

Multi-period fund performance evaluation: A dynamic network DEA approach with diversification and the directional distance function

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Abstract When analyzing the relative performance of mutual funds, current data envelopment analysis (DEA) models with diversification only consider risks and returns over the entire investment process, which ignore the performance change in consecutive periods. This paper introduces a novel multi-period network DEA approach with diversification and the directional distance function. The new approach decomposes the overall efficiency of a mutual fund in the whole investment interval into efficiencies at individual periods. At each period, mutual funds consume exogenous inputs and intermediate products produced from the preceding period to produce exogenous outputs and intermediate products for the next period to use. Efficiency decomposition reveals the time at which the inefficiency happens. The new model can provide expected inputs, outputs and intermediate variables at individual periods, which are helpful for fund managers to take effective ways to improve the fund performance. Under the assumption of discrete return distributions and a proper choice of inputs, outputs and intermediate variables, the proposed models can be transformed into linear programs. The applicability and reasonability of the proposed method are demonstrated by applying it to assess the relative performance of 40 open-ended funds in Chinese security markets.

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1 Introduction

Due to its academic and practical importance, mutual fund performance evaluation has been an important topic in financial management. Recently, data envelopment analysis (DEA) has been adopted for assessing the performance of mutual funds. DEA is a non-parametric approach for measuring the relative performance of homogeneous decision making units (DMUs) (Charnes et al. 1978). When applied to the fund performance evaluation, the DEA technique has the following advantages. Firstly, as a non-parametric analysis technique, DEA does not rely on any theoretical model as the measurement benchmark. Instead, DEA measures how well a fund performs with respect to efficient funds within the reference category. Secondly, DEA reveals the reason for the inefficiency of a fund and shows how to restore the fund to its optimum level of efficiency. Finally, different from other performance measures, the DEA approach can incorporate multiple factors that are associated with the fund performance.

In the earlier studies, the linear programming (LP) DEA models are directly applied to evaluate the performance of mutual funds. Typical examples include studies which consider factors such as standard deviation, transaction costs and expected return (Basso and Funari 2001; Chen and Lin 2006; Lin and Chen 2008; Murthi et al. 1997), standard semideviation, β -coefficient and the percentage of non-dominated subperiods (Basso and Funari 2001, 2005; Chen and Lin 2006), traditional performance indices, i.e., the Sharpe index, the Treynor index and the Jensen's α (Basso and Funari 2005; Lin and Chen 2008), lower and upper mean semi-skewness and semi-variance (Gregoriou et al. 2005), the percentage of periods with negative returns, skewness (Wilkens and Zhu, 2001), returns over different lengths of time periods (McMullen and Strong 1998), and the ethical score (Basso and Funari 2003). Table 1 gives a summary of these LP DEA approaches. All these approaches fail to take into account the effect of portfolio diversification. They use the linear combination of the risk measures of individual funds' returns to replace the risk measure of the benchmark returns, which are a linear combination of the funds' returns, and thus overestimate the efficiency of funds.

Recently, some DEA-like models considering the effect of portfolio diversification have been proposed in the literature. Table 2 presents a summary of typical DEA models with diversification. Based on the mean-variance (MV) approach, Morey and Morey (1999) propose two quadratic programming DEA models from a multi-period perspective. The difference between the two models is: one is aimed at augmenting the expected return with no increase in the variance, the other is aimed at contracting the variance with no decrease in the expected return. Although these two models are non-linear, their feasible regions are convex and the objective functions are linear, it is thus easy to obtain their global optimization solutions. Bricc and Kerstens (2009) propose another multi-period MV model based on the directional distance function (DDF) DEA model (Chambers et al. 1998). Different from the method in Morey and Morey (1999), the approach in Bricc and Kerstens (2009) simultaneously attempts to reduce variance and to increase expected return over all time periods along the direction of a given vector.

