Robust truss optimization using the sequential parametric convex approximation method

Alfredo Canelas · **Miguel Carrasco** · **Julio López**

Abstract We study the design of robust truss structures under mechanical equilibrium, displacements and stress constraints. Our main objective is to minimize the total amount of material, for the purpose of finding the most economic structure. A robust design is found by considering load perturbations. The nature of the constraints makes the mathematical program nonconvex. In order to solve this problem, we apply the sequential convex approximation method, which deals with the original problem through solving a sequence of differentiable convex ones. Additionally, we show the global convergence of this method using a Slater type hypothesis on the data.

Keywords truss optimization · robust design · stress constraints · sequential parametric convex approximation

1 Introduction

In a recent paper Beck et al (2010) have proposed the Sequential Parametric Convex Approximation method (SPCA), to find the solution of a differentiable nonconvex optimization problem by solving a sequence of differentiable convex ones. Let us consider the following optimization problem:

$$\begin{cases} \min_{x \in \mathbb{R}^n} f(x) \\ \text{s.t.} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m, \end{cases} \quad (P)$$

where g_i, $i = 1, \ldots, p$, are differentiable nonconvex functions, whilst f and g_i, $i = p + 1, \ldots, m$ are differentiable convex ones. Given $x_0 \in \mathbb{R}^n$, a feasible point of (P), for each $k \in \mathbb{N}$

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the SPCA solves iteratively the following differentiable convex optimization problem:

$$
(P_k) \quad
\begin{cases}
\min_{x \in \mathbb{R}^n} f(x) \\
\text{s.t.} \quad G_i(x, \psi_i(x_k)) \leq 0, \quad i = 1, \ldots, p, \\
\qquad\; g_i(x) \leq 0, \qquad\qquad i = p+1, \ldots, m,
\end{cases}
$$

where $G_i \colon \mathbb{R}^n \times Y \to \mathbb{R}$, $i = 1, \ldots, p$, are continuous functions such that $G_i(\cdot, y) \colon \mathbb{R}^n \to \mathbb{R}$ is differentiable convex for each fixed $y \in Y$, and $\psi_i \colon \mathbb{R}^n \to Y$, $i = 1, \ldots, p$, are continuous functions. The next iterate x_{k+1} is defined by the SPCA method as the optimal solution of (P_k). Function G_i is an upper estimate of the nonconvex g_i, and ψ_i is a control function of the parameters which is evaluated at each iteration. We denote by X the feasible set of (P), i.e.

$$
X := \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ i = 1, \ldots, m \}.
$$

Since G_i upper estimates g_i, (Assumption 1 below) we have directly that the sequence $\{x_k\}_{k \in \mathbb{N}}$ generated by the SPCA method belongs to X. Additionally, under appropriate assumptions the global convergence of the sequence generated by the SPCA method to Karush-Kuhn-Tucker (KKT) points of the original problem (P) is proven, see Section 2 for details. In this context, global convergence means monotone convergence of the objective function and KKT conditions satisfied for each accumulation point of the sequence generated by the SPCA method.

In this paper, we first study the global convergence of the SPCA method to KKT points of (P), considering weaker assumptions than the considered by Beck et al (2010). In particular, our global convergence result is obtained under a Slater type condition (Assumption 2 below), instead of the linear independence qualification for the constraints and the strict convexity of the objective function considered in (Beck et al 2010). We note that, the Slater type qualification condition assumed in this paper, depends on the data of the particular problem, but is absolute necessary if an interior point algorithm is used to solve the auxiliary problems (P_k).

The convergence analysis is done by viewing the SPCA method as the repeated application of a closed point-to-set map and by using the classical Zangwill theorem of convergence. Comments about the application of this approach to the non-differentiable case are also given. Then, in order to obtain a more efficient approach, we consider the case that (P_k) is not solved exactly at each iteration. For this purpose, we study the application of feasible and descent algorithms to (P_k).

The second part of this paper is devoted to the study of optimal designs of robust trusses under mechanical displacements and stress constraints. To obtain a robust design we assume that in addition to set of primary external loads, which are applied only at the nodes of the truss, there exists also a set of secondary loads that are uncertain in size and direction, which can be viewed as perturbations of the main loads. The objective is to find the truss that minimizes the total amount of material or weight, i.e. the most *economical* structure, satisfying stress and displacements constraints under the main loads and any possible load perturbation, i.e. we follow the worst-case formulation of the robust design problem.

More precisely, we study the following nonconvex semi-infinite mathematical programming problem

$$
\begin{cases}
\min\limits_{x\in\mathbb{R}^m} \sum\limits_{i=1}^{m} x_i \\
\text{s.t.} \quad \left|u_j(\xi,x)\right| \leq \bar{u}_j \ \ \forall \xi \in E, \ \ j \in J \subseteq N, \\
\qquad \left|\sigma_i(\xi,x)\right| \leq \bar{\sigma}_i \ \ \forall \xi \in E, \ \ i \in I \subseteq M, \\
\qquad \varepsilon \leq x \leq U,
\end{cases}
\qquad (P_w)
$$

where n is the number of nodes of the truss, $N = \{1,\ldots,n\}$, m is the number of potential bars, $M = \{1,\ldots,m\}$, x_i represents the volume of each bar which have the lower bound ε and upper bound U, u is the vector of displacements in mechanical equilibrium, i.e. the displacements satisfying $K(x)u(\xi,x) = f + \xi$, where $K(x)$ corresponds to the *stiffness matrix* of the truss, f is the primary load and ξ the secondary load (see Section 3 for details). The stress of the i-th bar is denoted by σ_i. A value $\varepsilon > 0$ makes sense here since ensures unique bounded displacement and stresses; however the case $\varepsilon = 0$ will be addressed as well. $E \subseteq \mathbb{R}^n$ is the set of all secondary loads, and $\bar{u}_j$ and $\bar{\sigma}_j$ are the upper bounds on the displacements and stresses, respectively. To address the infinite number of constraints we reformulate (P_w) as a mathematical program with a finite number of nonconvex constraints. This reformulation is obtained by assuming that the set E takes the form of a particular ellipsoid (Ben-Tal and Nemirovski 1997). The nonconvex program is then solved numerically by using the SPCA method.

Stress constrained problems like (P_w) above are computationally hard mathematical programming problems whose optimum could not be the best design because of the well-known stress singularity problem. In addition, the reformulation of the stress constraints to convex second order cone ones in the SPCA increase the number of constraints, making the mathematical program even more computationally expensive. The stress singularity problem is another limitation of the present approach. To handle this problem the ε-relaxation approach was proposed by Cheng and Guo (1997), see also (Rozvany 2001; Stolpe and Svanberg 2001, 2003). An alternative approach based on similar ideas was recently proposed by Bruggi (2008). In the design of continuous structures see (París et al 2009; Le et al 2010). Then, the present approach should be applied on top of a good initial design, e.g. obtained by solving a classical non-robust compliance model. Other mechanical constraints like global and local buckling constraints, natural frequency constraints, etc., are beyond the scope of this paper.

