In general, even if $s = c - \mathcal{A}^* y$ is such that $\mathrm{dist}\,(s, \mathcal{K})$ is small, there is no guarantee that $\mathrm{dist}\,(s, \mathcal{F}_{\mathrm{D}}^S)$ will also be small. In this case, the quantities $\mathrm{dist}\,(s, \mathcal{K})$ and $\mathrm{dist}\,(s, \mathcal{F}_{\mathrm{D}}^S)$ are sometimes called the *backward error* and *forward error*, respectively. The problem of bounding the forward error by the backward error is intrinsically connected with the notion of *error bounds*. See, for example, the fundamental work by Sturm [45] on error bounds for linear matrix inequalities, where he showed the importance of facial reduction in analyzing these questions. See also [20, 17] for some generalizations of Sturm's results to the so-called amenable cones and beyond.

2.2 Facial structure of $\mathcal{S}^n_+$ and a few remarks on $\mathcal{O}_{\mathrm{int}}$

The cone of positive semidefinite symmetric matrices has a very special structure and every face of $\mathcal{S}^n_+$ is linearly isomorphic to some $\mathcal{S}^r_+$ for $r \le n$. This is a well-known fact which we state as a proposition for future reference. For a proof, see [31]. See also Section 6 of [3]. Since it will be clear from the context, in what follows we use the convention that 0 always denotes a zero matrix of appropriate size.

Proposition 5. *Let $\mathcal{F}$ be a nonempty face of $\mathcal{S}^n_+$. There exists $r \le n$ and an orthogonal $n \times n$ matrix Q such that*

$$Q^\top \mathcal{F} Q = \left\{ \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}^n \;\middle|\; U \in \mathcal{S}^r_+ \right\} \tag{3}$$

Let $\mathcal{F}$ be as in Proposition 5, then $\mathcal{F}^*$ satisfies

$$Q \mathcal{F}^* Q^\top = (Q^\top \mathcal{F} Q)^* = \left\{ \begin{pmatrix} U & V \\ V^\top & W \end{pmatrix} \in \mathcal{S}^n \;\middle|\; U \in \mathcal{S}^r_+ \right\}. \tag{4}$$

Given an arbitrary nonempty face $\hat{\mathcal{F}}$ of $\mathcal{F}^*$, there is an orthogonal matrix $\hat{Q}$ such that

$$\hat{Q} \hat{\mathcal{F}} \hat{Q}^\top = \left\{ \begin{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} & V \\ V^\top & W \end{pmatrix} \in \mathcal{S}^n \;\middle|\; U \in \mathcal{S}^q_+ \right\}, \tag{5}$$

where $q \le r$. Then, $\hat{\mathcal{F}}^*$ satisfies

$$\hat{Q}^\top \hat{\mathcal{F}}^* \hat{Q} = \left\{ \begin{pmatrix} \begin{pmatrix} U & V \\ V^T & W \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}^n \;\middle|\; U \in \mathcal{S}^q_+ \right\}. \tag{6}$$

In the definition of $\mathcal{O}_{\text{int}}$, the affine space is contained in the space of $n \times n$ symmetric matrices and the optimization is carried over $\mathcal{S}^n_+$. Note that n is the same for both $\mathcal{S}^n_+$ and $\mathcal{S}^n$. However, for fixed $\mathcal{A}, b, c$ we might be interested in solving problems over a *face* of $\mathcal{S}^n_+$, the dual of a face of $\mathcal{S}^n_+$ or even over a face of the dual of a face as in (5).

In those cases, even if (Conic-D) and (Conic-P) are both primal and dual strongly feasible, it is not immediately clear how to use $\mathcal{O}_{\text{int}}$ to solve (Conic-D) and (Conic-P), since they are not exactly standard form SDPs. One possibility would be to consider a, *a priori*, stronger oracle that is also able to solve strongly feasible problems over faces of $\mathcal{S}^n_+$.

We will show that this is not necessary and, after some linear algebra, we can still solve (Conic-P) and (Conic-D) using $\mathcal{O}_{\text{int}}$. We register this fact as a proposition. Let $\mathcal{S}^{r,n}_+$ denote the face of $\mathcal{S}^n_+$ corresponding to the right-hand side of (3). Let $\mathcal{K}$ be a cone as in (3), (4), (5) or (6). We see that in all those cases, we have

$$R\mathcal{K}R^\top = \mathcal{S}^{r,n}_+ \oplus \mathcal{L}, \tag{7}$$

for some orthogonal matrix R, some $r \leq n$ and some linear subspace $\mathcal{L} \subseteq \mathcal{S}^n$ such that $\mathcal{L} \subseteq (\mathcal{S}^{r,n}_+)^\perp$. Here $\oplus$ denotes the Minkowski sum.

Proposition 6. *Let $\mathcal{K}$ be as in (7) with some orthogonal matrix R, some $r \leq n$, and some linear subspace $\mathcal{L} \subseteq \mathcal{S}^n$ such that $\mathcal{L} \subseteq (\mathcal{S}^{r,n}_+)^\perp$. Suppose that (Conic-P) and (Conic-D) are strongly feasible. Then, (Conic-P) and (Conic-D) are solvable with a single call to $\mathcal{O}_{int}$.*

The proof is elementary but quite cumbersome, so it is deferred to Appendix A.

3 Facial reduction with $\mathcal{O}_{\text{int}}$

A major obstacle for solving (SDP-D) with the oracle $\mathcal{O}_{\text{int}}$ is that, in general, (SDP-D) is not strongly feasible, i.e., Slater's condition might not hold. By using facial reduction, we are able to either detect infeasibility or to reformulate (SDP-D) as an SDP instance that is strongly feasible at *one side* of the problem. This will be an important step towards completely solving (SDP-D).

In this section, we discuss facial reduction for general conic linear programs and how it can be carried out by solving auxiliary problems that are ensured to be strongly feasible at both primal and dual sides. In particular, when the underlying cone $\mathcal{K}$ is $\mathcal{S}^n_+$, this will mean that facial reduction can be implemented through calls to $\mathcal{O}_{\text{int}}$. Although we will focus on problems formulated in the dual form (Conic-D), any analysis carried out for (Conic-D) can be translated back to (Conic-P). Here, we will follow the approach described in [51], which relies on the following key result.

Lemma 7 (The facial reduction lemma: Lemma 3.2 in [51])**.** *The following hold.*

- *(i)* (Conic-D) *is not strongly feasible (i.e., Slater's condition fails) if and only if there is $d \in \mathcal{K}^* \cap \ker \mathcal{A}$ such that:*
  - *(i)* $\langle c, d\rangle = 0$ *and* $d \notin \mathcal{K}^\perp$, *or*
  - *(ii)* $\langle c, d\rangle < 0$.
- *(ii)* (Conic-P) *is not strongly feasible (i.e., Slater's condition fails) if and only if there are $y \in \mathbb{R}^m, f \in \mathcal{K}$ such that $f = -\mathcal{A}^* y$ and*
  - *(i)* $\langle b, y\rangle = 0$ *and* $f \notin \mathcal{K}^{*\perp} = \operatorname{lin} \mathcal{K}$ *(item* (i) *of Lemma 3), or*
  - *(ii)* $\langle b, y\rangle > 0$.

Therefore, whenever (Conic-D) lacks a relative interior solution (i.e., $\mathcal{K} \cap (c + \operatorname{range} \mathcal{A}^*) = \emptyset$), it is either because (Conic-D) is infeasible (alternative *(ii)* together with Proposition 4) or because the set of dual feasible slacks $\mathcal{F}^S_{\mathrm{D}}$ is contained in $\mathcal{K} \cap \{d\}^\perp$ (alternative (i)).[1] If alternative (i) holds, since $d \notin \mathcal{K}^\perp$, we have

$$\mathcal{K} \cap \{d\}^\perp \subsetneq \mathcal{K}, \tag{8}$$

that is, the face $\mathcal{F}_2 \coloneqq \mathcal{K} \cap \{d\}^\perp$ is properly contained in $\mathcal{K}$. We then substitute $\mathcal{K}$ for $\mathcal{F}_2$ and repeat. As long as $(\operatorname{ri} \mathcal{F}_i) \cap (c + \operatorname{range} \mathcal{A}^*) = \emptyset$, we can find a new direction d.

We recall that if $\mathcal{F}$ is a face of $\mathcal{K}$, then $\mathcal{F} \subsetneq \mathcal{K}$ holds if and only if $\dim \mathcal{F} < \dim \mathcal{K}$. Therefore, (8) implies that after a finite number of iterations, we will either find some face $\mathcal{F}_\ell$ such that $(\operatorname{ri} \mathcal{F}_\ell) \cap (c + \operatorname{range} \mathcal{A}) \neq \emptyset$ or we will eventually find out that the problem is infeasible.

It turns out that $\mathcal{F}_\ell$ must be the smallest face $\mathcal{F}^{\mathrm{D}}_{\min}$ of $\mathcal{K}$ which contains $\mathcal{F}^S_{\mathrm{D}}$. This process is called *facial reduction* and it aims at finding $\mathcal{F}^{\mathrm{D}}_{\min}$. If $\mathcal{F}^S_{\mathrm{D}} = \emptyset$, we have $\mathcal{F}^{\mathrm{D}}_{\min} = \emptyset$ by convention. For the sake of preciseness, we will state the following definition.

Definition 8 (Reducing directions). *A* reducing direction for (Conic-D) *is an element $d \in \mathcal{K}^* \cap \ker \mathcal{A}$ such that $\langle c, d\rangle \leq 0$. A* reducing direction for (Conic-P) *is a pair (f, y) such that $f \in \mathcal{K}$, $f = -\mathcal{A}^* y$ (i.e., $f \in \operatorname{range} \mathcal{A}^*$) and $\langle b, y\rangle \geq 0$.*

Next, $\{d_1, \ldots, d_\ell\}$ is said to be a sequence of reducing directions for (Conic-D) *if*

$$d_i \in (\mathcal{K} \cap \{d_1\}^\perp \cap \cdots \cap \{d_{i-1}\}^\perp)^* \cap \ker \mathcal{A} \cap \{c\}^\perp, \text{ for } i = 1, \ldots, \ell - 1 \tag{9}$$

$$d_\ell \in (\mathcal{K} \cap \{d_1\}^\perp \cap \cdots \cap \{d_{\ell-1}\}^\perp)^* \cap \ker \mathcal{A}, \qquad \langle c, d_\ell\rangle \leq 0. \tag{10}$$

Analogously, $\{(f_1, y_1), \ldots, (f_\ell, y_\ell)\}$ is said to be a sequence of reducing directions for (Conic-P) *if*

$$f_i = -\mathcal{A}^* y_i, \quad y_i \in \{b\}^\perp, \quad f_i \in (\mathcal{K}^* \cap \{f_1\}^\perp \cap \cdots \cap \{f_{i-1}\}^\perp)^*, \text{ for } i = 1, \ldots, \ell - 1 \tag{11}$$

$$f_\ell = -\mathcal{A}^* y_\ell, \quad \langle b, y_\ell\rangle \geq 0, \quad f_\ell \in (\mathcal{K}^* \cap \{f_1\}^\perp \cap \cdots \cap \{f_{\ell-1}\}^\perp)^*. \tag{12}$$

Remark. *Liu and Pataki introduced in [19] the notion of* facial reduction cone*, see Definition 2 therein. The k-th facial reduction cone of $\mathcal{K}$ is given by*

$$\mathrm{FR}_k(\mathcal{K}) = \{(d_1, \ldots, d_k) \mid d_1 \in \mathcal{K}^*, d_i \in (\mathcal{K} \cap \{d_1\}^\perp \cap \cdots \cap \{d_{i-1}\}^\perp)^*, i = 2, \ldots, k\}.$$

With that, (9), (10) and (11), (12) imply that

$$(d_1, \ldots, d_\ell) \in \mathrm{FR}_\ell(\mathcal{K}), \quad (f_1, \ldots, f_\ell) \in \mathrm{FR}_\ell(\mathcal{K}^*).$$

The minimal face $\mathcal{F}^D_{\min}$ containing the feasible region of (Conic-D) also has the following well-known characterization, see for instance, Proposition 3.2.2 in [31].

Proposition 9 (Characterizations of the minimal face). *Let $\mathcal{F}$ be a face of $\mathcal{K}$ containing $\mathcal{F}^S_D$. Suppose $\mathcal{F}$ and $\mathcal{F}^S_D$ are both non-empty. Then the conditions below are equivalent.*

(i) $\mathcal{F}^S_D \cap \operatorname{ri} \mathcal{F} \neq \emptyset$.

(ii) $\operatorname{ri} \mathcal{F}^S_D \subseteq \operatorname{ri} \mathcal{F}$.

(iii) $\mathcal{F} = \mathcal{F}^D_{\min}$.

The computationally challenging part of facial reduction is computing d which requires, in general, solving another CLP. At first glance, it seems that we are again stuck solving an CLP that might also not be strongly feasible. However, *even if the original CLP is not strongly feasible, searching for d can always be done by solving problems that are primal and dual strongly feasible.* In particular, when $\mathcal{K} = \mathcal{S}^n_+$, finding reducing directions can be done with $\mathcal{O}_{\mathrm{int}}$.

[1] One must be careful that even if (Conic-D) is infeasible it might be the case that alternative (ii) is not satisfied at this stage. This happens, for instance, if (Conic-D) is weakly infeasible.

Lemma 10 (Finding a reducing direction through strongly feasible auxiliary problems)**.** *Let $e \in \operatorname{ri}\mathcal{K}$, $e^* \in \operatorname{ri}\mathcal{K}^*$ and consider the following pair of primal and dual problems.*

$$\begin{aligned} \inf_{x,t,w} \quad & t && && (P_{\mathcal{K}}) \\ \textit{subject to} \quad & -\langle c, x - te^*\rangle + t - w && = 0 && (13) \\ & \langle e, x\rangle + w && = 1 && (14) \\ & \mathcal{A}x - t\mathcal{A}e^* && = 0 && (15) \\ & (x,t,w) \in \mathcal{K}^* \times \mathbb{R}_+ \times \mathbb{R}_+ \end{aligned}$$

$$\begin{aligned} \sup_{y_1,y_2,y_3} \quad & y_2 && (D_{\mathcal{K}}) \\ \textit{subject to} \quad & cy_1 - ey_2 - \mathcal{A}^*y_3 \in \mathcal{K} && (16) \\ & 1 - y_1(1 + \langle c, e^*\rangle) + \langle e^*, \mathcal{A}^*y_3\rangle \geq 0 && (17) \\ & y_1 - y_2 \geq 0 && (18) \end{aligned}$$

The following properties hold.

(i) Both ($P_{\mathcal{K}}$) and ($D_{\mathcal{K}}$) are strongly feasible.

Let (x^*, t^*, w^*) be an optimal solution to ($P_{\mathcal{K}}$) and (y_1^*, y_2^*, y_3^*) be an optimal solution to ($D_{\mathcal{K}}$).

(ii) The primal optimal value $\theta_{P_{\mathcal{K}}}$ is zero if and only if $\mathcal{F}^D_{\min} \subsetneq \mathcal{K}$. In this case, one of the two alternatives below must hold:

(a) $\langle c, x^*\rangle < 0$ and $(c + \operatorname{range}\mathcal{A}^*) \cap \mathcal{K} = \emptyset$ (i.e., (Conic-D) is infeasible), or

(b) $\langle c, x^*\rangle = 0$ and $(c + \operatorname{range}\mathcal{A}^*) \cap \mathcal{K} \subseteq \mathcal{K} \cap \{x^*\}^\perp \subsetneq \mathcal{K}$.

(iii) The primal optimal value $\theta_{P_{\mathcal{K}}}$ is positive if and only if (Conic-D) is strongly feasible, i.e., $\mathcal{F}^D_{\min} = \mathcal{K}$. In this case, we have

$$c - \mathcal{A}^*\frac{y_3^*}{y_1^*} \in \operatorname{ri}\mathcal{K}.$$

Proof. (i) Let

$$t := \frac{1}{\langle e, e^*\rangle + 1}, \quad w := \frac{1}{\langle e, e^*\rangle + 1}, \quad x := \frac{e^*}{\langle e, e^*\rangle + 1}.$$

Then (x, t, w) is a strongly feasible solution to ($P_{\mathcal{K}}$), i.e.,

$$(x, t, w) \in \operatorname{ri}(\mathcal{K} \times \mathbb{R}_+ \times \mathbb{R}_+) = \operatorname{ri}\mathcal{K} \times \operatorname{ri}\mathbb{R}_+ \times \operatorname{ri}\mathbb{R}_+.$$

Next, we observe that $(y_1, y_2, y_3) := (0, -1, 0)$ is a feasible solution to ($D_{\mathcal{K}}$) such that (17), (18) are satisfied strictly and

$$cy_1 - ey_2 - \mathcal{A}^*y_3 = e \in \operatorname{ri}\mathcal{K}.$$

We have thus shown that both (Conic-P) and (Conic-D) are strongly feasible.

