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When are static and adjustable robust optimization with constraint-wise uncertainty equivalent?

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Abstract Adjustable Robust Optimization (ARO) yields, in general, better worst-case solutions than static Robust Optimization (RO). However, ARO is computationally more difficult than RO. In this paper, we derive conditions under which the worst-case objective values of ARO and RO problems are equal. We prove that if the uncertainty is constraint-wise and the uncertainty set is compact, then under one of the following sets of conditions robust solutions are optimal for the corresponding ARO problem: (i) the problem is fixed recourse, (ii) the problem is convex with respect to the adjustable variables and concave with respect to the parameters defining constraint-wise uncertainty, the adjustable variables lie in a convex and compact set and the uncertainty set is convex. Furthermore, if we have both constraint-wise and non-constraint-wise uncertainty, under similar sets of assumptions we prove that there is an optimal decision rule for the ARO problem that does not depend on the parameters defining constraint-wise uncertainty. Also, we show that for a class of problems, using affine decision rules that depend on both types of uncertain parameters yields the same optimal objective value as ones depending solely on the non-constraint-wise uncertain parameter. Additionally, we illustrate the usefulness of these results by applying them to several classes of problems, including facility location, inventory system, specific two-stage linear optimization, convex quadratic and conic quadratic problems.

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1 Introduction (rewritten)

Many of the real life optimization problems have parameters that are not exact. One way to deal with parameter uncertainty is Robust Optimization (RO), which enforces the constraints to hold for all uncertain parameter values in a user specified uncertainty region. All decision variables in RO represent "here and now" decisions, which means they should get specific numerical values as a result of solving the problem and before the actual uncertain parameter values "reveal themselves".

Ben-Tal et al. [5] introduce Adjustable Robust Optimization (ARO) as another approach dealing with uncertainties. In ARO, only part of the decision variables are "here and now" ones, while the remaining variables represent "wait and see" decisions. These last mentioned decisions are assigned numerical values when a part of the uncertain parameters have become known.

The advantage of using ARO lies in the fact that its worst-case objective value is not worse and most of the time better than the corresponding static RO. However, solving an ARO problem can be NP-hard even for linear cases [11]. Hence, many methods have been introduced in order to find a good approximation for an ARO problem. Ben-Tal et al. [5] mention that if the uncertainty is constraint-wise, under a few assumptions, RO and ARO have the same optimal objective values for linear problems with linear uncertainty and convex uncertainty set. Bertsimas et al. [9] show that even for specific non-constraint-wise uncertainty, the same result holds.

Bertsimas and Goyal [8] and Bertsimas et al. [9] show for some classes of problems how much conservative the RO solutions are for the ARO problem. In this paper, we determine some classes of problems for which the RO solutions are also optimal for the ARO problem.

Using affine decision rules [5] for "wait and see" variables appears to be very effective for many ARO problems. For fixed recourse linear ARO problems, using affine decision rules leads to a robust linear problem that is computationally tractable for many types of uncertainty sets, which is not the case for non-fixed recourse problems.

Bertsimas and Goyal [7] prove that for a linear ARO problem with right-hand side uncertainty and simplex uncertainty set, affine decision rules are optimal for the ARO problem. Also, in [10] the authors prove that affine decision rules are optimal for ARO problems with a specific objective function that is convex in the uncertain parameters and adjustable variables, box constraints for the variables, and box uncertainty set. Furthermore, Iancu et al. [16] prove that affine decision rules are optimal for an unconstrained multi-stage ARO problem under some structural assumptions on the uncertainty set and objective function. Bertsimas and Bidkhori [6] derive a bound for the gap between the objective value of the problem resulting from using affine decision rules and that of the ARO problem. Although substituting "wait and see" decision variables with affine functions seems to be really effective, this method needs adding many new variables. This is because for a problem with n adjustable variables and m uncertain parameters, using affine decision rules needs to substitute n adjustable variables with $n(m+1)$ non-adjustable variables, as the coefficient of the uncertain parameters and the constant vector.

The contribution of our paper is threefold:

1. We study fixed recourse constraint-wise problems with compact uncertainty set and show that the optimal value of ARO and RO problems are the same. This result extends the result in [5, Theorem 2.1], which is for problems that are linear in the variables and uncertain parameters, to nonlinear (possibly non-convex) cases with nonlinear uncertainty when the problems are fixed recourse. For our result we do not need any convexity assumption, neither for the feasible and uncertainty set, nor for the objective and constraint functions.
2. We study non-fixed recourse constraint-wise problems and prove that equality of ARO and RO optimal objective values is reached for some sets of conditions.
3. We study uncertain nonlinear problems containing both constraint-wise and non-constraint-wise uncertainties. In particular, we prove that for an ARO problem, under sets of conditions similar to the pure constraint-wise cases, there exists an optimal decision rule that depends only on non-constraint-wise uncertain parameters. Moreover, we show that for a specific class of problems there exists an affine decision rule that is only a function of the non-constraint-wise uncertain parameters and yields the same objective value as using an affine decision rule that is a function of all uncertain parameters.

The first two contributions mean that for such problems, there is no need to solve ARO ones. This has two outstanding merits: First, solving an RO problem is computationally much easier than solving an ARO one. Second, since ARO is an online approach, parts of the solution can only be implemented after knowing the values of the uncertain parameters. However, the RO approach is an offline one, and all preparations for implementing the solution can start promptly after solving the RO problem (for further discussion about online and offline approach see [18]).

The merit of the third contribution is that it reduces the number of variables in the problem using affine decision rules, because we know beforehand that the coefficients of the constraint-wise uncertain parameters are zero.

In the last part of the paper, we apply our theoretical results to some important classes of problems. We show that the second and third contributions are applicable to convex quadratic and/or conic quadratic problems, which can arise in e.g. multi-stage portfolio optimization. Moreover, for the facility location problem, we discuss two formulations that are equivalent in the deterministic case, and show, by using the first contribution, that the robust optimal value of one is better than the other. Also, we show that the third contribution results in a reduction in the complexity and the number of variables for inventory system problems with demand and cost uncertainty. Besides, for a specific class of two-stage linear optimization problems it is shown that a part of the results in [9, Section 4] can be derived easily using the third contribution of this paper.

We emphasize that the results obtained in this paper are with respect to the worst-case objective value of an ARO problem. We provide conditions under which the optimal RO solutions are also optimal for the ARO problem. However, in such cases another ARO optimal solution may yield a better average-case objective value [17].

The rest of the paper is organized as follows: Section 2 contains our main results. We derive some sets of conditions under which constraint-wise RO and ARO problems have the same optimal objective values. Moreover, for problems with both constraint-wise and non-constraint-wise uncertain parameters, we show that under similar sets of conditions, there exists an optimal decision rule that is independent of the constraint-wise uncertain parameters. Section 3 contains the application of our results to facility location problem, inventory system problem, a specific class of two-stage linear optimization problems that is also studied in [9], convex quadratic, and conic quadratic problems.

2 Main results

In this section, we derive the main results of the paper. The section starts with some definitions and preliminaries in Section 2.1. In Section 2.2 we derive sets of conditions for problems with constraint-wise uncertainty under which adjustable and static robust problems have the same optimal values. In Section 2.3 we study problems with both constraint-wise and non-constraint-wise uncertainties.

2.1 Preliminaries

Consider the following uncertain nonlinear optimization problem

$$\begin{aligned} \inf_{x\in\mathcal{X}} \inf_{y\in\mathcal{Y}(x)} & f(\zeta,x,y) \\ \text{s.t.}\quad & g_i(\zeta,x,y)\le 0, \qquad i=1,...,m, \end{aligned} \tag{1}$$

where $\zeta \in \mathcal{Z} \subseteq \mathbb{R}^l$ is an uncertain parameter and $\mathcal{Z}$ is a nonempty uncertainty set, $x \in \mathcal{X} \subseteq \mathbb{R}^r$ is a non-adjustable variable and $\mathcal{X}$ is a nonempty set defined by constraints depending only on x, $y \in \mathcal{Y}(x) \subseteq \mathbb{R}^n$ is an adjustable variable and $\mathcal{Y}(x)$ is defined by constraints independent of ζ. Also, we assume that $f(\zeta, x, y)$ and $g_i(\zeta, x, y)$, $i = 1, ..., m$, are continuous.

Corresponding to uncertain problem (1), we can define static and adjustable robust optimization problems.

Definition 1 (Static Robust Optimization) For problem (1), the static robust counterpart (RC) is defined by

$$
\begin{aligned}
(RC) \quad \inf_{x\in\mathcal{X}} \inf_{y\in\mathcal{Y}(x),\ t} \quad & t \\
\text{s.t.} \quad & f(\zeta, x, y) \le t \qquad \forall \zeta \in \mathcal{Z}, \\
& g_i(\zeta, x, y) \le 0, \quad \forall \zeta \in \mathcal{Z},\ i = 1, ..., m.
\end{aligned} \qquad \square
$$

Definition 2 (Adjustable Robust Optimization) For problem (1), there are two different definitions for the adjustable robust counterpart (ARC):

$$
\inf \left\{ t \;\middle|\; \exists x \in \mathcal{X} \ \ \forall \zeta \in \mathcal{Z} \ \ \exists y \in \mathcal{Y}(x) : \begin{array}{l} f(\zeta, x, y) \le t, \\ g_i(\zeta, x, y) \le 0, \ \ i = 1, ..., m \end{array} \right\} \tag{2}
$$

and,

$$
\begin{aligned}
(ARC) \quad \inf_{x\in\mathcal{X}} \sup_{\zeta\in\mathcal{Z}} \inf_{\substack{y(\zeta)\in\mathcal{Y}(x) \\ t(\zeta)}} \quad & t(\zeta) \\
\text{s.t.} \quad & f(\zeta, x, y(\zeta)) \le t(\zeta) \\
& g_i(\zeta, x, y(\zeta)) \le 0, \quad i = 1, ..., m.
\end{aligned} \qquad \square
$$

Takeda et al. [20] prove equivalency of problems (2) and (ARC). In this paper, we denote by $Opt(RC)$ and $Opt(ARC)$ the objective values of problems (RC) and (ARC), respectively.

We extend the definition of fixed recourse (ARC) for a linear problem with linear uncertainty in [5] to the nonlinear case (nonlinear problem with nonlinear uncertainty) in the following definition.

Definition 3 (Fixed Recourse Problem) (ARC) is called fixed recourse, whenever there are continuous functions $\tilde{f}, \tilde{g}_i : \mathbb{R}^{n+r} \to \mathbb{R}$, $\bar{f}, \bar{g}_i : \mathbb{R}^{l+r} \to \mathbb{R}$, for $i = 1, ..., m$, such that for all $\zeta \in \mathcal{Z} \subset \mathbb{R}^l$, $x \in \mathcal{X} \subset \mathbb{R}^r$ and $y \in \mathcal{Y}(x) \subset \mathbb{R}^n$,

$$
\begin{aligned}
f(\zeta, x, y) &= \tilde{f}(x, y) + \bar{f}(\zeta, x), \\
g_i(\zeta, x, y) &= \tilde{g}_i(x, y) + \bar{g}_i(\zeta, x), \quad i = 1, ..., m.
\end{aligned} \qquad \square
$$

In this paper, we mainly work with constraint-wise uncertainty, which is defined as follows.