The paper layout is as follows. Section 2 provides the convergence analysis of the SPCA method. Section 3 describes the proposed model for the optimal design of robust trusses and shows how to state a SPCA for this model. Section 4 presents some numerical results showing that the proposed formulation is effective to obtain a robust design under the considered set of load perturbations. Finally, Section 5 presents the conclusions.

2 General Algorithm

Given $x \in \mathbb{R}^n$, let (P_x) be the following differentiable convex minimization problem:

$$
\begin{cases}
\min\limits_{z\in\mathbb{R}^n} \ f(z) \\
\text{s.t.} \quad G_i(z,\psi_i(x)) \leq 0, \ \ i = 1,\ldots,p, \\
\qquad g_i(z) \leq 0, \qquad\quad i = p+1,\ldots,m.
\end{cases}
\qquad (P_x)
$$

Then, we can view the SPCA method as the repeated application of the map $A\colon X \rightrightarrows \mathbb{R}^n$ defined as:

$$A(x) := \text{Argmin } (P_x). \tag{1}$$

We assume that the functions G_i and ψ_i satisfy the following hypotheses:

Assumption 1 *Functions G_i and ψ_i, $i = 1, \ldots, p$, satisfy:*

$$g_i(x) \leq G_i(x,y) \qquad\qquad \textit{for every } x \in \mathbb{R}^n,\, y \in Y, \tag{2}$$
$$g_i(x) = G_i(x,\psi_i(x)) \qquad\qquad \textit{for every } x \in \mathbb{R}^n. \tag{3}$$

If (P_x) is solved using a feasible interior point method, it is necessary to assume that the inequalities of (P_x) satisfy the following Slater type constraint qualification:

Assumption 2 *For any feasible point x of (P) there exist a point $\bar{y}$ such that*

$$G_i(\bar{y}, \psi_i(x)) < 0, \qquad\qquad i = 1, \ldots, p, \tag{4}$$
$$g_i(\bar{y}) < 0, \qquad\qquad i = p+1, \ldots, m. \tag{5}$$

A direct consequence of the Slater constraint qualification is that any minimum point z^* of (P_x) is a KKT point, see e.g. (Bertsekas 1999, Proposition 3.3.9). In addition, as the following lemma shows, any minimum point x^* of the original problem (P) is a KKT point:

Lemma 1 *Under assumptions 1 and 2, any local minimum x^* of (P) is a KKT point of (P).*

Proof First note that, by Assumption 1, x^* being feasible for (P) it is also feasible for (P_{x^*}). Since x^* is a local minimum of (P), then it is also a local minimum of (P_{x^*}) and therefore by convexity is a global minimum. Hence, using the Slater's qualification condition, convexity and differentiability of (P_{x^*}) we have that x^* is a KKT point of (P_{x^*}), i.e. calling $y_i = \psi_i(x^*)$, there exist $\lambda_i \geq 0$, $i = 1, \ldots, m$, such that

$$\nabla f(x^*) + \sum_{i=1}^{p} \lambda_i \nabla_x G_i(x^*, y_i) + \sum_{i=p+1}^{m} \lambda_i \nabla g_i(x^*) = 0, \tag{6}$$

$$\lambda_i G_i(x^*, y_i) = 0, \quad i = 1, \ldots, p, \tag{7}$$
$$\lambda_i g_i(x^*) = 0, \quad i = p+1, \ldots, m. \tag{8}$$

We note that, as a direct consequence of Assumption 1 the complementarity condition for (P) holds, additionally, it is easy to see that x^* is a global minimum of the function $x \mapsto G_i(x, y_i) - g_i(x)$, which implies that

$$\nabla_x G_i(x^*, y_i) - \nabla g_i(x^*) = 0. \tag{9}$$

Equations (6)–(9) prove that x^* is a KKT point of (P).

Remark 1 From the proof of the previous lemma we easy see that if x^* is not a KKT point of (P) then x^* is not a KKT of (P_{x^*}). In the case that x^* is a KKT point of (P) then, by Assumption 1, it is also a KKT point of (P_{x^*}) and therefore, by convexity, it is a global minimum of (P_{x^*}).

Remark 2 Equation (9) was considered an independent assumption in the work by (Beck et al 2010, Property A Eq. 2.2). However, in the differentiable case (9) is a consequence of Assumption 1. The existence of G_i satisfying Assumption 1 is then very important in the theoretical analysis of (P). In the non-differentiable case a condition similar to (9), regarding the subdifferential sets of subgradients, must be independently assumed, since it cannot be obtained directly from Assumption 1; and therefore restricting the practical application of the SPCA method to the non-differentiable case, see Section 2.1 below for more details.

Consider the following definitions:

Definition 1 A point-to-set map $A\colon X \rightrightarrows Y$ is said to be closed if given $\{x_k\}_{k\in\mathbb{N}} \subseteq X$ with $x_k \to \bar{x}$, and $z_k \to \bar{z}$, with $z_k \in A(x_k)$, then we get $\bar{z} \in A(\bar{x})$.

Definition 2 $F\colon X \to \mathbb{R}$ is said to be a descent function for a point-to-set map $A\colon X \rightrightarrows X$ and a set $\Gamma \in X$ if, for all $x \in X$, it satisfies: (i) $x \notin \Gamma$ and $z \in A(x)$ then $F(z) < F(x)$; (ii) $x \in \Gamma$ and $z \in A(x)$ then $F(z) \leq F(x)$.

In order to prove the global convergence of the SPCA method, we consider the following theorem:

Theorem 1 (Zangwill) *Let $A\colon X \rightrightarrows X$ be a closed point-to-set map, $\Gamma \subseteq X$ a given solution set and $F\colon X \to \mathbb{R}$ a descent function for A and Γ. Assume that a sequence $\{x_k\}_{k\in\mathbb{N}}$ is generated by A, i.e. $x_{k+1} \in A(x_k)$, and that it is contained in a compact subset of X. Then, every accumulation point of the sequence $\{x_k\}_{k\in\mathbb{N}}$ belongs to Γ.*

Proof See (Luenberger 2003; Bazaraa et al 2006).

We note that in this theorem the terminology *global convergence* refers to the inclusion of the set of accumulation points of the sequence $\{x_k\}_{k\in\mathbb{N}}$ in the solution set. In the following, we assume that Assumptions 1 and 2 hold.

