On the unimodality of METRIC Approximation subject to normally distributed demands

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July 28th, 2015

Abstract

METRIC Approximation is a popular model for supply chain management. We prove that it has a unimodal objective function when the demands of the n retailers are normally distributed. That allows us to solve it with a convergent sequence. This optimization method leads us to a closed-form equation of computational complexity $\Theta(n)$. Its solutions are at most 0.001% above the optimum for all our instances. Our proof relies on a generic analytical rule that we introduce to prove unimodality, so quasi-concavity or quasi-convexity, of univariate functions.

1 Introduction

In literature devoted to optimization of distribution systems, the most studied configuration consists in several retailers which replenish from a common warehouse. In general, the latter receives goods from suppliers whose joint capacity is infinite. Regarding the retailers, they face stochastic demands, thus making the whole problem difficult. The simplest model for this system sticks to fixed transportation times, negligible orderings costs (no fixed costs) and both warehouse and retailers order 1 unit as soon as they are withdrawn from 1 unit (that is called a $(S-1, S)$ policy). The goal is generally to minimize the sum of the average holding and back-order costs, which are linear in the average on-hand and shorted inventories.

Authors often work on more complicated but more realistic variants. Sometimes, they focus on centralized policy control whereas the simplest problem manages independent decisions ([19] compares both). Certain authors tackle stochastic transportation times [35, 14], direct retailing by the warehouse [9] or ordering costs [8]. Most of them deal with (R, Q) policies (batch replenishments). Nevertheless, the basic system described above and the most famous method to address it are still widely used [29, 7]. The latter has been introduced in 1968 by Craig Sherbrooke [28] and, nowadays, it is usual to call it *METRIC Approximation*. In the eighteens, Vari-METRIC improved it by using a more accurate approximation of the retailer lead times [20]. But, for a start, we chose to focus on METRIC Approximation, popular, simple and not so bad [29].

1.1 Usual simplifications

Before exploring our simplified approach, we present an overview on usual ways of making supply chain problems tractable. A first idea is to explore the solution space with heuristics, as done by [19] and [24] among others. A different point of view consists in interpolation of precomputed tables, like [8] for a single-echelon problem and [11] for a two-echelon problem. On the opposite, when an exact solution is required, tight bounds on the solution have been stated to shrink the search space, for instance in [4]. Moreover, when looking for exact solutions, problem decompositions are a successful way of tractability. In [25], decomposition methods for serial systems as well as for distribution ones are stated. Thus, sub-problems with a fewer number of variables are obtained. For instance, [32, 33] decom-

pose a serial system into successive single-echelon problems, which results in a set of incomplete convolutions to be minimized recursively. Sometimes, these expensive recursions can be avoided, as done by the famous model of Clark and Scarf [13]. For decomposing two-echelon distribution systems, [2], [6] and [11] assign a virtual back-order cost to the warehouse, whose purpose is to account the impact of its decision to the rest of the system. By means of it, they consider the warehouse as a single-echelon problem, thus computing easily its optimal base-stock level. Consequently, the distribution of the retailer lead times is known and their inventories are optimized as single-echelon systems too. Another approach to simplify the problems consists in the use of normally distributed demands, which is very common [2, 5, 6, 11, 8, 19, 16, 14]. That allows for analytical simplifications and is faster to compute. Moreover, it allows for gradient-based searches [16]. At last, for some problems, other distributions can bring simplifications. Indeed, [32, 33] approximate the demands of a serial system by Erlang distributions, because they are trivial to convolute. One can cite also [14] who describes the stochastic warehouse lead time by an exponential distribution, although he still approximates the overall durations by normal distributions.

1.2 Overview of our approach

As shown in classical literature, when a model describes stochastic data by a Poisson distribution whose parameter λ tends to $+\infty$, the probabilities can be approximated by a normal distribution. Then, one translates the mean by $\mu = \lambda$ and the standard deviation by $\sigma = \sqrt{\lambda}$. In practice, $\lambda > 6$ is largely noticed to give accurate probabilities. Concretely, multiplying average demands by lead times has to give high values. For instance, our model is suitable for small parts (such as cereals, electronic components, screws). Furthermore, normal distributions match reality when continuous amounts of fluids are handled (such as water, sand, gas, fuel).

The replenishment policy of METRIC Approximation is $(S-1, S)$, which corresponds to a (R, Q) policy with $R = S - 1$ and $Q = 1$, meaning that both warehouse and retailers order 1 unit as soon as they are withdrawn from 1 unit, thus the inventory positions are constant. When adapting this model to continuous optimization, quantities are in $\mathbf{R}$. For any inventory of size S, an infinitesimal withdraw $\mathrm{d}S$ must trigger a replenishment in order to keep the position at S. As we will see, our solving method is equivalent to that of [2] and [6], but they do not state any univariate objective function for the model that we address. They build their method by a different approach. Convergence of the algorithm of [2] is proven only when back-order costs are greater than holding costs and they do not ensure that the first solution found is the global optimum, whereas our algorithm always reaches the global optimum independently of the parameters. In [6], a fast method is also stated but it is less accurate and optimum-uniqueness is not proven for any model. Other authors are not far from our approach. For instance, [19] rely on a more accurate model and approximate it by a formula close to ours. Nevertheless, their function is not proven to hold one optimum only. Consequently, they optimize it through a generic search algorithm instead of a fast devoted method. Indeed, we will show that our cost formula is quasi-convex and unimodal, and fast solving methods are suitable for such objective functions [27]. On this point, we noted that Sven Axsäter proposed a search method in $\mathrm{O}(|\mathrm{R}| \log S_0^* \log \hat{S}_r)$ for a model assumed to be convex, where $|\mathrm{R}|$ is the number of retailers, S_0^* is the optimal warehouse base-stock level and $\hat{S}_r$ is the greatest retailer base-stock level [5]. In contrast, our solving method converges so quickly that we suggest a closed-form solution in $\Theta(|\mathrm{R}|)$.

Companies may face very large problems, that is why they are in their best interest to have a fast solving method. Our approximation is accurate for large demands and is fast due to the suggested closed-form solution. In section 2, we translate METRIC Approximation into a model dealing with normally distributed demands and non-identical retailers, under the assumption that the average demand of any retailer is equal to the variance, like in the original model. Section 3 presents a solution approach for our model. Section 4 compares our closed-form solution to that

of [6] and exact solutions. At last, section 5 concludes and compares our approach to that assigning virtual back-order costs to the warehouse.

2 The model

Our model is built by considering METRIC Approximation [28] and replacing Poisson distributions of the retailer demands with normal distributions. We denote by 0 the warehouse and by $R = \{1 \ldots |R|\}$ the set of retailers. The base-stock levels S_0 and S_r $\forall r \in R$ are the decision variables. The warehouse is subject to a holding cost h_0, while a holding cost h_r and a back-order cost b_r are assigned to every retailer r. The average cost induced by stocked goods in the warehouse is $C_0(S_0)$. For each retailer r, the average cost due to back-orders and stocked goods is $C_r(S_r)$.

As shown in classical literature, any Poisson distribution of parameter λ can be approximated by the normal distribution when λ tends to $+\infty$ ($\lambda > 6$ allows already for a good approximation), by taking $\mu = \lambda$ and $\sigma = \sqrt{\lambda}$ as parameters. So, to consider that METRIC Approximation is under normally distributed demands, our model assumes that each retailer r faces a normal demand whose mean and variance are $\mu_r = \sigma_r^2 = \lambda_r$.

The problem is to minimize the average overall cost

$$C(S_0, \ldots, S_{|R|}) = C_0(S_0) + \sum_{r \in R} C_r(S_r). \quad (1)$$

The METRIC cost formulas are used without change:

$$C_0(S_0) = h_0 I_0(S_0) \quad (2)$$

and

$$C_r(S_r) = h_r I_r(S_r) + b_r B_r(S_r). \quad (3)$$

The average inventory levels $I_r(S_r)$ and the average back-orders $B_r(S_r)$ are translated into their respective continuous versions in a common way:

$$I_i(S_i) = \int_{-\infty}^{S_i} (S_i - x) f_i(x) \, dx, \quad (4)$$

$$B_i(S_i) = \int_{S_i}^{+\infty} (x - S_i) f_i(x) \, dx \quad (5)$$

with $f_i(x)$ the normal distribution of the demand (at the warehouse for $i = 0$, at a retailer for $i = r \in R$). For efficiency, formulas (36) and (41) in appendix B can be used instead of those integrals.

The difficulty of the problem comes from the computation of $C_r(S_r)$. According to equation (3), this cost depends on the expectations of the inventory level and the back-orders. In turn, they depend on the delay $\overline{L}_r$ between a replenishment order and its effective delivery. Indeed, the longer it is, the more back-orders occurs. In METRIC Approximation, this delay is taken into account through its influence on the parameters of the distribution f_r and is approximated by the transportation duration L_r and adding the average waiting time $\frac{1}{\lambda_0} B_0(S_0)$ due to stock-outs at the warehouse [28]. The latter expression comes from the well known Little's formula [26], which states the average waiting time in a queue with respect to the event rate λ_0 and the average number of events $B_0(S_0)$ (a simple proof is given in [30]). The average demand $\overline{\lambda}_r$ which occurs during the approximated delay $\overline{L}_r = L_r + \frac{1}{\lambda_0} B_0(S_0)$ being proportional to $\overline{L}_r$ itself, the model considers that

$$\overline{\lambda}_r = \lambda_r \left(L_r + \frac{1}{\lambda_0} B_0(S_0) \right).$$

We view our demand distribution as a normal approximation of the Poisson distribution. Consequently, the mean and the variance of the retailer demands look similar:

$$\overline{\mu_r} = \mu_r L_r + \frac{\mu_r}{\mu_0} B_0(S_0) \quad (6)$$

$$\overline{\sigma_r}^2 = \sigma_r^2 L_r + \frac{\sigma_r^2}{\mu_0} B_0(S_0). \quad (7)$$

Those parameters define the distributions used in the expressions of $I_r(S_r)$ and $B_r(S_r)$, thus making them intricate. We keep distinguishing μ_r from σ_r^2 in order to allow further work to extend our equations to the case $\sigma_r^2 \neq \mu_r$.