Definition 4 (**Constraint-wise Uncertainty**)[5] For problem (1), the uncertainty is constraint-wise when each uncertain parameter ζ can be split into blocks $\zeta = [\zeta_0, ..., \zeta_m]$ such that the data of the objective depends only on ζ_0, the data of the i-th constraint depends solely on ζ_i, and the uncertainty set $\mathcal{Z} = \mathcal{Z}_0 \times \mathcal{Z}_1 \times ... \times \mathcal{Z}_m$, where $\mathcal{Z}_j \subseteq \mathbb{R}^{l_j}$ is the uncertainty region of ζ_j, for some integers l_j, $j = 0, ..., m$. $\square$

Notice that problem (1) does not contain any equality constraint that depends on ζ. The usual way of dealing with such uncertain equalities in (ARC) is elimination of adjustable variables [14, Section 7]. This means that implicitly we force the adjustable variables that are eliminated, to obey specific decision rules. This is not allowed in (RC). We illustrate this issue in Example 4 Appendix D.

Intending to express conditions under which $Opt(RC) = Opt(ARC)$, we mention a collection of assumptions here and we use a subset of this collection in each theorem.

Assumptions.(rewritten)
All the assumptions are with respect to problem (1).
Throughout, we assume that

- there is no equality constraint in problem (1) that depends on ζ.
- the uncertainty set $\mathcal{Z}$ is compact.

In the theorems and corollaries we will use subsets of the following assumptions:

(i) **Constraint-wise uncertainty**: The uncertainty is constraint-wise (Definition 4).
(ii) **Convex uncertainty set**: The uncertainty set $\mathcal{Z} \subset \mathbb{R}^l$ is convex.
(iii) **Convex adjustable set**: $\mathcal{Y}(x)$ is a convex set, for each $x \in \mathcal{X}$.
(iv) **Compact adjustable set**: $\mathcal{Y}(x)$ is a compact set, for each $x \in \mathcal{X}$.
(v) **Fixed recourse**: (ARC) is fixed recourse (Definition 3).
(vi) **Concavity of functions in** ζ: $f(., x, y)$ and $g_i(., x, y)$ are concave for each $x \in \mathcal{X}$, $y \in \mathcal{Y}(x)$, and $i = 1, ..., m$.
(vii) **Convexity of functions in** y: $f(\zeta_0, x, .)$ and $g_i(\zeta_i, x, .)$ are convex for each $x \in \mathcal{X}$, $\zeta \in \mathcal{Z} = \mathcal{Z}_0 \times ... \times \mathcal{Z}_m$, and $i = 1, ..., m$.

With respect to Assumption (ii), for some of the results obtained in this paper, we do **not** assume that the uncertainty set is convex. This is useful since there are several cases in which the uncertainty set $\mathcal{Z}$ is not convex. For instance, consider the following constraint:

$$\sum_{i=1}^{l} \zeta_i f(z_i, x, y) \leq b, \ \forall (\zeta_1, ..., \zeta_l) \in \left\{ \zeta \in \mathbb{R}^l : \begin{array}{c} \sum_{i=1}^{l} z_i \zeta_i^2 - \left(\sum_{i=1}^{l} z_i \zeta_i \right)^2 \leq \sigma^2 \\ \sum_{i=1}^{l} \zeta_i = 1, \ \zeta \geq 0 \end{array} \right\},$$

where σ and z_i, $i = 1, ..., l$, are given. One can interpret this as an upper bound on the expectation of $f(z, x, y)$ for all possible probability distributions with variance at most σ^2, where z is a discrete random variable with the support $\{z_1, ..., z_m\}$. These types of constraints may arise in Distributionally Robust Optimization [12].

2.2 Constraint-wise uncertainty

In this subsection, we study problems with constraint-wise uncertainty and provide some sets of conditions, under which $Opt(RC)$ and $Opt(ARC)$ are equal.

2.2.1 Fixed recourse problems

The following theorem states that if the uncertainty is constraint-wise, then static and adjustable problems are equivalent, when (ARC) is fixed recourse. This theorem extends the result in [5, Theorem 2.1] even for linear problems with linear uncertainty.

Theorem 1 *If Assumptions (i)"Constraint-wise uncertainty" and (v)"Fixed recourse" hold, then* $Opt(RC) = Opt(ARC)$.

Proof (rewritten) First we suppose problem (1) does not contain any non-adjustable variable. According to the definitions of (RC) and (ARC), we have $Opt(ARC) \leq Opt(RC)$. It means that if (RC) is unbounded, then $Opt(RC) = Opt(ARC) = -\infty$. Now, if (RC) is not unbounded, we show that $Opt(ARC) \geq Opt(RC)$.

Since (ARC) is fixed recourse, we can simplify (RC) to the following problem:

$$\begin{array}{rl} \inf_{y\in\mathcal{Y},\ t} & t \\ \text{s.t.} & \tilde{f}(y) + \sup_{\zeta_0\in\mathcal{Z}_0}\bar{f}(\zeta_0) \leq t \\ & \tilde{g}_i(y) + \sup_{\zeta_i\in\mathcal{Z}_i}\bar{g}_i(\zeta_i) \leq 0, \quad i=1,...,m. \end{array} \tag{3}$$

Because uncertainty is constraint-wise, $\mathcal{Z}$ is non-empty and compact and continuity of $\bar{f}$ and $\bar{g}_i$, $i = 1, ..., m$, there is a point $\bar{\zeta} = [\bar{\zeta}_0, ..., \bar{\zeta}_m] \in \mathcal{Z}$ where $\bar{\zeta}_0$ is an optimal solution of $\sup_{\zeta_0\in\mathcal{Z}_0} \bar{f}(\zeta_0)$, and $\bar{\zeta}_i$ is an optimal solution of $\sup_{\zeta_i\in\mathcal{Z}_i} \bar{g}_i(\zeta_i)$, for all $i = 1, ..., m$. By (ARC) definition, $Opt(ARC) \geq q$, where

$$\begin{array}{rl} q := \inf_{\substack{y(\bar{\zeta})\in\mathcal{Y} \\ t(\bar{\zeta})}} & t(\bar{\zeta}) \\ & \tilde{f}(y(\bar{\zeta})) + \bar{f}(\bar{\zeta}_0) \leq t(\bar{\zeta}) \\ & \tilde{g}_i(y(\bar{\zeta})) + \bar{g}_i(\bar{\zeta}_i) \leq 0, \quad i=1,...,m, \end{array} \tag{4}$$

which is equivalent to (3). This implies that if (RC) is infeasible, so is (4), and therefore $Opt(ARC) = Opt(RC) = +\infty$. On the other hand, if (RC) is feasible, then $Opt(RC) = q \leq Opt(ARC)$. So, the equality of the optimal objective values of (ARC) and (RC) is proved.

Now, for the general case that (ARC) contains a non-adjustable variable x, we have to solve:

$$\begin{array}{l} \inf\limits_{x\in\mathcal{X}} \sup\limits_{\zeta\in\mathcal{Z}} \inf\limits_{y(\zeta)\in\mathcal{Y}(x)} f(\zeta, x, y(\zeta)) \\ \qquad\qquad g_i(\zeta, x, y(\zeta)) \leq 0, \quad i=1,...,m. \end{array} \tag{5}$$

According to the first part of the proof, for each $x \in \mathcal{X}$, the objective value of

$$\begin{array}{lll} \sup_{\zeta\in\mathcal{Z}} \inf_{y(\zeta)\in\mathcal{Y}(x)} & f(\zeta,x,y(\zeta)) & \\ \quad\quad\quad\text{s.t.} & g_i(\zeta,x,y(\zeta))\leq 0, \quad i=1,...,m, \end{array} \tag{6}$$

is equal to the objective value of

$$\begin{array}{ll} \inf_{y\in\mathcal{Y}(x)} \sup_{\zeta\in\mathcal{Z}} & f(\zeta_0,x,y) \\ \quad\text{s.t.} & g_i(\zeta_i,x,y)\leq 0, \quad \forall\zeta_i\in\mathcal{Z}_i, \quad i=1,...,m. \end{array} \tag{7}$$

It follows that the optimal objective value of problem (5) equals that of

$$\begin{array}{ll} \inf_{x\in\mathcal{X},y\in\mathcal{Y}(x)} \sup_{\zeta\in\mathcal{Z}} & f(\zeta_0,x,y) \\ \quad\text{s.t.} & g_i(\zeta_i,x,y)\leq 0, \quad \forall\zeta_i\in\mathcal{Z}_i,\ i=1,...,m. \end{array} \tag{8}$$

Therefore, $Opt(ARC) = Opt(RC)$. □

In Theorem 1, there is no convexity assumption. Therefore, it works for a huge class of problems containing Mixed Integer Nonlinear Programing problems.

In Appendix D, we provide several examples illustrating that if Assumption (v) "Fixed recourse" is removed, even if the other assumptions are satisfied, $Opt(ARC)$ and $Opt(RC)$ are not necessarily equal.

Remark 1 We point out that if the uncertainty set is not compact and (RC) is feasible, one can prove by Sion's minimax theorem [19] that $Opt(RC) = Opt(ARC)$ under the following conditions: Assumptions (i) "Constraint-wise uncertainty", (iii) "Convex adjustable set", (iv) "Compact adjustable set", (v) "Fixed recourse", and (vii) "Convexity of function in y".

2.2.2 Non-fixed recourse problems

The following theorem shows that $Opt(ARC) = Opt(RC)$ under another set of conditions that does not include Assumption (v) "Fixed recourse".

Theorem 2 *If Assumptions (i) "Constraint-wise uncertainty", (ii) "Convex uncertainty set", (iii) "Convex adjustable set", (iv) "Compact adjustable set", (vi) "Concavity of functions in ζ", and (vii) "Convexity of functions in y" hold, then $Opt(RC) = Opt(ARC)$.*

Proof The line of reasoning is the same as in [5, Theorem 2.1].

Case I: Suppose that (ARC) does not have any non-adjustable variable. First, we assume that (RC) is feasible. So, it is sufficient to show that whenever $\bar{t} \geq Opt(ARC)$ then $\bar{t} \geq Opt(RC)$ (feasibility of (RC) implies $Opt(ARC) < \infty$). According to the definitions, we have:

$$Opt(ARC) = \tag{9}$$
$$\inf\left\{t \;\middle|\; \forall\zeta\in\mathcal{Z}=\mathcal{Z}_0\times...\times\mathcal{Z}_m \;\; \exists y\in\mathcal{Y}: \begin{array}{l} f(\zeta_0,y)\leq t, \\ g_i(\zeta_i,y)\leq 0, \;\; i=1,...,m \end{array}\right\}$$

and,

$$Opt(RC) = \tag{10}$$
$$\inf \left\{ t \;\middle|\; \exists y \in \mathcal{Y} \;\; \forall \zeta \in \mathcal{Z} = \mathcal{Z}_0 \times ... \times \mathcal{Z}_m : \begin{array}{l} f(\zeta_0, y) \le t, \\ g_i(\zeta_i, y) \le 0, \;\; i = 1, ..., m \end{array} \right\}.$$

If $\mathcal{Y} = \emptyset$, it is clear that $Opt(ARC) = Opt(RC) = +\infty$. Now assume that $\mathcal{Y} \neq \emptyset$. By contradiction, suppose that there is a scalar $\bar{t}$ such that $\bar{t} \ge Opt(ARC)$ and $\bar{t} < Opt(RC)$. By (10), Assumption (i) "Constraint-wise uncertainty" and setting $\beta = (1, 0, 0, ..., 0)^T$, $G_0(\zeta_0, y) = f(\zeta_0, y)$ and $G_i(\zeta_i, y) = g_i(\zeta_i, y)$, for $i = 1, ..., m$, it follows that

$$\forall y \in \mathcal{Y} \;\; \exists \zeta^y \in \mathcal{Z} \; \exists i_y \in \{0, ..., m\} : G_{i_y}(\zeta^y_{i_y}, y) - \beta_{i_y} \bar{t} > 0.$$