Lemma 2 *Let $x_k \to \bar{x}$, $z_k \to \bar{z}$, with $z_k \in A(x_k)$ and $\hat{z}$ be such that $G_i(\hat{z}, \psi_i(\bar{x})) < 0$ for $i = 1,\ldots,p$, and $g_i(\hat{z}) \leq 0$ for $i = p+1,\ldots,m$. Then $f(\hat{z}) \geq f(\bar{z})$.*

Proof By continuity of G_i and ψ_i we get $G_i(\hat{z}, \psi_i(x_k)) \to G_i(\hat{z}, \psi_i(\bar{x}))$, therefore, there exists $\bar{k}$ such that $G_i(\hat{z}, \Psi(x_k)) \leq 0$ for all $k \geq \bar{k}$, $i = 1,\ldots,p$. Then, $\hat{z}$ is feasible for (P_{x_k}) for all $k \geq \bar{k}$, which implies that $f(\hat{z}) \geq f(z_k)$ for all $k \geq \bar{k}$. Taking the limit as $k \to \infty$, we obtain $f(\hat{z}) \geq f(\bar{z})$.

The main theorem of this paper is given below:

Theorem 2 *Let Γ be the set of KKT points of (P), and A the point-to-set map given by* (1). *Then we obtain:*

(a) f is a descent function for A and Γ.
(b) the map A is closed.

Proof Let x be a feasible point of (P). Then we have: (i) if $x \in \Gamma$, then by definition $f(z) \leq f(x)$ for all $z \in A(x)$; (ii) if $x \notin \Gamma$ then x is not a KKT point of (P) and therefore x is not a KKT point of (P_x), see Remark 1. Then x is not a minimum point of (P_x) so that $f(x) > f(z)$ for all $z \in A(x)$. From (i) and (ii) we have (a).

To prove (b) consider $\{x_k\}_{k\in\mathbb{N}} \subseteq X$ with $x_k \to \bar{x}$, and $z_k \to \bar{z}$, with $z_k \in A(x_k)$. By continuity of functions G_i, ψ_i, for $i = 1,\ldots,p$, and g_i, for $i = p+1,\ldots,m$, we get $\bar{x} \in X$. Take any

feasible point z of $(P_{\bar{x}})$, i.e., a point z satisfying $G_i(z, \psi_i(\bar{x})) \leq 0$ for $i = 1, \ldots, p$, and $g_i(z) \leq 0$ for $i = p+1, \ldots, m$. Take $\bar{y}$ be the point satisfying (4)–(5) in Assumption 2 with $x \equiv \bar{x}$. The sequence defined by $z_j = z + (1/j)(\bar{y} - z)$ is such that $G_i(z_j, \psi_i(\bar{x})) < 0$ for $i = 1, \ldots, p$, $g_i(z_j) < 0$ for $i = p+1, \ldots, m$, and satisfies $z_j \to z$ as $j \to \infty$. Using the result of Lemma 2, we get $f(z_j) \geq f(\bar{z})$ for all j, and conclude that $f(z) \geq f(\bar{z})$ for any feasible z. Therefore $\bar{z} \in A(\bar{x})$.

Remark 3 Under the assumption of compactness of X (the feasible set of (P)) and using Zangwill's Theorem 1 we get that any accumulation point of the sequence $\{x_k\}_{k \in \mathbb{N}}$ generated by SPCA is a KKT point of (P). Furthermore, as a consequence of Theorem 2 and the compactness of X, the monotonically decreasing sequence $\{f(x_k)\}_{k \in \mathbb{N}}$ converges.

2.1 The non-differentiable case

Let us assume in this case that f and g_i, for $i = 1, \ldots, m$, are possibly non-differentiable Lipschitz functions, as well as the upper convex approximation functions G_i, $i = 1, \ldots, p$. Let $\partial f(x)$ denote the Clark subdifferential of f (Clarke 1990). In addition to Assumptions 1 and 2 we have to consider the following assumption regarding the approximation functions G_i:

Assumption 3 *Given any feasible point x for (P), x is a KKT point of (P_x), in the sense that there exist real values $\lambda_i \geq 0, i = 1, \ldots, m$, such that*

$$0 \in \partial f(x) + \sum_{i=1}^{p} \lambda_i \partial_x G_i(x, y_i) + \sum_{i=p+1}^{m} \lambda_i \partial g_i(x), \tag{10}$$

with $y_i = \psi_i(x)$, only if x is a KKT point of the original problem (P), i.e., there exist real values $\lambda_i \geq 0, i = 1, \ldots, m$ satisfying

$$0 \in \partial f(x) + \sum_{i=1}^{m} \lambda_i \partial g_i(x). \tag{11}$$

Note that in the non-differentiable case we cannot prove that x is a KKT point of the original problem (P) whenever x is a KKT point of (P_x), since $\partial_x G_i(x, y) \neq \partial g_i(x)$ in the general case (in fact, from Assumption 1 we can prove that $\partial g_i(x) \subset \partial_x G_i(x, y)$, but the subdifferential $\partial g_i(x)$ could not contain all the subgradients in $\partial_x G_i(x, y)$). Assumption 3 is of main importance for the success of the SPCA method. If a point x were a KKT point of (P_x) but not a KKT point of (P), the SPCA method could get stuck at a point that is not a KKT point of (P). This situation cannot be accepted, see Figure 1.

Assumption 3 allow us to prove the non-differentiable version of Lemma 1:

Lemma 3 *If (P_x) satisfies Assumptions 1, 2 and 3, then any local minimum x^* of (P) is a KKT point of (P).*

Proof By Assumption 1, x^* is feasible for (P_{x^*}). Since x^* is a local minimum of (P), it is also a local minimum of (P_{x^*}). Thanks to Assumption 2, x^* is a KKT point of (P_{x^*}) (Clarke 1990, Theorem 6.4.4), and by Assumption 3, x^* must be KKT point of (P).

Remark 4 Note that Lemma 2 holds in the non-differentiable case as it is, since it does not recall on the differentiability of (P). Theorem 2 also holds in the non-differentible case if (P_x) satisfies the Assumption 3. Then, under Assumptions 1, 2 and 3, the sequence generated by the SPCA method converges globally to a KKT point of (P).
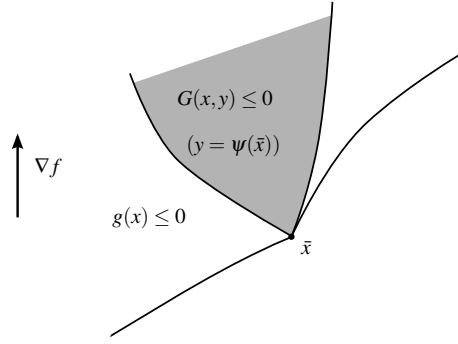
Fig. 1 Example with a single nonconvex non-differentiable constraint, where a point $\bar{x}$ is a KKT point of $(P_{\bar{x}})$ but not a KKT point of (P).

Remark 5 Note that Assumption 3 is the weaker possible condition in order that Lemma 3 hold. The disadvantage of Assumption 3 is that it is a rather difficult condition to be verified in a practical application. A more easy verifiable condition could be the assumption $\partial_x G_i(x, y_i) = \partial g_i(x)$, $i = 1, \ldots, p$, which implies Assumption 3 but has the disadvantage of being more restrictive.