Table 1 Summary of LP DEA models

Research	Inputs	Outputs
Murthi et al. (1997)	standard-deviation, expense ratio, turnover, loads	expected return
McMullen and Strong (1998)	standard-deviation, minimum initial investment, expense ratio, sales charge	1,3 and 5 year expected returns
Choi and Murthi (2001)	standard-deviation, expense ratio, turnover, loads	expected returns
Basso and Funari (2001)	standard deviation, β -coefficient, subscription and redemption costs	expected return, the percentage of non-dominated subperiods
Basso and Funari (2003)	standard-deviation, standard semideviation, β -coefficient, subscription and redemption costs	expected return, ethical score
Basso and Funari (2005)	standard-deviation, standard semideviation, β -coefficient, subscription and redemption costs	expected return, the percentage of non-dominated subperiods, Sharpe index, Treynor index, Jensen's α
Wilkins and Zhu, (2001)	standard-deviation, proportion of negative monthly returns during the year	expected return, skewness, minimum return
Gregoriou et al. (2005)	lower mean, lower semi-variance, lower semi-skewness	upper mean, upper semi-variance, upper semi-skewness
Chen and Lin (2006)	standard-deviation, standard semideviation, β -coefficient, VaR, CVaR, annually total cost	expected return, the percentage of non-dominated subperiods, Sharpe index, Treynor index, Jensen's α
Lin and Chen (2008)	standard-deviation, standard semideviation, β -coefficient, VaR, CVaR, turnover ratio, expense ratio, redemption fee, loads	expected return, Sharpe index, Treynor index, Jensen's α

Joro and Na (2006) develop a portfolio performance measure based on the mean-variance-skewness framework by utilizing the input-oriented DEA model under the assumption of non-increasing returns to scale (NIRS). Lozano and Gutiérrez (2008b) suggest three different DEA models, which are based on the mean-risk model and the additive DEA model. These three models are consistent with third-order stochastic dominance, but all of them can only deal with one risk measure and one return measure. The above studies account for diversification directly by using some nonlinear versions of DEA. Different with them, Lozano and Gutiérrez (2008a) combine the second-order stochastic dominance (SSD) criterion with the additive DEA model and propose six DEA models with diversification. All the six models are linear, but similar to the above three models, they can only handle one risk measure and one return measure.

Recently, Lamb and Tee (2012a) introduce a non-linear DEA model with diversification, which can deal with arbitrary number of risk measures and return measures. To avoid negative inputs, Lamb and Tee (2012a) use the positive parts of coherent risk measures as the inputs. However, this treatment loses significantly the information contained in the risk measures about the distribution of random parts (Branda 2015). Branda and Kopa (2014) transform negative risk measures into return measures and consider them as the outputs. Moreover, they show that the SSD efficiency can be equivalent to a radial DEA-risk efficiency under a suitable choice of inputs and outputs. However, their models have no fixed structure for all investigated investment opportunities, and thus cannot be used to compare inefficiency scores (Branda 2015). Branda (2013) suggests using a class of general deviation measures as the inputs. The general deviation measures are introduced by Rockafellar et al. (2006b) as a natural extension of standard deviation, which are convex and positive for non-constant investment opportunities, thus they are suitable as inputs for radial DEA models. Based on the DDF, Branda (2015)

proposes several diversification-consistent DEA models suitable for assessing the efficiency of investment opportunities. These models can adopt several risk measures as inputs and multiple return measures as outputs, which can take both positive and negative values. It should be noted that the DEA models proposed by Branda (2013), Branda and Kopa (2014) and Branda (2015) can be formulated as LP problems for discretely distributed returns.

Table 2 Summary of diversification DEA models

Research	Programming model	Inputs	Outputs
Morey and Morey (1999)	quadratic	covariance, variance	expected return
Briec and Kerstens (2009)	quadratic	covariance, variance	expected return
Joro and Na (2006)	non-linear	variance	expected return, skewness
Lozano and Gutiérrez (2008a)	non-linear	lower standard semideviation, second-order lower partial moment	expected return, above target mean return
Lozano and Gutiérrez (2008b)	linear	downside absolute semi-deviation, mean downside under-achievement, weighted absolute deviation from quantile, CVaR, below target semi-deviation	expected return, target mean return
Lamb and Tee (2012a)	non-linear	CVaR	expected return
Branda (2013)	non-linear	CVaR deviation	expected return
Branda and Kopa (2014)	non-linear	positive part of CVaR	negative part of CVaR, expected return
Branda (2015)	non-linear	CVaR	expected return

Except for those proposed by Morey and Morey (1999) and Briec and Kerstens (2009), all the above DEA models with diversification simply treat the whole investment process of mutual funds as a black-box and ignore the inner dynamic variation. Therefore, these DEA models with diversification could not provide the information on operations and performance variations at intermediate investment periods. In practice, most mutual fund investment problems are medium-term or long-term decision making problems. Ignoring the intermediate performance change might result in misleading results. For instance, a fund measured as efficient by these DEA models could be inefficient over most of times in the observed horizon. On the other hand, although Morey and Morey (1999) and Briec and Kerstens (2009) appraise the multi-horizon portfolio performance of mutual funds, both of them ignore the dynamic dependence among periods. In practice, there often exist some relationships between two adjacent investment periods. For example, if an investor holds a fixed number of a mutual fund during the whole investment process, the total value of this fund at the end of the current period can be viewed as the capital input at the beginning of the next period. Ignoring the inter-relationship among periods reduces greatly the information contained in the whole investment process.