(ii) First, let (x^*, t^*, w^*) be an optimal solution to (Conic-P) and suppose that $\theta_{P_{\mathcal{K}}}$ is zero. We have $t^* = 0$. Then, (13) and (15) together with $x^* \in \mathcal{K}^*$ and $w^* \geq 0$ imply that

$$x^* \in \ker\mathcal{A} \cap \mathcal{K}^*, \quad \langle c, x^*\rangle \leq 0. \tag{19}$$

Then, we have two possibilities.

(a) Suppose $\langle c, x^* \rangle < 0$. We will show that (Conic-D) must be infeasible. Let $s \in (c + \operatorname{range} \mathcal{A}^*)$, then (19) implies $\langle s, x^* \rangle < 0$. Since $x^* \in \mathcal{K}^*$, we conclude that s cannot belong to $\mathcal{K}$, because otherwise we would have $\langle x^*, s \rangle \geq 0$.

Therefore, (Conic-D) must be infeasible and $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K} = \emptyset$. In this case, we have $\mathcal{F}^{\mathrm{D}}_{\min} = \emptyset$ and, indeed, $\mathcal{F}^{\mathrm{D}}_{\min} \subsetneq \mathcal{K}$.

(b) Suppose $\langle c, x^* \rangle = 0$. This, together with (19) implies that

$$(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K} = \mathcal{F}^S_{\mathrm{D}} \subseteq \mathcal{K} \cap \{x^*\}^\perp.$$

Next, we will check that the inclusion $\mathcal{K} \cap \{x^*\}^\perp \subsetneq \mathcal{K}$ is indeed proper. We observe that since $t^* = 0$ and $\langle c, x^* \rangle = 0$, (13) implies that $w^* = 0$ too. Therefore, (14) implies that $\langle e, x^* \rangle = 1$. In particular, x^* does not belong to $\mathcal{K}^\perp$. In other words,

$$\mathcal{K} \cap \{x^*\}^\perp \subsetneq \mathcal{K}.$$

Since $\mathcal{F}^{\mathrm{D}}_{\min} \subseteq \mathcal{K} \cap \{x^*\}^\perp$, we also have $\mathcal{F}^{\mathrm{D}}_{\min} \subsetneq \mathcal{K}$.

Now, we will prove the converse. That is, we will suppose that $\mathcal{F}^{\mathrm{D}}_{\min} \subsetneq \mathcal{K}$ and we will show that $\theta_{P_\mathcal{K}} = 0$. We start by observing that since the objective function of ($P_\mathcal{K}$) is "t" and t is constrained to be nonnegative, if we exhibit a feasible solution for ($P_\mathcal{K}$) having $t = 0$ this would be enough to show that $\theta_{P_\mathcal{K}} = 0$.

Since $\mathcal{F}^{\mathrm{D}}_{\min} \subsetneq \mathcal{K}$, (Conic-D) is not strongly feasible. By Lemma 7, there exists some $x \in \mathcal{K}^* \cap \ker \mathcal{A}$ such that either *(a)* $\langle c, x \rangle = 0$ and $x \notin \mathcal{K}^\perp$ or *(b)* $\langle c, x \rangle < 0$. Let us check each case.

(a) Suppose $\langle c, x \rangle = 0$ and $x \notin \mathcal{K}^\perp$. Then the condition $x \notin \mathcal{K}^\perp$ implies that $\langle e, x \rangle > 0$, by item *(iii)* of Lemma 3. Then,

$$\left(\frac{x}{\langle e, x \rangle}, 0, 0 \right)$$

is a feasible solution for ($P_\mathcal{K}$), which shows that $\theta_{P_\mathcal{K}} = 0$.

(b) Suppose that $\langle c, x \rangle < 0$. We define

$$\alpha := \frac{1}{\langle e, x \rangle - \langle c, x \rangle}$$

and this is well-defined because $-\langle c, x \rangle > 0$ and $\langle e, x \rangle \geq 0$. Then $(x\alpha, 0, -\alpha\langle c, x \rangle)$ is a feasible solution to ($P_\mathcal{K}$), which also shows that $\theta_{P_\mathcal{K}} = 0$.

(iii) Since ($P_\mathcal{K}$) and ($D_\mathcal{K}$) are both strongly feasible, we have, in particular, that $\theta_{P_\mathcal{K}} = \theta_{D_\mathcal{K}}$ and there is an optimal solution to ($D_\mathcal{K}$) (y_1^*, y_2^*, y_3^*) satisfying $y_2^* = \theta_{P_\mathcal{K}}$. By item *(ii)*, we have that $\theta_{P_\mathcal{K}} = 0$ if and only if $\mathcal{F}^{\mathrm{D}}_{\min} \subsetneq \mathcal{K}$, which happens if and only if (Conic-D) is *not* strongly feasible, by Proposition 9. As $\theta_{P_\mathcal{K}}$ is always nonnegative (because t is constrained to be nonnegative), we conclude that (Conic-D) is strongly feasible if and only if $\theta_{P_\mathcal{K}}$ is positive.

Next, suppose that $\theta_{P_\mathcal{K}}$ is indeed positive. In this case we have that $y_2^* = \theta_{P_\mathcal{K}}$ is positive and that

$$ey_2^* \in \operatorname{ri} \mathcal{K},$$

since $e \in \operatorname{ri} \mathcal{K}$. This fact, together with (16) and item *(ii)* of Lemma 3, implies that

$$cy_1^* - \mathcal{A}^* y_3^* \in \operatorname{ri} \mathcal{K}.$$

To conclude, we observe that (18) implies that $y_1^* \geq y_2^* > 0$. Therefore,

$$c - \mathcal{A}^* \frac{y_3^*}{y_1^*} \in \operatorname{ri} \mathcal{K}$$

Using Proposition 9, we conclude that indeed $\mathcal{F}^{\mathrm{D}}_{\min} = \mathcal{K}$.

□

Remark. *Lemma 10 holds for any pair of* e, e^* *satisfying* $e \in \operatorname{ri} \mathcal{K}, e^* \in \operatorname{ri} \mathcal{K}^*$. *When* $\mathcal{K} = \mathcal{S}^n_+$, *we may take* e *and* e^* *to be, for example, both equal to the* $n \times n$ *identity matrix. If* $\mathcal{K}$ *is some face of* $\mathcal{S}^n_+$, *we can use Proposition 5 together with* (3) *and* (4) *to find* e *and* e^* *as follows. We take* $e = e^*$ *and let* e *be such that* $Q^\top e Q = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, *where* I_r *is the* $r \times r$ *identity matrix.*

For SDPs, we note that Cheung, Schurr and Wolkowicz also discuss an auxiliary problem that is primal and dual strongly feasible, see the problem (AP) in [6]. A key difference is that (AP) requires an additional second order cone constraint, whereas ($P_\mathcal{K}$) *and* ($D_\mathcal{K}$) *only use linear equalities/inequalities and cone constraints involving the original cone* $\mathcal{K}$ *and its dual.*

With the aid of Lemma 10, we now are able to state a facial reduction algorithm that can be easily adapted to use the oracle $\mathcal{O}_{\text{int}}$, when $\mathcal{K} = \mathcal{S}^n_+$, see Algorithm 1.

Algorithm 1: Facial reduction with strongly feasible auxiliary problems

Input : $\mathcal{K}, \mathcal{A}, c$

Output: Reducing directions $d_1, \ldots, d_\ell$ for (Conic-D) (Definition 8) together with `Feasible` or `Infeasible`. If `Feasible`, a pair (s, y) is also returned so that

$$s = c - \mathcal{A}^* y \in \operatorname{ri} (\mathcal{K} \cap \{d_1\}^\perp \cap \cdots \cap \{d_\ell\}^\perp).$$

1 $\mathcal{F}_1 \leftarrow \mathcal{K}$, $i \leftarrow 1$.

2 Replace $\mathcal{K}, \mathcal{K}^*$ by $\mathcal{F}_i, \mathcal{F}_i^*$ in ($D_\mathcal{K}$) and ($P_\mathcal{K}$), respectively, and solve the resulting pair of problems (associated with the cones $\mathcal{F}_i, \mathcal{F}_i^*$). Denote the obtained optimal solutions by (x^*, t^*, w^*) and (y_1^*, y_2^*, y_3^*).

3 **if** $t^* = 0$ **then**

4     $d_i \leftarrow x^*$ `/* Found a reducing direction */`

5   **if** $\langle c, x^* \rangle < 0$ **then**

6         $\mathcal{F}^{\text{D}}_{\min} \leftarrow \emptyset$ `/*` $\langle c, x^* \rangle < 0$ `attests that (Conic-D) is infeasible */`

7   **return** `Infeasible`, $\mathcal{F}^{\text{D}}_{\min}, d_1, \ldots, d_i$

8   **else**

9         $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap \{d_i\}^\perp$ `/* In this case we have` $\langle c, x^* \rangle = 0$ `*/`

10         $i \leftarrow i + 1$

11   **go to** line 2

12   **end**

13 **else**

14     $\mathcal{F}^{\text{D}}_{\min} \leftarrow \mathcal{F}_i$,   `/* Found the minimal face */`

15     $s \leftarrow c - \mathcal{A}^* \frac{y_3^*}{y_1^*}$ `/*` $s \in \operatorname{ri} \mathcal{F}_i$ `*/`

16   **return** `Feasible`, $\mathcal{F}^{\text{D}}_{\min}, d_1, \ldots, d_i, \left(s, \frac{y_3^*}{y_1^*}\right)$

17 **end**

Proposition 11 (Algorithm 1 is correct)**.** *Algorithm 1 correctly detects whether* (Conic-D) *is feasible or not. If* (Conic-D) *is feasible, Algorithm 1 correctly identifies the minimal face* $\mathcal{F}^D_{\min}$ *and the pair* (s, y) *returned by Algorithm 1 does indeed satisfy*

$$s = c - \mathcal{A}^* y \in \operatorname{ri} \mathcal{F}^D_{\min}.$$

Proof. The correctness of Algorithm 1 is a consequence of Lemma 10 and we will now explain some of the details. We have several claims.

Claim 1 For all i, $\mathcal{F}_i$ contains $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K}$ and $\mathcal{F}_{i+1}$ is strictly contained in $\mathcal{F}_i$

This claim holds by induction. When Algorithm 1 starts, we have $\mathcal{F}_1 = \mathcal{K}$. Now, suppose that for some i we have that $\mathcal{F}_i$ contains $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K}$. Given $\mathcal{F}_i$, we have that $\mathcal{F}_{i+1}$ is constructed by the relation

$$\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{d_i\}^\perp.$$

However, $\mathcal{F}_{i+1}$ is only computed if the optimal value of ($P_{\mathcal{K}}$) is 0 and $\langle c, x^* \rangle = 0$, see Lines 5 and 9. In this case, item $(ii)(b)$ of Lemma 10 ensures

$$(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{F}_i \subseteq \mathcal{F}_{i+1} \subsetneq \mathcal{F}_i. \tag{20}$$

Since $\mathcal{F}_i$ is a face (and, therefore, a subset) of $\mathcal{K}$, the hypothesis that $\mathcal{F}_i$ contains $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K}$ implies that, in fact,

$$(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{F}_i = (c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K},$$

which, combined with (20), implies that $\mathcal{F}_{i+1}$ must also contain $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K}$. This concludes the proof of **Claim 1**.

Claim 2 The minimal face of $\mathcal{F}_i$ containing $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K}$ coincides with $\mathcal{F}^{\mathrm{D}}_{\min}$

Claim 2 follows from **Claim 1** and the fact that if $\mathcal{F}$ is a face of $\mathcal{K}$ and $\hat{\mathcal{F}}$ is a face of $\mathcal{F}$, then $\hat{\mathcal{F}}$ is a face of $\mathcal{K}$.

Claim 3 For all i, $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{F}_i = \emptyset$ holds if and only if (Conic-D) is infeasible

Claim 3 is a consequence of **Claim 1**.

Now, **Claim 1** implies that whenever a new face $\mathcal{F}_{i+1}$ is computed, it must be strictly smaller than $\mathcal{F}_i$ and, therefore, the dimension must also be strictly smaller[2]. Since we cannot have an infinite strictly descending of faces, at some point, the optimal value of ($P_{\mathcal{K}}$) must become positive or a certificate that $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{F}_i = \emptyset$ will be found (see Lines 5 and 6). In the first case, **Claim 2** together with item (iii) of Lemma 10 (applied to $\mathcal{F}_i$) implies that $\mathcal{F}^{\mathrm{D}}_{\min} = \mathcal{F}_i$ and that

$$s = c - \mathcal{A}^* y \in \operatorname{ri} \mathcal{F}^{\mathrm{D}}_{\min},$$

where $y = y_3^*/y_1^*$. In the second case, **Claim 3** and item $(ii)(a)$ of Lemma 10 ensures that, indeed, (Conic-D) must be infeasible. □

Next, we examine the computational cost of Algorithm 1, following an analysis similar to other facial reduction approaches (e.g, [32, 51]). When Algorithm 1 is invoked, a chain of faces of $\mathcal{K}$ is constructed as follows

$$\mathcal{K} = \mathcal{F}_1 \supsetneq \cdots \supsetneq \mathcal{F}_\ell.$$

We recall that if $\mathcal{F}, \hat{\mathcal{F}}$ are faces of $\mathcal{K}$ such that $\mathcal{F} \subseteq \hat{\mathcal{F}}$, then $\mathcal{F} \neq \hat{\mathcal{F}}$ if and only if $\dim \mathcal{F} < \dim \hat{\mathcal{F}}$. As $\mathcal{K}$ is finite dimensional, we conclude that at most $\dim \mathcal{K} + 1$ faces will be found when Algorithm 1 is invoked. This estimate can be sharpened in several different ways. For example, let $\ell_{\mathcal{K}}$ denote the *longest chain of strictly decreasing non-empty faces of* $\mathcal{K}$. Then, the number of non-empty faces that will be found when Algorithm 1 is invoked is bounded above by $\ell_{\mathcal{K}}$. In particular, when $\mathcal{K} = \mathcal{S}^n_+$, we have

$$\dim \mathcal{K} = \frac{n(n+1)}{2}, \quad \ell_{\mathcal{S}^n_+} = n + 1.$$

This shows that, in some cases, $\ell_{\mathcal{K}}$ can be a much better bound than $\dim \mathcal{K}$. For a proof that $\ell_{\mathcal{S}^n_+} = n+1$ see, for example, Theorem 14 in [12] where it is shown that that whenever $\mathcal{K}$ is a symmetric cone (homogeneous self-dual cone), we have $\ell_{\mathcal{K}} = \operatorname{rank} \mathcal{K} + 1$, where $\operatorname{rank} \mathcal{K}$ is the Jordan algebraic rank of $\mathcal{K}$. We summarize this discussion in the next proposition.

Proposition 12 (Computational cost of Algorithm 1)**.** *The number of times that Algorithm 1 solves the pair* ($P_{\mathcal{K}}$) *and* ($D_{\mathcal{K}}$) *is bounded above by* $\ell_{\mathcal{K}}$. *In particular, when* $\mathcal{K} = \mathcal{S}^n_+$, *Algorithm 1 can be implemented by invoking* $\mathcal{O}_{int}$ *at most* $n + 1$ *times.*

Proof. In the proof of Proposition 11, we have shown that Algorithm 1 constructs a strictly nondecreasing chain of faces as follows

$$\mathcal{K} = \mathcal{F}_1 \supsetneq \cdots \supsetneq \mathcal{F}_\ell. \tag{21}$$

[2]The fact that $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ are faces is important, because, in general, $C_1 \subsetneq C_2$ does not imply $\dim C_1 < \dim C_2$.

We divide the proof in two cases. Suppose first that (Conic-D) is feasible. Then, $\mathcal{F}_\ell = \mathcal{F}^{\mathrm{D}}_{\min}$ by Proposition 11 and $\mathcal{F}^{\mathrm{D}}_{\min}$ is not empty. Finding a new face $\mathcal{F}_i$ in Algorithm 1 corresponds to solving the pair $(P_\mathcal{K})$ and $(D_\mathcal{K})$ once. So, after solving the pair $(P_\mathcal{K})$ and $(D_\mathcal{K})$ at most $\ell_\mathcal{K} - 1$ times, Algorithm 1 will set $\mathcal{F}_\ell$ to $\mathcal{F}^{\mathrm{D}}_{\min}$. Then, $(P_\mathcal{K})$ and $(D_\mathcal{K})$ will be solved one extra time in order to check that $\mathcal{F}_\ell$ is indeed the minimal face and to obtain $s \in \operatorname{ri} \mathcal{K}$, as in Lines 14 and 15. In total, $(P_\mathcal{K})$ and $(D_\mathcal{K})$ is solved at most $\ell_\mathcal{K}$ times.