Also, continuity implies

$$\forall y \in \mathcal{Y} \;\; \exists \epsilon^y > 0 \; \exists U_y \;\; \forall z \in U_y : G_{i_y}(\zeta^y_{i_y}, z) - \beta_{i_y} \bar{t} > \epsilon^y, \tag{11}$$

where U_y is the intersection of a 2-norm open ball with a strictly positive radius centered at y with $\mathcal{Y}$. Since $\mathcal{Y}$ is compact, there are $y^k \in \mathcal{Y}$, $k = 1, ..., N$ such that $\mathcal{Y} = \cup_{k=1}^N U_{y^k}$. So,

$$\forall k = 1, ..., N, \;\forall z \in \mathcal{Y} \quad \max_k G_{i_{y^k}}(\zeta^{y^k}_{i_{y^k}}, z) - \beta_{i_{y^k}} \bar{t} > \epsilon, \tag{12}$$

where $\epsilon = \min_k \epsilon^{y^k}$. As a simplification, we set $\zeta^k = \zeta^{y^k}_{i_{y^k}}$, $i_k = i_{y^k}$ and

$$f_k(z) = G_{i_k}(\zeta^k, z) - \beta_{i_k} \bar{t} \qquad \forall z \in \mathcal{Y}.$$

Since $\mathcal{Y}$ is convex and all $f_k(z)$ are convex and continuous on $\mathcal{Y}$ due to Assumption (vii) "Convexity of functions in y", and because $\max_k f_k(z) \ge \epsilon$ for each $z \in \mathcal{Y}$, there are nonnegative weights λ_k with $\sum_k \lambda_k = 1$ such that:

$$f(z) := \sum_k \lambda_k f_k(z) \ge \epsilon \qquad \forall z \in \mathcal{Y}. \tag{13}$$

We define

$$\begin{aligned} w_i &= \sum_{k: i_k = i} \lambda_k \quad\quad i = 0, ..., m \\ \bar{\zeta}_i &= \begin{cases} \sum_{k: i_k = i} \frac{\lambda_k}{w_i} \zeta^k, & w_i \neq 0 \\ \text{an arbitrary point in } \mathcal{Z}_i, & w_i = 0 \end{cases} \\ \bar{\zeta} &= \left[\bar{\zeta}_0, ..., \bar{\zeta}_m \right]. \end{aligned} \tag{14}$$

It is clear by convexity of $\mathcal{Z}$ that $\bar{\zeta} \in \mathcal{Z}$. Additionally, due to $\bar{t} \ge Opt(ARC)$, we have:

$$\exists t \le \bar{t} : \forall \zeta \in \mathcal{Z} \; \exists y \in \mathcal{Y}, \; \begin{array}{l} f(\zeta_0, y) \le t, \\ g_i(\zeta_i, y) \le 0, \;\; i = 1, ..., m, \end{array} \tag{15}$$

which means

$$\exists \bar{y} \in \mathcal{Y} : G_i(\bar{\zeta}_i, \bar{y}) - \beta_i \bar{t} \leq 0, \qquad i = 0, ..., m. \tag{16}$$

Also, we know that for each $i = 0, ..., m$, the functions $G_i(\zeta_i, \bar{y})$ are concave in ζ_i due to Assumption (vi) "Concavity of functions in ζ". Hence, for all $i = 0, ..., m$, and $w_i > 0$

$$\begin{aligned} G_i(\bar{\zeta}_i, \bar{y}) - \beta_i \bar{t} &= G_i\left(\sum_{k:i_k=i} \frac{\lambda_k}{w_i}\zeta^k, \bar{y}\right) - \beta_i \bar{t} \\ &\geq \sum_{k:i_k=i} \frac{\lambda_k}{w_i} G_i(\zeta^k, \bar{y}) - \beta_i \bar{t} = \sum_{k:i_k=i} \frac{\lambda_k}{w_i} f_k(\bar{y}). \end{aligned}$$

Summing over the indices results in

$$\sum_{\substack{i=1 \\ w_i \neq 0}}^{m} w_i \left(G_i(\bar{\zeta}_i, \bar{y}) - \beta_i \bar{t}\right) \geq \sum_{k=1}^{N} \lambda_k f_k(\bar{y}). \tag{17}$$

By applying (13) and (16), the above inequality contradicts $\epsilon > 0$.
Now, we consider the case where (RC) is not feasible, which means $Opt(RC) = +\infty$. To prove equality of (RC) and (ARC) with respect to the worst-case objective value, it is sufficient to show that there is no $\bar{t} \in \mathbb{R}$ such that $\bar{t} \geq Opt(ARC)$. So, the same argument used in the previous part implies that $Opt(ARC) = +\infty$.

Case II: Now, we consider a general case, where (ARC) contains non-adjustable variable x. As it is proved in Case I, for any $x \in \mathcal{X}$,

$$\begin{array}{ll} \sup_{\zeta \in \mathcal{Z}} \inf_{y(\zeta) \in \mathcal{Y}(x)} & f(\zeta, x, y(\zeta)) \\ \qquad\qquad \text{s.t.} & g_i(\zeta, x, y(\zeta)) \leq 0, \quad i = 1, ..., m, \end{array} \tag{18}$$

and

$$\begin{array}{ll} \inf_{y \in \mathcal{Y}(x)} & \sup_{\zeta \in \mathcal{Z}} f(\zeta_0, x, y) \\ \quad \text{s.t.} & g_i(\zeta_i, x, y(\zeta_0, ..., \zeta_m)) \leq 0, \ \forall \zeta_i \in \mathcal{Z}_i, \ i = 1, ..., m, \end{array} \tag{19}$$

have the same optimal value. Therefore, taking the infimum over all $x \in \mathcal{X}$ results in $Opt(RC) = Opt(ARC)$. □

Theorem 2 extends the results for linear problems, obtained by Ben-Tal et al. [5, Theorem 2.1], to nonlinear ones. In Appendix A, we replace Assumption (iv) "Compact adjustable set" in Theorem 2 with two assumptions in order to derive another set of conditions under which $Opt(RC) = Opt(ARC)$.

2.3 Non-constraint-wise uncertainty

Section 2.2 focuses on constraint-wise uncertainty. The question is what can be concluded when we have two different types of uncertainties (constraint-wise and non-constraint-wise). Consider the following problem:

$$
\begin{aligned}
(HRC) \quad \inf_{x\in\mathcal{X}} \inf_{y\in\mathcal{Y}(x),\ t} \quad & t \\
\text{s.t. } & f(\zeta_0,\alpha,x,y) \le t && \forall \alpha\in\mathcal{A}, \zeta_0\in\mathcal{Z}_0,\\
& g_i(\zeta_i,\alpha,x,y) \le 0, \quad i=1,...,m, && \forall \alpha\in\mathcal{A}, \zeta_i\in\mathcal{Z}_i,
\end{aligned}
$$

where $\zeta=(\zeta_0,...,\zeta_m)\in\mathcal{Z}=\mathcal{Z}_0\times...\times\mathcal{Z}_m$ and $\alpha\in\mathcal{A}\subseteq\mathbb{R}^d$ are uncertain parameters (ζ is constraint-wise and α is non-constraint-wise). This problem has a hybrid uncertainty, so we cannot use the aforementioned results to deduce equality of the optimal values of the hybrid robust counterpart (HRC) and the corresponding hybrid adjustable robust counterpart $(HARC)$. However, the following corollary states that if in such a case the same sets of conditions as in Theorems 1 or 2 hold with respect to the constraint-wise uncertain parameters, then there exists an optimal decision rule that is a function of only the non-constraint-wise uncertain parameters. In other words, the following two problems:

$$
\begin{aligned}
(HARC) \quad \inf_{x\in\mathcal{X}} \sup_{\substack{\zeta\in\mathcal{Z}\\ \alpha\in\mathcal{A}}} \inf_{\substack{y(\zeta,\alpha)\in\mathcal{Y}(x)\\ t(\zeta,\alpha)}} \quad & t(\zeta,\alpha) \\
\text{s.t. } & f(\zeta_0,\alpha,x,y(\zeta,\alpha)) \le t(\zeta,\alpha),\\
& g_i(\zeta_i,\alpha,x,y(\zeta,\alpha)) \le 0, \qquad i=1,...,m,
\end{aligned}
$$

and,

$$
\begin{aligned}
(HARC_\alpha) \quad \inf_{x\in\mathcal{X}} \sup_{\alpha\in\mathcal{A}} \inf_{\substack{y(\alpha)\in\mathcal{Y}(x)\\ t(\alpha)}} \quad & t(\alpha) \\
\text{s.t. } & f(\zeta_0,\alpha,x,y(\alpha)) \le t(\alpha) && \forall \zeta_0\in\mathcal{Z}_0,\\
& g_i(\zeta_i,\alpha,x,y(\alpha)) \le 0, \quad i=1,...,m, && \forall \zeta_i\in\mathcal{Z}_i,
\end{aligned}
$$

have the same optimal objective values. We denote by $Opt(HARC)$ and $Opt(HARC_\alpha)$ the optimal objective value of $(HARC)$ and $(HARC_\alpha)$, respectively.

Corollary 1 *Suppose that for all $\alpha\in\mathcal{A}$ the assumptions of Theorems 1 or 2 hold with respect to $\zeta,\ x,\ y$. Then, $Opt(HARC)=Opt(HARC_\alpha)$.*

Proof By fixing $\alpha\in\mathcal{A}$, $x\in\mathcal{X}$ and applying Theorems 1 or 2, the optimal objective value of

$$
\begin{aligned}
\sup_{\zeta\in\mathcal{Z}} \inf_{\substack{y(\zeta,\alpha)\in\mathcal{Y}(x)\\ t(\zeta,\alpha)}} \quad & t(\zeta,\alpha) \\
\text{s.t. } & f(\zeta_0,\alpha,x,y(\zeta,\alpha)) \le t(\zeta,\alpha),\\
& g_i(\zeta_i,\alpha,x,y(\zeta,\alpha)) \le 0, \qquad i=1,...,m,
\end{aligned}
$$

and

$$\begin{aligned}
\inf_{\substack{y(\alpha)\in\mathcal{Y}(x)\\ t(\alpha)}} \quad & t(\alpha) \\
\text{s.t.} \quad & f(\zeta_0,\alpha,x,y(\alpha)) \le t(\alpha) && \forall \zeta_0 \in \mathcal{Z}_0, \\
& g_i(\zeta_i,\alpha,x,y(\alpha)) \le 0, \qquad i=1,...,m, && \forall \zeta_i \in \mathcal{Z}_i,
\end{aligned}$$

are equal. By taking the supremum over $\alpha \in \mathcal{A}$ and infimum over $x \in \mathcal{X}$, the result follows. □

Corollary 1 may be used to reduce the complexity of solving adjustable robust optimization problems. Because, in order to solve $(HARC)$, one needs to find an optimal decision rule with respect to both α and ζ, but by applying this corollary, we are ensured about the existence of an optimal decision rule that only depends on α. In Section 3.2, an application of Corollary 1 on inventory system problems is described.