Similarly, we inherit from the additive property of the Poisson distribution in order to compute the distribution parameters of the total demand viewed by the warehouse:

$$\mu_0 = \sum_{r \in R} \mu_r, \quad (8)$$

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$$\sigma_0^2 = \sum_{r \in \mathrm{R}} \sigma_r^2 \,. \tag{9}$$

Like in the original model, they are extended over the transportation time L_0 between the warehouse and its supplier by

$$\overline{\mu_0} = L_0 \, \mu_0 \,, \tag{10}$$

$$\overline{\sigma_0}^2 = L_0 \, \sigma_0^2 \,. \tag{11}$$

Those parameters define the distribution used in the expressions of $I_0(S_0)$ and $B_0(S_0)$.

Because METRIC is a decentralized model, each retailer r is a single echelon system. As shown by lemma 15 in appendix B, the optimal base-stock level of such a system with normally distributed demands is known to be $S_r^* = F_r^{-1}\left(\frac{b_r}{h_r + b_r}\right)$ where F_r^{-1} is the inverse cumulative distribution function of the retailer. Therefore, the objective function (1) of our problem can be written with respect to one variable only. Our model will be solved by

$$\min_{S_0 \in \mathbf{R}_+} \left\{ C_0(S_0) + \sum_{r \in \mathrm{R}} C_r\left(F_r^{-1}\left(\tfrac{b_r}{h_r + b_r}\right)\right) \right\} \,. \tag{12}$$

As it is, the equation is still complicated, since we saw that the parameters of C_r and F_r^{-1} depends on S_0. The next section shows how to tackle it in a simple way and without loss of optimality.

3 Analysis and solution approach

Our problem, summarized by equation (12), can be written as a simpler function of one variable (namely S_0). In theorem 1, we state it. We show in theorem 2 that it is strictly unimodal, meaning that it holds one global optimum and no other stationary point. This property allows theorem 3 to state a convergent sequence which always reaches the optimal solution. We use it as a search algorithm and it turns out to converge so fast that we suggest a closed form approximation based on the first two iterations. The reader will find in appendix proofs of the lemmas and methods that we use here. We advice non-specialists to read the appendix first. There, tools are introduced in an pedagogical and linear structure, without cross-references to the article.

The following theorem brings our model an objective function of one variable, thus optimization becomes easier.

Theorem 1. For a given warehouse base-stock level S_0, the optimal overall cost of the complete model can be computed independently of the optimal retailer base-stock levels. It is

$$C^*(S_0) =$$
$$h_0 \, I_0(S_0) + \sum_{r \in \mathrm{R}} (h_r + b_r) \gamma_r \sqrt{L_r + \frac{1}{\mu_0} B_0(S_0)}, \tag{13}$$

with

$$\gamma_r = \frac{\sigma_r}{\sqrt{2\pi}} \, e^{-s_r^2} \tag{14}$$

and

$$s_r = \mathrm{erf}^{-1}\left(\frac{b_r - h_r}{b_r + h_r}\right) \tag{15}$$

two constants related to every retailer r. Some readers may prefer to consider $\gamma_r = \sigma_r \, \phi\big(\Phi^{-1}(\alpha_r)\big)$, where ϕ is the standard normal distribution, Φ^{-1} is the inverse of its cumulative distribution, and $\alpha_r = \frac{b_r}{b_r + h_r}$.

Proof. According to equation (1), for a given S_0, the optimal overall cost is

$$C(S_0, S_1^*, \dots, S_{|\mathrm{R}|}^*) = h_0 \, I_0(S_0) + \sum_{r \in \mathrm{R}} C_r(S_r^*),$$

where $S_1^* \dots S_{|\mathrm{R}|}^*$ are the optimal base-stock levels of the retailers. Lemma 16 in appendix B states that

$$C_r(S_r^*) = (h_r + b_r) \frac{\overline{\sigma_r}}{\sqrt{2\pi}} \, e^{-s_r^2}$$

where $\overline{\sigma_r}$ is replaced by $\sigma_r \sqrt{L_r + \frac{1}{\mu_0} B_0(S_0)}$ according to equation (7). $\square$

Due to the following theorem, any algorithm finding a stationary point on function (13) is ensured to have reached the global optimum. By a straightforward extension of the traditional notion of strict unimodality (defining functions having one maximum and no other stationary point), we say that this theorem state that our univariate objective function is strictly unimodal.

Theorem 2. Function (13) has one minimum and no other stationary point.

Proof. We follow the flowcharts of the rule presented in appendix D, but we will test positivity of functions instead of negativity. Indeed, we are proving uniqueness of the minimum.

According to step 2 of the main procedure, we derive equation (13):

$$h_0 \, F_0(S_0)$$
$$+ \frac{F_0(S_0) - 1}{2\mu_0} \sum_{r \in \mathrm{R}} \gamma_r \frac{h_r + b_r}{\sqrt{L_r + \frac{1}{\mu_0} B_0(S_0)}} \,, \quad (16)$$

where $F_0(S_0)$ and $F_0(S_0) - 1$ are the respective derivatives of $I_0(S_0)$ and $B_0(S_0)$ according to lemmas 6 and 11 in appendix B. For conciseness, let us denote

$$\Sigma_1(S_0) = \sum_{r \in \mathrm{R}} \gamma_r \frac{h_r + b_r}{\sqrt{L_r + \frac{1}{\mu_0} B_0(S_0)}}$$

and

$$\Sigma_3(S_0) = \sum_{r \in \mathrm{R}} \gamma_r \frac{h_r + b_r}{\sqrt{L_r + \frac{1}{\mu_0} B_0(S_0)}^3} \,.$$

As allowed by step 2, equation (16) is divided by $h_0 \left(1 - F_0(S_0) \right)$, thus obtaining

$$\frac{F_0(S_0)}{1 - F_0(S_0)} - \frac{1}{2\mu_0 h_0} \Sigma_1(S_0) \,. \quad (17)$$

At step 3, the latter is derived:

$$\frac{f_0(S_0)}{\left(1 - F_0(S_0) \right)^2} - \frac{1 - F_0(S_0)}{4\mu_0{}^2 h_0} \Sigma_3(S_0) \,. \quad (18)$$

According to step 5 of the sub-procedure, we extract $\frac{1}{2\mu_0 h_0}$ from equation (17) :

$$\frac{1}{2\mu_0 h_0} = \frac{F_0(S_0)}{1 - F_0(S_0)} \frac{1}{\Sigma_1(S_0)}$$

and we substitute it in equation (18) :

$$\frac{f_0(S_0)}{\left(1 - F_0(S_0) \right)^2} - \frac{F_0(S_0)}{2\mu_0} \frac{\Sigma_3(S_0)}{\Sigma_1(S_0)} \,. \quad (19)$$

At step 2, we perform the first three transformations allowed. Regarding the dominance transformation, a lower function is considered because we are proving positivity. We suggest to replace

$\frac{\Sigma_3(S_0)}{\Sigma_1(S_0)}$ with a greater function. Let L be the parameter L_r of the retailer r having the greatest quotient

$$\gamma_r \frac{h_r + b_r}{\sqrt{L_r + \frac{1}{\mu_0} B_0(S_0)}^3} \left(\gamma_r \frac{h_r + b_r}{\sqrt{L_r + \frac{1}{\mu_0} B_0(S_0)}} \right)^{-1} \,.$$

The latter is simply equal to $\frac{1}{L_r + \frac{1}{\mu_0} B_0(S_0)}$ and, due to theorem 17 in appendix C, we have

$$\frac{1}{L + \frac{1}{\mu_0} B_0(S_0)} \geq \frac{\Sigma_3(S_0)}{\Sigma_1(S_0)} \,.$$

Furthermore, $\frac{1}{L + \frac{1}{\mu_0} B_0(S_0)}$ is dominated by $\frac{\mu_0}{B_0(S_0)}$ due to non-negativity of parameters L_r. Thus, $\frac{\Sigma_3(S_0)}{\Sigma_1(S_0)}$ is replaced with the latter in equation (19). As allowed by step 2, we multiply the result by $2 \frac{B_0(S_0)}{\overline{\sigma_0} f_0(S_0)}$. That leads us to

$$\frac{2B_0(S_0)}{\overline{\sigma_0} \left(1 - F_0(S_0) \right)^2} - \frac{F_0(S_0)}{\overline{\sigma_0} f_0(S_0)} \,.$$

At last, we compose that equation with $\overline{\mu_0} + \overline{\sigma_0} z$, as allowed by step 2. By denoting the standard normal distribution by $\phi(z)$ and its cumulative distribution by $\Phi(z)$, and from definition of B_0 (equation (42) in appendix B), we obtain

$$u(z) = 2 \frac{\phi(z) - z \left(1 - \Phi(z) \right)}{\left(1 - \Phi(z) \right)^2} - \frac{\Phi(z)}{\phi(z)} \,. \quad (20)$$

At this point, it is noticed that the whole problem data disappeared. The main principle of our rule to prove unimodality, namely simplifying the function to study, succeeds.

Step 3 must be passed by proving that $u(z)$ is positive everywhere. For a start, we show that $u(z)$ tends to $+\infty$ as x approaches $\pm\infty$. When $x \to -\infty$, the left fraction of $u(z)$ is trivially equivalent to $-z$ and the right fraction is showed to be equivalent to $-1/z$ by means of the well known l'Hôpital's rule (take $\phi'(z) = -z\phi(z)$). Hence

$$\lim_{z \to -\infty} u(z) = \lim_{z \to -\infty} 2|z| + \frac{1}{z} = +\infty \,.$$

The shortest way to obtain the limit for $z \to +\infty$ consists in 2 steps. First, the limit of the quotient

of the fractions is computed. Note that multiplying the first fraction by $\phi(z)/\Phi(z)$ can be done by replacing $\Phi(z)$ with its limit, namely 1. Thus, we look for the limit of

$$\frac{\phi(z)}{1-\Phi(z)} \cdot \frac{\phi(z) - z\big(1-\Phi(z)\big)}{1-\Phi(z)} \,.$$

In that equation, we apply once l'Hôpital's rule to the left fraction and independently twice to the right one. $z \cdot \frac{1}{z} = 1$ is obtained, meaning that both fractions of $u(z)$ are equivalent. The first one is then replaced by $\Phi(z)/\phi(z)$, hence

$$\lim_{z\to+\infty} u(z) \;=\; \lim_{z\to+\infty} 2\frac{1}{\phi(z)} - \frac{1}{\phi(z)} \;=\; +\infty \,.$$

Then, lemma 21 in appendix D tells us that $u(z)$ has a minimum. Due to the limits, it is sufficient to check positivity of the minimums.