2.2 Using other closed and descent algorithms

In practical applications algorithm (1) cannot be implemented analytically, and an inexact search must be accomplished, i.e., an optimization algorithm must be used to obtain an approximate solution to (P_x). In fact, this procedure strictly falls outside the framework studied up to this section, since Algorithm (1) is actually replaced by other algorithm for which we have to prove independently the descent and closedness properties to ensure global convergence to a KKT point of (P).

Hence, the application of the SPCA to practical problems involves the execution of a nested iteration, where a complete iteration to solve (P_x) is performed at each iteration of the main algorithm. The inner iteration must then be performed very efficiently in order to allow the application of the SPCA. Note that global convergence to a KKT point of the original problem (P) will be obtained provided the approximate solution to (P_x) is found performing one or more iterations of a descent and closed algorithm. We show here for a simple problem that we can efficiently solve the original problem reducing the inner iteration to just one iteration of a simple closed and descent optimization algorithm, and this simple procedure can even overcome Algorithm (1).

Let us consider here the following problem:

$$
\begin{cases}
\min_{x \in \mathbb{R}^2} f(x) = (x_1 - a)^2 + (x_2 - a)^2 \\
\text{s.t.} \quad g(x) = x_1 x_2 \leq 1, \\
\qquad 0.01 \leq x_1, x_2 \leq 100,
\end{cases}
\tag{12}
$$

where a is a real parameter. The same problem with $a = 2$ was considered in (Beck et al 2010). Since the value $a = 2$ makes (12) to be ill-conditioned (the Hessian of the Lagrangian restricted to the tangent plane at the solution is singular), we also considered the value

$a = 1.5$. The SPCA version of (12) is

$$G(x,\lambda) = \frac{\lambda}{2}x_1^2 + \frac{1}{2\lambda}x_2^2, \qquad \psi(x) = \frac{x_2}{x_1}. \tag{13}$$

Table 1 gives the number of main iterations required to obtain the approximate $\bar{x}$ to the exact solution $x^* = (1,1)$, when starting from the point $x_0 = (5,0.02)$, for five different algorithms: SPCA-Exact: SPCA with an exact search; FDIPA: algorithm *FDIPA* applied to the original problem (P), see (Herskovits et al 2005); fmincon: solver *fmincon* of MATLAB applied to the original problem (P); SPCA-FDIPA: SPCA with an inexact search corresponding to one iteration of the *FDIPA* algorithm; SPCA-fmincon: SPCA with an inexact search corresponding to one iteration of the *fmincon* algorithm. The algorithms were stopped if $f(\bar{x}) - f(x^*) \le 1.0 \times 10^{-8}$, or 30 main iterations were performed.

Note that even thought the SPCA-Exact finds the best solution for the auxiliary problem (P_k), it is the slowest version in the overall process to solve (P). FDIPA, fmincon, SPCA-FDIPA and SPCA-fmincon perform much better than SPCA-Exact, being FDIPA and fmincon slightly better than SPCA-FDIPA and SPCA-fmincon. This last fact is probably due to the reinitialization of the Lagrange multipliers.

The results obtained for this example suggest that for a general problem (P) and having the numerical algorithm 'A', the better strategy should be (i) if A can solve (P_x) but not (P) then use SPCA-A with a large tolerance to reduce the number of inner iterations; (ii) if A can directly handle the original problem (P) then use A alone.

Table 1 Performance of the algorithms.

	Iter	$\|\bar{x} - x^*\|$	$f(\bar{x}) - f(x^*)$
a=1.5			
SPCA-Exact	24	1.06×10^{-4}	8.40×10^{-9}
FDIPA	11	2.19×10^{-9}	2.17×10^{-10}
fmincon	9	7.29×10^{-11}	1.11×10^{-16}
SPCA-FDIPA	15	3.85×10^{-8}	4.20×10^{-9}
SPCA-fmincon	7	7.20×10^{-9}	1.88×10^{-9}
a=2			
SPCA-Exact	30	2.60×10^{-1}	1.15×10^{-3}
FDIPA	30	1.95×10^{-2}	3.81×10^{-8}
fmincon	30	6.50×10^{-2}	1.69×10^{-4}
SPCA-FDIPA	30	7.81×10^{-2}	2.94×10^{-5}
SPCA-fmincon	30	1.64×10^{-1}	1.90×10^{-4}

3 Structural optimization model

This section presents an application of the proposed algorithm to the topology design of robust trusses. Trusses are two or three-dimensional mechanical structures that consist of an ensemble of L nodes joint by $m \ge 2$ bars which are made of a linear elastic, isotropic and homogeneous material. Long bars overlapping small ones are not allowed, and therefore $m \le L(L-1)/2$ for a mesh full of bars, see e.g. (Achtziger 1997) and (Bendsøe 1995) for details about truss structures. Trusses are designed to support one or more external *loadings*,

each one consisting of one or more forces applied to the nodes of the structure. A largely employed model for truss topology optimization considers structures submitted to a set of *primary* loadings, and looks for the volume of each of the bars that minimizes a given performance functional, e.g. the total amount of material or the structural compliance, subject to positivity constraints regarding the bar volumes, mechanical constraints, among others, see (Bendsøe 1995). The initial set of nodes and potential bars and supports is called the *ground structure*. If we allow the bars to reach a zero volume, the optimal structure will contain a strict subset of that potential nodes, bars, and supports, so that the topology of the optimal structure is initially unknown.

It is well known that optimal structures obtained by some models are unstable from the mechanical point of view, and great effort have been made in order to obtain formulations having robust optimal trusses, see e.g. (Achtziger et al 1992; Ben-Tal and Nemirovski 1997; Alvarez and Carrasco 2005).

The model considered here, additionally to the primary loadings, includes a set of small *secondary* loadings that are uncertain in size and direction and can eventually act over the structure.