Next, suppose that (Conic-D) is infeasible. In this case, the last face $\mathcal{F}_\ell$ will be empty (see Line 6), but all faces up to $\ell - 1$ will be nonempty. Therefore, $\ell - 1 \leq \ell_\mathcal{K}$. As in the previous case, each face in the chain (21) corresponds to solving the pair $(P_\mathcal{K})$ and $(D_\mathcal{K})$ once. In summary, after solving $(P_\mathcal{K})$ and $(D_\mathcal{K})$ at most $\ell_\mathcal{K} - 1$ times, Algorithm 1 will find the last nonempty face $\mathcal{F}_{\ell-1}$ and, then, $(P_\mathcal{K})$ and $(D_\mathcal{K})$ will be solved once more in order to set $\mathcal{F}_\ell$ to "$\emptyset$".

To conclude, we suppose that $\mathcal{K} = \mathcal{S}^n_+$. Then, Algorithm 1 successively solves the problem $(P_\mathcal{K})$ and $(D_\mathcal{K})$ over $\mathcal{S}^n_+$ and its faces at most $\ell_{\mathcal{S}^n_+} = n + 1$ times. By Lemma 10 and Proposition 6, these are strongly feasible problems that can be solved by invoking $\mathcal{O}_{\mathrm{int}}$ a single time. □

We mention in passing that the minimal number of facial reduction steps needed to find the minimal face of (Conic-D) is often called the *singularity degree* of (Conic-D). The singularity degree is also bounded by $\ell_\mathcal{K}$, but sharper estimates can be obtained by considering facial reduction strategies that take into account the existence of polyhedral faces of $\mathcal{K}$ as in the case of the FRA-Poly algorithm in [23].

To conclude this section, we quickly review some variants of facial reduction. The search for efficient ways of doing facial reduction and computing the reducing directions is an area of active research. Permenter, Friberg and Andersen have recently shown that reducing directions can be obtained naturally if we have access to relative interior solutions to a certain self-dual homogeneous model of (Conic-P) and (Conic-D), see Theorem 3.2 and Section 4 of [36].

It is also possible to relax the search criteria in order to make the problem of finding d more tractable by considering, for example, polyhedral approximations as in the Partial Facial Reduction approach of Permenter and Parrilo [37] or relaxing the definition of reducing direction as in the approach by Friberg [8] by removing the conic constraints. See also the work of Zhu, Pataki and Tran-Dinh for a heuristic facial reduction algorithm for SDPs in primal standard format [27]. In the case of [37] and [27], the drawback is that facial reduction might end with a face other than $\mathcal{F}^{\mathrm{D}}_{\min}$, although their experiments show that many interesting instances become easier to solve nonetheless. For the approach in [8], there are some representability issues affecting the cones obtained by intersecting $\mathcal{K}$ with the hyperplanes defined by the reducing directions, see Sections 4 and 6 therein. In [23], we proposed "FRA-Poly", a two-phase facial reduction algorithm that takes into consideration the presence of polyhedral faces in the face lattice of $\mathcal{K}$. Instead of performing facial reduction until Slater's condition is satisfied, Phase 1 of the algorithm in [23] regularizes the problem until the so-called *partial polyhedral Slater's condition* is satisfied. Then, in Phase 2, the algorithm jumps directly to the minimal face. An extension of Lemma 10 appropriate for FRA-Poly is proved in Lemma 3 of [23].

4 Double facial reduction

In Section 3, we saw how to perform facial reduction by solving auxiliary problems that are always primal and dual strongly feasible. However, as we remarked previously, facial reduction only guarantees that one side of the problem will be strongly feasible, after reformulating the problem over the minimal face. In order to finally obtain a problem where both the primal and dual are strongly feasible, we only need to do facial reduction *twice*, which is mildly surprising. We call this *double facial reduction.*

In this section, we discuss technical aspects related to double facial reduction and how it can be used to compute the optimal value of (Conic-D). Double facial reduction will also enable us to compute almost optimal solutions when the optimal value of (Conic-D) is not attained, as we will see in Section 4.2. We will also show how to obtain almost feasible solutions when (Conic-D) is weakly infeasible, see Section 4.3.

4.1 Computing the optimal value of (Conic-D)

The first step towards computing the optimal value θ_{D} of (Conic-D) is to apply facial reduction to (Conic-D). Then, if (Conic-D) is feasible, we obtain the following pair of CLPs:
[**Primal-dual pair obtained after applying facial reduction to** (Conic-D)]

$$\begin{aligned} \inf_{x} \quad & \langle c, x\rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in (\mathcal{F}^{\mathrm{D}}_{\min})^* \end{aligned} \tag{$\hat{\mathrm{P}}$}$$

$$\begin{aligned} \sup_{y} \quad & \langle b, y\rangle \\ \text{subject to} \quad & c - \mathcal{A}^*y \in \mathcal{F}^{\mathrm{D}}_{\min}. \end{aligned} \tag{$\hat{\mathrm{D}}$}$$

Here, ($\hat{\mathrm{D}}$) is strongly feasible, but it could still be the case that ($\hat{\mathrm{P}}$) is not strongly feasible. Therefore, when $\mathcal{K} = \mathcal{S}^n_+$, the pair ($\hat{\mathrm{P}}$) and ($\hat{\mathrm{D}}$) might still not be solvable with $\mathcal{O}_{\mathrm{int}}$. To remedy this issue, if $\mathcal{F}^{\mathrm{D}}_{\min} \neq \emptyset$, it is reasonable to consider applying facial reduction to ($\hat{\mathrm{P}}$), which leads to the following pair of problems.
[**Primal-dual pair obtained after applying facial reduction to** ($\hat{\mathrm{P}}$)]

$$\begin{aligned} \inf_{x} \quad & \langle c, x\rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{F}^{\hat{\mathrm{P}}}_{\min} \end{aligned} \tag{P*}$$

$$\begin{aligned} \sup_{y} \quad & \langle b, y\rangle \\ \text{subject to} \quad & c - \mathcal{A}^*y \in (\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*. \end{aligned} \tag{D*}$$

Here, $\mathcal{F}^{\hat{\mathrm{P}}}_{\min}$ is the minimal face of $(\mathcal{F}^{\mathrm{D}}_{\min})^*$ which contains the feasible region of ($\hat{\mathrm{P}}$). Now, if $\mathcal{F}^{\mathrm{D}}_{\min}$ and $\mathcal{F}^{\hat{\mathrm{P}}}_{\min}$ are non-empty, both (P*) and ($\hat{\mathrm{D}}$) are ensured to be strongly feasible. However, it is not obvious at all whether (D*) still satisfies strong feasibility, since $\mathcal{F}^{\mathrm{D}}_{\min} \subseteq (\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*$. After all, $C_1 \subseteq C_2$ does not imply $\mathrm{ri}\, C_1 \subseteq \mathrm{ri}\, C_2$ in general.

Nevertheless, we will show in this section that, in fact, if (Conic-D) and (Conic-P) are both feasible, then (D*) will still be strongly feasible. In addition, if $\mathcal{F}^{\hat{\mathrm{P}}}_{\min}$ is empty, then it is because $\theta_{\mathrm{D}} = +\infty$.

In essence, the question boils down to understanding the possible ways that the feasibility properties of (Conic-D) might change when a single facial reduction step is performed at (Conic-P) and $\mathcal{K}$ is replaced by $(\mathcal{K}^* \cap \{f\}^\perp)^*$ in (Conic-D), for some $f \in \mathcal{K} \cap \mathrm{range}\, \mathcal{A}^*$. First, we need a few auxiliary results.

Lemma 13. *Let $u \in \mathrm{ri}\, \mathcal{K}$, $d \in \mathcal{K}$ and $v \in T_d\mathcal{K}$, where $T_d\mathcal{K}$ is the tangent space of $\mathcal{K}$ at d. Then, there is $t > 0$ such that*

$$u + v + td \in \mathrm{ri}\, \mathcal{K}.$$

The intuition for Lemma 13 is as follows. If $v + td$ were a point in $\mathcal{K}$, then it would be clear that $u + v + td \in \mathrm{ri}\, \mathcal{K}$, by item (*ii*) of Lemma 3. Unfortunately, this does not happen in general. However, as t increases, $v + td$ gets closer and closer to $\mathcal{K}$, so adding u will eventually drag everything to the relative interior.

Proof. Let

$$C = \{u + v + td \mid t \geq 0\}.$$

To prove the lemma, it is enough to show that $\mathrm{ri}\, C \cap \mathrm{ri}\, \mathcal{K} \neq \emptyset$. Suppose, for the sake of obtaining a contradiction, that $\mathrm{ri}\, C \cap \mathrm{ri}\, \mathcal{K} = \emptyset$. This implies that there is a separating hyperplane

$$H = \{w \in \mathcal{E} \mid \langle z, w\rangle = \theta\},$$

such that H properly separates C and $\mathcal{K}$, see Theorem 11.3 in [43]. We recall that *proper separation* means that C and $\mathcal{K}$ lie in opposite closed half-spaces defined by H and H *does not contain both sets at the same time*. Without loss of generality, we may assume that C and $\mathcal{K}$ lie in the "lower" and "upper" closed halfspaces defined by H, respectively. Therefore, we have

$$\langle u, z\rangle + \langle v, z\rangle + \langle td, z\rangle \leq \theta \leq \langle w, z\rangle, \quad \forall t \geq 0,\ \forall w \in \mathcal{K}. \tag{22}$$

For (22) to hold, we must have $z \in \mathcal{K}^*$ and $\theta \leq 0$ (since $0 \in \mathcal{K}$). Furthermore, because $d \in \mathcal{K}$ (by assumption) and $z \in \mathcal{K}^*$, we have $\langle d, z \rangle \geq 0$. However, in view of (22), it must be the case that

$$\langle d, z \rangle = 0, \tag{23}$$

since t can be taken to be any nonnegative number. By item (*v*) of Lemma 3, we conclude that $z \in \mathcal{F}(d, \mathcal{K})^{\Delta}$.

From item (*vii*) of Lemma 3, we have $T_d\mathcal{K} = \mathcal{F}(d, \mathcal{K})^{\Delta\perp}$. Therefore,

$$\langle v, z \rangle = 0. \tag{24}$$

From (22), (23), (24) and recalling that $\theta \leq 0$, we obtain $\langle u, z \rangle = 0$. This implies that $C \subseteq H$ and $\theta = 0$. By item (*iv*) of Lemma 3, we have $\mathcal{K} \subseteq H$ as well, since $u \in \operatorname{ri} \mathcal{K}$ by assumption. This contradicts the properness of the separation. □

We are now ready to state a result on the conservation of feasibility after one facial reduction step. For what follows, we recall that (Conic-D) *is in weak status* if it is weakly feasible or weakly infeasible. We also recall the following basic facts. A face $\mathcal{F}$ of $\mathcal{K}$ always satisfies $\mathcal{F} = \mathcal{K} \cap \operatorname{span} \mathcal{F}$, therefore we have

$$\mathcal{F}^* = \operatorname{cl}(\mathcal{K}^* + \mathcal{F}^\perp). \tag{25}$$

Also, if C_1 and C_2 are two convex sets we have $\operatorname{ri}(C_1 + C_2) = \operatorname{ri} C_1 + \operatorname{ri} C_2$, $\operatorname{ri}(\operatorname{cl} C_1) = \operatorname{ri} C_1$.

Proposition 14 (Conservation of feasibility)**.** *Let $f \in \mathcal{K} \cap \operatorname{range} \mathcal{A}^*$ and let $\mathcal{F} := \mathcal{K}^* \cap \{f\}^\perp = \mathcal{F}(f, \mathcal{K})^{\Delta}$ (see item (v) of Lemma 3). Let (D') be the problem obtained by replacing $\mathcal{K}$ by $\mathcal{F}^*$ in (Conic-D), i.e.,*

$$\sup_{y} \quad \langle b, y \rangle \qquad \text{(D')}$$
$$\textit{subject to} \quad c - \mathcal{A}^* y \in \mathcal{F}^*.$$

We have the following relations:

(i) (Conic-D) is strongly feasible if and only if (D') is;

(ii) (Conic-D) is strongly infeasible if and only if (D') is;

(iii) (Conic-D) is in weak status if and only if (D') is.

Proof. (*i*) First, since $\mathcal{F}^* = \operatorname{cl}(\mathcal{K} + \mathcal{F}^\perp)$ and $\operatorname{ri} \mathcal{F}^\perp = \mathcal{F}^\perp$, we have

$$\operatorname{ri} \mathcal{F}^* = \operatorname{ri}(\operatorname{cl}(\mathcal{K} + \mathcal{F}^\perp)) = (\operatorname{ri} \mathcal{K}) + \mathcal{F}^\perp. \tag{26}$$

Now, suppose that (Conic-D) is strongly feasible. Since $\operatorname{ri} \mathcal{K} \subseteq \operatorname{ri} \mathcal{K} + \mathcal{F}^\perp$, we conclude that (D') must be strongly feasible as well.

Conversely, suppose that (D') is strongly feasible and let $s = c - \mathcal{A}^* y$ be such that $s \in \operatorname{ri} \mathcal{F}^*$. By (26), we have

$$s = u + v,$$

where $u \in \operatorname{ri} \mathcal{K}$ and $v \in \mathcal{F}^\perp$. By items (*v*) and (*vii*) of Lemma 3, we have

$$\mathcal{F}^\perp = \mathcal{F}(f, \mathcal{K})^{\Delta\perp} = T_f\mathcal{K}.$$

Invoking Lemma 13 we conclude that there exists $t > 0$ such that

$$u + v + tf \in \operatorname{ri} \mathcal{K}.$$

Since $f \in \operatorname{range} \mathcal{A}^*$, there exists $\hat{y}$ such that $f = -\mathcal{A}^* \hat{y}$. We conclude that

$$s + tf = u + v + tf = c - \mathcal{A}^*(y + \hat{y}) \in \operatorname{ri} \mathcal{K},$$

thus showing that (Conic-D) is strongly feasible.

(*ii*) Since $\mathcal{K} \subseteq \mathcal{F}^*$, if (D') is strongly infeasible, (Conic-D) must be strongly infeasible as well.

Conversely, suppose that (Conic-D) is strongly infeasible. Then, Proposition 4 implies the existence of x satisfying

$$\langle c, x \rangle = -1, \quad x \in \mathcal{K}^* \cap \ker \mathcal{A}.$$

Since $x \in \ker \mathcal{A}$ and $f \in \operatorname{range} \mathcal{A}^*$, we have $\langle x, f \rangle = 0$. So, in fact, $x \in \mathcal{F}$. Therefore, by Proposition 4, the same x attests that (D') is strongly infeasible.

(*iii*) First, we recall that the four feasibility statuses described in Section 2.1 are mutually exclusive. Next, suppose that (Conic-D) is in weak status, i.e., it is either weakly feasible or weakly infeasible. By items (*i*) and (*ii*) proved so far, (D') cannot be strongly infeasible nor strongly feasible because that would imply that (Conic-D) has that same feasibility status. Therefore, (D') must be in weak status as well.

Conversely, suppose that (D') is in weak status. Again, items (*i*) and (*ii*) imply that the only two possibilities for (Conic-D) are weak infeasibility or weak feasibility.

□

We can now state and prove our main result on the preservation of feasibility status after facial reduction is performed on (Conic-P). Intuitively, Theorem 15 means the following. Whenever facial reduction is applied to, say, (Conic-P), we obtain a new problem which is ensured to be strongly feasible, if (Conic-P) is feasible. This new problem will also have a dual problem whose feasibility properties might be different than the original dual problem (Conic-D). However, Theorem 15 says that no drastic changes are allowed, i.e., if (Conic-D) was strongly feasible to begin with, it will stay strongly feasible. The only possible room for change is that a weakly feasible/infeasible problem might become weakly infeasible/feasible. Theorem 15 also contains the relatively surprising fact that strong feasibility of the new dual implies strong feasibility of (Conic-D).

Theorem 15 (Preservation of feasibility under facial reduction)**.** *Let $\mathcal{F}^P_{\min}$ denote the minimal face of $\mathcal{K}^*$ that contains the feasible region of* (Conic-P) *and suppose that $\mathcal{F}^P_{\min} \neq \emptyset$. Consider the problem obtained by replacing $\mathcal{K}$ by $(\mathcal{F}^P_{\min})^*$ in* (Conic-D)*, i.e.,*

$$\begin{aligned} \sup_{y} \quad & \langle b, y \rangle \\ \textit{subject to} \quad & c - \mathcal{A}^* y \in (\mathcal{F}^P_{\min})^*, \end{aligned} \tag{Conic-D-FP}$$

The following hold.