The derivative of $u(z)$ is

$$4\phi(z)\frac{\phi(z) - z\big(1-\Phi(z)\big)}{\big(1-\Phi(z)\big)^3} - \frac{2}{1-\Phi(z)} - \frac{\Phi(z)}{\phi(z)}z - 1$$

from which we state

$$2\frac{\phi(z) - z\big(1-\Phi(z)\big)}{\big(1-\Phi(z)\big)^2} = \frac{\big(1-\Phi(z)\big)\big(1+\frac{\Phi(z)}{\phi(z)}z\big) + 2}{2\phi(z)}$$

at stationary points. Consequently, we substitute the first fraction of $u(z)$ through that equality in order to obtain a function having the same value as $u(z)$ for all stationary points. Since we are not interested in the values but only in the signs, we simplify the function by multiplying it by $\phi^2(z)$, thus obtaining

$$\frac{1}{2}\big(1-\Phi(z)\big)\big(\phi(z) + z\Phi(z)\big) + \phi(z)\big(1-\Phi(z)\big) \,.$$

$\phi(z) + z\Phi(z)$ is identified with $\int_{-\infty}^{z} \Phi(x)\,\mathrm{d}x$. Therefore, every term is positive. $\qquad\square$

The following sequence is an efficient way of solving the problem. It starts at any value $x_0 > 1$ and is iterated until it reaches its fixed point x_∞. The optimal solution S_0^* to the objective function (13) is then deduced from x_∞.

Theorem 3. The sequence

$$x_{k+1} = 1 + \frac{1}{2\mu_0 h_0}\sum_{r\in\mathrm{R}}\gamma_r\frac{h_r + b_r}{\sqrt{L_r + \frac{1}{\mu_0}I_0\big(F_0^{\text{-}1}(x_k^{\,-1})\big)}} \tag{21}$$

converges to a unique fixed point x_∞ whose basin of attraction is $]1\,;+\infty[$. The minimum of function (13) is at

$$S_0^* \;=\; 2\,\overline{\mu_0} - F_0^{\text{-}1}\left(\frac{1}{x_\infty}\right). \tag{22}$$

$F_0^{\text{-}1}$ is the inverse of the cumulative normal distribution whose parameters are given by equations (10) and (11).

Proof. Theorem 2 implies that there exists a unique point S_0^* where equation (16) is zero. At this point, we can divide the equation by any non-zero function, the result will remain 0. Let us divide by $h_0\big(F_0(S_0) - 1\big)$. Then, for $S_0 = S_0^*$,

$$\frac{F_0(S_0)}{1-F_0(S_0)} \;=\; \frac{1}{2\mu_0 h_0}\sum_{r\in\mathrm{R}}\gamma_r\frac{h_r + b_r}{\sqrt{L_r + \frac{1}{\mu_0}B_0(S_0)}} \,. \tag{23}$$

Let us denote by $Q(S_0)$ the left part and by $\Sigma(S_0)$ the right part. For clarity, we might also denote $F = F_0(S_0)$ and $Q = Q(S_0)$. The equation $Q = \frac{F}{1-F}$ is trivial to rewrite $F = 1 - \frac{1}{1+Q}$ so $S_0 = Q^{-1}(Q) = F_0^{\text{-}1}\left(1 - \frac{1}{1+Q}\right)$ that we write, according to lemma 2 in appendix A,

$$S_0 \;=\; 2\,\overline{\mu_0} - F_0^{\text{-}1}\left(\frac{1}{1+Q}\right). \tag{24}$$

Because equation (13) has a minimum at S_0^* and no other stationary point, equation (16) is negative for any $S_0 < S_0^*$, which implies $Q(S_0) < \Sigma(S_0)\ \forall S_0 < S_0^*$. Similarly, $Q(S_0) > \Sigma(S_0)\ \forall S_0 > S_0^*$. The right-hand side of equation (24) can be easily shown to be monotonically increasing with respect to Q. Hence $S_0 < Q^{\text{-}1}\big(\Sigma(S_0)\big)\ \forall S_0 < S_0^*$ and $S_0 > Q^{\text{-}1}\big(\Sigma(S_0)\big)\ \forall S_0 > S_0^*$. Therefore, the sequence

$$S_0^{k+1} = Q^{\text{-}1}\big(\Sigma(S_0^k)\big)$$

converges to S_0^*.

Now we show that equations (21) and (22) are a shortened writing of this sequence. Let us apply

function Σ to both sides of the sequence. We obtain $\Sigma(S_0^{k+1}) = \Sigma(Q^{-1}(\Sigma(S_0^k)))$ and, by denoting $x_k - 1 = \Sigma(S_0^k)$, it becomes

$$x_{k+1} - 1 = \Sigma\left(Q^{-1}(x_k - 1)\right). \qquad (25)$$

In the expression of the right-hand side,

$$B_0\left(2\overline{\mu_0} - F_0^{-1}\left(\tfrac{1}{1+(x_k-1)}\right)\right)$$

is replaced with

$$I_0\left(F_0^{-1}\left(\tfrac{1}{x_k}\right)\right)$$

according to lemma 10 in appendix B. Note that the convergence is kept because $\Sigma(S_0)$ is monotonic due to the monotonicity of $B_0(S_0)$ stated by lemma 11. $\qquad \square$

We advice to compute $I_0(S_0)$ by means of the efficient formula (36) given in appendix B. Once the optimum S_0^* is found, each optimal retailer base-stock level can be obtained in O(1) by means of $S_r^* = F_r^{-1}(\alpha_r)$, the latter being defined by the parameters stated in equations (6) and (7). Thus the problem is entirely solved. Experiments will show that the sequence above converges quickly, so we suggest to view the first two iterations as a closed form equation approaching the optimal warehouse base-stock level. Starting from $x_0 = +\infty$, we decompose the iterations in 4 steps:

$$x_1 = 1 + \frac{1}{2\mu_0 h_0} \sum_{r \in R} \gamma_r \frac{h_r + b_r}{\sqrt{L_r}}$$

$$\Delta = \frac{1}{\mu_0} I_0\left(F_0^{-1}\left(\tfrac{1}{x_1}\right)\right)$$

$$x_2 = 1 + \frac{1}{2\mu_0 h_0} \sum_{r \in R} \gamma_r \frac{h_r + b_r}{\sqrt{L_r + \Delta}}$$

$$S_0^* \approx 2\overline{\mu_0} - F_0^{-1}\left(\tfrac{1}{x_2}\right).$$

For x_1, L_r is alone in the square root because $\lim_{\alpha \to 0} I_0\left(F_0^{-1}(\alpha)\right) = \lim_{S_0 \to -\infty} I_0(S_0)$ and the latter is 0 according to equation (36) in appendix B.

4 Experiments

We restrict the test to 5 identical customers. The default data is $b_r = 20$, $\mu_r = 5$, $h_0 = 5$, $h_r = 10$, $L_0 = 10$, $L_r = 5$. On figure 1, each graph reports the relative error of our convergent sequence for various data values and iteration depths. The data varies on the X-axis and the relative error is reported on the Y-axis. The latter has a logarithmic scale.

We notice that our sequence converges in linear time in the sense of complexity theory (in number of correct decimal digits). Indeed, for each graph and for a given data value, the distance between the curves (iteration depths) is quite constant. This behavior is noticed for all depths but we report the first three iterations only for readability.