Let us consider a two or a three-dimensional ground structure with L nodes, m initial potential bars, and n degrees of freedom. For the sake of simplicity, let us consider a single primary loading $f \in \mathbb{R}^n$. We propose the robust structural design model defined by the following nonconvex semi-infinite mathematical programming problem:

$$
(P_w) \quad
\begin{cases}
\min\limits_{x \in \mathbb{R}^m} \sum\limits_{i=1}^{m} x_i \\
\text{s.t.} \quad \left| u_j(\xi,x) \right| \le \bar{u}_j \quad \forall \xi \in E, \quad j \in J \subseteq N, \\
\qquad \left| \sigma_i(\xi,x) \right| \le \bar{\sigma}_i \quad \forall \xi \in E, \quad i \in I \subseteq M, \\
\qquad \varepsilon \le x \le U.
\end{cases}
$$

In (P_w), x is the vector of bar volumes, $u_j(x,\xi)$ and $\sigma_i(x,\xi)$ denote, respectively, the displacement corresponding to the j-th degree of freedom and the stress in the i-th bar, when the external force $f + \xi$ is acting over the structure defined geometrically by x. Note that m positivity constraints are defined by the real parameter ε that we initially consider positive (displacements and stresses may be undefined for configurations with zero bar volumes). The vector $\xi \in \mathbb{R}^n$ represents a small perturbation that belongs to the set $E \subseteq \mathbb{R}^n$ of secondary loadings. For given x and ξ, the displacement vector $u(x,\xi)$ is the unique solution to the following mechanical equilibrium equation:

$$
K(x)u = f + \xi,
$$

where $K(x)$ is the stiffness matrix given by

$$
K(x) = \sum_{i=1}^{m} x_i K^i,
$$

with $K^i = b^i(b^i)^\top$ denoting the stiffness matrix corresponding to the i-th bar of unitary volume and b^i is a vector that depends of the coordinates of the bar and its material properties. As usual, to obtain a well posed optimization problem we assume that the sum $\sum_{i=1}^{m} K^i$ is positive definite. The values $\bar{u}_j$ and $\bar{\sigma}_i$ are the bounds for the displacements and the stresses, respectively.

Note that the displacement $u_j(x,\xi)$ and the stress $\sigma_i(x,\xi)$ are given by similar expressions:

$$u_j(x,\xi) = (e^j)^\top u(x,\xi), \quad \sigma_i(x,\xi) = \sqrt{E_i}\,(b^i)^\top u(x,\xi),$$

where e^j is the canonical vector, and E_i corresponds to the *Young modulus* of the i-th bar, see e.g. (Landau and Lifshitz 1986, Chapter 1) and (Popov and Balan 1998, Chapter 2) for details. In order to simplify the notation we denote by $C = \{1,\dots,c\}$ with $c = |I| + |J|$ the total number of displacement and stress constraints. Then, (P_w) can be conveniently expressed as

$$\begin{cases} \min\limits_{x\in\mathbb{R}^m} \sum\limits_{i=1}^m x_i \\ \text{s.t.} \quad |(v^j)^\top u(x,\xi)| \le \bar{v}_j \;\; \forall \xi \in E,\; j \in C, \\ \quad\quad \varepsilon \le x \le U. \end{cases} \tag{P'_w}$$

To address the infinite number of constraints, we reformulate (P'_w) as the following non-convex bilevel mathematical programming problem:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^m} \sum\limits_{i=1}^m x_i \\ \text{s.t.} \quad \max\{(v^j)^\top u(x,\xi) \mid \xi \in E\} \le +\bar{v}_j,\; j \in C, \\ \quad\quad \min\{(v^j)^\top u(x,\xi) \mid \xi \in E\} \ge -\bar{v}_j,\; j \in C, \\ \quad\quad \varepsilon \le x \le U, \end{cases} \tag{P_B}$$

where we have implicitly assumed that the internal problems have optimal solutions, which is true if we consider a compact set of secondary loadings. In the following we will consider the ellipsoid $E = \{Qe \mid \|e\| \le 1\}$, where $Q \in \mathbb{R}^{n\times d}$ is a full-rank matrix and d is the dimension of E. The idea of using an ellipsoid of secondary loadings is taken from the work of Ben-Tal and Nemirovski (1997). However, each element of E in this paper can be viewed as a small perturbation of the main load f, so that this approach is different of the proposed in (Ben-Tal and Nemirovski 1997) where a large ellipsoid containing the convex hull of all the main loadings is considered.

In this case, since the displacement vector is linear with respect to e, the internal problems of (P_B) can be solved analytically. In fact, we have

$$u(x,e) = K(x)^{-1}(f + Qe), \tag{14}$$

and the unique optimal solutions of the internal problems are obtained by solving the KKT conditions:

$$e^j_{max} = \frac{+\nabla_e[(v^j)^\top u(x,e)]}{\|\nabla_e[(v^j)^\top u(x,e)]\|} = \frac{+Q^\top K(x)^{-1} v^j}{\|Q^\top K(x)^{-1} v^j\|}, \tag{15}$$

$$e^j_{min} = \frac{-\nabla_e[(v^j)^\top u(x,e)]}{\|\nabla_e[(v^j)^\top u(x,e)]\|} = \frac{-Q^\top K(x)^{-1} v^j}{\|Q^\top K(x)^{-1} v^j\|}. \tag{16}$$

A direct calculation, using (14) and (15)-(16), give us the inequality constraints of (P_B) in the form:

$$f^\top K(x)^{-1} v^j + \|Q^\top K(x)^{-1} v^j\| \le +\bar{v}_j,\; j \in C,$$
$$f^\top K(x)^{-1} v^j - \|Q^\top K(x)^{-1} v^j\| \ge -\bar{v}_j,\; j \in C,$$

which can be equivalently expressed as

$$\left| f^\top K(x)^{-1} v^j \right| + \| Q^\top K(x)^{-1} v^j \| \leq \bar{v}_j, \; j \in C.$$

Then, (P_B) can be formulated as the following nonconvex finite-dimensional model:

$$\begin{cases} \displaystyle\min_{x \in \mathbb{R}^m} \sum_{i=1}^m x_i \\ \text{s.t.} \quad \left| f^\top K(x)^{-1} v^j \right| + \| Q^\top K(x)^{-1} v^j \| \leq \bar{v}_j, \; j \in C, \\ \quad\quad \varepsilon \leq x \leq U. \end{cases} \tag{P_B'}$$

Finally, it is convenient to introduce the additional scalar variables τ_j^1 and τ_j^2, the vector variable $r^j \in \mathbb{R}^d$, and the set $D = \{1, \ldots, d\}$ to formulate (P_B') as:

$$\begin{cases} \displaystyle\min_{x, \tau_j^1, \tau_j^2, r^j} \sum_{i=1}^m x_i \\ \quad\quad \| r^j \| \leq \tau_j^2, & j \in C, \\ \quad\quad \tau_j^1 + \tau_j^2 \leq \bar{v}_j, & j \in C, \\ \quad\quad \left| f^\top K(x)^{-1} v^j \right| \leq \tau_j^1, & j \in C, \\ \quad\quad \left| (q^\ell)^\top K(x)^{-1} v^j \right| \leq r_\ell^j, & j \in C, \; \ell \in D, \\ \quad\quad \varepsilon \leq x \leq U, \end{cases} \tag{P_B''}$$

where q^ℓ is the ℓ-th column of Q.

We note that the previous formulation considers only one main force f and one ellipsoid of perturbations. However, the model can be easily extended to consider several independent loadings and ellipsoids, with the consequent increase of the problem dimension and number of constraints.