(*i*) (Conic-D) *is strongly feasible if and only if* (Conic-D-FP) *is strongly feasible;*

(*ii*) (Conic-D) *is strongly infeasible if and only if* (Conic-D-FP) *is strongly infeasible;*

(*iii*) (Conic-D) *is in weak status if and only if* (Conic-D-FP) *is in weak status.*

Proof. Applying facial reduction to (Conic-P) (e.g., Algorithm 1), we see that $\mathcal{F}^{\mathrm{P}}_{\min}$ can be written as

$$\mathcal{F}^{\mathrm{P}}_{\min} = \mathcal{K}^* \cap \{f_1\}^\perp \cap \cdots \cap \{f_\ell\}^\perp,$$

where each f_i satisfies

$$f_i \in \left(\mathcal{K}^* \cap \{f_1\}^\perp \cap \cdots \{f_{i-1}\}^\perp\right)^* \cap \operatorname{range} \mathcal{A}^*.$$

Now, denote by (D_i) the problem obtained by replacing $\mathcal{K}$ by $\left(\mathcal{K}^* \cap \{f_1\}^\perp \cap \cdots \{f_{i-1}\}^\perp\right)^*$ in (Conic-D). We observe the following:

1. (D_1) and $(D_{\ell+1})$ are precisely (Conic-D) and (Conic-D-FP), respectively.

2. f_i is a reducing direction for (D_i), so Proposition 14 applies to f_i, (D_i) and (D_{i+1}), for $i = 1, \ldots, \ell$.

By induction, we conclude that items (*i*), (*ii*) and (*iii*) hold. □

We are now in position to state our main result on double facial reduction.

Theorem 16 (Double facial reduction)**.** *Suppose $\mathcal{F}^{D}_{\min} \neq \emptyset$ and consider the problems* ($\hat{\mathrm{P}}$) *and* ($\hat{\mathrm{D}}$) *above.*

Let $\mathcal{F}^{\hat{P}}_{\min}$ be the minimal face of $(\mathcal{F}^{D}_{\min})^*$ that contains the feasible region of ($\hat{\mathrm{P}}$)*. Consider the pair of problems* (P*) *and* (D*)*, which we repeat below for convenience.*

$$\begin{aligned} \inf_{x} \quad & \langle c, x\rangle \\ \textit{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{F}^{\hat{P}}_{\min} \end{aligned} \qquad \text{(P*)}$$

$$\begin{aligned} \sup_{y} \quad & \langle b, y\rangle \\ \textit{subject to} \quad & c - \mathcal{A}^*y \in (\mathcal{F}^{\hat{P}}_{\min})^*. \end{aligned} \qquad \text{(D*)}$$

The following hold.

(*i*) *The optimal value of* (Conic-D) *(θ_{D}) is finite if and only if $\mathcal{F}^{\hat{P}}_{\min} \neq \emptyset$. In this case,* (P*) *and* (D*) *are both strongly feasible and*

$$\theta_{\mathrm{D}} = \theta_{P*} = \theta_{D*}.$$

(*ii*) *$\theta_{\mathrm{D}} = +\infty$ if and only if $\mathcal{F}^{\hat{P}}_{\min} = \emptyset$.*

Proof. (*i*) Suppose that the optimal value of (Conic-D) is finite. Then, by Proposition 2, the optimal value of ($\hat{\mathrm{P}}$) must be equal to θ_{D}, since ($\hat{\mathrm{D}}$) is strongly feasible. In particular, ($\hat{\mathrm{P}}$) must be feasible and, therefore, $\mathcal{F}^{\hat{\mathrm{P}}}_{\min} \neq \emptyset$. Since $\mathcal{F}^{\hat{\mathrm{P}}}_{\min}$ is the minimal face of $(\mathcal{F}^{\mathrm{D}}_{\min})^*$ that contains the feasible region of ($\hat{\mathrm{P}}$), (P*) is strongly feasible and its optimal value must coincide with the optimal value of ($\hat{\mathrm{P}}$), which is θ_{D}.

Next, since (P*) is strongly feasible and has finite optimal value, (D*) must have the same optimal value. Therefore, as stated, we have

$$\theta_{\mathrm{D}} = \theta_{\mathrm{P*}} = \theta_{\mathrm{D*}}.$$

By item (*i*) of Theorem 15, substituting $\mathcal{F}^{\mathrm{D}}_{\min}$ by $(\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*$ preserves strong feasibility, so (P*) and (D*) are both strongly feasible.

Conversely, suppose that $\mathcal{F}^{\hat{\mathrm{P}}}_{\min} \neq \emptyset$. This means that (P*) is feasible. So ($\hat{\mathrm{P}}$) must be feasible as well, because any feasible solution to (P*) must be a feasible solution to ($\hat{\mathrm{P}}$). Since we are assuming that $\mathcal{F}^{\mathrm{D}}_{\min} \neq \emptyset$, ($\hat{\mathrm{D}}$) must be feasible as well. Therefore, ($\hat{\mathrm{P}}$) and ($\hat{\mathrm{D}}$) are feasible primal and dual problems sharing the same optimal value, which must be finite. Since ($\hat{\mathrm{D}}$) shares the same optimal value with (Conic-D), we conclude that θ_{D} is indeed finite.

(*ii*) It follows from item (*i*). □

The conclusion is that, when θ_{D} is finite, the pair of problems (P*) and (D*) are both strongly feasible. When $\mathcal{K} = \mathcal{S}^n_+$, they can indeed be solved by $\mathcal{O}_{\mathrm{int}}$ in order to obtain θ_{D} by Proposition 6. At this stage, even though θ_{D} might have been unattained for (Conic-D), (D*) is never hindered by unattainment.

The problem, however, is that a feasible solution to (D*) might not be feasible to (Conic-D). And, indeed, if θ_{D} is finite but not attained, even though (D*) has an optimal solution, (Conic-D) will not have optimal solutions. When θ_{D} is finite but not attained, the best we can do is to compute some solution y_ϵ satisfying $\langle b, y_\epsilon\rangle \geq \theta_{\mathrm{D}} - \epsilon$, for some arbitrary $\epsilon > 0$. We will discuss this issue in the next subsection.

Before we move on, we give an intuitive explanation of why unattainment disappears when doing facial reduction. The "dual" explanation is that strong feasibility is satisfied at (P*) (which is the Lagrangian dual of (D*)), so, of course, (D*) must be attained when θ_{D} value is finite.

We now give an "primal" explanation of why unattainment disappears. Suppose that the optimal value θ_{D} of (Conic-D) is finite but not attained. Then, there exists a sequence $\{y^k\}$ such that $\langle b, y_k\rangle \to \theta_{\mathrm{D}}$ and

$$s^k := c - \mathcal{A}^*y^k \in \mathcal{K}, \quad \forall k.$$

As we are assuming unattainment and $\mathcal{F}_{\mathrm{D}}^{S}$ is closed, $\{y^k\}$ cannot be bounded. Passing to a subsequence if necessary, we may assume that $\|y^k\| \to \infty$ and $y^k/\|y^k\|$ converges to some $y \in \mathbb{R}^m$ and that $\theta_{\mathrm{D}} \geq \langle b, y^k\rangle \geq \theta_{\mathrm{D}} - 1$ for every k. Dividing $c - \mathcal{A}^* y^k \in \mathcal{K}$ and $\theta_{\mathrm{D}} \geq \langle b, y^k \rangle \geq \theta_{\mathrm{D}} - 1$ by $\|y^k\|$ and taking limits, we conclude that $f := -\mathcal{A}^* y$ satisfies

$$f \in \mathcal{K}, \quad f \in \operatorname{range} \mathcal{A}^*, \quad \langle b, y \rangle = 0$$

and y must be nonzero. This shows that (f, y) is a reducing direction for (Conic-P), see Definition 8. In fact, with more effort, we can show that there must be at least one pair (f, y) as above satisfying $f \not\in \operatorname{lin} \mathcal{K}$, see Lemma 7. In other words, a necessary condition for unnatainment of (Conic-D) is the existence of (f, y) as above with $f \not\in \operatorname{lin} \mathcal{K}$. Informally speaking, (f, y) acts as a "recession direction" for the problem (Conic-D). This suggests that one possible way of fixing unattainment is by preventing f from becoming a recession direction. This is accomplished, for instance, by substituting $\mathcal{K}$ by $\operatorname{cl}(\mathcal{K} + \operatorname{span}\{f\})$, so that $f \in \operatorname{lin}(\operatorname{cl}(\mathcal{K} + \operatorname{span}\{f\}))$. However, $\operatorname{cl}(\mathcal{K} + \operatorname{span}\{f\})$ is equal to $(\mathcal{K}^* \cap \{f\}^\perp)^*$, which corresponds to a single facial reduction step done at (Conic-P).

In other words, from the point of view of (Conic-D), facial reduction done at (Conic-P) removes recession directions that affect attainment. We remark that Abrams [1] also proposed a regularization procedure that removes recession directions, in order to fix unattainment in convex programming.

4.2 Obtaining feasible almost optimal solutions

The pair of problems (D*) and (P*) are strongly feasible and their common optimal value is θ_{D}. However, an optimal solution to (D*) is unlikely to be feasible for (Conic-D). In fact, it may happen that θ_{D} is not attained, in which case (Conic-D) has no optimal solution at all.

Nevertheless, we will show how to construct feasible solutions that are almost optimal for (Conic-D) using the directions obtained when calling Algorithm 1 with ($\hat{\mathrm{P}}$) as input, see Algorithm 2.

Algorithm 2: Finding an ϵ-optimal solution to (Conic-D)

Input:

1. Reducing directions for ($\hat{\mathrm{P}}$) (Definition 8): $(f_1, y_1), \ldots, (f_{\ell_2}, y_{\ell_2})$.
2. $\hat{y}$ such that $c - \mathcal{A}^* \hat{y} \in \operatorname{ri} \mathcal{F}_{\min}^{\mathrm{D}}$,
3. an optimal solution y^* to (D*).
4. $\epsilon > 0$

Output: A feasible solution y_ϵ to (Conic-D) satisfying $\langle b, y_\epsilon \rangle \geq \theta_{\mathrm{D}} - \epsilon$.

if $\langle b, \hat{y} \rangle \geq \theta_{\mathrm{D}} - \epsilon$ **then**
    return $\hat{y}$
else
    $\beta \leftarrow \frac{\theta_{\mathrm{D}} - \langle b, \hat{y}\rangle - \epsilon}{\theta_{\mathrm{D}} - \langle b, \hat{y}\rangle}$
    $w_{\ell_2+1} \leftarrow \beta y^* + (1-\beta)\hat{y}$
    $\mathcal{F}_{\ell_2+1} \leftarrow (\mathcal{F}_{\min}^{\mathrm{D}})^* \cap \{f_1\}^\perp \cap \cdots \cap \{f_{\ell_2}\}^\perp$
    for $i = \ell_2$ **to** 1 **do**
        $\mathcal{F}_i \leftarrow (\mathcal{F}_{\min}^{\mathrm{D}})^* \cap \{f_1\}^\perp \cap \cdots \cap \{f_{i-1}\}^\perp$
        Find α_i positive such that $c - \mathcal{A}^*(w_{i+1} + \alpha_i y_i) = c - \mathcal{A}^* w_{i+1} + \alpha_i f_i \in \operatorname{ri} \mathcal{F}_i^*$
        $w_i \leftarrow w_{i+1} + \alpha_i y_i$
    end
    return w_1
end

Note that the inner loop in Algorithm 2 goes from ℓ_2 to 1. This is because we start from a relative interior solution to $\mathcal{F}_{\ell_2+1}$ and we have to work our way until the bottom of the chain $\mathcal{F}^{\mathrm{D}}_{\min}$. The tricky part is ensuring that at each step there is indeed an α_i as in Line 9. If there is at least one α_i, then any number larger than α_i will work. Therefore, it is enough to keep trying larger and larger numbers until the condition in Line 9 is met. We will now show that an appropriate α_i always exists and that Algorithm 2 is indeed correct. For that, we need a few auxiliary results. First, suppose that $f \in \mathcal{F}^*$, for $\mathcal{F}$ a closed convex cone. We have by items (*v*) and (*vii*) of Lemma 3 and (25) that

$$(\mathcal{F} \cap \{f\}^\perp)^* = (\mathcal{F}(f, \mathcal{F}^*)^\Delta)^* = \mathrm{cl}\,(\mathcal{F}^* + \mathcal{F}(f, \mathcal{F}^*)^{\Delta\perp}) = \mathrm{cl}\,(\mathcal{F}^* + T_f\mathcal{F}^*). \tag{27}$$

Lemma 17. *Suppose that $s \in (c + \mathrm{range}\,\mathcal{A}^*) \cap \mathrm{ri}\,\mathcal{K}$ and let $\mathcal{F}^P_{\min}$ be the minimal face of $\mathcal{K}^*$ that contains the feasible region of* (Conic-P)*. If $\mathcal{F}^P_{\min} \neq \emptyset$, then $s \in \mathrm{ri}\,((\mathcal{F}^P_{\min})^*)$.*

Proof. It is a consequence of the proof of Theorem 15. Nevertheless, we will work out the details here. As in the proof of Theorem 15, applying facial reduction to (Conic-P) (e.g., Algorithm 1), we see that $\mathcal{F}^{\mathrm{P}}_{\min}$ can be written as

$$\mathcal{F}^{\mathrm{P}}_{\min} = \mathcal{K}^* \cap \{f_1\}^\perp \cap \cdots \cap \{f_\ell\}^\perp,$$

where each f_i satisfies

$$f_i \in \left(\mathcal{K}^* \cap \{f_1\}^\perp \cap \cdots \{f_{i-1}\}^\perp\right)^* \cap \mathrm{range}\,\mathcal{A}^*.$$

Now, let $\mathcal{F}_i \coloneqq \left(\mathcal{K}^* \cap \{f_1\}^\perp \cap \cdots \cap \{f_{i-1}\}^\perp\right)$. We observe the following:

1. $\mathcal{F}_1 = \mathcal{K}^*$ and $\mathcal{F}_{\ell+1} = \mathcal{F}^{\mathrm{P}}_{\min}$,
2. $\mathcal{F}_i \leftarrow \mathcal{F}_{i-1} \cap \{f_{i-1}\}^\perp$, for all $i > 1$.

By hypothesis, we have $s \in \mathrm{ri}\,\mathcal{F}_1^*$ and by (27) we have

$$\begin{aligned} \mathcal{F}_2^* &= \mathrm{cl}\,(\mathcal{F}_1^* + T_{f_1}\mathcal{F}_1^*) \\ \mathrm{ri}\,\mathcal{F}_2^* &= (\mathrm{ri}\,\mathcal{F}_1^*) + T_{f_1}\mathcal{F}_1^*. \end{aligned}$$

Therefore, $s \in \mathrm{ri}\,\mathcal{F}_2^*$ as well. At the i-th step, we have:

$$\begin{aligned} \mathcal{F}_i^* &= \mathrm{cl}\,(\mathcal{F}_{i-1}^* + T_{f_{i-1}}\mathcal{F}_{i-1}^*) \\ \mathrm{ri}\,\mathcal{F}_i^* &= (\mathrm{ri}\,\mathcal{F}_{i-1}^*) + T_{f_{i-1}}\mathcal{F}_{i-1}^*. \end{aligned}$$

By induction, we conclude that $s \in \mathrm{ri}\,\mathcal{F}_i^*$ for every i. In particular, $s \in \mathrm{ri}\,((\mathcal{F}_{\ell+1})^*) = \mathrm{ri}\,((\mathcal{F}^{\mathrm{P}}_{\min})^*)$. ☐

Theorem 18. *Algorithm 2 is correct, that is, the output y_ϵ is indeed a feasible solution to* (Conic-D) *satisfying $\langle b, y_\epsilon\rangle \geq \theta_{\mathrm{D}} - \epsilon$.*

Proof. y_ϵ is ϵ-optimal. By construction, $\langle b, w_{\ell_2+1}\rangle \geq \theta_{\mathrm{D}} - \epsilon$. Moreover, all the y_i satisfy $\langle b, y_i\rangle = 0$. Therefore, $\langle b, y_\epsilon\rangle \geq \theta_{\mathrm{D}} - \epsilon$.

y_ϵ is feasible for (Conic-D). If the algorithm stops before Line 5, y_ϵ is feasible because $\mathcal{F}^{\mathrm{D}}_{\min} \subseteq \mathcal{K}$. So suppose that we have reached Line 5. Since $\mathcal{F}_1^* = \mathcal{F}^{\mathrm{D}}_{\min}$, if Line 9 is correct, then y_ϵ is feasible for (Conic-D). We now show that Line 9 is indeed correct.

Let

$$\begin{aligned} \hat{s} &\coloneqq c - \mathcal{A}^*\hat{y}, \\ s_i &\coloneqq c - \mathcal{A}^* w_i, \text{ for } i = 1, \ldots, \ell_2 + 1. \end{aligned}$$

We have $\hat{s} \in \mathrm{ri}\,\mathcal{F}^{\mathrm{D}}_{\min}$ and, by Lemma 17, $\hat{s} \in \mathrm{ri}\,((\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*)$ as well. Note that s_{ℓ_2+1} is a strict convex combination of $c - \mathcal{A}^*y^*$ and $\hat{s}$, see Line 5. These points belong to $(\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*$ and $\mathrm{ri}\,((\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*)$, respectively, so s_{ℓ_2+1} must belong to $\mathrm{ri}\,((\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*)$ as well. In addition, s_{ℓ_2+1} is a feasible slack for (D*).