Iteration 1 refers to the solution computed by [6] when their method is applied to the univariate objective function that we give in theorem 1. Iteration 2 refers to our closed-form solution. Each iteration improves by a factor 100 or 1000 the relative precision. In particular, our closed-form solution is never further than 0.001% from the optimal solution. Since we proved that the sequence converges always to the optimum, such a closeness is not surprising, although there is no formal guarantee within 2 iterations.
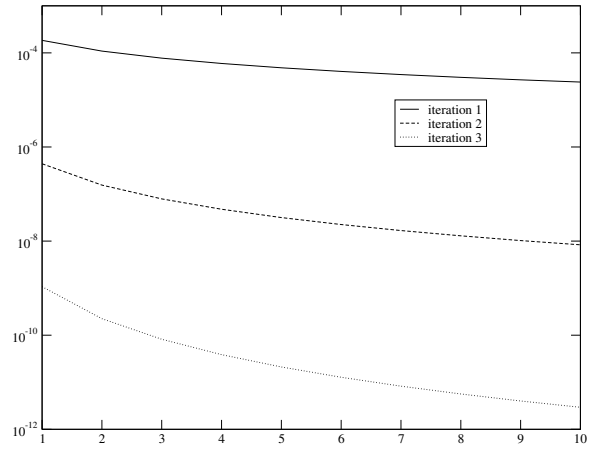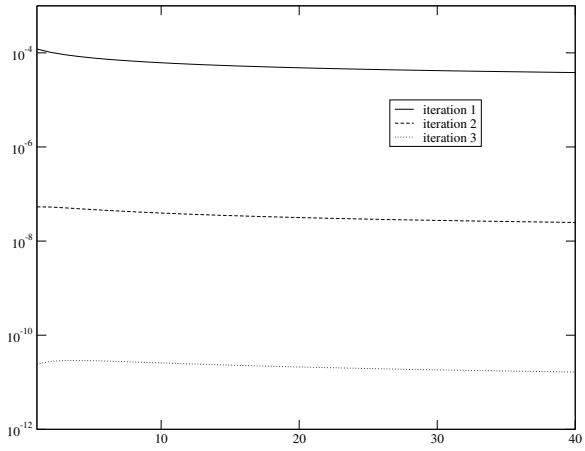
On the left, b_r varies. On the right, μ_r varies.
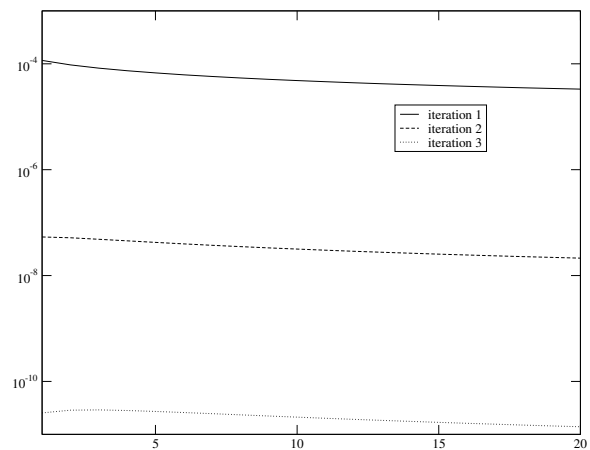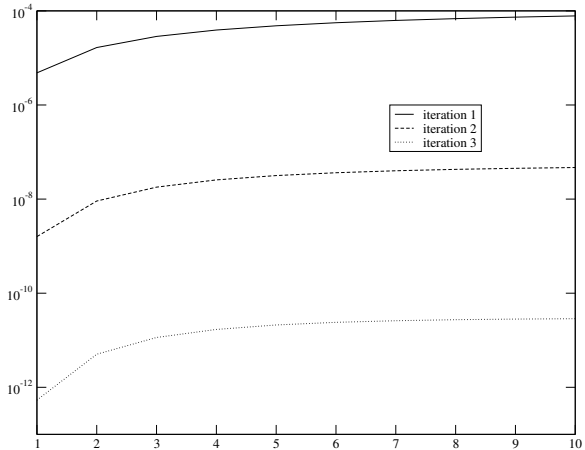
On the left, h_0 varies. On the right, h_r varies.
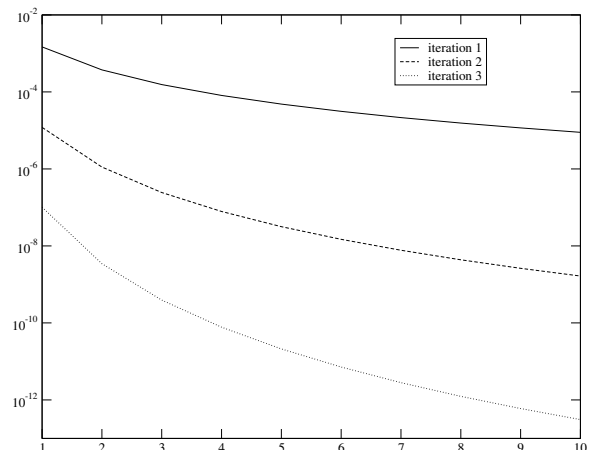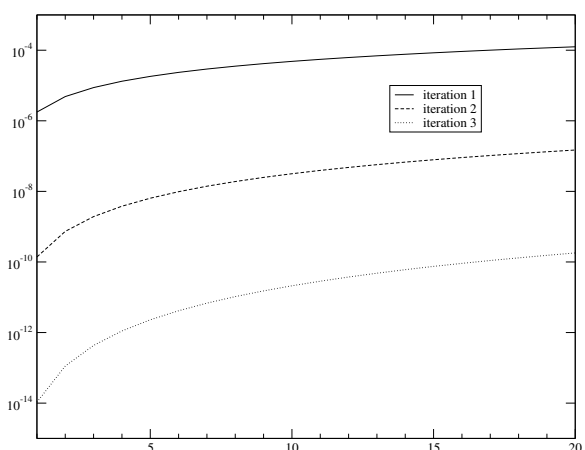
On the left, L_0 varies. On the right, L_r varies.

Figure 1: Relative error of our convergent sequence.

8

5 Conclusion

The details of our approach being stated, we discuss its contribution when compared to similar works.

An approach of [19] consists in approximating the cost function by means of normal distributions. They state a cost formula close to ours, but they distinguish the variance of the warehouse inventory from the mean, as suggested by [20], thus leading to a more accurate solution. They do not prove that the obtained function is unimodal. In such a case, a convergent sequence could be stated, perhaps leading to a good closed-form solution to Vari-METRIC with normally distributed demands.

The convergent sequence introduced in theorem 3 relies on equation (23). Under the notations used there, a little algebra shows that the equation is equivalent to $F_0(S_0^*) = \frac{\Sigma(S_0^*)}{1+\Sigma(S_0^*)}$. Then, by denoting

$$b_0(S_0) = h_0 \, \Sigma(S_0) \, ,$$

we can write

$$S_0^* = F_0^{-1} \left(\frac{b_0(S_0^*)}{h_0 + b_0(S_0^*)} \right) \, .$$

Concretely, we identify here the optimal solution to a single-echelon problem (see lemma 15 in appendix B), where $b_0(S_0)$ is a virtual back-order cost at the warehouse. Thus, we meet the work of [2], which seems to be the first to assign a virtual back-order cost to the warehouse in two-echelons distribution systems. As a result, it is sufficient to solve the warehouse inventory as if it was a single-echelon problem. Their procedure for estimating the virtual back-order cost is iterative and proven to converge. However, they prove only that it converges to a local optimum and they have to assume that back-order costs are greater than holding costs.

A few years later, one of the authors simplified that method by using the first iteration only [6]. To estimate the virtual cost, he assumes that no stock-outs occur at the warehouse (meaning that he initializes the method of [2] with $S_0 = +\infty$). No univariate function like ours (see theorem 1) is given, approaching the global optimum is not proven and, since one iteration only is done, it is less accurate than our solution.

A recent work simplifies the solving by means of pre-computations [11]. Up to the practitioner, it suggests to estimate the virtual back-order cost either by reading precomputed tables or by the use of closed-form formulas whose parameters are precomputed. In both cases, when all parameters of the problem match the table, no interpolation is done and the given cost is optimal. That improves the solving time of [2] (iterations are no longer needed) and improves the accuracy of [6]. It might be interesting to compare to the accuracy of our closed-form equation.

Acknowledgements

Appendix

A   Probability density functions

We recall here common definitions and straightforward properties used over the article.

A.1   Definitions

Except in the last appendix, $f(x)$ denotes a probability density function and $F(x)$ is its cumulative distribution function. For any x where $f(x)$ is not defined, our formulas will consider that $f(x)$ is 0.

For any cumulative distribution function $F(x)$,

$$F^{\text{-}1}(\alpha) \qquad (26)$$

is the real number x such that $F(x) = \alpha$.

We say that a probability density function $f(x)$ is symmetric about μ when

$$f(2\mu - x) = f(x). \qquad (27)$$

For instance, it is clear that uniform and normal distributions are symmetric.

When mentioned, some lemmas are restricted to the normal distribution

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (28)$$

and the cumulative normal distribution

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \, \mathrm{d}t. \qquad (29)$$

where μ is the mean and σ the standard deviation.

A.2   Lemmas

Lemma 1. For any probability density function symmetric about μ, the complementary of its cumulative distribution function is

$$1 - F(x) = F(2\mu - x). \qquad (30)$$

Proof. Since $\int_{\mathbf{R}} f(t) \, \mathrm{d}t = 1$,

$$1 - F(x) = \int_{x}^{+\infty} f(t) \, \mathrm{d}t.$$

By symmetry, it is equal to

$$\int_{x}^{+\infty} f(2\mu - t) \, \mathrm{d}t$$

and the change of variable $t' = 2\mu - t$ gives

$$-\int_{2\mu-x}^{-\infty} f(t') \, \mathrm{d}t' = F(2\mu - x).$$

$\square$

Lemma 2. Given a probability density function symmetric about μ and a complementary probability $1 - \alpha$, the inverse of the cumulative distribution function is

$$F^{\text{-}1}(1 - \alpha) = 2\mu - F^{\text{-}1}(\alpha). \qquad (31)$$

Proof. In equation (30), we change $F(x)$ into α and, therefore, x into $F^{\text{-}1}(\alpha)$. We obtain

$$1 - \alpha = F\big(2\mu - F^{\text{-}1}(\alpha)\big)$$

and take the inverse of the cumulative distribution function for both parts. $\square$

The following lemma is usually known for the special case where $a \to -\infty$. Our version allows the reader to adapt the rest of the paper when negative demands are neglected ($a = 0$).

Lemma 3. The mean of a normally distributed value over an interval $[a \, ; b]$ is

$$\int_{a}^{b} x f(x) \, \mathrm{d}x = \mu \left[F(b) - F(a) \right] - \sigma^2 \left[f(b) - f(a) \right]. \qquad (32)$$

Proof. Let $f(x)$ be defined by equation (28). We notice that

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mu - x}{\sigma^2} f(x).$$

We reorder it to get

$$x f(x) = \mu f(x) - \sigma^2 \frac{\mathrm{d}f}{\mathrm{d}x}(x)$$

whose integral is equation (32). $\square$

Lemma 4. The inverse function of the cumulative normal distribution is

$$F^{\text{-}1}(\alpha) \;=\; \mu + \sqrt{2}\sigma\,\text{erf}^{\text{-}1}(2\alpha - 1)\,. \qquad (33)$$

Proof. It is well known that

$$F(x) \;=\; \frac{1}{2} + \frac{1}{2}\,\text{erf}\left(\frac{x - \mu}{\sqrt{2}\sigma}\right)\,. \qquad (34)$$

Indeed, the change of variable $t' = \frac{t-\mu}{\sqrt{2}\sigma}$ in equation (29) leads to

$$F(x) \;=\; \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x-\mu}{\sqrt{2}\sigma}} e^{-t'^2}\,\text{d}t'$$

which is decomposed into

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-t'^2}\,\text{d}t' \;+\; \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-\mu}{\sqrt{2}\sigma}} e^{-t'^2}\,\text{d}t'\,.$$

The first integral is half the well known Gaussian integral, equal to $\sqrt{\pi}$. The second term is by definition half the function $\text{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)$. Hence (34). The inverse of the latter is

$$x \;=\; \mu + \sqrt{2}\sigma\,\text{erf}^{\text{-}1}(2F(x) - 1)$$

where we identify $F(x)$ with α and, therefore, x with $F^{\text{-}1}(\alpha)$. $\qquad\square$

B The single echelon problem

We noticed that most papers dealing with the single echelon problem subject to normally distributed demands use statistical formalisms and properties spread in literature, thus making them less approachable for non-specialists and students. Consequently, we decided to gather here some common properties along with some original or rare claims. Above all, the formalism is legible for non-statisticians and everything is proven in a consistent, clear and short chain of lemmas. When literature gives a different proof, we cite it.