3.1 SPCA approximation of the structural model

Note that (P_B'') have some second-order cone (SOC) constraints, some linear constraints and some nonlinear nonconvex constraints of the form $\left| q^\top K(x)^{-1} v \right| \leq \tau$. We recall that a SOC constraint has the form

$$\| Ax + b \| \leq c^\top x + d,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, see e.g. (Alizadeh and Goldfarb 2003).

The SPCA approximation of (P_B'') is based on the convex upper estimate function $F(x, \lambda, h)$ proposed in (Beck et al 2010) for the function $H(x) = \left| q^\top K(x)^{-1} v \right|$:

$$F(x, \lambda, h) = \frac{\lambda}{2} q^\top K(x)^{-1} q + \frac{1}{2\lambda} v^\top K(x)^{-1} v +$$
$$+ \frac{1}{\lambda} v^\top h + \frac{1}{2\lambda} h^\top K(x) h. \tag{17}$$

Theorem 3 *For given $\lambda > 0$ and $h \in \mathbb{R}^n$ satisfying $q^\top h = 0$, the function $F(\cdot, \lambda, h)$ defined in the open convex set $S = \{ x \in \mathbb{R}^m \mid x > 0 \}$ is a convex upper estimate of H. In addition, at any x such that $H(x) \neq 0$, we have $F(x, \lambda(x), h(x)) = H(x)$, where the functions $\lambda(x)$ and $h(x)$ are given by*

$$\lambda(x) = \left| \theta(x) \right|, \quad h(x) = K(x)^{-1} (\theta(x) q - v),$$
$$\theta(x) = \frac{q^\top K(x)^{-1} v}{q^\top K(x)^{-1} q}.$$

Proof See Beck et al (2010).

Using the above theorem, the SPCA approach presented in Section 2 can be applied to (P''_B) as follows: given $x_0 \in \mathbb{R}^m$ a feasible point of (P''_B), we generate iteratively the sequence $\{x_k\}_{k \in \mathbb{N}}$ by solving the convex minimization problem (P''_{Bk}). This problem is obtained by replacing each one of the constraints of the form $\left|q^\top K(x)^{-1}v\right| \leq \tau$ in (P''_B) by:

$$\frac{\lambda(x_k)}{2}q^\top K(x)^{-1}q + \frac{1}{2\lambda(x_k)}v^\top K(x)^{-1}v +$$

$$+ \frac{1}{\lambda(x_k)}v^\top h(x_k) + \frac{1}{2\lambda(x_k)}h(x_k)^\top K(x)h(x_k) \leq \tau. \tag{18}$$

The next iterate x_{k+1} corresponds to a solution of (P''_{Bk}).

Defining the additional scalar variables α_1 and α_2, the inequality constraint (18) can be expressed as:

$$q^\top K(x)^{-1}q \leq \alpha_1, \quad v^\top K(x)^{-1}v \leq \alpha_2, \tag{19}$$

$$\frac{\lambda(x_k)\alpha_1}{2} + \frac{\alpha_2}{2\lambda(x_k)} + \frac{v^\top h(x_k)}{\lambda(x_k)} +$$

$$+ \frac{h(x_k)^\top K(x)h(x_k)}{2\lambda(x_k)} \leq \tau. \tag{20}$$

Note that (20) is a linear constraint on the variables α_1, α_2 and x, while (19) can be equivalently expressed as a set of linear and SOC constraints, as shown in the following proposition.

Proposition 1 *If $\varepsilon \geq 0$, the constraints $x \geq \varepsilon$ and $q^\top K(x)^{-1}q \leq \tau$ can be equivalently expressed introducing vector variables $s \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^m$ as:*

$$\sum_{i=1}^m s_i b^i = q,$$

$$(s_i)^2 \leq x_i \beta_i \quad \Leftrightarrow \quad \|(2s_i, x_i - \beta_i)\| \leq x_i + \beta_i, \ i \in M,$$

$$\sum_{i=1}^m \beta_i \leq \tau,$$

$$x \geq \varepsilon.$$

Proof See Alizadeh and Goldfarb (2003, Sec. 2) and also (Beck et al 2010; Ben-Tal and Nemirovski 2001).

Finally, applying Theorem 3 and Proposition 1, after some computations, we obtain

$$
(P''_{Bk}) \quad
\begin{cases}
\displaystyle \min_{x,\tau_j^1,\tau_j^2,r^j} \sum_{i=1}^{m} x_i \\
\text{s.t.} \quad \|r^j\| \le \tau_j^2, \ j \in C, \\
\qquad \tau_j^1 + \tau_j^2 \le \bar{v}_j, \ j \in C, \\
\qquad (s_i)^2 \le x_i \beta_i, \ i \in M, \\
\qquad (p_{ij})^2 \le x_i \gamma_{ij}, \ i \in M, \ j \in C, \\
\qquad (z_{i\ell})^2 \le x_i \sigma_{i\ell}, \ i \in M, \ \ell \in D, \\
\qquad \dfrac{\lambda_j(x_k)\alpha_1}{2} + \dfrac{\alpha_2^j}{2\lambda_j(x_k)} + \dfrac{(v^j)^\top h_j(x_k)}{\lambda_j(x_k)} + \\
\qquad \quad + \dfrac{h_j(x_k)^\top K(x)h_j(x_k)}{2\lambda_j(x_k)} \le \tau_j^1, \ j \in C, \\
\qquad \dfrac{\tilde{\lambda}_\ell^j(x_k)\alpha_3^\ell}{2} + \dfrac{\alpha_2^j}{2\tilde{\lambda}_\ell^j(x_k)} + \dfrac{(v^j)^\top \tilde{h}_\ell^j(x_k)}{\tilde{\lambda}_\ell^j(x_k)} + \\
\qquad \quad + \dfrac{\tilde{h}_\ell^j(x_k)^\top K(x)\tilde{h}_\ell^j(x_k)}{2\tilde{\lambda}_\ell^j(x_k)} \le r_\ell^j, \ j \in C, \ \ell \in D, \\
\displaystyle \sum_{i=1}^{m} s_i b^i = f, \\
\displaystyle \sum_{i=1}^{m} p_{ij} b^i = v^j, \ j \in C, \\
\displaystyle \sum_{i=1}^{m} z_{i\ell} b^i = q^l, \ \ell \in D, \\
\displaystyle \sum_{i=1}^{m} \beta_i \le \alpha_1, \\
\displaystyle \sum_{i=1}^{m} \gamma_{ij} \le \alpha_2^j, \ j \in C, \\
\displaystyle \sum_{i=1}^{m} \sigma_{i\ell} \le \alpha_3^\ell, \ \ell \in D, \\
\qquad \varepsilon \le x \le U.
\end{cases}
$$

where:

$$
\lambda_j(x) = |\theta_j(x)|, \quad h_j(x) = K(x)^{-1}(\theta_j(x)f - v^j),
$$
$$
\tilde{\lambda}_\ell^j(x) = |\tilde{\theta}_\ell^j(x)|, \quad \tilde{h}_\ell^j(x) = K(x)^{-1}(\tilde{\theta}_\ell^j(x)q^\ell - v^j),
$$
$$
\theta_j(x) = \frac{f^\top K(x)^{-1} v^j}{f^\top K(x)^{-1} f}, \quad \tilde{\theta}_\ell^j(x) = \frac{(q^\ell)^\top K(x)^{-1} v^j}{(q^\ell)^\top K(x)^{-1} q^\ell}.
$$