It is important to note that if we restrict ourselves to a class of decision rules, e.g., affine decision rules, as is usually done, then Corollary 1 does not necessarily guarantee that there exists an optimal decision rule that only depends on α. The following theorem states, however, that if the problem is fixed recourse with respect to the constraint-wise uncertain parameter and we use a specific class of decision rules that are separable with respect to ζ and α, then the optimal one depends only on α.

Let us denote by $\bar{y}_\omega(\alpha)$ a function of α that belongs to a specific class parametrized by ω. One of the examples for $\bar{y}_\omega(\alpha)$ is a polynomial. In this case, ω could be the vector of coefficients for the monomials.

Theorem 3 *Assume that (HARC) is fixed recourse with respect to the constraint-wise uncertain parameter, i.e.,*

$$g_i(\zeta_i,\alpha,x,y) = \tilde{g}_i(\zeta_i,x) + \bar{g}_i(\alpha,x,y), \quad i=0,...,m,$$

where $g_0(\zeta_0,\alpha,x,y) = f(\zeta_0,\alpha,x,y)$, and $\tilde{g}_i(\zeta_i,x)$ and $\bar{g}_i(\alpha,x,y)$ are continuous, for $i=0,...,m$. Also, assume that we restrict the decision rules to be in the form of $y(\zeta)+\bar{y}_\omega(\alpha)$, where $y:\mathbb{R}^l \longrightarrow \mathbb{R}^n$. Then the optimal objective value of (HRC) when using this decision rule is equal to that of using decision rule $y+\bar{y}_\omega(\alpha)$.

Proof Consider the following problem

$$\begin{aligned}
\inf_{x\in\mathcal{X},\omega} \sup_{\zeta\in\mathcal{Z}} \inf_{y(\zeta),\, t(\zeta)} \quad & t(\zeta) \\
\text{s.t.} \quad & \tilde{g}_0(\zeta_0,x) + \bar{g}_0(\alpha,x,y(\zeta)+\bar{y}_\omega(\alpha)) \le t(\zeta), \quad \forall \alpha \in \mathcal{A}, \\
& \tilde{g}_i(\zeta_i,x) + \bar{g}_i(\alpha,x,y(\zeta)+\bar{y}_\omega(\alpha)) \le 0, \quad \forall \alpha \in \mathcal{A}, \quad i=1,...,m, \\
& y(\zeta)+\bar{y}_\omega(\alpha) \in \mathcal{Y}(x), \qquad \forall \alpha \in \mathcal{A}.
\end{aligned} \tag{20}$$

By defining $\bar{\mathcal{Y}}(x,\omega) = \cap_{\alpha\in\mathcal{A}}\left[\mathcal{Y}(x) - \bar{y}_\omega(\alpha)\right]$ and

$$\hat{g}_i(x,\omega,y(\zeta)) = \sup_{\alpha\in\mathcal{A}} \bar{g}_i(\alpha, x, y(\zeta) + \bar{y}_\omega(\alpha)), \quad i = 0,...,m,$$

the optimal objective value of (20) is equal to the optimal objective value of

$$\begin{aligned} \inf_{x\in\mathcal{X},\omega} \ \sup_{\zeta\in\mathcal{Z}} \ \inf_{\substack{y(\zeta)\in\bar{\mathcal{Y}}(x,\omega),\\ t(\zeta)}} \ & t(\zeta) \\ \text{s.t. } & \tilde{g}_0(\zeta_0,x) + \hat{g}_0(x,\omega,y(\zeta)) \le t(\zeta), \\ & \tilde{g}_i(\zeta_i,x) + \hat{g}_i(x,\omega,y(\zeta)) \le 0, \quad i = 1,...,m, \end{aligned} \tag{21}$$

which is the adjustable robust counterpart related to the following robust problem:

$$\begin{aligned} \inf_{x\in\mathcal{X},\omega} \ \inf_{y\in\bar{\mathcal{Y}}(x,\omega),\ t} \ & t \\ \text{s.t. } & \tilde{g}_0(\zeta_0,x) + \hat{g}_0(x,\omega,y) \le t, \quad \forall\zeta_0\in\mathcal{Z}_0, \\ & \tilde{g}_i(\zeta_i,x) + \hat{g}_i(x,\omega,y) \le 0, \quad \forall\zeta_i\in\mathcal{Z}_i, \ i = 1,...,m. \end{aligned} \tag{22}$$

Now, because the assumptions of Theorem 1 hold, (21) and (22) have the same optimal objective value. Using the definitions of $\bar{\mathcal{Y}}(x,\omega)$ and $\hat{g}_i(x,\omega,y)$, $i = 0,...,m$, we can easily see that the optimal objective value of (22) is equal to the optimal objective value of

$$\begin{aligned} \inf_{x\in\mathcal{X},\omega} \ \inf_{y,\ t} \ & t \\ \text{s.t. } & \tilde{g}_0(\zeta_0,x) + \bar{g}_0(\alpha,x,y+\bar{y}_\omega(\alpha)) \le t, \ \forall\alpha\in\mathcal{A}, \ \forall\zeta_0\in\mathcal{Z}_0, \\ & \tilde{g}_i(\zeta_i,x) + \bar{g}_i(\alpha,x,y+\bar{y}_\omega(\alpha)) \le 0, \ \forall\alpha\in\mathcal{A}, \ \forall\zeta_i\in\mathcal{Z}_i, \ i = 1,...,m, \\ & y + \bar{y}_\omega(\alpha) \in \mathcal{Y}(x), \quad \forall\alpha\in\mathcal{A}. \end{aligned} \tag{23}$$

So, we proved that the optimal objective value of (20) and (23) are the same. It means that using $y(\zeta)+\bar{y}_\omega(\alpha)$, and $y+\bar{y}_\omega(\alpha)$, as the form of decision rules, yields the same approximation of the optimal objective values. □

In Theorem 3, $y(\zeta)$ is a general function. For instance, if we assume that $\bar{y}_\omega(\alpha)$ lies in the class of affine functions, even for a general $y(\zeta)$, the optimal objective value is independent from ζ. The other example is when both $y(\zeta)$ and $\bar{y}_\omega(\alpha)$ are affine, which means that the decision rule is affine. We consider this case in the next corollary.

Corollary 2 *Suppose that problem (HARC) is fixed recourse with respect to constraint-wise uncertain parameter ζ. Then using an affine decision rule, $y(\alpha) = u + W\alpha$ or $y(\zeta,\alpha) = u + V\zeta + W\alpha$ yields the same approximate optimal value, where $u\in\mathbb{R}^n$, $V\in\mathbb{R}^{n\times l}$, and $W\in\mathbb{R}^{n\times d}$.* □

In Corollary 2, two different problems are mentioned to approximate $(HARC)$, one is considering $y(\alpha) = u + W\alpha$ as the form of decision rule and the other is considering $y(\zeta, \alpha) = u + V\zeta + W\alpha$. We denote by $Opt(AARC_\alpha)$ and $Opt(AARC_{\zeta,\alpha})$ the optimal objective values of the former affinely adjustable robust counterparts, respectively. Then, in general for problem (HRC), we have

$$Opt(HARC) \leq Opt(AARC_{\zeta,\alpha}) \leq Opt(AARC_\alpha) \leq Opt(HRC). \qquad (24)$$

In this section, we discussed different conditions that turn inequalities in (24) into equalities. Theorems 1 and 2 provide sets of conditions under which all of the inequalities can be replaced by equalities. Besides, under similar sets of conditions to these theorems, Corollary 2 ensures us that the middle inequality in (24) turns into equality. Other sets of conditions for which $Opt(HARC) = Opt(AARC_{\zeta,\alpha})$ are proposed in [7,10,16]. In Appendix D, we provide some examples to show that these inequalities can be strict.

3 Applications (new)

In this section, we present some applications of the results obtained in Section 2. In Section 3.1 we show by using Theorem 1 that between two deterministic equivalent formulations of facility location problems with binary adjustable variables, one is better than the other with respect to the robust optimal value. In Section 3.2 we use Corollary 1 to reduce the complexity and the size of the affinely adjustable robust counterpart. In Section 3.3 we show that for a class of two-stage linear optimization problems, a part of the results in [9] is a direct consequence of Corollary 1. Finally, for problems with uncertain convex quadratic and conic quadratic constraints, we use Theorem 2 and Corollary 1 to show that if the uncertainty in the quadratic constraints is constraint-wise, then there exist optimal solutions for the adjustable variables that are independent of the constraint-wise uncertain parameters.

3.1 Facility location problem with uncertain demands (new)

In this subsection, we briefly recall the result in [1, Theorem 1] and show that this result also holds when parts of the adjustable variables are binary.

Assume that in a horizon with T periods, a facility should be assigned to some candidate locations in order to satisfy demands in different customers' locations. The goal is to find the best allocation that satisfies the demands and maximizes the profit. Ardestani-Jaafari and Delage [1] study two equivalent formulations of this problem and show by using [5, Theorem 2.1] that if the demands are uncertain and the uncertainty set is a box, then the optimal value of the robust counterpart of one of the formulations is better than the other (the full description of the facility location problem with both formulations are in Appendix B).

In both formulations a binary decision variable $I_{i\tau}$ denotes whether the facility should be open or closed in the ith candidate location for period τ. Contrary to [1] , we assume that $I_{i\tau}$ is an adjustable variable. Even though in this case we cannot use [5, Theorem 2.1] anymore to prove which formulation is better with respect to the robust optimal value, by using Theorem 1 in Section 2.2 one can prove this. We postpone the proof to Appendix B.

3.2 Inventory system problem with demand and cost uncertainty (new)

We now apply the result of Section 2 to the inventory system problem in [5, Section 5], in which the authors only consider uncertainty in the demand and propose an affine decision rule to approximate the adjustable robust counterpart. If the cost is uncertain in addition to the demand, then the problem is not fixed recourse anymore and using an affine decision rule leads to a non-concave robust problem in the uncertain parameters. In this subsection, we show that by using Corollary 1, the affinely adjustable robust counterpart of the inventory system problem with demand and cost uncertainty can be reformulated to a linear optimization problem.

To describe the inventory system problem in [5, Section 5], let $I, T \in \mathbb{N}$ be the number of producers and the length of the horizon, respectively. Assume that during the time period τ, the i-th producer produces $p_{i\tau}$ units with per-unit cost $c_{i\tau} \in \mathbb{R}_+$. Producer i has a production capacity $P_{i\tau} \in \mathbb{R}_+$ in period τ and overall capacity $Q_i \in \mathbb{R}_+$. Let v be the amount of the product in the warehouse at the beginning. Besides, assume that at period τ, the demand is $d_\tau \in \mathbb{R}_+$ and inventory has a minimal and maximal restriction of $v_{min} \in \mathbb{R}_+$ and $v_{max} \in \mathbb{R}_+$, respectively. The goal is to minimize the total cost. The problem described in Ben-Tal et al. [5] is as follows:

$$\begin{aligned}
\min_{p\in\mathbb{R}^{I\times T}} \quad & \sum_{\tau=1}^{T}\sum_{i=1}^{I} c_{i\tau}p_{i\tau} \\
\text{s.t.} \quad & 0 \le p_{i\tau} \le P_{i\tau}, \ \ \sum_{\tau=1}^{T} p_{i\tau} \le Q_i, \ \ i=1,...,I, \ \tau=1,...,T, \\
& v_{min} \le v + \sum_{s=1}^{\tau}\sum_{i=1}^{I} p_{is} - \sum_{s=1}^{\tau} d_s \le v_{max}, \ \ \tau=1,...,T.
\end{aligned} \tag{25}$$

Consider c and d as the uncertain parameters with box uncertainty for both parameters. The uncertainty set for d and $c_{i\tau}$ is denoted as D and $[\underline{c_{i\tau}}, \overline{c_{i\tau}}]$, $i = 1, ..., I$, $\tau = 1, ..., T$, respectively. Because (25) is not fixed recourse, using affine decision rules in c and d for the adjustable variable $p_{i\tau}$, leads to an optimization problem with a non-concave quadratic objective function in the uncertain parameter c.