Now suppose that we have shown that $s_{i+1} \in \operatorname{ri} \mathcal{F}^*_{i+1}$, for some i. By (27), we have

$$\mathcal{F}^*_{i+1} = (\mathcal{F}_i \cap \{f_i\}^\perp)^* = \operatorname{cl}(\mathcal{F}^*_i + T_{f_i}\mathcal{F}^*_i)$$
$$\operatorname{ri} \mathcal{F}^*_{i+1} = (\operatorname{ri} \mathcal{F}^*_i) + T_{f_i}\mathcal{F}^*_i.$$

Therefore, $s_{i+1} = u_i + v_i$ for some $u_i \in \operatorname{ri} \mathcal{F}^*_i$ and $v_i \in T_{f_i}\mathcal{F}^*_i$. We can apply Lemma 13 to $u_i, v_i, f_i, \mathcal{F}^*_i$ and conclude the existence of positive α_i such that $s_{i+1} + \alpha f_i$ belongs to $\operatorname{ri} \mathcal{F}^*_i$. Therefore,

$$c - \mathcal{A}^*(w_{i+1} + \alpha y_i) \in \operatorname{ri} \mathcal{F}^*_i.$$

In other words, $s_i = c - \mathcal{A}^* w_i \in \operatorname{ri} \mathcal{F}^*_i$.

By induction, we conclude that at each iteration it is possible to find α_i as stated in Line 9. □

4.2.1 Computational aspects of Algorithm 2

Having proved the correctness of Algorithm 2 in Theorem 13, we discuss the computation of α_i in Line 9, which is the most computationally expensive part of the algorithm. As remarked previously, the existence of α_i follows from Lemma 13. So, we will discuss the computation of t as in Lemma 13.

Let u, v, d be as in Lemma 13, i.e., $u \in \operatorname{ri} \mathcal{K}$, $d \in \mathcal{K}$ and $v \in T_d\mathcal{K}$. As we remarked before Lemma 17, if $t > 0$ is such that $u + v + td \in \operatorname{ri} \mathcal{K}$, then any $\hat{t} \geq t$ will also work. So, the simplest algorithm for computing t starts with some arbitrary positive value and keeps doubling it, until $u + v + td \in \operatorname{ri} \mathcal{K}$. Whether this is computationally reasonable or not depends on how hard it is to decide membership in $\mathcal{K}$ and $\operatorname{ri} \mathcal{K}$. If $\mathcal{K} = \mathcal{S}^n_+$ or $\mathcal{K}$ is as in Proposition 6, the membership problem is not too expensive in contrast to the situation where $\mathcal{K}$ is, say, a completely positive cone.

We also recall that given $s \in \mathcal{S}^n$, its minimum eigenvalue $\lambda_{\min}(s)$ satisfies

$$\lambda_{\min}(s) = \sup \{t \mid s - tI_n \in \mathcal{S}^n_+\} = \sup \{-t \mid s + tI_n \in \mathcal{S}^n_+\},$$

where I_n is the $n \times n$ identity matrix. Accordingly, the line search problem of finding t with $u + v + td \in \operatorname{ri} \mathcal{K}$ seems quite akin to a minimum eigenvalue computation. In the context of semidefinite programming, although one could use $\mathcal{O}_{\text{int}}$ to solve the membership problem, it seems more reasonable to solve it directly via minimum eigenvalue computations and/or factorizations.

Nevertheless, for the sake of completeness, we show that for arbitrary $\mathcal{K}$, we can obtain t by solving a pair of primal and dual strongly feasible problems. First, we consider the following pair of problems:

$$\begin{array}{lll} \inf\limits_{x} & \langle u + v, x \rangle & (P_d) \\ \text{subject to} & \langle d, x \rangle = 1 & \\ & x \in \mathcal{K}^* & \end{array} \qquad \begin{array}{lll} \sup\limits_{t} & -t & (D_d) \\ \text{subject to} & u + v + td \in \mathcal{K}. & \end{array}$$

Lemma 13 guarantees that (D_d) is strongly feasible, so we can apply Lemma 10 to the pair (P_d) and (D_d) by replacing b, c by -1 and $u + v$ respectively and $\mathcal{A}^*$ by the map that takes t to $-td$. From item (*iii*) of Lemma 10, if we solve the pair ($P_\mathcal{K}$) and ($D_\mathcal{K}$) we will obtain t such that $u + v + td \in \operatorname{ri} \mathcal{K}$. If t turns out to be negative, we can just set it to zero.

Moving on, we also notice the following curious feature of Algorithm 2. Except for the problem of finding α_i in Line 9, the complexity of Algorithm 2 does not depend on ϵ. Decreasing ϵ, however, might lead to larger α_i in Algorithm 2.

Finally, we compare Algorithm 2 to an elementary method for computing an ϵ-optimal solution. Namely, once θ_D is known and given some fixed $\epsilon > 0$, the naive approach is to directly apply a feasibility algorithm to the problem of finding a point in the set

$$\{y \in \mathbb{R}^m \mid c - \mathcal{A}^* y \in \mathcal{F}^\text{D}_{\min}, \langle b, y \rangle \geq \theta_\text{D} - \epsilon\}. \tag{28}$$

The set in (28) can be expressed as the feasible set of a conic linear program over $\mathcal{F}^{\mathrm{D}}_{\min} \times \mathbb{R}_+$:

$$\begin{aligned} \sup_{y} \quad & 0 && \text{(Naive)} \\ \text{subject to} \quad & c - \mathcal{A}^* y \in \mathcal{F}^{\mathrm{D}}_{\min}, \\ & \epsilon - \theta_{\mathrm{D}} + \langle b, y \rangle \in \mathbb{R}_+. \end{aligned}$$

Since ($\hat{\mathrm{D}}$) is strongly feasible, there is y_ϵ such that $c - \mathcal{A}^* y_\epsilon \in \operatorname{ri} \mathcal{F}^{\mathrm{D}}_{\min}$ and $\langle b, y_\epsilon \rangle > \theta_{\mathrm{D}} - \epsilon$, thus showing that (Naive) is strongly feasible[3]. Then, if we solve the auxiliary problems ($P_{\mathcal{K}}$), ($D_{\mathcal{K}}$) in Lemma 10 associated to (Naive), we obtain a feasible solution to (Naive) by item *(iii)* of Lemma 10. A feasible solution to (Naive) is precisely an ϵ-optimal solution to (Conic-D).

The drawback of this naive approach is that for every ϵ we need to, at the very least, solve one extra conic linear program. However, the approach using Algorithm 2 only requires solving cone membership problems which might be significantly cheaper depending on $\mathcal{K}$.

4.3 Distinguishing between weak and strong infeasibility

When (Conic-D) is infeasible, it can be either strongly or weakly infeasible. Strong infeasibility is relatively straightforward to analyze. Indeed, by Proposition 4, if we wish to show that (Conic-D) is strongly infeasible, it is enough to exhibit some $x \in \mathcal{K} \cap \ker \mathcal{A}$ such that $\langle c, x \rangle = -1$. Therefore, in order to prove that (Conic-D) is strongly infeasible we need to solve an CLP feasibility problem. In particular, when $\mathcal{K} = \mathcal{S}^n_+$, this can be done in at most $n+1$ calls to $\mathcal{O}_{\mathrm{int}}$, by Proposition 12.

When (Conic-D) is weakly infeasible, the situation is far more complicated. In order to prove that (Conic-D) is weakly infeasible, we have to prove that (Conic-D) is infeasible (which can also be done by Algorithm 1) and that the feasibility problem associated to strong infeasibility is infeasible, i.e., we have to show that there is no solution to

$$\text{find} \quad x \in \mathcal{K}^* \cap \{x \in \ker \mathcal{A} \mid \langle c, x \rangle = -1\}.$$

In this subsection, we will use the techniques of Section 4.2 to analyze weak infeasibility. This is not surprising because weak infeasibility and non-attainment of optimal solutions are closely related as we will see in a moment. In fact, let $e \in \operatorname{ri} \mathcal{K}$ and consider the following problem and its primal counterpart.

$$\begin{aligned} \inf_{x} \quad & \langle c, x \rangle && \text{(P-Feas)} \\ \text{subject to} \quad & \mathcal{A} x = 0 \\ & \langle e, x \rangle = 1 \\ & x \in \mathcal{K}^* \end{aligned}$$

$$\begin{aligned} \sup_{t,y} \quad & t && \text{(D-Feas)} \\ \text{subject to} \quad & c - te - \mathcal{A}^* y \in \mathcal{K}. \end{aligned}$$

Before we proceed, we need two preliminary results.

Proposition 19. *If $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K} \neq \emptyset$, then (D-Feas) is strongly feasible. If $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K} = \emptyset$, then (Conic-D) is strongly infeasible.*

Proof. Suppose that $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K} \neq \emptyset$ and let y and s be such that

$$s = c - \mathcal{A}^* y \in (c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K}.$$

Then, since $e \in \operatorname{ri} \mathcal{K}$, there exists $\alpha > 0$ such that $e + \alpha s \in \operatorname{ri} \mathcal{K}$. Since $\mathcal{K}$ is a cone, we have $e/\alpha + s \in \operatorname{ri} \mathcal{K}$. Therefore, $(t, y) := (-\alpha, y)$ is a solution for (D-Feas) for which the corresponding slack $c - te - \mathcal{A}^* y$ belongs to $\operatorname{ri} \mathcal{K}$, thus showing that (D-Feas) is strongly feasible.

[3] A way to see that this must be true is through the correctness of Algorithm 2. Lines 2 and 9 ensure that Algorithm 2 returns an ϵ-optimal y_ϵ for which the slack $c - \mathcal{A}^* y_\epsilon$ is a relative interior point of the minimal face of (Conic-D).

Next, suppose that $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K} = \emptyset$. Because $c + \operatorname{range} \mathcal{A}^*$ and $\operatorname{span} \mathcal{K}$ are polyhedral sets, this implies that

$$\operatorname{dist}(c + \operatorname{range} \mathcal{A}^*, \operatorname{span} \mathcal{K}) > 0,$$

e.g., see Corollary 19.3.3 and Theorem 11.4 in [43]. In particular, we must have $\operatorname{dist}(c + \operatorname{range} \mathcal{A}^*, \mathcal{K}) > 0$ as well, thus showing that (Conic-D) is strongly infeasible. □

Lemma 20. *If* $\operatorname{dist}(c + \operatorname{range} \mathcal{A}^*, \mathcal{K}) = 0$. *Then,* $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K} \neq \emptyset$ *and*

$$\operatorname{dist}((c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K}, \mathcal{K}) = 0.$$

Proof. For simplicity of notation, let $\mathcal{L} := \operatorname{range} \mathcal{A}^*$. Since $\operatorname{dist}(\mathcal{L} + c, \mathcal{K}) = 0$, we also have $\operatorname{dist}(\mathcal{L} + c, \operatorname{span} \mathcal{K}) = 0$. However, because $\mathcal{L} + c$ and $\operatorname{span} \mathcal{K}$ are polyhedral sets, this implies that $(\mathcal{L} + c) \cap \operatorname{span} \mathcal{K} \neq \emptyset$, see Corollary 19.3.3 and Theorem 11.4 in [43]. So, let $\hat{c} \in (\mathcal{L} + c) \cap \operatorname{span} \mathcal{K}$. We have

$$(\mathcal{L} + c) \cap \operatorname{span} \mathcal{K} = (\mathcal{L} \cap \operatorname{span} \mathcal{K}) + \hat{c}.$$

For the sake of obtaining a contradiction, assume that $\operatorname{dist}((\mathcal{L} + c) \cap \operatorname{span} \mathcal{K}, \mathcal{K}) > 0$. By item (*ii*) of Proposition 4, there exists x such that

$$\langle \hat{c}, x \rangle = -1, \quad x \in \mathcal{K}^* \cap ((\mathcal{L} \cap \operatorname{span} \mathcal{K})^\perp).$$

Therefore, x satisfies $x = u + v$, where $u \in \mathcal{L}^\perp$ and $v \in \mathcal{K}^\perp$. Recall that, since $\hat{c} \in \mathcal{L} + c$, there exists $l \in \mathcal{L}$ such that $\hat{c} = l + c$. We have

$$-1 = \langle \hat{c}, x \rangle = \langle l + c, u + v \rangle = \langle c, u \rangle,$$

because $l + c \in \operatorname{span} \mathcal{K}$, $v \in \mathcal{K}^\perp$ and $u \in \mathcal{L}^\perp$. Furthermore, $u \in \mathcal{K}^*$, because $u = x - v$ and $\mathcal{K}^\perp \subseteq \mathcal{K}^*$. Gathering all we have shown, we obtain

$$\langle c, u \rangle = -1, \quad u \in \mathcal{K}^* \cap \mathcal{L}^\perp.$$

Again, by item (*ii*) of Proposition 4, we conclude that $\operatorname{dist}(\mathcal{L} + c, \mathcal{K}) > 0$, which contradicts our assumptions. □

Proposition 21. *Denote by* θ_{D-Feas} *the optimal value of* (D-Feas). *Then,*

(*i*) $\theta_{D-Feas} > 0$ *if and only if* (Conic-D) *is strongly feasible.*

(*ii*) $\theta_{D-Feas} = 0$ *if and only if* (Conic-D) *is in weak status (i.e., either weakly infeasible or weakly feasible).*

(*iii*) $\theta_{D-Feas} = 0$ *and is not attained if and only if* (Conic-D) *is weakly infeasible*

(*iv*) $\theta_{D-Feas} < 0$ *if and only if* (Conic-D) *is strongly infeasible.*

Proof. (*i*) First, suppose that (Conic-D) is strongly feasible and let s, y be such that

$$s = c - \mathcal{A}^* y \in \operatorname{ri} \mathcal{K}.$$

By hypothesis, we have $e \in \operatorname{ri} \mathcal{K}$. By item (*iii*) of Lemma 3, there exists $\alpha > 1$ such that

$$\alpha s + (1 - \alpha) e \in \operatorname{ri} \mathcal{K}.$$

Therefore,

$$c - te - \mathcal{A}^* y \in \mathcal{K},$$

where $t = (\alpha - 1)/\alpha$. This shows that $\theta_{\mathrm{D-Feas}} > 0$.

Conversely, if $\theta_{\mathrm{D-Feas}} > 0$, there exists (t, y) such that $c - te - \mathcal{A}^* y \in \mathcal{K}$ with $t > 0$. By item (*ii*) of Lemma 3, we have $c - \mathcal{A}^* y \in \operatorname{ri} \mathcal{K}$.

(*ii*) Suppose that (Conic-D) is in weak status. If (Conic-D) is weakly feasible, then there is y such that $(0, y)$ is feasible for (D-Feas). Therefore, $\theta_{\text{D-Feas}} \geq 0$. By item (*i*), we must have $\theta_{\text{D-Feas}} = 0$.

Next, we suppose that (Conic-D) is weakly infeasible. From item (*i*), we must have $\theta_{\text{D-Feas}} \leq 0$. By Lemma 20, we have

$$(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K} \neq \emptyset.$$

so Proposition 19 implies that (D-Feas) must be strongly feasible. In particular, $\theta_{\text{D-Feas}}$ is finite. By Proposition 2, (P-Feas) has optimal value equal to $\theta_{\text{D-Feas}}$ and there exists a feasible solution x to (P-Feas) satisfying

$$\langle c, x \rangle = \theta_{\text{D-Feas}}, \quad \mathcal{A}x = 0, \quad x \in \mathcal{K}^*.$$

If $\theta_{\text{D-Feas}} < 0$, we would have that (Conic-D) is strongly infeasible by Proposition 4. Since this would contradict the weak infeasibility of (Conic-D), we must have $\theta_{\text{D-Feas}} = 0$. This concludes the first half of item (*ii*).

Now suppose that $\theta_{\text{D-Feas}} = 0$. Then, there are sequences $\{t_k\}$, $\{y_k\}$ such that $t_k \to 0$ and (t_k, y_k) is feasible for (D-Feas). Then, since $c - t_k e - \mathcal{A}^* y_k \in \mathcal{K}$ holds for every k, we have

$$\operatorname{dist}(c + \operatorname{range} \mathcal{A}^*, \mathcal{K}) \leq \|(c - \mathcal{A}^* y_k) - (c - t_k e - \mathcal{A}^* y_k)\| \leq \|t_k e\|, \quad \forall k.$$

Since $t_k \to 0$, this shows that $\operatorname{dist}(c + \operatorname{range} \mathcal{A}^*, \mathcal{K}) = 0$ and, therefore, (Conic-D) is in weak status.

(*iii*) From item (*ii*), we know that (Conic-D) is in weak status if and only if $\theta_{\text{D-Feas}} = 0$. In particular, if (Conic-D) is weakly infeasible then the optimal value of (D-Feas) cannot be attained because, otherwise, we would obtain a feasible solution to (Conic-D). Conversely, if $\theta_{D-Feas} = 0$ and is not attained, then (Conic-D) is in weak status and cannot be weakly feasible, so it must be weakly infeasible.