4 Numerical Examples

In this section we present some examples to illustrate the solution of formulation (P'_B) considering $Q = 0$, namely *non-robust* formulation, and $Q \ne 0$ that we call *robust* formulation. For the examples considered here we bound the stress of each bar, so we take $\bar{v}_j = \bar{v}$ and $v^j = \sqrt{E}b^j$, for $j = 1, \dots, m$. The ellipsoid Q represents perturbations not greater than 5% of the original force. Additionally, once we obtain the optimized structure we measure the
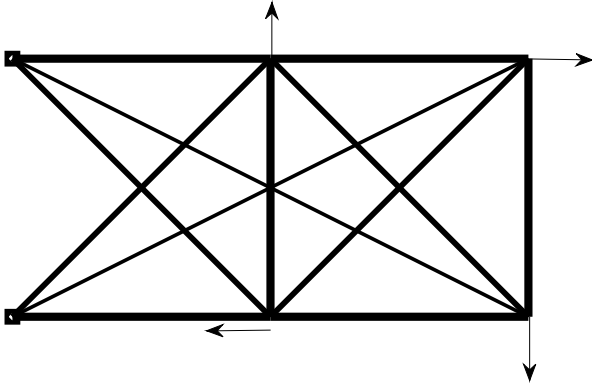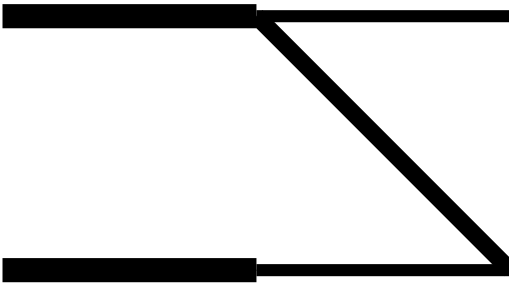
Fig. 2 Example 1: ground structure.

Fig. 3 Example 1: optimized structure, non-robust formulation.

maximal absolute displacement u_{max} among all nodes, and the maximal absolute stress σ_{max}, among all bars, computed by considering the worst perturbation given by (15)–(16). In the pictures presenting the optimized solutions, Figs. 3-4, 6-7 and 9-10, the bars that resulted with a volume lower than 0.01% of the largest bar of the optimized structure are not depicted.

Example 1 We take this example from (Ben-Tal and Nemirovski 1997). In the reference configuration, the left nodes are fixed, while the main loading corresponds to four forces as shown in Fig. 2. We consider $\bar{v} = 45$ and $E = 1$. The optimized structure for the non-robust formulation is depicted in Fig. 3, while Fig. 4 shows the robust solution when the force at the bottom-right node is perturbed by a ball of secondary loads. Table 2 shows for each formulation the total amount of material of the optimized structure, the maximal displacement u_{max}, and the maximal stress σ_{max}. As the table shows, the non-robust solution presents high displacements and stresses when small perturbations on the bottom-right node are applied.

Example 2 The second example corresponds to a cantilever truss structure, similar to Example 5.1 in (Beck et al 2010). The left nodes of the structure are fixed, and a force is applied at the bottom-right node as shown in Fig. 5. We take $\bar{v} = 0.22$, $E = 1$. The optimized truss for the non-Robust formulation is depicted in Fig. 6, while Fig. 7 depicts the solution for the robust formulation when the main load is perturbed considering a ball of secondary
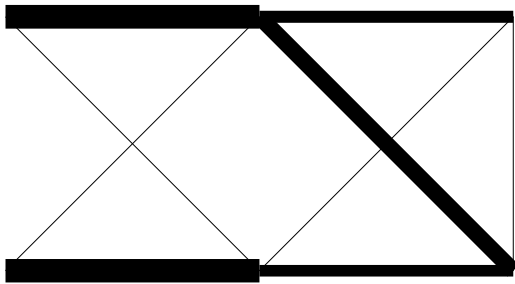
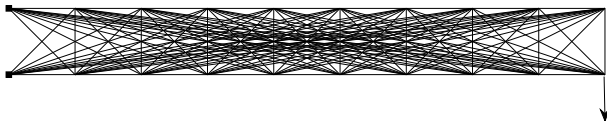**Fig. 4** Example 1: optimized structure, robust formulation.
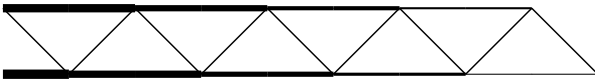

Fig. 5 Example 2: ground structure.

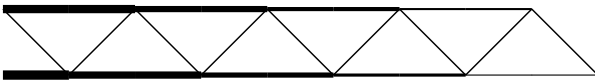**Fig. 6** Example 2: optimized structure, non-robust formulation.

Fig. 7 Example 2: optimized structure, robust formulation.

loads. The results of Table 2 show that, in this particular case, the non-robust solution behaves reasonably well under the effect of the secondary loads, even though it was obtained considering one simple loading case. The robust formulation provides, in this case, a design with the same topology and slightly different volumes of the bars, correcting the slightly infeasible maximal stress of the non-robust solution.

Example 3 The four-story dome of this example was considered by Alvarez and Carrasco (2005), see also (Achtziger 1997). It has a vertical load applied just on the top, see Fig. 8. We take $\bar{v} = 155$, $E = 1$ and perturb the load on the top by considering a three-dimensional ball of secondary loads. The solution obtained for the non-robust formulation is shown in Fig. 9, the robust counterpart is shown in Fig. 10. Table 2 shows that in this example the non-robust solution present high displacements and stresses when submitted to the worst secondary load, that are reduced considerably by using the robust formulation.