Because c appears only in the objective function and for each realization of d in D, the problem is linear with the compact feasible and uncertainty set, so the assumptions of Corollary 1 hold. Therefore, the adjustable robust counterpart of (25) equals

$$\begin{aligned}
\max_{d\in D}\ \min_{p(d)\in\mathbb{R}^{I\times T}}\ \max_{\underline{c_{i\tau}}\,\le\, c_{i\tau}\,\le\,\overline{c_{i\tau}}} & \sum_{\tau=1}^{T}\sum_{i=1}^{I} c_{i\tau}p_{i\tau}(d) \\
\text{s.t.}\quad & 0\le p_{i\tau}(d)\le P_{i\tau},\ \ \sum_{\tau=1}^{T} p_{i\tau}(d)\le Q_i,\ \ i=1,...,I,\ \tau=1,...,T, \\
& v_{min}\le v+\sum_{s=1}^{\tau}\sum_{i=1}^{I} p_{is}(d)-\sum_{s=1}^{\tau} d_s\le v_{max},\ \ \tau=1,...,T,
\end{aligned}$$

in which the objective function is equivalent to $\sum_{\tau=1}^{T}\sum_{i=1}^{I}\overline{c_{i\tau}}p_{i\tau}(d)$. So, we can approximate the above problem using affine decision rules only in d. This reduces the complexity and the number of variables compared to the problem acquired by using affine decision rules in c and d.

3.3 Two-stage linear optimization problems

Another application of Corollary 1 is for specific two-stage linear optimization problems with uncertainty in both the constraints and the objective. In [9] a bound is derived for this class of problems and the authors show that if the uncertainty in the objective is independent of that in the constraints then this bound does not depend on the objective uncertainty. In this subsection, we show that this result is a direct consequence of Corollary 1. As in [9], we consider the adjustable robust counterpart corresponding to a linear optimization problem (ARC_{LP})

$$\begin{aligned}
(ARC_{LP})\qquad \min c^T x + \max_{(B,d)\in\mathcal{Z}}\ \min_{y(B,d)}\ & d^T y(B,d) \\
\text{s.t.}\ \ & Ax+By(B,d)\le h, \\
& x\in\mathbb{R}^r_+, \\
& y(B,d)\in\mathcal{Y}(x),
\end{aligned}$$

where $A\in\mathbb{R}^{m\times r}$, $c\in\mathbb{R}^r_+$, $h\in\mathbb{R}^m$, $\mathcal{Y}(x)\subset\mathbb{R}^n_+$ is a polytope, B is an uncertain second-stage constraint matrix, and d is an uncertain objective coefficient vector. Also, let $\mathcal{Z}=\mathcal{Z}^B\times\mathcal{Z}^d\subseteq\mathbb{R}^{m\times n}_+\times\mathbb{R}^n_+$ be a convex compact uncertainty set. In addition, we suppose that $\mathcal{Z}^d$ is a polytope, as well. In [9], for problems with deterministic objective coefficient d, it is shown that

$$Opt(ARC_{LP})\ge\rho(\mathcal{Z})Opt(RC_{LP}), \tag{26}$$

where

$$\begin{aligned}
\rho(\mathcal{Z}) &= \max\left\{\kappa\left(T\left(\mathcal{Z},h\right)\right)\mid h>0\right\}, \\
T\left(\mathcal{Z},h\right) &= \left\{B^T\mu \,\middle|\, h^T\mu=1,\ B\in\mathcal{Z},\ \mu\ge 0\right\},
\end{aligned}$$

$$\kappa\left(T\left(\mathcal{Z},h\right)\right) = \min\left\{\alpha \mid conv(T\left(\mathcal{Z},h\right)) \subseteq \alpha T\left(\mathcal{Z},h\right)\right\},$$

and (RC_{LP}) is the robust counterpart corresponding to (ARC_{LP}). Then, the authors in [9] show separately that for problem (ARC_{LP}), which has uncertainty on objective coefficient d and second-stage matrix coefficient B, the lower bound is independent of the objective uncertainty, i.e., they show that $Opt(ARC_{LP}) \geq \rho(\mathcal{Z}^B)Opt(RC_{LP})$.

Here, we show that the latter result is a direct consequence of Corollary 1 and (26). To see that, consider the following problem

$$\begin{aligned}\min c^T x + \max_{B\in\mathcal{Z}^B} \min_{y(B),\ t(B)}\ & t(B) \\ \text{s.t.}\ & \max_{d\in\mathcal{Z}^d} d^T y(B) \leq t(B), \\ & Ax + By(B) \leq h, \\ & x \in \mathbb{R}^r_+, \\ & y(B) \in \mathcal{Y}(x).\end{aligned} \tag{27}$$

It is clear that all assumptions of Corollary 1 hold. So, applying this corollary to (ARC_{LP}) implies that (ARC_{LP}) and (27) have the same optimal objective values. Now, assume that $d^j \in \mathcal{Z}^d$, for $j = 1, ..., K$, are the extreme points of the polytope $\mathcal{Z}^d$. Then, the optimal objective value of (27) is equal to that of

$$\begin{aligned}\min_x c^T x + \max_{B\in\mathcal{Z}^B} \min_{y(B),\ t(B)}\ & t(B) \\ \text{s.t.}\ & {d^j}^T y(B) \leq t(B), \quad j = 1, ..., K, \\ & Ax + By(B) \leq h, \\ & x \in \mathbb{R}^r_+, \\ & y(B) \in \mathcal{Y}(x).\end{aligned} \tag{28}$$

Defining

$$\bar{B} = \begin{pmatrix} {d^1}^T \\ \vdots \\ {d^K}^T \\ B \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A \end{pmatrix}, \quad \bar{h} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix},$$

we rewrite (28) as

$$(\overline{ARC_{LP}}) \qquad \begin{aligned}\min_x c^T x + \max_{\bar{B}\in\bar{\mathcal{Z}}} \min_{y(\bar{B}),\ t(\bar{B})}\ & t(\bar{B}) \\ \text{s.t.}\ & \bar{A}x - \beta t(\bar{B}) + \bar{B}y(\bar{B}) \leq \bar{h}, \\ & x \in \mathbb{R}^r_+, \\ & y(\bar{B}) \in \mathcal{Y}(x),\end{aligned}$$

where $\bar{\mathcal{Z}} = [{d^1}^T, ..., {d^K}^T]^T \times \mathcal{Z}^B$. Consequently, we get

$$Opt(ARC_{LP}) = Opt(\overline{ARC_{LP}}) \geq \rho(\mathcal{Z}^B)Opt(RC_{LP})$$

by applying (26) and the fact that $\rho(\bar{\mathcal{Z}}) = \rho(\mathcal{Z}^B)$.

It is worth mentioning that if the uncertainty set $\mathcal{Z}^B$ in (ARC_{LP}) is a Cartesian product of the uncertainty region of B^j with another set, where B^j is the j-th row of B, then it can be proved analogously that the bound is independent of the uncertainty in B^j. Even though this is not an extension of the results in [9], it gives an intuition behind why the bound is independent of constraint-wise uncertainty.

We emphasize that the proofs in [9] are for polytopal uncertainty sets. However, the authors provide us additional proofs for general uncertainty sets (private communications).

3.4 Uncertain problems containing convex quadratic and/or conic quadratic constraints (new)

One application of the results derived in Section 2 is for the following problem:

$$\begin{aligned} \inf_{x\in\mathcal{X}} \inf_{y\in\mathcal{Y}(x)} \quad & f(x,y) \\ \text{s.t.} \quad & g_j(\alpha,x,y) \le 0,\ \forall\alpha\in\mathcal{A},\ j=1,...,m, \\ & h_i(\zeta_i,x,y) \le 0,\ \forall\zeta_i\in\mathcal{Z}_i,\ i=1,...,I, \\ & p_k(\theta_k,x,y) \le 0,\ \theta_k\in\mathcal{T}_k,\ k=1,...,K, \end{aligned}$$

where $g_j(\alpha,x,y)$ is a continuous function, $j=1,...,m$, the convex quadratic function h_i is defined as

$$h_i(\zeta_i,x,y) = \begin{pmatrix} x \\ y \end{pmatrix}^T A_i(\zeta_i) \begin{pmatrix} x \\ y \end{pmatrix} + b_i(\zeta_i)^T \begin{pmatrix} x \\ y \end{pmatrix} + c_i(\zeta_i),$$

and the conic quadratic function p_k is defined as

$$p_k(\theta_k,x,y) = \sqrt{\begin{pmatrix} x \\ y \end{pmatrix}^T B_k(\theta_k) \begin{pmatrix} x \\ y \end{pmatrix}} + d_k(\theta_k)^T \begin{pmatrix} x \\ y \end{pmatrix} + e_k(\theta_k),$$

where $\alpha \in \mathcal{R}^l$, $\zeta_i \in \mathcal{R}^{l_i}$, and $\theta_k \in \mathcal{R}^{l_{I+k}}$ are the uncertain parameters, for some integers l, l_{i+k}, $i=1,...,I$, $k=1,...,K$, x and y are non-adjustable and adjustable variables, respectively. We assume that the matrices $A_i(\zeta_i)$ and $B_k(\theta_k)$ are positive semi-definite, for all $\zeta_i \in \mathcal{Z}_i$ and $\theta_k \in \mathcal{T}_k$, $i=1,...,I$, $k=1,...,K$. Also, we assume that $A_i(\zeta_i)$, $b_i(\zeta_i)$, $c_i(\zeta_i)$, $B_k(\theta_k)$, $d_k(\theta_k)$ and $e_k(\theta_k)$ are affine in ζ_i and θ_k, $i=1,...,I$, $k=1,...,K$, respectively.

This type of problem arises for example when a part of the problem is related to multi-stage mean-variance portfolio optimization [15], in which the asset return mean and covariance matrix are uncertain and these uncertainties only occur in the objective function (hence the problem has constraint-wise uncertainty).

If the uncertainty over α is constraint-wise and $g_j(\alpha,x,y)$ is concave in α and convex in y, $j=1,...,m$, $\mathcal{A}$, $\mathcal{Z}_i$ and $\mathcal{T}_k$ are convex, $i=1,...,I$, $k=1,...,K$, and $\mathcal{Y}(x)$ is compact and convex for all $x \in \mathcal{X}$, then by Theorem 2 the optimal values of the corresponding static and adjustable robust problems are equal, because h_i and p_k are convex in y and concave in ζ_i and θ_k, $i = 1, ..., I$, $k = 1, ..., K$, respectively. Moreover, if the uncertainty over α is not constraint-wise, then by Corollary 1, there exists an optimal y for the corresponding adjustable robust counterpart that is independent of ζ_i and θ_k, $i = 1, ..., I$, $k = 1, ..., K$.

4 Conclusion

In this paper, we show that for some classes of constraint-wise uncertain optimization problems with compact uncertainty set, the robust optimal solution is also optimal for the adjustable robust problem. These classes include: i) Fixed recourse problems, ii) Problems that are convex with respect to the adjustable variables and concave with respect to the uncertain parameters, and that have a convex uncertainty set and adjustable variables lie in a convex compact set.