(*iv*) Suppose that (Conic-D) is strongly infeasible. By items (*i*), (*ii*) and (*iii*), $\theta_{D-Feas} \geq 0$ implies that (Conic-D) is either strongly feasible or in weak status. Therefore, it must be the case that $\theta_{D-Feas} < 0$.

Conversely, if $\theta_{D-Feas} < 0$, items (*i*), (*ii*) and (*iii*) imply that (Conic-D) is neither strongly feasible nor in weak status. Therefore, it must be strongly infeasible.

□

Remark. *Item (iv) of Proposition 21 includes the possibility that $\theta_{\text{D-Feas}} = -\infty$, i.e., (D-Feas) might be infeasible. This happens, for example, when $\mathcal{K}$ is a subspace and $c + \operatorname{range} \mathcal{A}^*$ does not intersect $\mathcal{K}$. However, under the hypothesis that $\mathcal{K}$ is full-dimensional (i.e., $\operatorname{span} \mathcal{K} = \mathcal{E}$), (D-Feas) must always be feasible, see Proposition 19.*

From Proposition 21 we see that if (Conic-D) is weakly infeasible, we can obtain almost feasible solutions to (Conic-D) by constructing almost optimal solution solutions to (D-Feas), which can be done through the discussion in Section 4.2 and Algorithm 2. For future reference, we register this fact as a proposition.

Proposition 22 (From almost optimality to almost feasibility)**.** *Suppose that $\epsilon > 0$ and that (t_ϵ, y_ϵ) is a feasible solution to (D-Feas) satisfying $0 \geq t_\epsilon \geq -\epsilon$. Then,*

$$\operatorname{dist}(c - \mathcal{A}^* y_\epsilon, \mathcal{K}) \leq \epsilon \|e\|.$$

Proof. Since (t_ϵ, y_ϵ) is feasible for (D-Feas) we have $c - t_\epsilon e - \mathcal{A}^* y_\epsilon \in \mathcal{K}$. Therefore,

$$\operatorname{dist}(c - \mathcal{A}^* y_\epsilon, \mathcal{K}) \leq \|c - \mathcal{A}^* y_\epsilon - (c - t_\epsilon e - \mathcal{A}^* y_\epsilon)\| \leq \epsilon \|e\|.$$

□

To conclude this section, we present an algorithm for handling infeasibility of (Conic-D), see Algorithm 3. The algorithm is able to distinguish between weak and strong infeasibility and, for weakly infeasible problems, it returns almost feasible solutions. During the algorithm's run, we will need to apply facial reduction to (P-Feas). For convenience, denote by $\mathcal{F}_{\min}^{\text{P-Feas}}$ the minimal face of $\mathcal{K}^*$ that contains the feasible region of (P-Feas). Applying facial reduction to (P-Feas) leads to the following pair of problems:

$$\begin{aligned} \inf_{x} \quad & \langle c, x\rangle \\ \text{subject to} \quad & \mathcal{A}x = 0 \\ & \langle e, x\rangle = 1 \\ & x \in \mathcal{F}_{\min}^{\mathrm{P-Feas}} \end{aligned} \tag{$\hat{\text{P}}$-Feas}$$

$$\begin{aligned} \sup_{t,y} \quad & t \\ \text{subject to} \quad & c - te - \mathcal{A}^* y \in (\mathcal{F}_{\min}^{\mathrm{P-Feas}})^*. \end{aligned} \tag{$\hat{\text{D}}$-Feas}$$

The pair ($\hat{\text{P}}$-Feas) and ($\hat{\text{D}}$-Feas) satisfy the following property.

Proposition 23. *Suppose* (D-Feas) *is feasible and* $\mathcal{F}_{\min}^{\mathrm{P-Feas}} \neq \emptyset$*, then the pair* ($\hat{\text{P}}$-Feas), ($\hat{\text{D}}$-Feas) *are both strongly feasible and their common optimal value is equal to* θ_{D-Feas}*. Moreover, if* $\mathcal{F}_{\min}^{\mathrm{P-Feas}} = \emptyset$ *then* (Conic-D) *is strongly feasible.*

Proof. Under the assumption that (D-Feas) is feasible, (D-Feas) must be, in fact, strongly feasible, by Proposition 19. Therefore, the minimal face of $\mathcal{K}$ that contains the feasible region of (D-Feas) is $\mathcal{K}$ itself. That is,

$$\mathcal{F}_{\min}^{\mathrm{D-Feas}} = \mathcal{K}.$$

Applying Theorem 16 to (D-Feas) we conclude that $\theta_{\mathrm{D-Feas}}$ is finite if and only if $\mathcal{F}_{\min}^{\mathrm{P-Feas}} \neq \emptyset$. We also obtain from item (i) of Theorem 16 that, if indeed $\mathcal{F}_{\min}^{\mathrm{P-Feas}} \neq \emptyset$ holds, then ($\hat{\text{P}}$-Feas), ($\hat{\text{D}}$-Feas) are both strongly feasible and their common optimal value must coincide with $\theta_{\mathrm{D-Feas}}$. Alternatively, from item (ii) of Theorem 16, we conclude that $\mathcal{F}_{\min}^{\mathrm{P-Feas}} = \emptyset$ if and only if $\theta_{\mathrm{D-Feas}} = +\infty$, in which case (Conic-D) must be strongly feasible by Proposition 21. ☐

Algorithm 3: Determining the infeasibility status of (Conic-D)

Input: $\mathcal{K}, \mathcal{A}, b, c, \epsilon$ ((Conic-D) is assumed to be infeasible)
Output: `Weakly Infeasible` or `Strongly Infeasible`. If `Weakly Infeasible` then y_ϵ such that $\operatorname{dist}(c - \mathcal{A}^* y_\epsilon, \mathcal{K}) \leq \epsilon$ is also returned.

if $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K} = \emptyset$ **then**
  return `Strongly Infeasible` `/* Proposition 19 */`
end
Apply Algorithm 1 to (P-Feas) and let $(f_1, y_1), \ldots, (f_{\ell_2}, y_{\ell_2})$ be the obtained corresponding reducing directions.
if *Algorithm 1 returned* `Infeasible` *(i.e.,* $\mathcal{F}_{\min}^{\mathrm{P-Feas}} = \emptyset$*)* **then**
  return `Strongly Feasible` `/* Proposition 23, impossible by assumption */`
else
  Solve the pair ($\hat{\text{P}}$-Feas) and ($\hat{\text{D}}$-Feas) and denote the optimal value by θ and the optimal solution of ($\hat{\text{D}}$-Feas) by (t^*, y^*).
  if $\theta < 0$ **then**
    return `Strongly Infeasible` `/* Proposition 21 */`
  else if $\theta = 0$ **then**
    Let $\hat{t}, \hat{y}$ be such that $c - \hat{t}e - \mathcal{A}^* \hat{y} \in \operatorname{ri} \mathcal{K}$.[4]
    Apply Algorithm 2 to (D-Feas) using as input $(f_1, y_1), \ldots, (f_{\ell_2}, y_{\ell_2})$, $(\hat{t}, \hat{y})$, (t^*, y^*), $\epsilon/\|e\|$. Let y_ϵ be the output of Algorithm 2.
    return y_ϵ and `Weakly Infeasible`
  else if $\theta > 0$ **then**
    return `Strongly Feasible` `/* Proposition 21, impossible by assumption */`
  end
end

Proposition 24 (Algorithm 3 is correct)**.** *The following hold.*

(i) Assuming that (Conic-D) is infeasible, Algorithm 3 correctly identifies whether (Conic-D) is strongly or weakly infeasible.

(ii) When (Conic-D) is weakly infeasible, the output of Algorithm 3 is indeed an ϵ-feasible solution.

(iii) When $\mathcal{K} = \mathcal{S}^n_+$, Algorithm 3 is implementable with $O(n)$ calls to $\mathcal{O}_{int}$.

Proof. The correctness of Algorithm 3 follows from Proposition 23 and the correctness of Algorithm 2. We will now explain some details.

By Propositions 21 and 23, to distinguish between weak and strong infeasibility it is enough to check the following three items: whether (D-Feas) is feasible or not; whether $\mathcal{F}_{\min}^{P-Feas}$ is empty or not; whether the optimal value of the pair ($\hat{\text{P}}$-Feas) and ($\hat{\text{D}}$-Feas) is negative or zero. These three items are checked at Lines 1, 5, 9 and 11 of Algorithm 3.

At Line 1, if $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{span} \mathcal{K} = \emptyset$, then the optimal value of (D-Feas) is $-\infty$ and (Conic-D) is strongly infeasible by Proposition 19.

However, if we progress until the check of Line 5, (D-Feas) must be strongly feasible, also by Proposition 19. By this point, facial reduction is applied to (P-Feas) and if $\mathcal{F}_{\min}^{P-Feas} = \emptyset$, then Proposition 23 tells us that (Conic-D) is strongly feasible. As we are assuming that (Conic-D) is infeasible, this should not happen.

If the algorithm reaches Line 9 then (D-Feas) is feasible, $\mathcal{F}_{\min}^{P-Feas} \neq \emptyset$ and, therefore, Proposition 23 applies. By Proposition 21, if $\theta < 0$ it must be the case that (Conic-D) is strongly infeasible. If $\theta = 0$ then (Conic-D) is weakly infeasible and the correctness of Algorithm 2 shows that (t_ϵ, y_ϵ) is indeed an $\epsilon/\|e\|$ optimal solution to (D-Feas) which, by Proposition 22 implies that y_ϵ must be an ϵ-feasible solution to (Conic-D).

Finally, suppose that $\mathcal{K} = \mathcal{S}^n_+$. The only lines where SDPs need to be solved are when Algorithm 1 is invoked and at Line 8. Algorithm 1 and Algorithm 2 require at most $n + 1$ calls to $\mathcal{O}_{\text{int}}$ each so, we only need to check that we can indeed solve the SDP at Line 8 with $\mathcal{O}_{\text{int}}$. We note that if we reach Line 8, then (D-Feas) is feasible and $\mathcal{F}_{\min}^{P-Feas} \neq \emptyset$ which, by Proposition 23 implies that the pair ($\hat{\text{P}}$-Feas) and ($\hat{\text{D}}$-Feas) are both strongly feasible and can indeed be solved by $\mathcal{O}_{\text{int}}$. Therefore, Algorithm 3 can be implemented with $O(n)$ calls to $\mathcal{O}_{\text{int}}$. □

To conclude, we note that, when $\mathcal{K} = \mathcal{S}^n_+$, the problem at Line 12 can be solved by invoking $\mathcal{O}_{\text{int}}$ (which would not affect the $O(n)$ complexity of Algorithm 3), but that is not necessary. Let $\lambda_{\min}(\cdot)$ denote the minimum eigenvalue function and recall $\lambda_{\min}(U + V) \geq \lambda_{\min}(U) + \lambda_{\min}(V)$ always holds for $U, V \in \mathcal{S}^n$. With that, if α is positive then

$$\alpha > -\frac{\lambda_{\min}(c - \mathcal{A}^*\hat{y})}{\lambda_{\min}(e)} \Rightarrow c - \mathcal{A}^*\hat{y} + \alpha e \in \mathcal{S}^n_+.$$

So, with two minimum eigenvalue computations, we can solve the problem in Line 12 of Algorithm 3. As in Section 4.2.1, a strategy of starting with some negative t and doubling it at each step would also work.

5 Completely solving (Conic-D)

Using the techniques described in Sections 3 and 4, we can now present a general algorithm for completely solving (Conic-D), in the sense of Definition 1. In particular, when $\mathcal{K} = \mathcal{S}^n_+$, we can completely solve (SDP-D) through polynomially many calls to $\mathcal{O}_{\text{int}}$. For ease of reference, we write down below again some of the auxiliary problems that are referenced in Algorithm 4 below.

$$\begin{aligned} \inf_x \quad & \langle c, x\rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in (\mathcal{F}^{\text{D}}_{\min})^* \end{aligned} \tag{$\hat{\text{P}}$}$$

$$\begin{aligned} \sup_y \quad & \langle b, y\rangle \\ \text{subject to} \quad & c - \mathcal{A}^*y \in \mathcal{F}^{\text{D}}_{\min}. \end{aligned} \tag{$\hat{\text{D}}$}$$

$$\begin{aligned} \inf_{x} \quad & \langle c, x\rangle & \text{(P*)} \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{F}^{\hat{\mathrm{P}}}_{\min} \end{aligned} \qquad \begin{aligned} \sup_{y} \quad & \langle b, y\rangle & \text{(D*)} \\ \text{subject to} \quad & c - \mathcal{A}^* y \in (\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*. \end{aligned}$$

Here, we recall that $\mathcal{F}^{\mathrm{D}}_{\min}$ is the minimal face of $\mathcal{K}$ that contains $(c + \operatorname{range} \mathcal{A}^*) \cap \mathcal{K}$ and $\mathcal{F}^{\hat{\mathrm{P}}}_{\min}$ is the minimal face of $(\mathcal{F}^{\mathrm{D}}_{\min})^*$ that contains the feasible region of ($\hat{\mathrm{P}}$). We also recall that, by Theorem 16, θ_{D} is finite if and only if (D-Feas) is feasible and $\mathcal{F}^{\hat{\mathrm{P}}}_{\min} \neq \emptyset$. In this case, (P*) and (D*) are both strongly feasible and, when $\mathcal{K} = \mathcal{S}^n_+$, they can be solved by invoking $\mathcal{O}_{\mathrm{int}}$. By doing so, we are able to obtain the dual optimal value θ_{D}. Checking whether θ_{D} is attained can be done by solving the following feasibility problem.

$$\begin{aligned} \text{find} \quad & y & \text{(D-OPT)} \\ \text{subject to} \quad & c - \mathcal{A}^* y \in \mathcal{F}^{\mathrm{D}}_{\min}, \quad \langle b, y\rangle = \theta_{\mathrm{D}}. \end{aligned}$$

Let $L : \mathbb{R}^m \to \mathbb{R}^m$ be an affine map (that is, $L - u$ is linear for some $u \in \mathbb{R}^m$) such that

$$\operatorname{range} L = \{y \mid \langle b, y\rangle = \theta_{\mathrm{D}}\}$$

In particular, $\langle b, L(\hat{y})\rangle = \theta_{\mathrm{D}}$ holds for every $\hat{y}$. With that we can put (D-OPT) in "dual standard format" as follows

$$\begin{aligned} \text{find} \quad & \hat{y} & \text{(D-OPT-STD)} \\ \text{subject to} \quad & c - \mathcal{A}^*(L(\hat{y})) \in \mathcal{F}^{\mathrm{D}}_{\min}. \end{aligned}$$

We observe that (D-OPT) is feasible if and only if (D-OPT-STD) is feasible[5]. Once (D-OPT-STD) is solved (for example, with Algorithm 1) and a solution $\hat{y}^*$ is obtained, a solution to (D-OPT) is obtained by letting $y^* = L\hat{y}^*$. Of course, it might be the case (D-OPT) is not feasible in the first place. Nevertheless, we now have all pieces in place, see Algorithm 4.

Theorem 25 (Algorithm 4 is correct)**.** *Algorithm 4 completely solves* (Conic-D)*. That is, it correctly determines whether* (Conic-D) *is feasible or not. If* (Conic-D) *is infeasible, Algorithm 4 distinguishes between weak and strong infeasibility and, in case of weak infeasibility, an ϵ-feasible solution is returned. If* (Conic-D) *is feasible, Algorithm 4 computes the optimal value of* (Conic-D)*. If the optimal value is finite and attained, an optimal solution is returned. If the optimal value is finite but not attained, an ϵ-optimal solution is returned. If the optimal value is $+\infty$, a feasible solution is returned.*

In addition, if $\mathcal{K} = \mathcal{S}^n_+$, then (Conic-D) *can be implemented with $O(n)$ calls to the oracle $\mathcal{O}_{int}$.*

Proof. To prove the result we gather everything we have done so far. We consider the following cases.

1. (Conic-D) *is infeasible.* The correctness of Facial Reduction and Algorithm 1 implies that if (Conic-D) is infeasible, then this will be correctly detected after Line 1. Furthermore, the correctness of Algorithm 3 (Proposition 24) ensures that weak infeasibility and strong infeasibility will be correctly detected. And, in case of weak infeasibility, an ϵ-feasible solution will be returned.

2. (Conic-D) *is feasible but unbounded.* If the algorithm advances until Line 7, it is because (Conic-D) is feasible and, in particular, $\mathcal{F}^{\mathrm{D}}_{\min}$ is not empty. In this case, we are under the hypothesis of Theorem 16. By item (*ii*) of Theorem 16, we have $\theta_{\mathrm{D}} = +\infty$ if and only if $\mathcal{F}^{\hat{\mathrm{P}}}_{\min} = \emptyset$.

3. (Conic-D) *is feasible, θ_{D} is finite but not attained.* In that case, when Algorithm 1 is invoked at Line 11, it is correctly detected that (D-OPT-STD) is infeasible and Algorithm 2 correctly constructs an ϵ-optimal solution.