4.1 Performance of the solution strategies

In this section we compare the performance of four solution strategies. The first strategy, referenced by "fmincon" in Tables 3 and 4, corresponds to the *fmincon* solver applied to
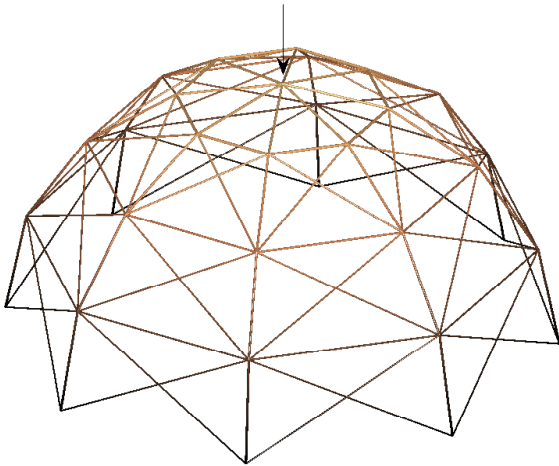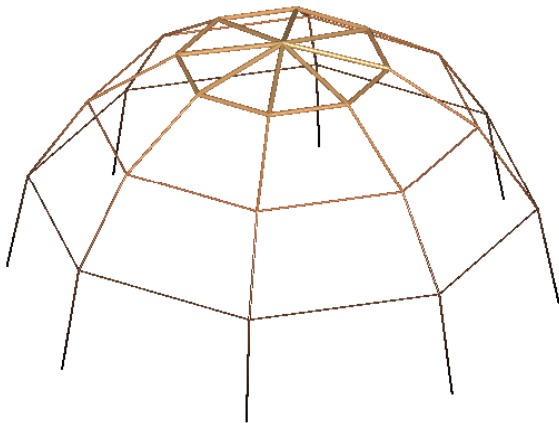
Fig. 8 Example 3: ground structure.

Fig. 9 Example 3: optimal structure Non-robust case.

Table 2 Results

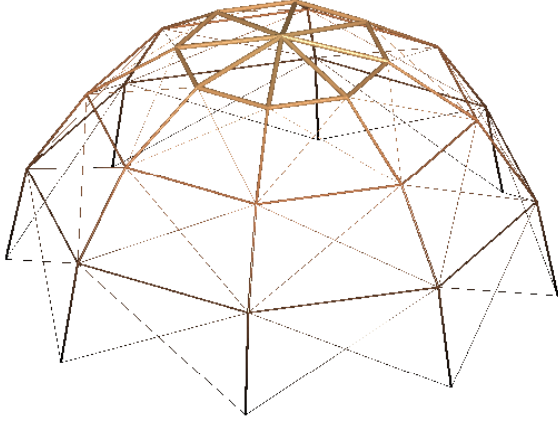| Example	model	w^*	u_{max}	σ_{max}
Example 1	non-robust	0.711	$3.25e7$	$1.75e9$
	robust	0.758	2.387	45
Example 2	non-robust	450	22.87	0.236
	robust	472	21.77	0.22
Example 3	non-robust	0.0077	418	8650
	robust	0.0084	0.51	155

Fig. 10 Example 3: optimal structure Robust case.

Problem (P'_B). The strategy "SPCA-fmincon 1 iter" is to solve (P''_B) using the SPCA where each convex auxiliary problem (P''_{Bk}) is solved by performing just one iteration of *fmincon*. The third strategy "SPCA-fmincon" solves (P''_B) using the SPCA where each convex auxiliary problem (P''_{Bk}) is solved using *fmincon* up to its default tolerance. The last strategy "SPCA-SeDuMi" is the same as the third one, but each convex auxiliary problem is solved using *SeDuMi* solver (Sturm 1999) up to its default tolerance.

To find an initial feasible point we check if $x_i^0 = 1$, $i = 1, \ldots, m$, is feasible. If it is feasible then we set $x_i^0 = 1/j$, with $j \in \mathbb{N}$ the largest integer such that x^0 is feasible. If $x_i^0 = 1$ is infeasible then we set $x_i^0 = j$ with $j \in \mathbb{N}$ the smallest integer such that x^0 is feasible.

Tables 3 and 4 present the results obtained by using the four strategies of solution. The fmincon solver applied to problem (P'_B) without the use of the SPCA strategy presented the best performance. It solved all the problems with higher accuracy and requiring less iterations than the other strategies, even though problem (P'_B) has non-differentiable constraints. The other strategies performed reasonably well for the non-robust and robust formulations, with the exception of SPCA-fmincon 1 iter, which in example 2 of the non-robust formulation and examples 1 and 2 in the robust formulation required a large number of iterations and still did not obtain an accurate solution. The SPCA-fmincon 1 iter approach also could not obtain a reasonable solution for the third example in the case of the robust formulation.

5 Conclusions

In this paper we have shown the global convergence of the SPCA method to solve non-convex problems. The proof presented is based on Zangwill's theorem and requires weaker hypotheses than the proof in Beck et al (2010). The extension of the theorem to the non-differentiable case was also discussed. By considering a simple example we have shown that the approximated solution of the auxiliary problem must be preferred over the exact solution. Indeed, we have shown that just one iteration of a descent algorithm can outperform the exact solution strategy by reducing the number of iterations required to solve the convex auxiliary problems and even by reducing the number of main iterations, i.e. the number of auxiliary problems considered to obtain the final solution.

Table 3 Performance for the non-robust formulation

Example/Algorithm	iter	w^*
Example 1		
fmincon	5	0.7111
SPCA-fmincon 1 iter	65	0.7112
SPCA-fmincon	114	0.7191
SPCA-SeDuMi	40	0.7193
Example 2		
fmincon	34	450
SPCA-fmincon 1 iter	11220	454.23
SPCA-fmincon	316	450.58
SPCA-SeDuMi	21	450.57
Example 3		
fmincon	11	0.0077
SPCA-fmincon 1 iter	19	0.0077
SPCA-fmincon	21	0.0078
SPCA-SeDuMi	23	0.0078

Table 4 Performance for the robust formulation

Example/Algorithm	iter	w^*
Example 1		
fmincon	11	0.7582
SPCA-fmincon 1 iter	14580	0.769
SPCA-fmincon	46	0.762
SPCA-SeDuMi	41	0.767
Example 2		
fmincon	43	472.85
SPCA-fmincon 1 iter	18468	475.39
SPCA-fmincon	26	475.51
SPCA-SeDuMi	27	473.09
Example 3		
fmincon	44	0.0084
SPCA-fmincon 1 iter	–	–
SPCA-fmincon	35	0.0085
SPCA-SeDuMi	39	0.0085

In Section 3 we have presented a semi-infinite nonconvex optimization model to design robust trusses. This model leads to a nonconvex mathematical problem, which was applied to find the optimal truss by using the *fmincon* solver of MATLAB. Numerical examples are presented illustrating that the structures obtained by solving the robust model are always mechanically stable.

In addition we presented a SPCA reformulation of the nonconvex mathematical problem. The convex auxiliary problems are in the form of convex second-order cone constrained problems which can be solved by using efficient interior-point algorithms like *SeDuMi*. However, the *fmincon* solver of MATLAB used without the SPCA presented a better performance. In addition, for this structural optimization problem we could not obtain a reduction of main iterations by solving the convex auxiliary problems of the SPCA with less accuracy as was observed in Section 2.2. Then, in spite of the strong theoretical properties of the SPCA regarding the global convergence to optimal solutions, further studies are required to

obtain a practical tool based in the SPCA for solving the structural optimization problem here considered.

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