These results do not hold when a problem has both constraint-wise and non-constraint-wise uncertainties, but under sets of assumptions similar to the pure constraint-wise cases, we can prove that there exists an optimal decision rule that does not depend on the constraint-wise uncertain parameters. Also, we show that for a class of problems, restricting decision rules to be affine and independent of the constraint-wise uncertain parameters, yields the same optimal objective value as the case in which the decision rules are affine and depending on both the constraint-wise and non-constraint-wise uncertain parameters.

Lastly, we apply our results to several classes of problems, such as facility location, inventory system, specific two-stage linear optimization and convex quadratic and/or conic quadratic problems. We prove that using our results for the facility location problems with box uncertainty, one can generalize a part of the result in [1] to the case where a part of the adjustable variables is binary. Besides, we show that when affine decision rules are used in an inventory system problem with cost and demand uncertainty, using the results in this paper reduces not only the complexity of the problem but also the number of variables. For a specific class of two-stage linear optimization problem that is studied in [9], we show that part of the results in [9] is a direct consequence of one of the results derived in this paper. Furthermore, we prove that for adjustable robust optimization problems with convex quadratic and/or conic quadratic constraints, if the uncertainty in the quadratic constraints is constraint-wise, then there exists optimal adjustable variables that are independent of the constraint-wise uncertain parameters.

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References

1. Amir Ardestani-Jaafari and Erick Delage. The value of flexibility in robust location-transportation problem. *Les Cahiers du GERAD G–2014–83, GERAD, HEC Montréal*, 2014.
2. Opher Baron, Joseph Milner, and Hussein Naseraldin. Facility location: A robust optimization approach. *Production and Operations Management*, 20(5):772–785, 2011.
3. Mokhtar S. Bazaraa, Hanif D. Sherali, and Chitharanjan Marakada Shetty. *Nonlinear programming: theory and algorithms*. John Wiley & Sons, 2013.
4. Aharon Ben-Tal, Dick den Hertog, and Jean-Philippe Vial. Deriving robust counterparts of nonlinear uncertain inequalities. *Mathematical Programming*, 149(1-2):265–299, 2015.
5. Aharon Ben-Tal, Alexander Goryashko, Elana Guslitzer, and Arkadi Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2004.
6. Dimitris Bertsimas and Hoda Bidkhori. On the performance of affine policies for two-stage adaptive optimization: a geometric perspective. *Mathematical Programming*, 153(2):577–594, 2015.
7. Dimitris Bertsimas and Vineet Goyal. On the power and limitations of affine policies in two-stage adaptive optimization. *Mathematical Programming*, 134(2):491–531, 2012.
8. Dimitris Bertsimas and Vineet Goyal. On the approximability of adjustable robust convex optimization under uncertainty. *Mathematical Methods of Operations Research*, 77(3):323–343, 2013.
9. Dimitris Bertsimas, Vineet Goyal, and Brian Y. Lu. A tight characterization of the performance of static solutions in two-stage adjustable robust linear optimization. *Mathematical Programming*, 150(2):281–319, 2015.
10. Dimitris Bertsimas, Dan A. Iancu, and Pablo A. Parrilo. Optimality of affine policies in multistage robust optimization. *Mathematics of Operations Research*, 35(2):363–394, 2010.
11. Uriel Feige, Kamal Jain, Mohammad Mahdian, and Vahab Mirrokni. Robust combinatorial optimization with exponential scenarios. In Matteo Fischetti and David P. Williamson, editors, *Integer Programming and Combinatorial Optimization*, volume 4513 of *Lecture Notes in Computer Science*, pages 439–453. Springer Berlin Heidelberg, 2007.
12. Joel Goh and Melvyn Sim. Distributionally robust optimization and its tractable approximations. *Operations research*, 58(4-part-1):902–917, 2010.
13. Bram L. Gorissen and Dick den Hertog. Robust nonlinear optimization via the dual. *Optimization Online*, 2015.
14. Bram L. Gorissen, İhsan Yanıkoğlu, and Dick den Hertog. A practical guide to robust optimization. *Omega*, 53:124–137, 2015.
15. Nalan Gülpınar and Berç Rustem. Worst-case robust decisions for multi-period mean–variance portfolio optimization. *European Journal of Operational Research*, 183(3):981–1000, 2007.
16. Dan A. Iancu, Mayank Sharma, and Maxim Sviridenko. Supermodularity and affine policies in dynamic robust optimization. *Operations Research*, 61(4):941–956, 2013.
17. Dan A. Iancu and Nikolaos Trichakis. Pareto efficiency in robust optimization. *Management Science*, 60(1):130–147, 2013.
18. Michal Melamed, Aharon Ben-Tal, and Boaz Golany. On the average performance of the adjustable RO and its use as an offline tool for multi-period production planning under uncertainty. *Computational Management Science*, 13(2):293–315, 2016.

19. Maurice Sion. On general minimax theorems. *Pacific Journal of Mathematics*, 8(1):171–176, 1958.
20. Akiko Takeda, Shunsuke Taguchi, and Reha H Tütüncü. Adjustable robust optimization models for a nonlinear two-period system. *Journal of Optimization Theory and Applications*, 136(2):275–295, 2008.

Appendices

The appendix is divided into four parts. In the first part, we provide a set of conditions under which $Opt(ARC) = Opt(RC)$. In the second part, we describe the two formulations of the facility location problem studied in [1] and use Theorem 1 to prove that the robust optimal value of one is better than the other even when parts of the adjustable variables are binary. In the third part, we provide some examples that are illustrating the results from Section 2. The fourth part includes several examples to show that the assumptions in Theorems 1 and 2 are essential.

A Another case under which $Opt(ARC) = Opt(RC)$

In this section, we prove that under a set of condition different than those provided in Theorems 1 and 2, $Opt(ARC) = Opt(RC)$. In this theorem, without loss of generality, we assume that (RC) is:

$$\begin{aligned} \inf_{x\in\mathcal{X}} \inf_{y\in\mathcal{Y}(x)} \quad & c^T y \\ \text{s.t.} \quad & g_i(\zeta_i, x, y) \le 0, \quad i = 0, ..., m, \quad \forall \zeta_i \in \mathcal{Z}_i, \end{aligned} \tag{29}$$

where $c \in \mathbb{R}^r$ is certain and for $i = 0, ..., m$,

$$\mathcal{Z}_i = \{\zeta_i : \ h_{ik}(\zeta_i) \le 0, \ k = 1, ..., K_i\}.$$

In what follows, the relative interior of a set S and domain of a function $f(.)$ are denoted by $\text{relint}(S)$ and $\text{dom}(f(.))$, respectively.

Theorem 4 *Assume that for problem (29) the following assumptions hold:*

(a) $h_{ik}(.)$ *is convex,* $i = 0, ..., m$, $k = 1, ..., K_i$,
(b) there exists $(\zeta_0, ..., \zeta_m)$ *such that* $h_{ik}(\zeta_i) < 0$, *for all* $i = 0, ..., m$, $k = 1, ..., K_i$,
(c) for each $x \in \mathcal{X}$ *and* $\zeta \in \mathcal{Z}$,

$$\bigcap_{i=1}^{m} \mathit{relint}\left(\mathit{dom}\left(g_i(\zeta_i, x, .)\right)\right) \bigcap \mathit{relint}(\mathcal{Y}(x)) \neq \emptyset.$$

Additionally, if Assumptions (iii) "Convex adjustable set", (vi) "Concavity of functions in ζ*", (vii) "Convexity of functions in* y*" hold, then* $Opt(ARC) = Opt(RC)$.

Proof (rewritten) Consider the (ARC) corresponding to (29):

$$\begin{aligned} \inf_{x\in\mathcal{X}} \sup_{[\zeta_0,...,\zeta_m]\in\mathcal{Z}} \inf_{y(\zeta)\in\mathcal{Y}(x)} \quad & c^T y(\zeta) \\ \text{s.t. } & g_i(\zeta_i, x, y(\zeta)) \leq 0, \quad i = 0, ..., m. \end{aligned} \tag{30}$$

By [4, Lemma 9] (because of Assumptions (iii) "Convex adjustable set", (vii) "Convexity of functions in y", and assumption (c)), the optimal value of (30) is equal to

$$\begin{aligned} \inf_{x\in\mathcal{X}} \sup_{\substack{u\,\in\,\mathbb{R}^{m+1}_{\leq} \\ \{v^i\},\, v^{m+1}}} \sup_{\zeta=[\zeta_0,...,\zeta_m]} \quad & \sum_{i=0}^{m} u_i g_i^* \left(\zeta_i, x, \frac{v^i}{u_i}\right) + u_{m+1}\delta^*_{\mathcal{Y}(x)}\left(\frac{v^{m+1}}{u_{m+1}}\right) \\ \text{s.t.} \quad & \sum_{i=0}^{m+1} v^i = c, \\ & h_{ik}(\zeta_i) \leq 0, \quad i = 0, ..., m, \ k = 1, ..., K_i, \end{aligned} \tag{31}$$

where $\delta^*_{\mathcal{Y}(x)}\left(\frac{v^{m+1}}{u_{m+1}}\right) = \sup_{y\in\mathcal{Y}(x)} \frac{y^T v^{m+1}}{u_{m+1}}$, and

$$g_i^*\left(\zeta_i, x, \frac{v^i}{u_i}\right) = \sup_{y\in dom(g_i(\zeta_i,x,.))} \left\{\frac{y^T v^i}{u_i} - g_i\left(\zeta_i, x, y\right)\right\}.$$

Problem (31) has the same optimal objective value as

$$\begin{aligned} \inf_{x\in\mathcal{X}} \sup_{\substack{u\,\in\,\mathbb{R}^{m+1}_{\leq} \\ \{v^i\},\, v^{m+1}}} \sup_{w^i} \quad & \sum_{i=0}^{m} u_i g_i^* \left(\frac{w^i}{u_i}, x, \frac{v^i}{u_i}\right) + u_{m+1}\delta^*_{\mathcal{Y}(x)}\left(\frac{v^{m+1}}{u_{m+1}}\right) \\ \text{s.t.} \quad & \sum_{i=0}^{m+1} v^i = c, \\ & - u_i h_{ik}(\frac{w_i}{u_i}) \leq 0, \quad i = 0, ..., m, \ k = 1, ..., K_i, \end{aligned}$$

which is the dual of (29) with the same optimal objective values according to [13, Theorem 1] (because the uncertainty is constraint-wise and assumptions (a), (b), Assumptions (vi) "Concavity of functions in ζ", (vii) "Convexity of functions in y" hold). So, $Opt(ARC) = Opt(RC)$. □

B Facility location problem with demand uncertainty(new)

In this part, we describe two formulations of a facility location problem that are studied in [1] and also [2], which are equivalent in the deterministic case.