[5] If y is feasible for (D-OPT), then $y \in \operatorname{range} L$, so there exists $\hat{y}$ such that $L(\hat{y}) = y$. Reciprocally, if $\hat{y}$ is feasible for (D-OPT-STD) then $L\hat{y}$ is feasible for (D-OPT).

Algorithm 4: Completely Solving (Conic-D)

Input: $\mathcal{K}, \mathcal{A}, b, c, \epsilon$

1 Apply Algorithm 1 to (Conic-D) and let $d_1, \ldots, d_{\ell_1}$ be the corresponding reducing directions.
2 **if** *Algorithm 1 returned* ***Infeasible*** **then**
3   Invoke Algorithm 3, **return** its outputs.
4 **else**
5   Let $\hat{s}$ be such that $\hat{s} \in \operatorname{ri} \mathcal{F}^{\mathrm{D}}_{\min}$ (see output 2. of Algorithm 1) and $\hat{y}$ such that $c - \mathcal{A}^*\hat{y} = \hat{s}$.
6   Apply Algorithm 1 to ($\hat{\mathrm{P}}$), obtain reducing directions $(f_1, y_1), \ldots, (f_{\ell_2}, y_{\ell_2})$.
7   **if** $\mathcal{F}^{\hat{P}}_{\min} = \emptyset$ **then**
8   **return** $\hat{s}$ and `Feasible Unbounded`.
9   **else**
10   Solve (P*) and (D*) and obtain θ_{D} and optimal solutions y^*, x^* to (D*) and (P*), respectively.
11   Apply Algorithm 1 to (D-OPT-STD).
12   **if** *Algorithm 1 returned* ***Infeasible*** **then**
13   Use Algorithm 2 with $f_1, \ldots, f_{\ell_1}, \hat{y}, y^*, \epsilon$ as inputs and **return** y_ϵ (the output of Algorithm 2), θ_{D} and `Feasible Unattained`.
14   **else**
15   Let (y, s) be the feasible solution returned by Algorithm 1.
16   **return** y, θ_{D} and `Feasible Attained`.
17   **end**
18   **end**
19 **end**

4. (Conic-D) *is feasible,* θ_{D} *is finite and attained.* In that case, when Algorithm 1 is invoked at Line 11, a feasible solution to (D-OPT-STD) will be obtained, which corresponds to an optimal solution to (Conic-D).

For the last part of the proof, suppose that $\mathcal{K} = \mathcal{S}^n_+$. Algorithm 4 directly invokes facial reduction (Algorithm 1) at most 3 times (Lines 1, 6 and 11). It also directly invokes Algorithm 2 and Algorithm 3 at most one time, each. The only other time where SDPs need to be solved is at Line 10 where we need to solve the SDPs (P*) and (D*), which are both strongly feasible (item (i) of Theorem 16) and therefore can be solved by a single call to $\mathcal{O}_{\mathrm{int}}$. By Propositions 12, 24 and the discussion in Section 4.2.1 we have that Algorithms 1, 2 and 3 can also be implemented with $O(n)$ calls to $\mathcal{O}_{\mathrm{int}}$, so the same must be true of Algorithm 4. ☐

See Appendix B for an example where Algorithm 4 is applied to an instance that has a finite nonzero duality gap and unattainment at both primal and dual sides.

6 Comparison with other approaches

One of the features of this work is the interior point oracle $\mathcal{O}_{\mathrm{int}}$ and its application to the complete solvability of semidefinite programs. Related to this task, the usage of reducing directions to the construction of almost optimal solutions and almost feasible solutions for general conic linear programs seems to be new, although some of those ideas were present in our previous works on semidefinite programming [22] and second order cone programming [24]. Technical results on facial reduction and double facial reduction such as Proposition 14, Theorems 15 and 16 seem to be novel as well. Nevertheless, the idea of completely solving a problem in some sense is not necessarily new and, in this section, we compare our approach with other proposals in the literature that had similar goals.

In section 5.10 of [7], de Klerk, Terlaky and Roos have described a possible sequence of steps to solve (SDP-D). Their tool of choice is a self-dual embedding strategy of the original pair (SDP-P) and (SDP-D).

As we mentioned before, in the absence of both primal and dual strong feasibility, the embedded problem might fail to reveal the optimal value of the original problem or detect infeasibility/nonattainment. To account for that, they go for a second step, where they consider an embedded problem using Ramana's dual. The Ramana's dual (P_R) is a substitute for (SDP-P) and they consider the pair formed by (P_R) and its dual (D_{cor}), which is a "corrected" version of (SDP-D). The pair (P_R, D_{cor}) can then be solved by their embedding strategy to find θ_{D}. As the embedded problem is both primal and dual strongly feasible, it is possible to invoke $\mathcal{O}_{\text{int}}$ to solve it. However, if the solution given by $\mathcal{O}_{\text{int}}$ is not of maximum rank at both steps, their strategy might not work. We should mention that they do show in detail how to build an interior point method suitable for their approach. Our analysis, on the other hand, is completely agnostic to the inner workings of the interior point oracle and no assumption is made on the optimal solutions returned by $\mathcal{O}_{\text{int}}$.

As our approach does not rely on Ramana's dual, our analysis is easily generalizable to other classes of cones. Indeed, Algorithm 4 is valid and correct for any closed convex cone $\mathcal{K}$. We remark that although there is a strong connection between Ramana's dual and facial reduction [42, 32], no similar construction is known for any other class of cones. For example, following Pataki's approach in [32], one could formulate an alternative dual system for a second order cone programming problem. Such a system would have many of the properties that Ramana's dual has, but it is not clear whether that system can be expressed via second order cone constraints.

Permenter, Friberg and Andersen present in [36] a very elegant approach for general conic linear programming based on self-dual embeddings and they are able to achieve most of the goals included in Definition 1. They showed that the relative interior of the set of solutions to a certain self-dual embedding of the pair (Conic-P) and (Conic-D) will reveal reducing directions for (Conic-P) and (Conic-D) under certain circumstances, see Corollary 3.3 in [36]. They used this property to present an algorithm for solving (Conic-P) and (Conic-D) while identifying several pathologies, see Algorithms 1 and 2 in [36].

We remark that even if (Conic-P) and/or (Conic-D) are not strongly feasible, it could still be the case that the duality gap is zero and both problems are attained. In this case, certain self-dual embeddings might recover optimal solutions to the pair (Conic-P), (Conic-D) even in the absence of strong feasibility. Indeed, a crucial advantage of the approach in [36] is that a facial reducing step is performed only if it strictly necessary in order to recover zero duality gap and attainment, see item 1. of Theorem 4.1 therein. As such, the approach in [36] regularizes a problem only if needed.

However, the main drawback in [36] seems to be the fact that it requires a *relative interior* solution to their self-dual embedding, which is a stronger requirement than our assumption of having access to $\mathcal{O}_{\text{int}}$, since $\mathcal{O}_{\text{int}}$ is allowed to return any optimal solution. While relative interior optimal solutions might be obtainable via interior point algorithms, this is not necessarily true for other methods. Nevertheless, although our algorithm is more general, the approach in [36] seems to be more likely to lead to a practical implementation than ours, especially in conjunction with interior point methods. Indeed, the numerical experiments in Section 5 of [36] suggest that even when reducing directions are computed inexactly there are cases where they are still useful for analyzing the problem, although sometimes these approximate directions can also lead to incorrect conclusions.

Finally, we remark on the difference between the double reformulation proposed by Pataki [34] and our double facial reduction of Section 4. In Definition 1 of [34], Pataki defined that a reformulation of (SDP-P) and (SDP-D) corresponds to the SDP primal and dual pair obtained by applying certain elementary operations to (SDP-P) and (SDP-D). These elementary operations preserve the properties of the original problem such as duality gaps and whether the optimal value is attained or not. In simplified terms, Pataki showed in Theorem 4 of [34] that (SDP-D) can be "doubly reformulated" as

$$\begin{aligned} \sup_{y} \quad & \langle b', y \rangle \\ \text{subject to} \quad & c' - \sum_{i=1}^{m} \mathcal{A}'_i y_i \in \mathcal{S}^n_+, \end{aligned} \tag{SDP-Ref}$$

where $\mathcal{A}_i \in \mathcal{S}^n$ for all i and in such a way that c' belongs to $\operatorname{ri} \mathcal{F}^{\text{D}}_{\min}$. Furthermore, for some ℓ, $(c', \mathcal{A}'_1, \ldots, \mathcal{A}'_\ell)$ can be used to obtain the minimal face associated to the so-called "homogeneous dual" of (SDP-Ref). So, this double reformulation, in a sense, reveal both the minimal face of (SDP-Ref) and the minimal face associated to a homogenized version of the corresponding dual problem of (SDP-Ref).

As far as we could see, the homogeneous dual of (SDP-Ref) is related but it is quite different from either ($\hat{\mathrm{P}}$) or (P*) even when $\mathcal{K} = \mathcal{S}^n_+$. We note that the homogeneous dual in Theorem 4 of [34] is computed with respect the cone $\mathcal{S}^n_+$ and not with respect the minimal face associated to (SDP-Ref). Therefore, the closest analogous of the double reformulation in our setting would be if we applied the second facial reduction to (Conic-P) instead of applying to ($\hat{\mathrm{P}}$). In conclusion, it seems to us that double facial reduction and Pataki's double reformulation serve different purposes.

7 Concluding remarks

In this paper, we have discussed how to use facial reduction and double facial reduction to completely solving (Definition 1) a general conic linear program, under the assumption that certain auxiliary problems can be solved, see Algorithm 4 and Theorem 25. When specialized to the particular case of semidefinite programming, these results imply that an arbitrary semidefinite program over $n \times n$ matrices can be completely solved by invoking at most $O(n)$ times an oracle that only return solutions to primal and dual strongly feasible SDPs. We also provided technical results on facial reduction and double facial reduction that might be of independent interest, see Sections 3 and 4.

For limitations, drawbacks and comparison to other approaches, see Sections 1.3 and 6. In particular, as discussed in Section 1.3, in our analysis we assumed that the oracle $\mathcal{O}_{\mathrm{int}}$ returns an exact solution. An interesting topic of future research would be to consider the effects of impreciseness in the solutions returned by $\mathcal{O}_{\mathrm{int}}$.

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A The proof of Proposition 6

In this appendix we discuss and prove Proposition 6, which we restate below.

Proposition. *Let $\mathcal{K}$ be as in (7) with some orthogonal matrix R, some $r \le n$, and some linear subspace $\mathcal{L} \subseteq \mathcal{S}^n$ such that $\mathcal{L} \subseteq (\mathcal{S}^{r,n}_+)^\perp$. Suppose that (Conic-P) and (Conic-D) are strongly feasible. Then, (Conic-P) and (Conic-D) are solvable with a single call to $\mathcal{O}_{int}$.*

Proof. In what follows, let $\mathcal{A}_1, \ldots, \mathcal{A}_m \in \mathcal{S}^n$ be such that

$$c - \mathcal{A}^* y = c - \sum_{i=1}^m \mathcal{A}_i y_i, \text{ and } \quad \mathcal{A}x = b \Leftrightarrow \langle \mathcal{A}_i, x \rangle = b_i, i = 1, \ldots, m.$$

Since $\mathcal{K}$ is as in (7), we have

$$R\mathcal{K}R^\top = \mathcal{S}^{r,n}_+ \oplus \mathcal{L}, \quad \operatorname{ri}(R\mathcal{K}R^\top) = (\operatorname{ri}\mathcal{S}^{r,n}_+) \oplus \mathcal{L}, \tag{29}$$

$$(R\mathcal{K}R^\top)^* = \mathcal{S}^{r,n}_+ \oplus (\mathcal{L}^\perp \cap (\mathcal{S}^{r,n}_+)^\perp), \quad \operatorname{ri}(R\mathcal{K}R^\top)^* = (\operatorname{ri}\mathcal{S}^{r,n}_+) \oplus (\mathcal{L}^\perp \cap (\mathcal{S}^{r,n}_+)^\perp). \tag{30}$$

Furthermore,

$$\operatorname{ri}\mathcal{S}^{r,n}_+ = \left\{ \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}^n \;\middle|\; U \in \mathcal{S}^r_{++} \right\}, \tag{31}$$

where $\mathcal{S}^r_{++}$ denotes the set of $r \times r$ real symmetric positive definite matrices. Let $\mathcal{S}^{r,n}$ denote the span of $\mathcal{S}^{r,n}_+$. The proof is divided in four cases and we will show that each case leads back to the previous one.

Case 1. $\mathcal{L} = (\mathcal{S}^{r,n})^\perp$, i.e., $\mathcal{K} = (\mathcal{S}^{r,n}_+)^*$. *$R$ is the identity matrix.*

Let $\pi_r : \mathcal{S}^n \to \mathcal{S}^r$ be the orthogonal projection that maps $x \in \mathcal{S}^n$ on its upper left $r \times r$ block. We can reformulate (Conic-P) and (Conic-D) as follows

$$\begin{aligned} \inf_{\hat{x}} \quad & \langle \pi_r(c), \hat{x} \rangle && \text{(P)} \\ \text{subject to} \quad & \langle \pi_r(\mathcal{A}_i), \hat{x} \rangle = b_i, \quad i = 1, \ldots, m \\ & \hat{x} \in \mathcal{S}^r_+ \end{aligned}$$

$$\begin{aligned} \sup_{y} \quad & \langle b, y \rangle && \text{(D)} \\ \text{subject to} \quad & \pi_r(c) - \sum_{i=1}^m \pi_r(\mathcal{A}_i) y_i \in \mathcal{S}^r_+. \end{aligned}$$

We note that $s \in \mathcal{K}$, if and only if the upper-left $r \times r$ block of s is positive semidefinite. Therefore, (D) and (Conic-D) are equivalent in the sense that they share the same feasible solutions and have the same optimal value. From (29), we see that (Conic-D) is strongly feasible if and only if (D) is strongly feasible.

Similarly, $x \in \mathcal{K}^*$ if and only the upper-left $r \times r$ block of x is positive semidefinite and the other entries are zero. Therefore,

$$\langle A, x \rangle = \langle \pi_r(A), \pi_r(x) \rangle, \quad \forall A \in \mathcal{S}^n.$$

Accordingly, if x is a feasible solution to (Conic-P) such that $x \in \operatorname{ri} \mathcal{K}^*$, then $\pi_r(x) \in \operatorname{ri} \mathcal{S}^r_+$ and $\pi_r(x)$ is feasible for (P). Conversely, if $\hat{x}$ is feasible for (P), then $\pi_r^*(\hat{x})$ is feasible for (P). Furthermore, $\hat{x} \in \mathcal{S}^{r,n}_+ \Leftrightarrow \pi_r^*(\hat{x}) \in \mathcal{K}^*$, where π_r^* is the adjoint of π_r. We conclude that (Conic-P) is strongly feasible if and only if (P) is strongly feasible. Also, both problems have the same optimal values.

Therefore, if (Conic-D) and (Conic-P) are both strongly feasible, the same is true for the pair (P) and (D). Since (P) and (D) are *bona fide* SDPs, they can be solved with $\mathcal{O}_{\text{int}}$. The preceding discussion shows how to recover optimal solutions to (Conic-D) and (Conic-P) from the solutions to (D) and (P).

Case 2. $\mathcal{L} = \{0\}$, i.e., $\mathcal{K} = \mathcal{S}^{r,n}_+$. *$R$ is the identity matrix.*

First, we recall that the role of primal and dual problems in conic linear programming is interchangeable in the following sense. Given $\mathcal{A}, b, c, \mathcal{K}$, we can find $\hat{\mathcal{A}}, \hat{b}, \hat{c}$ such that (Conic-D) is "equivalent" to

$$\begin{aligned} \inf_{s} \quad & \langle \hat{c}, s \rangle && \text{(P-D)} \\ \text{subject to} \quad & \langle \hat{\mathcal{A}}_i, s \rangle = \hat{b}_i, \quad i = 1, \ldots, m \\ & s \in \mathcal{K}, \end{aligned}$$

which is a problem in "primal format", where the feasible solutions correspond to the feasible slacks of (Conic-D). Only linear algebra is needed to find $\hat{\mathcal{A}}, \hat{b}, \hat{c}$. For example, we can take $\hat{\mathcal{A}}, \hat{b}$ to be such that

$$s \in c + \operatorname{range} \mathcal{A}^* \Leftrightarrow \hat{A}s = \hat{b}.$$

Next, let $\hat{c}$ be such that $\mathcal{A}\hat{c} = b$. Here, $\hat{c}$ is not required to be feasible for (Conic-P), so no SDPs need to be solved in order to obtain $\hat{c}$. Nevertheless, if y is a feasible solution to (Conic-D) and $s = c - \mathcal{A}^* y$, we have

$$-\langle b, y \rangle = \langle \hat{c}, -c + c - \mathcal{A}^* y \rangle = -\langle \hat{c}, c \rangle + \langle \hat{c}, s \rangle. \tag{32}$$

From (32), we draw two conclusions. The first is that, provided that the linear system $Ax = b$ has a solution[6], then $\langle b, y \rangle = \langle b, y' \rangle$ if y, y' are associated to the same slack of (Conic-D). Therefore, without ambiguity we can associate an objective value to an slack s of (Conic-D). The second conclusion is that the optimal values of (P-D) and (Conic-D) differ by the constant $\langle \hat{c}, c \rangle$.