B.1 Average inventory level

For any probability density function $f(x)$, the average amount of goods available from an inventory whose base-stock level is S is

$$I(S) \;=\; \int_{-\infty}^{S} (S - x)\,f(x)\,\text{d}x\,. \qquad (35)$$

Lemma 5. For the normal distribution, the average amount of goods available from an inventory whose base-stock level is S is

$$I(S) \;=\; (S - \mu)\,F(S) + \sigma^2 f(S)\,. \qquad (36)$$

Proof. Equation (35) can be decomposed into

$$I(S) \;=\; S \int_{-\infty}^{S} f(x)\,\text{d}x \;-\; \int_{-\infty}^{S} x f(x)\,\text{d}x\,. \qquad (37)$$

The first term is $SF(S)$ by definition. The second one is translated through equation (32) by noting that $\lim\limits_{a\to-\infty} f(a) \;=\; \lim\limits_{a\to-\infty} F(a) \;=\; 0$. $\qquad\square$

Lemma 6. For any probability density function, the derivative of $I(S)$ with respect to S is

$$\frac{\text{d}I}{\text{d}S}(S) \;=\; F(S)\,. \qquad (38)$$

Proof. The derivative of equation (37) is

$$\frac{\text{d}}{\text{d}S}\, S \int_{-\infty}^{S} f(x)\,\text{d}x \;-\; \frac{\text{d}}{\text{d}S} \int_{-\infty}^{S} x f(x)\,\text{d}x\,.$$

Due to the rule $(uv)' = u'v + uv'$, the first term is $F(S) + S f(S)$. The second term is $S f(S)$. Therefore, only $F(S)$ remains. $\qquad\square$

Lemma 7. For the normal distribution, $I(S)$ meets the differential equation

$$\sigma^2 \frac{\text{d}^2 I}{\text{d}S^2}(S) \;+\; (S-\mu)\frac{\text{d}I}{\text{d}S}(S) \;-\; I(S) \;=\; 0\,. \quad (39)$$

Proof. In equation (36), $F(S)$ is identified with equation (38) and $f(S)$ is viewed as the derivative of the latter. $\qquad\square$

B.2 Average back-order

For any probability density function $f(x)$, the average amount of goods which cannot be delivered immediately from an inventory whose base-stock level is S is

$$B(S) = \int_S^{+\infty} (x - S)\, f(x)\, \mathrm{d}x\,. \qquad (40)$$

Lemma 8. For any probability density function, the average amount of goods which cannot be delivered immediately from an inventory whose base-stock level is S is

$$B(S) = I(S) - S + \mu\,. \qquad (41)$$

Proof. First, we notice that

$$\int_{-\infty}^{+\infty} (S - x)\, f(x)\, \mathrm{d}x$$
$$= S \int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x - \int_{-\infty}^{+\infty} x f(x)\, \mathrm{d}x = S - \mu\,.$$

On the other hand, the sum can be decomposed:

$$\int_{-\infty}^{+\infty} (S - x)\, f(x)\, \mathrm{d}x$$
$$= \int_{-\infty}^{S} (S - x)\, f(x)\, \mathrm{d}x + \int_S^{+\infty} (S - x)\, f(x)\, \mathrm{d}x\,.$$

On the right, we identify equation (35) added to the opposite of equation (40). Therefore, that equality is

$$S - \mu = I(S) - B(S)$$

which is reordered to obtain equation (41). $\quad\square$

Lemma 9. For the normal distribution, the average amount of goods which cannot be delivered immediately from an inventory whose base-stock level is S is

$$B(S) = (S - \mu)\,[F(S) - 1] + \sigma^2 f(S)\,. \quad (42)$$

Proof. According to equation (41), we subtract S from equation (36) and add μ, thus obtaining

$$B(S) = (S - \mu)\, F(S) + \sigma^2 f(S) - (S - \mu)\,,$$

which is factorized into equation (42).

A different proof is given by [14]. $\quad\square$

Lemma 10. For any probability density function symmetric about μ, the average back-order and the average inventory level are linked by the relations

$$B(S) = I(2\mu - S) \qquad (43)$$

and

$$B(2\mu - S) = I(S)\,. \qquad (44)$$

Proof. For a start, let us prove the first relation. In equation (35), we change S into $2\mu - S'$, thus obtaining

$$\int_{-\infty}^{2\mu - S'} (2\mu - x - S')\, f(x)\, \mathrm{d}x\,.$$

The change of variable $x' = 2\mu - x$ gives

$$-\int_{+\infty}^{S'} (x' - S')\, f(2\mu - x')\, \mathrm{d}x'$$

where, by symmetry, $f(2\mu - x')$ is replaced with $f(x')$. Then, the integral is identified with $B(S')$ according to equation (40).

Equation (44) is a straightforward corollary obtained by replacing S with $2\mu - S'$ in equation (43). $\quad\square$

Lemma 11. For any probability density function, the derivative of $B(S)$ with respect to S is

$$\frac{\mathrm{d}B}{\mathrm{d}S}(S) = F(S) - 1\,. \qquad (45)$$

Proof. That is the derivative of equation (41), where the derivative of $I(S)$ is given by equation (38). $\quad\square$

Lemma 12. For the normal distribution, $B(S)$ meets the differential equation

$$\sigma^2 \frac{\mathrm{d}^2 B}{\mathrm{d}S^2}(S) + (S - \mu)\frac{\mathrm{d}B}{\mathrm{d}S}(S) - B(S) = 0\,. \ (46)$$

Proof. In equation (42), $F(S) - 1$ is identified with equation (45) and $f(S)$ is viewed as the derivative of the latter. $\quad\square$

B.3 Average overall cost

For any probability density function, the average cost for holding and back-ordering is

$$C(S) \;=\; h\,I(S) \;+\; b\,B(S)\,. \qquad (47)$$

Due to equation (41) it is also

$$C(S) \;=\; (h+b)\,I(S) \;-\; bS \;+\; b\mu\,. \qquad (48)$$

Lemma 13. For any probability density function, the derivative of $C(S)$ with respect to S is

$$\frac{\mathrm{d}C}{\mathrm{d}S}(S) \;=\; (h+b)\,F(S) \;-\; b\,. \qquad (49)$$

Proof. That is the derivative of equation (48), in which the derivative of $I(S)$ is given by equation (38).

A different proof is given by [7]. $\qquad\square$

Lemma 14. For the normal distribution, $C(S)$ meets the differential equation

$$\sigma^2\frac{\mathrm{d}^2 C}{\mathrm{d}S^2}(S) \;+\; (S-\mu)\frac{\mathrm{d}C}{\mathrm{d}S}(S) \;-\; C(S) \;=\; 0\,. \;(50)$$

Proof. According to equation (47), $C(S)$ is a linear combination of $I(S)$ and $B(S)$. Both of them match the differential equation, as stated by equations (39) and (46). $\qquad\square$

Lemma 15. For any probability density function, $C(S^*)$ is optimal if and only if the inventory base-stock level is

$$S^* \;=\; F^{\text{-}1}\left(\frac{b}{h+b}\right)\,. \qquad (51)$$

For any probability density function symmetric about μ, it is also

$$S^* \;=\; 2\mu - F^{\text{-}1}\left(\frac{h}{h+b}\right)\,. \qquad (52)$$

Proof. For a start, let us prove the first equality. We look for the base-stock level S^* for which the derivative of $C(S^*)$ is 0. According to equation (49), that means

$$F(S^*) = \frac{b}{h+b}\,.$$

The second derivative of $C(S^*)$ is positive, since the derivative of equation (49) is $(h+b)\,f(S)$ whose every term is positive. So it is a minimum. A different proof is given by [7].

To obtain the second equality, we notice that $\frac{b}{h+b} = 1 - \frac{h}{h+b}$, to which we apply equation (31). $\qquad\square$

Lemma 16. For the normal distribution, the optimal cost $C(S^*)$ is

$$C^* \;=\; (h+b)\,\frac{\sigma}{\sqrt{2\pi}}\,e^{-s^2} \qquad (53)$$

with

$$s \;=\; \text{erf}^{\text{-}1}\left(\frac{b-h}{b+h}\right)\,.$$

Proof. Due to optimality conditions, $\frac{\mathrm{d}C}{\mathrm{d}S}(S^*) = 0$, so differential equation (50) becomes

$$\sigma^2\frac{\mathrm{d}^2 C}{\mathrm{d}S^2}(S^*) \;-\; C(S^*) \;=\; 0\,.$$

The second derivative of C is $(h+b)\,f(S)$ according to equation (49), thus giving

$$C(S^*) \;=\; (h+b)\,\sigma^2 f(S^*)\,. \qquad (54)$$

By definition, $f(S^*) \;=\; \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(S^*-\mu)^2}{2\sigma^2}\right)$, where S^* is replaced with equation (51), namely $\mu + \sqrt{2}\sigma\,\text{erf}^{\text{-}1}\left(2\frac{b}{h+b} - 1\right)$ according to equation (33). We notice that $2\frac{b}{h+b} - 1 = \frac{b-h}{b+h}$. Thus,

$$f(S^*) \;=\; \frac{1}{\sqrt{2\pi}\sigma}\,e^{-\left[\text{erf}^{\text{-}1}\frac{b-h}{b+h}\right]^2}$$

and σ simplifies with σ^2 in equation (54).