In order to describe the two formulations, we follow the description and notations from [2]. Let $T, L, N \in \mathbb{N}$, be the length of the horizon, the number of candidate locations to which a facility can be assigned, and the number of locations that have a demand for the facility, respectively. Let $\eta \in \mathbb{R}_+$ be the unit price of goods, and $c_i, C_i, K_i \in \mathbb{R}_+$ be the cost per unit of production, the cost per unit of capacity, and the cost of the opening a facility at location i, respectively, for $i = 1, ..., L$. Moreover, let $d_{ij} \in \mathbb{R}_+$ be the cost of shipping units from location i to j, and $D_{j\tau} \in \mathbb{R}_+$ be the demand in period τ at location j, $i = 1, ..., L$, $j = 1, ..., N$, $\tau = 1, ..., T$. Decision variable $X_{ij\tau}$ represents the proportion of the demand at location j in period τ that is satisfied by facility i, and $P_{i\tau}$ represents the amount of good that is produced at facility i during the period τ. For each facility i, the decision variable $I_{i\tau}$ denotes whether the facility in location i is open or closed in period τ by taking 1 or 0 respectively, and $Z_{i\tau}$ denotes the capacity of the facility in this location and period in case it is open. Using these notations, a deterministic facility location problem is described by the following mixed integer linear optimization [2]:

$$
\begin{aligned}
\max_{\substack{X \,\in\, \mathbb{R}^{L\times N\times T} \\ I,\,Z,\,P \,\in\, \mathbb{R}^{L\times T}}} \quad & \sum_{\tau=1}^{T}\sum_{i=1}^{L}\sum_{j=1}^{N}(\eta - d_{ij})X_{ij\tau}D_{j\tau} - \sum_{\tau=1}^{T}\sum_{i=1}^{L}\left(c_iP_{i\tau} + C_iZ_{i\tau} + K_iI_{i\tau}\right) \\
\text{s.t.} \quad & \sum_{i=1}^{L} X_{ij\tau} \le 1, \;\; j = 1, ..., N, \; \tau = 1, ..., T, \\
& \sum_{j=1}^{N} X_{ij\tau}D_{j\tau} \le P_{i\tau}, \;\; i = 1, ..., L, \; \tau = 1, ..., T, \\
& X_{ij\tau} \ge 0, \;\; i = 1, ..., L, \; j = 1, ..., N, \; \tau = 1, ..., T, \\
& P_{i\tau} \le Z_{i\tau}, \; Z_{i\tau} \le MI_{i\tau}, \;\; i = 1, ..., L, \; \tau = 1, ..., T, \\
& I \in \{0, 1\}^{L\times T},
\end{aligned}
$$

where, M is a large enough constant. We call the above formulation, *proportion-shipping* formulation.

In [1] another equivalent formulation for the facility problem is studied, in which $X_{ij\tau}D_{j\tau}$ is replaced with $Y_{ij\tau}$ for all i, j and τ, where $Y_{ij\tau}$ represents how much good is shipped in the period τ from location i to j. We call the model with $Y_{ij\tau}$ *total-shipping* formulation.

Let the demands $D_{j\tau}$, $j = 1, ..., N$, $\tau = 1, ..., T$, be the uncertain parameters with interval uncertainty sets. In [1] the authors assume that I, Z are non-adjustable variables, whereas X, P in the *proportion-shipping* formulation, and Y, P in the *total-shipping* formulation are adjustable variables. However, we assume here that parts of I, Z are non-adjustable and the rest are adjustable variables.

Let us denote the optimal values of the robust counterpart of *proportion-shipping*, *total-shipping* formulations and their adjustable robust counterparts by $Opt(RCproportion)$, $Opt(RCtotal)$, $Opt(ARCproportion)$ and $Opt(ARCtotal)$, respectively.

We show that the robust counterpart of the *total-shipping* formulation is better than the robust counterpart of the *proportion-shipping* i.e., $Opt(RCtotal) \ge$

$Opt(RCproportion)$. It is clear that the *proportion-shipping* and *total-shipping* formulations are equivalent for each realization of the demand. Therefore, the corresponding adjustable robust counterparts are equivalent as well, so $Opt(ARCtotal) = Opt(ARCproportion)$.

The uncertainty in the *total-shipping* formulation is constraint-wise, because the uncertain parameters appear only in $\sum_{i=1}^{L} Y_{ij\tau} \leq D_{j\tau}$, which is constructed from the first constraint in *proportion-shipping* formulation after the replacement $X_{ij\tau}D_{j\tau} = Y_{ij\tau}$, for each $i = 1, ..., L$, $j = 1, ..., N$, $\tau = 1, ..., T$. Therefore, $Opt(RCtotal) = Opt(ARCtotal)$, by Theorem 1. So, we have shown that

$$Opt(RCtotal) = Opt(ARCtotal) = Opt(ARCproportion) \geq Opt(RCproportion),$$

where the right inequality holds because the problem is a maximization one and the optimal objective value of the adjustable robust counterpart is not less than the optimal objective value of the robust counterpart. Hence, $Opt(RCtotal) \geq Opt(RCproportion)$.

In [1], the authors show that $Opt(RCtotal) \geq Opt(RCproportion)$ in [1, Theorem 1] for the case in which I, Z are non-adjustable variable. As observed in [1], the *proportion-shipping* formulation can be obtained from the *total-shipping* formulation by using a special decision rule $Y_{ij\tau} = X_{ij\tau}D_{j\tau}$, $i = 1, ..., L$, $j = 1, ..., N$ and $\tau = 1, ..., T$. One might think therefore that $Opt(RCtotal) \leq Opt(RCproportion)$. However, the contrary is true. This is caused by the fact that the decision rule does not have any constant term.

C Illustrative examples

Example 1 **(Illustrating Theorem 1)** Consider the uncertain problem

$$\begin{aligned} \min \ & y^2 + x^3 \\ \text{s.t.} \ & y^3 + \zeta^3 x \leq 0, \\ & y^2 + x^2 \leq 8, \\ & |x| \leq 1, \end{aligned}$$

where $\zeta \in \mathcal{Z} = [-2, 2]$ is an uncertain parameter, y is an adjustable and x is a non-adjustable variable. For this problem,

$$\mathcal{X} = [-1, 1], \ \ \mathcal{Y}(x) = \left\{ y \ \middle| \ y^2 + x^2 \leq 8 \right\}, \ \ \forall x \in \mathcal{X}.$$

First, we use Theorem 1 to calculate $Opt(ARC)$ because its assumptions hold for this problem. According to this theorem, $Opt(ARC) = Opt(RC)$.

Since ζ^3 is an increasing function, (RC) is equivalent to

$$\begin{aligned} \min\ & y^2 + x^3 \\ \text{s.t.}\ & y^3 + 8x \leq 0, \\ & y^3 - 8x \leq 0, \\ & y^2 + x^2 \leq 8, \\ & |x| \leq 1. \end{aligned}$$

It is easy to verify that $Opt(RC) = 0$. Now, we solve the (ARC) problem directly:

$$\begin{aligned} \min_{x\in\mathcal{X}}\ \max_{\zeta\in\mathcal{Z}}\ \min_{y(\zeta)}\ & y(\zeta)^2 + x^3 \\ \text{s.t.}\ & y(\zeta)^3 + \zeta^3 x \leq 0, \\ & y(\zeta)^2 + x^2 \leq 8. \end{aligned}$$

First, we solve

$$\begin{aligned} z^*(\zeta, x) := \min_{y(\zeta)}\ & y(\zeta)^2 \\ \text{s.t.}\ & y(\zeta)^3 + \zeta^3 x \leq 0, \\ & y(\zeta)^2 + x^2 \leq 8, \end{aligned}$$

for each $\zeta \in \mathcal{Z}$ and $x \in \mathcal{X}$. It is clear that

$$z^*(\zeta, x) = \begin{cases} \left(\sqrt[3]{-\zeta^3 x}\right)^2, & \zeta x \geq 0, \\ 0, & \text{o.w.} \end{cases}$$

Hence, $Opt(ARC) = \min_{x\in\mathcal{X}} x^3 + \max_{\zeta\in\mathcal{Z}} z^*(\zeta, x)$. Therefore, we need the optimal objective value of $\max_{\zeta\in[-2,2]}\ z^*(\zeta, x)$, for each $x \in \mathcal{X}$. By checking the different values of x, we find $4\sqrt[3]{x^2}$ as its optimal objective value. Hence, $Opt(ARC) = min_{x\in[-1,1]} x^3 + 4\sqrt[3]{x^2} = 0$. □

After considering a fixed recourse (ARC), we are considering a convex optimization problem.

Example 2 **(Illustrating Theorem 2)**(rewritten) Consider the following problem:

$$\begin{aligned} \min\ & y_1 + y_2 \\ \text{s.t.}\ & \ln(\zeta) y_1^2 + y_2^2 \leq 3, \\ & y_1^2 + y_2^2 \leq 4, \end{aligned}$$

where $\zeta \in \mathcal{Z} = [1, 4]$ is an uncertain parameter, and $y = (y_1, y_2)$ is an adjustable variable.

For this example, $Opt(RC) = Opt(ARC)$ because by defining

$$\mathcal{Y} = \left\{ y \mid y_1^2 + y_2^2 \leq 4 \right\},$$

the assumptions of Theorem 2 hold for this problem. Since $\ln(\zeta)$ is an increasing function, (RC) is as follows:

$$
\begin{aligned}
\min \quad & y_1 + y_2 \\
\text{s.t.} \quad & \ln(4) y_1^2 + y_2^2 \leq 3, \\
& y_1^2 + y_2^2 \leq 4,
\end{aligned}
$$

with the optimal value of $-\sqrt{\frac{3\ln(4)}{1+\ln(4)}}\left(\frac{1}{\ln(4)}+1\right)$.

Even though $Opt(RC) = Opt(ARC)$, by using the symmetry bound introduced in [8], which is $(1+\rho)\, Opt(RC) \leq Opt(ARC) \leq Opt(RC)$, where

$$
\rho = \min \left\{ \alpha \geq 0 \,\middle|\, \mathcal{Z} - (1-\alpha)\frac{5}{2} \subset \mathbb{R}_+ \right\} = \frac{3}{5},
$$

one gets, $\left(\frac{8}{5}\right) Opt(RC) \leq Opt(ARC) \leq Opt(RC)$.

This example shows that the symmetry bound is not tight in the presence of constraint-wise uncertainty, even when there is only one uncertain parameter in the problem. □

Hitherto, we studied examples regarding constraint-wise uncertainty. Now, we consider an example possessing both constraint-wise and non-constraint-wise uncertainties.

Example 3 **(Hybrid uncertainty)** Consider the following uncertain problem,

$$
\begin{aligned}
\min_{y,x} \quad & -x \\
\text{s.t.} \quad & (1-2\alpha)x + y \geq \zeta, \\
& \alpha x - y \geq 0, \\
& x \leq 1,
\end{aligned}
\tag{32}
$$

where $\alpha \in [0,1]$ is a non-constraint-wise and $\zeta \in [-1,0]$ a constraint-wise uncertain parameter, y is an adjustable and x is a non-adjustable variable.

Corollary 1 shows that there exists an optimal decision rule for $(HARC)$ that is independent of ζ. In this example, we check the inequalities in (24). First, we find the optimal objective values of the static and adjustable robust counterparts corresponding to (32), and after that we discuss the dependency of the optimal decision rules on the uncertain parameters in the adjustable robust optimization problem.

To calculate the optimal value of the robust counterpart corresponding to (32), it is sufficient to solve the following problem,

$$
\begin{aligned}
q^*_{RC} = \min_{y,x} \quad & -x \\
\text{s.t.} \quad & x + y \geq 0, \\
& -x + y \geq 0, \\
& -y \geq 0, \\
& x - y \geq 0, \\
& x \leq 1,
\end{aligned}
$$

because the constraints in (3) are linear with respect to the uncertain parameters α and ζ. It means that $(0,0)$ is the only robust feasible solution of (32). Hence, $q^*_{RC} = 0$.