We can now explain in which sense (Conic-D) and (P-D) are equivalent. Every feasible slack of (Conic-D) is a feasible solution to (P-D) (and vice-versa), and their objective values differ by $\langle \hat{c}, c \rangle$. In particular, given

[6]Which it does since we are assuming (Conic-P) is (strongly) feasible.

an optimal solution s^* to (P-D), any y^* satisfying $s^* = c - \mathcal{A}^* y^*$ will be optimal to (Conic-D). Furthermore, (Conic-D) is strongly feasible if and only if (P-D) is strongly feasible.

The dual counterpart of (P-D) is equivalent to (Conic-P) in an analogous fashion and the similarly cumbersome details are omitted.

In **Case 2**, $\mathcal{K}^*$ corresponds to the cone $\mathcal{K}$ in **Case 1**. The overall conclusion is that in order to return to **Case 1**, it is enough to reformulate (Conic-D) as a problem in primal format.

Case 3. $\mathcal{L} \subseteq (\mathcal{S}^{r,n})^\perp$ is arbitrary. R is the identity matrix.

Let $E_1, \ldots, E_\ell$ be a basis for $\mathcal{L}$. In view of (29), (30), we have that (Conic-D) and (Conic-P) are equivalent to the following pair of problems.

$$\begin{array}{lll} \inf\limits_{x} & \langle c, x \rangle & \\ \text{subject to} & \langle \mathcal{A}_i, x \rangle = b_i, & i = 1, \ldots, m \\ & \langle E_j, x \rangle = 0, & j = 1, \ldots, \ell \\ & x \in (\mathcal{S}_+^{r,n})^* & \end{array} \tag{P}$$

$$\begin{array}{ll} \sup\limits_{y,t} & \langle b, y \rangle \\ \text{subject to} & c - \sum_{i=1}^m \mathcal{A}_i y_i - \sum_{j=1}^{\ell} E_j t_j \in \mathcal{S}_+^{r,n} \end{array} \tag{D}$$

In particular, the feasible solutions and the optimal value of (P) and (Conic-P) are the same. Furthermore, y is feasible for (Conic-D) if and only if there exists t such that (y, t) is feasible for (D).

The pair of problems (P) and (D) are in the format described in **Case 2**.

Case 4. $\mathcal{L} \subseteq (\mathcal{S}^{r,n})^\perp$ is arbitrary, R is an arbitrary orthogonal matrix.

We observe that

$$c - \sum_{i=1}^m \mathcal{A}_i y_i \in R^\top (\mathcal{S}_+^{r,n} \oplus \mathcal{L}) R \quad \Leftrightarrow \quad RcR^\top - \sum_{i=1}^m R\mathcal{A}_i R^\top y_i \in \mathcal{S}_+^{r,n} \oplus \mathcal{L}.$$

Therefore, replacing the c, $\mathcal{A}_1, \ldots, \mathcal{A}_m$ by $RcR^\top$, $R\mathcal{A}_1 R^\top, \ldots, R\mathcal{A}_m R^\top$ in (Conic-D), (Conic-P) leads to an equivalent problem that falls under **Case 3**. □

B An example

In this appendix, we show an example of the application of Algorithm 4 to a problem that has a nonzero duality gap and such that both primal and dual optimal values are not attained.

In the following, we denote the $r \times r$ zero matrix and identity matrix by 0_r and I_r, respectively. Similar to Section 2.2, when it is clear from context, we use 0 to denote a zero matrix of appropriate size. Furthermore, when an entry of a matrix is omitted, it is assumed to be zero.

Let us consider the following problem.

$$\begin{array}{ll} \sup\limits_{y \in \mathbb{R}^8} & -y_4 - 2y_6 - 2y_7 \\ \text{subject to} & \begin{pmatrix} y_1 & & & & & & & y_3 - 1 \\ & y_1 & & & & & & y_5 - 1 \\ & & y_2 & y_3 & & & & \\ & & y_3 & y_4 - y_5 & & & & \\ & & & & y_4 & -0.5y_8 + 0.5 & y_6 & \\ & & & & -0.5y_8 + 0.5 & y_8 & y_7 & \\ & & & & y_6 & y_7 & 0 & \\ y_3 - 1 & y_5 - 1 & & & & & & 0 \end{pmatrix} \in \mathcal{S}_+^8. \end{array} \tag{D}$$

We also select $c, \mathcal{A}_1, \ldots, \mathcal{A}_8 \in \mathcal{S}^8$ such that $(y_1, \ldots, y_8)$ is feasible for (D) if and only if

$$c - \sum_{i=1}^8 \mathcal{A}_i y_i \in \mathcal{S}_+^n. \tag{33}$$

The corresponding primal of (D) is

$$
\begin{aligned}
\inf_{x} \quad & -2x_{18} - 2x_{28} + x_{56} && \text{(P)}\\
\text{subject to} \quad & -x_{11} - x_{22} = 0\\
& -x_{33} = 0\\
& -2x_{18} - 2x_{34} = 0\\
& -x_{44} - x_{55} = -1\\
& -2x_{28} + x_{44} = 0\\
& -2x_{57} = -2\\
& -2x_{67} = -2\\
& x_{56} - x_{66} = 0\\
& x \in \mathcal{S}^8_+.
\end{aligned}
$$

B.1 The optimal values of (P) and (D) and their unattainment

We check that $\theta_{\mathrm{D}} = -1$, $\theta_{\mathrm{P}} = 0$ and that neither (D) nor its primal are attained.

Let y be a feasible solution for (D). At the dual side (D), the $(7,7)$ and $(8,8)$ entries are 0, therefore any feasible y must satisfy

$$y_3 = y_5 = 1, \quad y_6 = y_7 = 0.$$

In addition, we have

$$y_4 - y_5 = y_4 - 1 \geq 0,$$

since the $(4,4)$ entry must be nonnegative. Similarly, we have $y_8 \geq 0$. It follows that $\theta_{\mathrm{D}} \leq -1$. However, for every $\epsilon > 0$,

$$y_\epsilon = (0, 1/\epsilon, 1, 1+\epsilon, 1, 0, 0, 0)$$

is a feasible solution that has value equal to $-1-\epsilon$. This shows that $\theta_{\mathrm{D}} = -1$.

We will now show that θ_{D} is not attained. For the sake of obtaining a contradiction, suppose that we had $y_4 = 1$ for some feasible y. Since $y_5 = 1$, this would imply that the $(4,4)$ entry is zero. This forces y_3 to be zero and, consequently, the $(1,8)$ entry must -1, which contradicts positive semidefiniteness.

Next, we move on the primal problem (P). Let x be a feasible solution to (P). The first and second equality constraints force

$$x_{11} = x_{22} = x_{33} = 0,$$

which implies that $x_{18} = x_{28} = 0$. Then, the fifth constraint implies that $x_{44} = 0$. Therefore, $x_{55} = 1$ by the fourth constraint. The last equality constraint forces $x_{56} = x_{66}$, which implies that $\theta_{\mathrm{P}} \geq 0$. Putting everything together we have that x is feasible for (P) if and only if x is positive semidefinite and has the following shape

$$\begin{pmatrix} 0_4 & & & & \\ & 1 & x_{56} & x_{57} & x_{58} \\ & x_{56} & x_{56} & 1 & x_{68} \\ & x_{57} & 1 & x_{77} & x_{78} \\ & x_{58} & x_{68} & x_{78} & x_{88} \end{pmatrix}. \tag{34}$$

Then, because $x_{67} = 1$, we can never assign zero to $x_{66} = x_{56}$. However, if $x_{66} = x_{56}$ is small but positive, we can construct feasible points by taking

$$x_{57} = x_{58} = x_{68} = x_{78} = 0$$

and taking x_{77} and x_{88} to be very large. This shows that $\theta_{\mathrm{P}} = 0$ but is not attained.

B.2 Applying Algorithm 4

We run Algorithm 4 with (D) as input. The first step is to apply facial reduction (Algorithm 1) to (D).

A possible reducing direction is

$$d_1 = \begin{pmatrix} 0_6 & 0 \\ 0 & I_2 \end{pmatrix}.$$

Let $\mathcal{F} = \mathcal{S}^8_+ \cap \{d_1\}^\perp$, then

$$\mathcal{F} = \mathcal{S}^{6,8}_+ = \left\{ \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}^8 \;\middle|\; U \in \mathcal{S}^6_+ \right\}.$$

Recalling (31), the relative interior of $\mathcal{F}$ correspond to the matrices in $\mathcal{F}$ for which U is positive definite. Therefore, we see that $\mathcal{F}$ is the minimal face $\mathcal{F}^D_{\min}$ of (D) by constructing the following feasible solution

$$\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4, \hat{y}_5, \hat{y}_6, \hat{y}_7, \hat{y}_8) = (1, 2, 1, 2, 1, 0, 0, 1). \tag{35}$$

Indeed, $\hat{s} = c - \mathcal{A}^*\hat{y}$ is a matrix with nonzero entries

$$\hat{s}_{11} = \hat{s}_{22} = \hat{s}_{44} = \hat{s}_{66} = 1,\ \hat{s}_{33} = \hat{s}_{55} = 2,\ \hat{s}_{34} = \hat{s}_{43} = 1,$$

which is a relative interior point of $\mathcal{F}$. Let $(\hat{\mathrm{D}})$ be the problem obtained by replacing $\mathcal{S}^8_+$ by $\mathcal{F}^D_{\min} = \mathcal{F}$ in (D) (see also Section 4.1). Let $(\hat{\mathrm{P}})$ be the "dual" problem of $(\hat{\mathrm{D}})$, which is obtained by replacing $\mathcal{S}^8_+$ by $(\mathcal{F}^D_{\min})^*$ in (P). Then $(\mathcal{F}^D_{\min})^*$ satisfies

$$(\mathcal{F}^D_{\min})^* = (\mathcal{S}^{6,8}_+)^* = \left\{ \begin{pmatrix} U & V \\ V^\top & W \end{pmatrix} \in \mathcal{S}^8 \;\middle|\; U \in \mathcal{S}^6_+ \right\}. \tag{36}$$

This has the effect of relaxing the constraint "$x \in \mathcal{S}^8_+$" to merely requiring that the upper left 6×6 block of x be positive semidefinite. The relative interior of $(\mathcal{F}^D_{\min})^*$ correspond to the matrices in (36) for which U is positive definite.

At this stage, $(\hat{\mathrm{D}})$ is strongly feasible and shares the same optimal value with (D). Because of this, the duality gap between $(\hat{\mathrm{P}})$ and $(\hat{\mathrm{D}})$ is zero and $(\hat{\mathrm{P}})$ has an optimal solution of value -1. Indeed, let x be the symmetric matrix having

$$x_{56} = x_{66} = -1, x_{57} = x_{67} = x_{55} = 1$$

and let the other entries be zero. Then $x \in (\mathcal{F}^D_{\min})^*$ and is an optimal solution to $(\hat{\mathrm{P}})$. Though it possesses an optimal solution, $(\hat{\mathrm{P}})$ may not be strongly feasible. We also remark that $(\hat{\mathrm{D}})$ does not have an optimal solution, since the feasible regions of (D) and $(\hat{\mathrm{D}})$ are the same and (D) does not have an optimal solution as we saw previously.

The next step in Algorithm 4 is applying facial reduction to $(\hat{\mathrm{P}})$. We observe that $(\hat{\mathrm{P}})$ is not strongly feasible. The first two equality constraint in (P) and (36) yields that any feasible solution to $(\hat{\mathrm{P}})$ must have the upper left 3×3 block equal to zero.

Let $y^1 \in \mathbb{R}^8$ be such that $y^1_1 = y^1_2 = 1$ and the other entries are zero. A reducing direction to $(\hat{\mathrm{P}})$ is given by the following matrix

$$f_1 = \begin{pmatrix} I_3 & 0 \\ 0 & 0_5 \end{pmatrix} = -\sum_{i=1}^{8} \mathcal{A}_i y^1_i = -\mathcal{A}_1 - \mathcal{A}_2, \tag{37}$$

where $\mathcal{A}_i$ is as in (33). Let $\hat{\mathcal{F}} := (\mathcal{F}^D_{\min})^* \cap \{f_1\}^\perp$. $\hat{\mathcal{F}}$ is a face of $(\mathcal{F}^D_{\min})^*$ that can be described as follows.

$$\hat{\mathcal{F}} = \left\{ \begin{pmatrix} 0_3 & 0 & V_1 \\ 0 & U & V_2 \\ V_1^\top & V_2^\top & W \end{pmatrix} \in \mathcal{S}^8 \;\middle|\; U \in \mathcal{S}^3_+ \right\}. \tag{38}$$

The relative interior of $\hat{\mathcal{F}}$ consists of the matrices in (38) for which $U \in \mathcal{S}^3_+$ is positive definite. We see that $\hat{\mathcal{F}}$ is the minimal face of $(\hat{\mathrm{P}})$ by letting $x \in \hat{\mathcal{F}}$ such that

$$x_{45} = x_{46} = 0, \quad x_{57} = x_{67} = 1, \quad x_{56} = x_{66} = 0.25, \quad x_{44} = x_{55} = 0.5, \quad x_{28} = 0.25,$$

Next, let (P*) denote the problem obtained by replacing $\mathcal{S}^8_+$ by $\hat{\mathcal{F}}$ in (P). Let (D*) be the corresponding dual problem, which is obtained by replacing $\mathcal{S}^8_+$ by $\hat{\mathcal{F}}^*$ in (D). We have

$$\hat{\mathcal{F}}^* = \left\{ \begin{pmatrix} W & V_1 & V_2 \\ V_1^\top & U & 0 \\ V_2^\top & 0 & 0_2 \end{pmatrix} \in \mathcal{S}^8 \;\middle|\; U \in \mathcal{S}^3_+ \right\}. \tag{39}$$

By Theorem 16, we have $\theta_{\rm D} = \theta_{\rm P^*} = \theta_{\rm D^*} = -1$. And, indeed, the following solution y^* is optimal to (D*) with $\langle b, y^* \rangle = \theta_D = -1$.

$$y^* = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (0, 0, 1, 1, 1, 0, 0, 1). \tag{40}$$

The nonzero entries of the slack matrix $s^* = c - \mathcal{A}^* y^*$ are

$$s^*_{55} = s^*_{66} = 1, s^*_{34} = s^*_{43} = 1.$$

We note that $s^* \in \hat{\mathcal{F}}^*$ but $s^* \notin \mathcal{F}^D_{\min}$ and $s^* \notin \mathcal{S}^8_+$. Thus, y^* is not a feasible solution to ($\hat{\rm D}$) nor (D).

At this stage, (P*) and (D*) are both strongly feasible and they can be solved by the interior point oracle, as in Step 10 of Algorithm 4. Knowing that the common optimal value of (P*) and (D*) is -1, Algorithm 4 then checks whether there is an optimal solution to (D) by solving (D-OPT-STD).

Since we already know that (D) is not attained, we skip this step and go to construction of an almost optimal solution to (D). Let $\hat{y}, (f_1, y^1), y^*$ be as in (35), (37), (40), respectively. We feed $(f_1, y^1), \hat{y}, y^*$ and some $\epsilon > 0$ to Algorithm 2.

Suppose that, say, $\epsilon = 0.1$. Recall that $\langle b, \hat{y} \rangle = -2$. Let

$$\beta = \frac{\theta_{\rm D} - \langle b, \hat{y} \rangle - \epsilon}{\theta_{\rm D} - \langle b, \hat{y} \rangle} = 0.9.$$

The computation done in Algorithm 2 boils down to finding $\alpha > 0$ such that

$$c - \sum_{i=1}^{8} \mathcal{A}_i(\beta y^*_i + (1-\beta)\hat{y} + \alpha y^1) = \beta s^* + (1-\beta)\hat{s} + \alpha f_1 \in \mathrm{ri}\, \mathcal{F}^{\rm D}_{\min}.$$

Since $\langle b, y^1 \rangle = 0$, moving in the direction of y^1 does not change the objective value, and since f_1 is positive semidefinite, it does not violate the cone constraint, no matter how large α is. Taking $\alpha = 10$ is enough for the purpose, so, we see that

$$\tilde{y} = \beta y^*_i + (1-\beta)\hat{y} + 10 y^1$$

is an ϵ-optimal solution to (D). Indeed, $\tilde{s} = c - \mathcal{A}^* \tilde{y}$ is a positive semidefinite matrix whose nonzero entries are

$$\tilde{s}_{11} = \tilde{s}_{22} = 10.1,\ \tilde{s}_{33} = 10.2,\ \tilde{s}_{55} = 0.1,\ \tilde{s}_{66} = 1,\ \tilde{s}_{34} = \tilde{s}_{43} = 1.$$