An equivalent formula is proven by [19]. $\qquad\square$

C   Algebra

The fact introduced here is exploited in the proof of theorem 2.

Theorem 17. Let $\text{N} = \{1\ldots n\}$ be a set of n indexes. Let $\{a_i\,\forall i \in \text{N}\}$ and $\{b_i\,\forall i \in \text{N}\}$ be two sets of real numbers. All $b_i \neq 0$ and all b_i have the same sign.

Let $\frac{a_M}{b_M}$ be the greatest fraction of $\{\frac{a_i}{b_i} \, \forall i \in \mathrm{N}\}$ and let $\frac{a_m}{b_m}$ be the lowest one. Then,

$$\frac{a_M}{b_M} \geq \frac{\sum_{i \in \mathrm{N}} a_i}{\sum_{i \in \mathrm{N}} b_i} \geq \frac{a_m}{b_m}. \qquad (55)$$

By denoting by S_n the set of all permutations of N, tighter bounds are

$$\min_{\sigma \in \mathrm{S}_n} \max_{i \in \mathrm{N}} \frac{a_{\sigma(i)}}{b_i} \geq \frac{\sum_{i \in \mathrm{N}} a_i}{\sum_{i \in \mathrm{N}} b_i} \geq \max_{\sigma \in \mathrm{S}_n} \min_{i \in \mathrm{N}} \frac{a_{\sigma(i)}}{b_i}. \qquad (56)$$

Proof. For a start, we prove relation (55) by induction.

Let us denote by σ a permutation of N such that

$$\frac{a_{\sigma(i)}}{b_{\sigma(i)}} \geq \frac{a_{\sigma(i+1)}}{b_{\sigma(i+1)}} \quad \forall i \in \mathrm{N} \backslash \{n\}. \qquad (57)$$

For conciseness, we denote $a'_i = a_{\sigma(i)}$ and $b'_i = b_{\sigma(i)}$. Due to commutativity of the sum, we have

$$\frac{\sum_{i \in \mathrm{N}} a'_i}{\sum_{i \in \mathrm{N}} b'_i} = \frac{\sum_{i \in \mathrm{N}} a_i}{\sum_{i \in \mathrm{N}} b_i},$$

so proving the theorem over σ will prove it over N. Let us denote $A'_i = \sum_{k=1}^{i} a'_k$ and $B'_i = \sum_{k=1}^{i} b'_k$. Due to relation (57), we have

$$\frac{a'_1}{b'_1} = \frac{A'_1}{B'_1} \geq \frac{a'_2}{b'_2}. \qquad (58)$$

That inequality is the basis of our induction. Our induction hypothesis is

$$\frac{A'_{i-1}}{B'_{i-1}} \geq \frac{a'_i}{b'_i}. \qquad (59)$$

Let us rewrite it $A'_{i-1} b'_i \geq B'_{i-1} a'_i$ and add arbitrarily $A'_{i-1} B'_{i-1}$ to both sides, thus obtaining $A'_{i-1}(B'_{i-1} + b'_i) \geq B'_{i-1}(A'_{i-1} + a'_i)$, so $A'_{i-1} B'_i \geq B'_{i-1} A'_i$, that we rewrite

$$\frac{A'_{i-1}}{B'_{i-1}} \geq \frac{A'_i}{B'_i}. \qquad (60)$$

Similarly, adding arbitrarily $a'_i b'_i$ to both sides of $A'_{i-1} b'_i \geq B'_{i-1} a'_i$ leads to

$$\frac{A'_i}{B'_i} \geq \frac{a'_i}{b'_i}. \qquad (61)$$

Moreover, according to relation (57), $\frac{a'_i}{b'_i} \geq \frac{a'_{i+1}}{b'_{i+1}}$, so equation (61) implies

$$\frac{A'_i}{B'_i} \geq \frac{a'_{i+1}}{b'_{i+1}}. \qquad (62)$$

The latter relation turns out to be the induction hypothesis (59) at $i+1$. Then, due to the basis relation (58), the hypothesis is true for all $i \in \mathrm{N} \backslash \{1\}$. Therefore, relation (60) is true for all $i \in \mathrm{N} \backslash \{1\}$ and relation (61) is true for all $i \in \mathrm{N}$. Then, from relation (60), we deduce

$$\frac{A'_1}{B'_1} \geq \cdots \geq \frac{A'_i}{B'_i} \geq \cdots \geq \frac{A'_n}{B'_n}$$

so

$$\frac{a'_1}{b'_1} \geq \frac{A'_n}{B'_n},$$

which is the first inequality of relation (55). A particular case of relation (61) is

$$\frac{A'_n}{B'_n} \geq \frac{a'_n}{b'_n},$$

which is the second inequality of relation (55).

At last, relation (56) is proven by the fact that, for a given i, the claim does not assume any relation between a_i and b_i, therefore all permutations are suitable, in particular those leading to tighter bounds. $\qquad \square$

The following corollary is not required in this paper but we mention it for elegance of the formula and potential utility.

Corollary 18. Let N be a set of indexes. Let $\{a_i \, \forall i \in \mathrm{N}\}$ be a set of non-negative real numbers and $\{b_i \, \forall i \in \mathrm{N}\}$ be a set of positive real numbers. Then

$$\frac{\sum_{i \in \mathrm{N}} a_i}{\sum_{i \in \mathrm{N}} b_i} \leq \sum_{i \in \mathrm{N}} \frac{a_i}{b_i}. \qquad (63)$$

Proof. Let $\frac{a_M}{b_M}$ be the greatest fraction in $\{\frac{a_i}{b_i} \, \forall i \in \mathrm{N}\}$. Then, according to equation (55),

$$\frac{\sum_{i \in \mathrm{N}} a_i}{\sum_{i \in \mathrm{N}} b_i} \leq \frac{a_M}{b_M}.$$

All fractions are positive, so that inequality remains true after adding any $\frac{a_i}{b_i}$ to its right part. $\qquad \square$

D   A rule to show unimodality

Unimodality is an important property in optimization. When an objective function is proven to be unimodal, most algorithms will find the global optimum. In particular, there exist specialized and fast optimal procedures. Furthermore, economists use such functions widely because of the simpler models they allow. Unfortunately, literature does not provide any general rule to show analytically that a function is unimodal. After a brief review of history and definitions of unimodality, we introduce a rule for twice differentiable univariate functions. It constitutes a sufficient condition, meaning that, if it succeeds, the function is unimodal. Otherwise, nothing is proven.

In 1938, while working on probability distributions, Alexander KHINTCHINE defined the notion of unimodality [22]. When (strict) unimodality is generalized to univariate functions, it means that function $f(x)$ reaches a maximum at $x = x^*$ and is monotonically (strictly) increasing for all $x < x^*$ and monotonically (strictly) decreasing for all $x > x^*$. Any monotonic shape is allowed on both sides, so such functions are not necessarily concave. In other words, one has a single maximum and no minimum – bounds of the domain are not considered among the possible optimums – and, in case of strict unimodality, no other stationary point. Originally, unimodality addressed the uniqueness of a maximum, but considering $-f$ instead of f extends the notion to functions having a minimum. Later, unimodality has been generalized to multivariate functions [1].

Quasi-concavity

Unimodality is a particular case of quasi-concavity, so proving the former proves the latter and we will see the condition to add to prove the former through the latter. Several variants of the notion of quasi-concavity has been studied [15, 31]. Quasi-concavity, formalized by [17], is the fundamental one. It has several equivalent definitions [10, 15, 12]. We review here a few points of view, representative of the variety.

The earliest one says that a function $f : \mathrm{X}^n \subseteq \mathbf{R}^n \mapsto \mathrm{Y} \subseteq \mathbf{R}$ is quasi-concave if and only if, for any $l \in \mathrm{Y}$, the upper level set $\{\boldsymbol{x} \in \mathrm{X}^n \,|\, f(\boldsymbol{x}) \geq l\}$ is convex. This elegant definition is intuitive with two variables, since it means that the surface enclosed by any contour line is convex. Another definition says that f is quasi-concave if and only if, for every couple $\boldsymbol{x} \in \mathrm{X}^n$ and $\boldsymbol{x}' \in \mathrm{X}^n$, $\min\{f(\boldsymbol{x}), f(\boldsymbol{x}')\} \leq f\big(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{x}'\big) \,\forall \lambda \in [0\,;1]$. Both definitions are suitable for univariate functions.

An interesting definition for multivariate functions is due to Jacques FERLAND [18] and relies on the n determinants of what is called the *bordered Hessian sub-matrices* of f. It was originally stated for quasi-convexity but has been extended straightforwardly to quasi-concavity and a summary can be found in [12]. Note that it is unsuitable for univariate functions.

To conclude the review, we suggest to regard two well known theorems as sufficient conditions of quasi-concavity. First, if a function can be decomposed into a convolution of two symmetric and quasi-concave functions, then it is quasi-concave [34]. Second, let g be a log-concave and quasi-concave function and f the function to check; if $f * g$ is quasi-concave, then f is quasi-concave [21].

On the interest of a rule to prove unimodality

Proving that a function is unimodal is most of the time difficult. It may be done by means of the first definition given in the introduction, provided that a unique increasing part and a unique decreasing part can be shown, which is rarely feasible in practice. Instead, one could check the quasi-concavity conditions that we reviewed. If one test is passed and if the function has a maximum (see, for instance, lemma 22 below), then the function is unimodal. Unfortunately, those quasi-concavity definitions are too formal to be usable in general. The first one requires to check an infinite number of values l. Similarly, with the second one, an infinite number of values of λ and couples (x, x') must satisfy the inequality. Concerning the two sufficient conditions that we selected, they require to check the properties of other functions instead, and a convolution prod-

uct is often impossible to, respectively, reverse or state.

Thus, there is no general method to show unimodality of non-trivial functions. Consequently, this property is sometimes ignored or assumed [19]. The first interest in such functions concerns economists, to model real behaviors as well as obtain simpler equations [3]. Another benefit is the guarantee that certain algorithms always find the global optimum. On this point, efficient, simple and devoted search procedures exist. Indeed, while unimodal functions can make general algorithms bounce many times from a slope to its opposite, dedicated algorithms are not troubled. For instance, Golden Section Search runs in $O(D + \log \Delta x)$, where Δx is the search window given by two bounds surrounding the optimal solution and D is the asked number of correct decimal digits. It has been introduced in 1953 [23] and a modern description can be found in [27]. For more speed, it can be hybridized with quadratic convergence algorithms.