The adjustable robust counterpart corresponding to (32) is as follows:

$$\begin{aligned} q^*_{ARC} = \min_{x} \quad \max_{(\alpha,\zeta)\in\mathcal{Z}} \quad \min_{y(\alpha,\zeta)} \quad & -x \\ \text{s.t.} \quad & (1-2\alpha)x + y(\alpha,\zeta) \geq \zeta, \\ & \alpha x - y(\alpha,\zeta) \geq 0, \\ & x \leq 1, \end{aligned} \tag{33}$$

where (α,ζ) is the uncertain parameter and $\mathcal{Z} = [0,1]\times[-1,0]$ is the uncertainty set. According to the last constraint, $q^*_{ARC} \geq -1$. Fixing $x = 1$, we have

$$\zeta + 2\alpha - 1 \leq y(\alpha,\zeta) \leq \alpha, \tag{34}$$

which means that $q^*_{ARC} = -1$ by choosing $y^*(\alpha,\zeta) = \zeta + 2\alpha - 1$ as the optimal decision rule, which depends on both α and ζ. However, $y^{**}(\alpha,\zeta) = \alpha$ is another optimal decision rule for (33), which is independent of ζ. By this discussion, we showed that for (32) there is only one strict inequality in (24):

$$-1 = Opt(HARC) = Opt(AARC_{\zeta,\alpha}) = Opt(AARC_{\alpha}) < Opt(HRC) = 0. \quad \square$$

D Counterexamples when one of the conditions is not satisfied

In this section, we consider examples in which all of the assumptions of Theorem 1 or 2 are satisfied except one. We name each example to the assumption that is not satisfied.

Example 4 **(Problem without equality constraints)**(rewritten) One of the assumptions we use in this paper is having no equality constraint that is dependent on ζ. Consider the following problem, which has an equality constraint,

$$\begin{aligned} \min \quad & -y_1 \\ \text{s.t.} \quad & \zeta y_1 + y_2 = 1, \\ & 0 \leq y_1, y_2 \leq 10, \end{aligned} \tag{35}$$

where $\zeta \in [1,2]$. It is clear that $Opt(RC) = 0$, since $(0,1)$ is the only robust feasible solution.

To calculate the optimal value of the corresponding (ARC), we eliminate the equality constraint in (35) and reach to the following adjustable robust problem:

$$\begin{aligned} \max_{\zeta\in\mathcal{Z}} \quad \min_{y_1(\zeta)} \quad & -y_1(\zeta) \\ \text{s.t.} \quad & 0 \leq y_1(\zeta) \leq \frac{1}{\zeta}. \end{aligned} \tag{36}$$

It is clear that the optimal value of (36) is $\max_{\zeta\in[1,2]} -\frac{1}{\zeta} = -\frac{1}{2}$. Hence, $Opt(ARC) < Opt(RC)$. These optimal values are different because in the elimination we use the decision rule $y_2 = 1 - \zeta y_1$, which is not allowed in the corresponding (RC). □

Example 5 **(Compact uncertainty set)** In this paper, we assume that the uncertainty set is compact. Consider the following problem

$$\begin{aligned} \min \;\; & -y^2 \\ \text{s.t.} \;\; & y \leq \zeta, \end{aligned} \tag{37}$$

where $\zeta \leq 0$. It is clear that $Opt(RC) = +\infty$ because (RC) is infeasible. However, (ARC) is feasible and $Opt(ARC) = -\infty$. □

Example 6 **("Constraint-wise uncertainty" in Theorem 1)** Ben-Tal et al. [5] consider the following uncertain problem:

$$\begin{aligned} \min \;\; & -x \\ s.t. \;\; & (1-2\zeta)x + y \geq 0, \\ & \zeta x - y \geq 0, \\ & 0 \leq x \leq 1, \\ & |y| \leq 2, \end{aligned}$$

where $\zeta \in [0,1]$ is an uncertain parameter, y is an adjustable variable and x is a non-adjustable variable. It is easy to check that all assumptions hold except "Constraint-wise uncertainty" . The corresponding (RC) can be reformulated as

$$\begin{aligned} \min \;\; & -x \\ s.t. \;\; & x + y \geq 0, \\ & x - y \geq 0, \\ & -x + y \geq 0, \\ & -y \geq 0, \\ & 0 \leq x \leq 1, \\ & |y| \leq 2. \end{aligned}$$

It can easily be checked that $Opt(RC) = 0$. The corresponding (ARC) is as follows:

$$\begin{aligned} \min_{x} \; \max_{\zeta} \; \min_{y(\zeta)} \;\; & -x \\ s.t. \;\; & (1-2\zeta)x + y(\zeta) \geq 0, \\ & \zeta x - y(\zeta) \geq 0, \\ & 0 \leq x \leq 1, \\ & |y(\zeta)| \leq 2. \end{aligned} \tag{38}$$

Similar to the discussion in Example 3, we can verify that $Opt(ARC) = -1$, which means $Opt(ARC) < Opt(RC)$. □

We emphasize that for the next examples, Assumption (v) "Fixed recourse" does not hold.

Example 7 **("Convex uncertainty set" in Theorem 2)** Consider the problem $\min_{y\in\mathcal{Y}} \zeta y$, where $\zeta \in \mathcal{Z}$ is the uncertain parameter, $\mathcal{Z} = \{-1, 2\}$ is the uncertainty set, and $\mathcal{Y} = [-1, 1]$. It is clear that all the assumptions of Theorem 2 hold except (ii) "Convex uncertainty set". It is straightforward that

$$\begin{aligned} Opt(RC) &= \min_{y\in\mathcal{Y}} \max_{\zeta\in\mathcal{Z}} \; \zeta y = \min\{ \min_{y\in[0,1]} \max_{\zeta\in\mathcal{Z}} \; \zeta y, \; \min_{y\in[-1,0]} \max_{\zeta\in\mathcal{Z}} \; \zeta y\} \\ &= \min\{ \min_{y\in[0,1]} 2y, \; \min_{y\in[-1,0]} -y\} \\ &= 0, \end{aligned}$$

and

$$Opt(ARC) = \max_{\zeta\in\mathcal{Z}} \min_{y(\zeta)\in X} \; \zeta y(\zeta) = \max\{\min_{y\in X} \; -y, \min_{y\in X} \; 2y\} = \max\{-1, -2\} = -1.$$

So, $Opt(ARC) < Opt(RC)$. However, if we replace $\mathcal{Z}$ with $Conv(\mathcal{Z})$ then $Opt(RC)$ remains the same but $Opt(ARC)$ becomes zero, which shows that convexity of $\mathcal{Z}$ is crucial to get $Opt(ARC) = Opt(RC)$. □

Example 8 **("Convex adjustable set" in Theorem 2)** As a counterexample for the case that Assumption (iii) "Convex adjustable set" is not satisfied, we can use the problem in Example 7 with $\mathcal{Z} = [-1, 2]$ and $\mathcal{Y} = \{-1, 1\}$. Then, $Opt(ARC) = 0 < Opt(RC) = 1$. □

Example 9 **("Concavity of functions in ζ" in Theorem 2)** Consider the problem

$$\begin{aligned} \min \quad & -y_1 - y_2 \\ s.t. \quad & \zeta^2 + (1-\zeta)y_1 + (1+\zeta)y_2 \leq 3, \\ & |y_i| \leq 3, \quad i = 1, 2, \end{aligned} \tag{39}$$

where $\zeta \in [-1, 1]$ is an uncertain parameter, $y = (y_1, y_2)$ is an adjustable variable. It is clear that (39) is not concave in ζ, but convex in y_1 and y_2. Also, $\mathcal{Z} = [-1, 1]$ and $\mathcal{Y} = \{(y_1, y_2) : \; |y_i| \leq 3, \; i = 1, 2\}$ are compact and convex, and the uncertainty is constraint-wise. The (RC) corresponding to (39) is as follows:

$$\begin{aligned} \min \quad & -y_1 - y_2 \\ s.t. \quad & \max_{\zeta\in[-1,1]} \left[\zeta^2 + (1-\zeta)y_1 + (1+\zeta)y_2\right] \leq 3, \\ & |y_i| \leq 3 \quad i = 1, 2. \end{aligned}$$

Due to the fact that the maximum value of a convex function over a convex set is attained at one of the extreme points [3, Theorem 3.4.7], (RC) is equivalent to the following problem whose optimal objective value is -2,

$$\begin{aligned} \min \quad & -y_1 - y_2 \\ s.t. \quad & y_1 \leq 1, \\ & y_2 \leq 1, \\ & |y_i| \leq 3, \quad i = 1, 2. \end{aligned}$$

To get an upper bound for $Opt(ARC)$, we choose $y_1(\zeta) = \frac{3}{2}(1+\zeta)$ and $y_2(\zeta) = \frac{3}{2}(1-\zeta)$ as a decision rule and it is easy to check feasibility of $(y_1(\zeta), y_2(\zeta))$. Hence, an upper bound for $Opt(ARC)$ is

$$\max_{\zeta\in[-1,1]} -y_1(\zeta) - y_2(\zeta) \;=\; \max_{\zeta\in[-1,1]} -3 \;=\; -3.$$

So, $Opt(ARC) \leq -3 < -2 = Opt(RC)$. $\square$

Example 10 **("Convexity of functions in y" in Theorem 2)** Consider the problem

$$\begin{aligned} \min \;\; & t \\ s.t. \;\; & |y_1| \leq t, \\ & |y_2| \leq t, \\ & -(y_1-\zeta_1)^2 - (y_2-\zeta_1)^2 \leq -4 - 2\zeta_1^2, \\ & -(y_1-\zeta_2)^2 - (y_2-\zeta_2)^2 \leq -4 - 2\zeta_2^2, \\ & |y_i| \leq 5, \quad i = 1,2, \end{aligned} \tag{40}$$

where $\zeta_1 \in [-1,2]$ and $\zeta_2 \in [-2,1]$ are the uncertain parameters, and $y = (y_1, y_2)$ is an adjustable variable. It is easy to check that (40) is concave (and more precisely it is linear) in the uncertain parameter ζ, and the uncertainty is constraint-wise. Also, $\mathcal{Z} = [-1,2] \times [-2,1]$ and $\mathcal{Y} = [-5,5] \times [-5,5]$ are convex and compact. However, the problem is not convex in the adjustable variable $y = (y_1, y_2)$. The (RC) corresponding to (40) is equivalent to

$$\begin{aligned} \min \;\; & \|y\|_\infty \\ s.t. \;\; & (y_1+1)^2 + (y_2+1)^2 \geq 6, \\ & (y_1-2)^2 + (y_2-2)^2 \geq 12, \\ & (y_1-2)^2 + (y_2+2)^2 \geq 12, \\ & (y_1+1)^2 + (y_2-1)^2 \geq 6, \\ & |y_i| \leq 5, \quad i = 1,2. \end{aligned} \tag{41}$$

It is easy to verify that the optimal solution is $y_1 = -\frac{2+\sqrt{14}}{5} \approx -1.15$ and $y_2 = \frac{5+\sqrt{127+6\sqrt{14}}}{5} \approx 3.44$ with the approximated objective value 3.44 for the problem. We choose

$$y_1(\zeta) = \begin{cases} -1.7, & \zeta_2 \leq 0.3 \\ 1.6, & o.w. \end{cases} \quad , \qquad y_2(\zeta) = \begin{cases} 2.2, & \zeta_2 \leq 0.3 \\ -1.6, & o.w. \end{cases}$$

as a decision rule to find an upper bound for $Opt(ARC)$. The feasibility of the decision rule can easily be checked and it implies that $Opt(ARC) \leq 2.2 < 3.44 \approx Opt(RC)$. $\square$