Overview

By means of our rule, some functions can be proven to be unimodal. Its philosophy consists in decomposing functions into *easy* and *hard* parts and substituting and bounding the latter by simplifying functions. The allowed transformations rely on the fact that studying the function at its stationary points is sufficient.

Properties resulting from the transformations are hopefully easier to check and, when met, imply unimodality of the original function. Nevertheless, our rule can amplify the variations of the functions, thus taking them away from the desired properties. Hence, non-unimodality cannot be proven.

D.1   Prerequisites

Several claims of this section look straightforward, but they have practical purposes beside proving the rule. In particular, lemmas 21 and 22 can be viewed as a brief handbook helping practitioners to progress in the flowchart, as done for

example in the proof of Theorem 2. For a function f, we will denote by f' its first derivative and by f'' the second one.

Definition 19. Let f be a twice differentiable function defined on X $\subseteq \mathbf{R}$.
$a \in$ X is called an optimum of f if and only if $f'(a) = 0$ and $f''(a) \neq 0$.

Definition 20. Let f be a twice differentiable function defined on X $\subseteq \mathbf{R}$.
f is (strictly) unimodal if and only if there exists one maximum $a \in$ X and no other optimum (and no other stationary point).

Lemma 21. Let f be a once differentiable function defined on X $\subseteq \mathbf{R}$. Denote by a and b, $a < b$, the bounds of X.

If either

(1) $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to b} f(x) = -\infty,$

(2) $f(t) > \lim\limits_{x \to a} f(x)$ and $f(t) > \lim\limits_{x \to b} f(x)$
$\forall t \in]a, b[$,

(3) $\lim\limits_{x \to a} f'(x) > 0$ and $\lim\limits_{x \to b} f'(x) < 0.$

then

- There is at least one maximum on $]a, b[$.

- The number of optimums on $]a, b[$ cannot be 2.

- If the number of optimums on $]a, b[$ is at least 3, then a local minimum is between two local maximums.

Proof. f defined on X means that $f(x)$ is finite for any $x \in$ X. Therefore, condition (1) implies condition (2) and there exists $c \in$ X such that $f(c) \geq f(x) \ \forall x \in$ X. By contradiction, assume that exactly 2 optimums exist on $]a, b[$. Denote the other optimum by t. Since c is a maximum, t is a minimum. Therefore, for $t < c$, $f(x) \geq f(t)$ $\forall x < t$, including $x \to a$, which contradicts (2). For $t > c$, $f(x) \geq f(t) \ \forall x > t$, including $x \to b$, which contradicts (2). Let us assume that at least 3 optimums exist. As just seen, the first one and the last one cannot be minimums, so a minimum is between two maximums.

To prove (3), note that there exists at least one $x \in$ X such that $f'(x) = 0$, due to *Darboux's theorem*. For each optimum x, the sign of $f'(x^-)$ changes to the sign of $f'(x^+)$. According to (3), $\text{sign}\big(f'(a^+)\big) \neq \text{sign}\big(f'(b^-)\big)$, which implies that the number of changes is odd. If only 1 change occurs at c, $\lim_{x \to a} f'(x) > \lim_{x \to b} f'(x)$ implies that c is a maximum. Let us assume that at least 3 changes occur. Since $\lim_{x \to a} f'(x) > 0$, the first change consists in triggering positivity of f' to negativity, so the first optimum is a maximum. Then, the second change corresponds to a minimum and, consequently, the third change to a maximum. $\qquad\square$

Lemma 22. Let f be a twice differentiable function defined on X $\subseteq \mathbf{R}$. Denote by a and b, $a < b$, the bounds of X.

f has a local maximum if either

(1) $\exists y \in$ X such that $f(y) > \lim_{x \to a} f(x)$ and
$f(y) > \lim_{x \to b} f(x)$,

(2) $\exists x, y, z \in$ X such that $x < y < z$ and
$f(x) < f(y)$ and $f(y) > f(z)$,

(3) $\exists x \in$ X such that $f'(x) = 0$ and $f''(x) < 0$,

(4) $\exists x, y \in$ X such that $x < y$ and $f'(x) > 0$ and
$f'(y) < 0$,

(5) a condition of lemma 21 is met.

Proof. Due to continuity, (1) implies (2) and (2),(3) are the definition of a maximum. (4) implies condition (3) of lemma 21 over $[x, y]$. $\quad\square$

Lemma 23. Given X, Y $\subseteq \mathbf{R}$, let $f :$ X $\mapsto$ Y be a function which meets conditions of lemma 22. Let $v_1 :$ Z $\mapsto \mathbf{R}_+$ and $v_2 :$ Z $\mapsto \mathbf{R}_+$ be positive and once differentiable functions, with Z $\subseteq \mathbf{R}$. Let $w :$ Z $\mapsto$ X be a monotonically increasing and once differentiable function. Denote $g(t) = v_1(t) \cdot f' \circ w(t)$.

If $v_2(t)g'(t) < 0 \ \forall t \in$ Z then f is strictly unimodal.

Proof. By contraposition, let us prove that non-strict unimodality of f implies $\exists t \in$ Z$|v_2(t)g'(t) \geq$

0. Due to lemma 22, there exists a maximum so non-strict unimodality implies that there exists $a \in$ X such that $f'(a) = 0$ and $f''(a) \geq 0$. Let t be such that $w(t) = a$. Then, $f'(a) = 0$ implies $v_1'(t) \cdot f' \circ w(t) = 0$. Moreover, due to positivity of v_1 and because w is increasing, $f''(a) \geq 0$ implies $v_1(t)w'(t) \cdot f'' \circ w(t) \geq 0$. Therefore, $v_1'(t) \cdot f' \circ w(t) + v_1(t)w'(t) \cdot f'' \circ w(t) \geq 0$, whose left side is $g'(t)$. Due to positivity of v_2, that implies $v_2(t)g'(t) \geq 0$. $\qquad\square$

Lemma 24. Let $f(x) = \phi\big(x, \boldsymbol{g}(x)\big)$ be a function which meets conditions of lemma 21. Let $\boldsymbol{h}(x)$ be a vector function whose domain is or include that of $f(x)$, such that $f'(x) = 0 \Rightarrow \boldsymbol{g}(x) = \boldsymbol{h}(x)$.

If $\phi\big(x, \boldsymbol{h}(x)\big)$ is unimodal, then $f(x)$ is unimodal.

Proof. By contraposition, let us prove that non-unimodality of $\phi\big(x, \boldsymbol{g}(x)\big)$ implies non-unimodality of $\phi\big(x, \boldsymbol{h}(x)\big)$. According to lemma 21, $f(x)$ has a local minimum surrounded by two local maximums. Formally, let us denote the minimum by b and the maximums by a and c. Then $f(a) > f(b)$, $f(b) < f(c)$ and $f'(a) = f'(b) = f'(c) = 0$. The latter implies $\boldsymbol{g}(x) = \boldsymbol{h}(x) \ \forall x \in \{a, b, c\}$. Hence $\phi\big(a, \boldsymbol{h}(a)\big) > \phi\big(b, \boldsymbol{h}(b)\big)$ and $\phi\big(b, \boldsymbol{h}(b)\big) < \phi\big(c, \boldsymbol{h}(c)\big)$, meaning that $\phi\big(x, \boldsymbol{h}(x)\big)$ has a local minimum between a and c. $\qquad\square$

Lemma 25. Let f, v_2, g and Z all be as defined by lemma 23. Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be vector functions whose domains are or include Z, such that $v_2(t)g'(t) = \phi\big(t, \boldsymbol{u}_1(t)\big)$ and $g(t) = 0 \Rightarrow \boldsymbol{u}_1(t) = \boldsymbol{u}_2(t)$.

If $\phi\big(t, \boldsymbol{u}_2(t)\big) < 0 \ \forall t \in$ Z then f is strictly unimodal.

Proof. By contraposition, let us prove that non-strict unimodality of f implies $\exists t \in$ Z $|\ \phi\big(t, \boldsymbol{u}_2(t)\big) \geq 0$. As seen in the proof of lemma 23, in case of non-strict unimodality, there exists $t \in$ Z such that $f' \circ w(t) = 0$ and $v_2(t)g'(t) \geq 0$. The equality implies $g(t) = 0$, so $\boldsymbol{u}_1(t) = \boldsymbol{u}_2(t)$. The inequality implies $\phi\big(t, \boldsymbol{u}_1(t)\big) \geq 0$. So $\phi\big(t, \boldsymbol{u}_2(t)\big) \geq 0$. $\qquad\square$

D.2 The rule

Although simple, applying our rule can take long. Indeed, the main steps offer a wide range of possibilities in general, and a search tree is deployed in some cases. The operations performed may be categorized into derivations, transformations, extractions and substitutions. On flowcharts of figures 2 and 3, the numbers in parentheses refer to the transformations of section D.2.1. Arrows nm mean "not managed". Such paths must be chosen when the practitioner cannot decide. Section D.2.2 illustrates the variety of possible extractions and substitutions through some examples.

For a start, we explain the main procedure of the rule. It is described on figure 2. Denote by f_i the function to check, with $i = 1$ when starting the study.

At step ①, we ensure that f_i has at least one maximum by means of lemma 22. Indeed, lemmas 23 and 25 will be used next and they rely on lemma 22.

Steps ② and ③ prepare the use of lemma 23. At step ②, g_i denotes $g(t)$ in lemma 23. At step ③, h_i denotes $v_2(t)g'(t)$ in lemma 23. The transformations allowed at those steps should make h_i simpler to study than f_i. At step ④, we check whether h_i is negative everywhere. The sub-procedure (figure 3) can help to decide. If h_i is negative everywhere, then f_i is strictly unimodal as lemma 23 claims, so f_1 is unimodal.

Instead of the sub-procedure, typically because it failed, the use of lemma 24 can be attempted. Since it relies on lemma 21, we check the conditions of the latter at step ⑥. At step ⑧, we state $f_i'(x) = 0$ in order to rise an equivalence between a term of $f_i'(x)$ and the rest. Then, according to lemma 24, we can substitute part(s) of $f_i(x)$ by their equivalents, in order to create a simpler function to study. The new function is named f_{i+1} at step ⑧. Step ⑨ looks recursively for unimodality, as allowed by lemma 24. Note that strict unimodality of f_{i+1} does not ensure that unimodality of f_i is strict.

Now, we explain the sub-procedure of the rule. It is described on figure 3. The goal is to decide whether h_i is negative everywhere, unless it is undecidable.

At step ②, we are allowed to transform the function in order to obtain a function u_i simpler to study. It is obvious that transformations (1) and (2) do not change the sign of the function, whereas validity of transformations (3) and (4) is also clear.

Note that tricks not detailed in this general flowchart can help to decide at step ③. For instance, at the end of the proof of theorem 2, $u_i(x)$ is such that it is sufficient to study the sign of a further transformed function whose value is that of $u_i(x)$ at its stationary points.

At step ⑤, we state $g_i(x) = 0$ in order to rise an equivalence between a term of $g_i(x)$ and the rest. Then, according to lemma 25, we can substitute part(s) of h_i by their equivalents, in order to create a function u_i simpler to study.

D.2.1 Transformations

Here is the set of possible transformations to apply to a given equation:

(1) multiplication by a positive and once differentiable function,

(2) composition with a monotonically increasing and once differentiable function,

(3) replacement with a dominating function,

(4) change of variable.

Those transformations are applied when specified in figures 2 and 3, but are not mandatory. The practitioner can try them or not and experience will play a role. However, they are part of the founding principles of our rule: their purpose is to simplify equations.

Typically, transformation (1) is used to remove a denominator and transformation (2) may lighten the equation. On the other hand, those transformations can produce more intricate terms, but such that further steps will end successfully.

At some point, after having transformed over and over, one might obtain a formula whose variable, everywhere it appears, is applied to a certain function. Then, transformation (4) allows for

substituting all occurrences of the function by a variable. A simpler formula remains.

D.2.2 Extractions and substitutions

Here, we consider an example to illustrate the wide range of possibilities. Let us assume that the equation to extract from is of the form

$$x + p(x) \qquad (64)$$

and the equation whose terms can be substituted is

$$x^2 + 2q(x) \qquad (65)$$

where p and q are arbitrary functions.

We extract

$$x = -p(x) \qquad (66)$$

which implies

$$x^2 = p^2(x).$$

As a result, equation (65) can be replaced with $x^2 + 2q\big(-p(x)\big)$ or $p^2(x) + 2q(x)$.

Freedom is often large. Among others, multiplying equation (66) by x allows to state

$$x^2 = -x\, p(x).$$

Note that the right-hand side might be negative wherever equation (64) is not zero. It is not a issue for the founding lemmas of the rule. It is also valid to take its absolute value if that helps for further steps. We can also deduce

$$x = \text{sign}(x)\sqrt{|xp(x)|},$$

which gives further substitution candidates for terms of equation (65).

If $p(x)$ has a closed-form inverse then

$$x = p^{-1}(-x),$$

which implies

$$x^2 = \big(p^{-1}(-x)\big)^2$$

and even

$$x^2 = x\, p^{-1}(-x)$$

so

$$x = \text{sign}(x)\sqrt{|x\, p^{-1}(-x)|}.$$

Among further possibilities, one can also suggest

$$x^2 = -p(x)\, p^{-1}(-x).$$

Those examples show that various substituting terms can be extracted in general. The only limitation is that they must be implied by setting to 0 the equation to extract from.

Note that constants can be also extracted. From equation (64),

$$0 = x + p(x)$$

holds. Hence, in our paradigm, equation (65) is equivalent to $x^2 + x + p(x) + 2q(x)$, $\big(2x + p(x)\big)^2 + 2q(x)$ or $x^2 + 2q\big(2x + p(x)\big)$.

If $p(x)$ is never 0 or if that value can be studied separately, equation (64) allows for extracting

$$1 = -\frac{x}{p(x)}.$$

Thus, terms of equation (65) can be multiplied neutrally. That gives $x^2 - 2x\frac{q(x)}{p(x)}$ or even $x^2 + 2q\left(-\frac{x^2}{p(x)}\right)$ among other possibilities, which might bring great simplifications.

Furthermore, by adding 1 to that substituting term, we extract

$$2 = 1 - \frac{x}{p(x)}$$

which could lead to replace equation (65) with $x^2 + q(x) - x\frac{q(x)}{p(x)}$.

Most examples look likely useless but the point is the variety of possibilities to try, whose respective success depends on the studied function. In each example, we substitute one term only for readability. Nevertheless, note that several terms of equation (65) can be substituted and that can be done from different extracted terms.
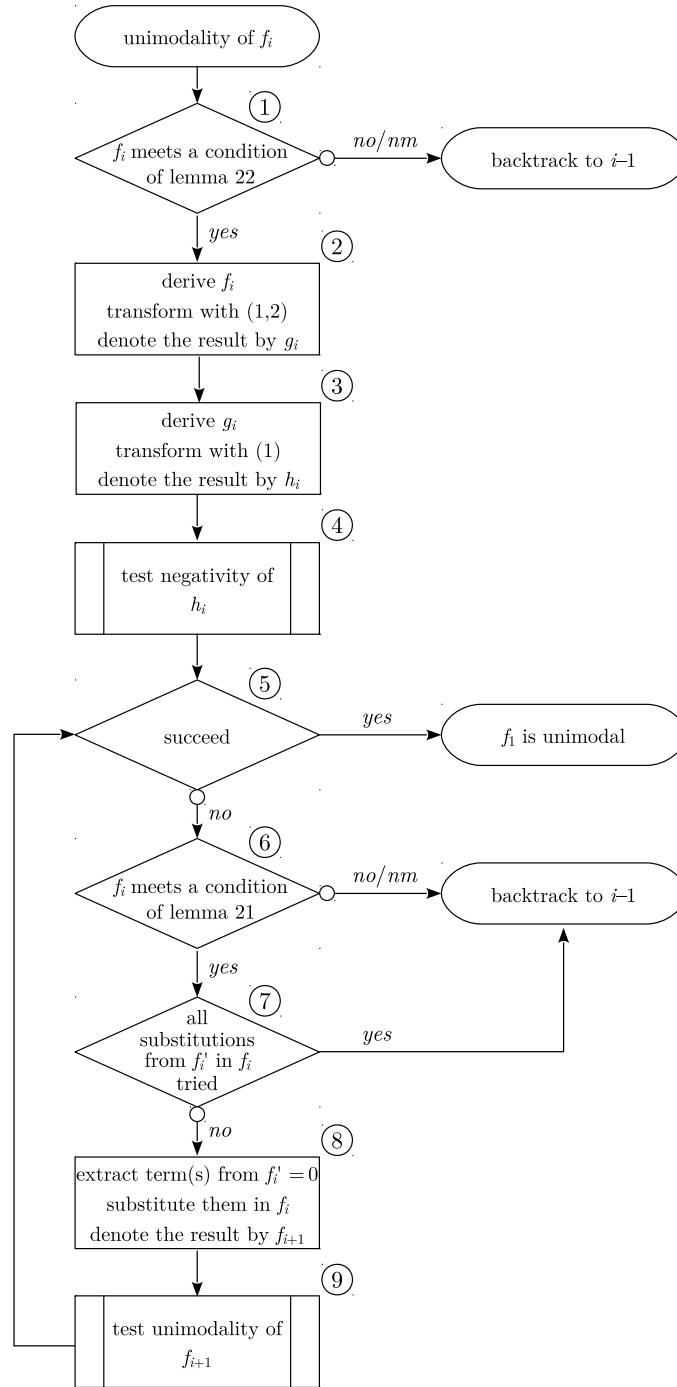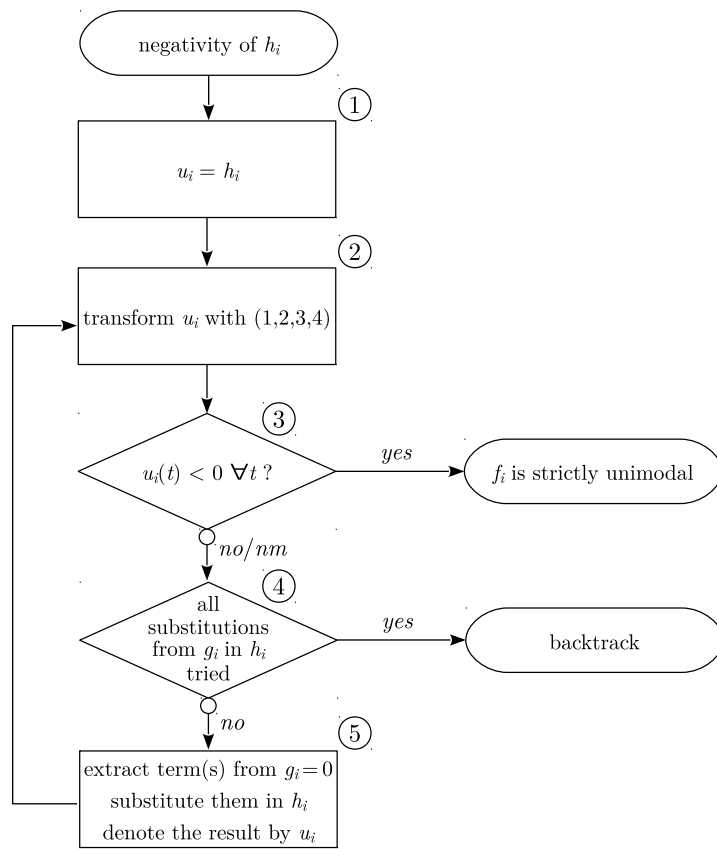
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Figure 2: Main procedure

Figure 3: Sub-procedure

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