The traditional DEA technique is proposed to measure the performance of a DMU in a static manner in a specified period, without considering its internal variations. When several investment periods with inter-dependence are involved, the efficiency over the whole investment process must be measured in a dynamic manner, taking into account the inter-relationship between consecutive periods. This means that one should use DEA models to describe the inter-relationships among individual periods and apply the associated solution methods to calcu-

late the relative efficiency for a set of multi-period DMUs (Kao 2013). Network DEA is a technique to measure the performance of a DMU in a dynamic manner. Färe and Grosskopf (1996) introduce the dynamic aspects of production into the conventional DEA model when multi-outputs are involved. They propose several inter-temporal models, which are the basis for many later studies on DEA. Färe and Grosskopf (2000) propose three network DEA models, which can be applied to a variety of situations: intermediate products, the allocation of budgets or fixed factors and certain (time separable) dynamic systems. In the last few years, network DEA has become a hot research topic in the field of DEA, which results in the introduction of the relational network DEA approach (Kao 2009, 2014a,c, 2013; Kao and Hwang 2010, 2008), the weighted additive efficiency decomposition approach (Cook et al. 2010; Chen et al. 2009); the slacks-based network DEA approach (Kao 2014b; Tone and Tsutsui 2009, 2010, 2014; Lozano 2015) and other models (Fukuyama and Weber 2010; Li et al. 2012; Maghbouli et al. 2014). In addition to theoretical studies, a growing number of applications have been developed in different fields, such as incineration plants (Chen et al. 2012), finance (Premachandra et al. 2012), regional innovation systems (Chen and Guan 2012) and transportation (Lozano et al. 2013).

Although there are many network DEA approaches, none of them is suitable for measuring the performance of a fund since they do not take into account the effect of portfolio diversification. In this paper, we investigate the relative performance of mutual funds by introducing a novel multi-period network DEA model with diversification, where the overall efficiency of a mutual fund over the entire investment horizon is decomposed into stage-wise efficiencies at individual periods. Main contributions of this paper include:

- (i) We decompose the overall efficiency of a mutual fund over the entire investment horizon into efficiencies at individual periods in the chronological order. The proposed DEA method can assess both the overall efficiency over the entire horizon and stage-wise efficiencies at individual periods.
- (ii) Any two adjacent periods are connected by an intermediate variable. In our study, the net asset value is chosen as the intermediate variable.
- (iii) Unlike existing studies on network DEA, we consider diversification in our multi-period network DEA model, which is very important for financial applications. Therefore, our approach broadens the application of the network DEA method in the financial management area.
- (iv) Our approach not only considers risk measures and return measures, but also takes into account other important inputs and/or outputs without diversification, such as transaction costs.
- (v) Our method can reveal the periods at which the inefficiency of a fund happens and can provide expected inputs, outputs and intermediate variables which make an inefficient fund efficient over the entire investment horizon.
- (vi) Different from previous DEA models with diversification, our method is inconsistent with stochastic dominance. This is due to that the stochastic dominance approach cannot treat intermediate variables, which act as inputs as well as outputs.
- (vii) For discretely distributed returns and an appropriate choice of inputs, outputs and intermediate variables, the proposed models can be transformed into LPs.

The rest of the paper is organized as follows. Section 2 introduces a generic diversification DEA model based on the direction distance function, which assesses a fund's performance in terms of reducing inputs and expanding outputs simultaneously. Section 3 decomposes the whole investment process into a number of successive periods and then, based on the proposed diversification DEA model, develops a novel multi-period network DEA method. Section 4 describes the sample data from the Chinese fund market, selects input-output variables and directions, and presents empirical results. Some concluding remarks are presented in Section 5.

2 Generic diversification DEA model based on the directional distance function

Suppose there are n funds. We denote the return of the j th ($j = 1, \dots, n$) fund as R_j , which is a random variable defined on the probability space (Ω, \mathcal{F}, P) . According to Branda (2013), we choose the following set of investment opportunities:

$$\mathcal{R} = \left\{ \sum_{j=1}^n \lambda_j R_j : \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n \right\}, \quad (1)$$

where λ_j s are weight variables. \mathcal{R} allows the full diversification of a portfolio across all funds. Other choices of the investment opportunity set are also possible, e.g., the set which allows short sales, or restricts the number of investments in specific funds (Branda 2013; Briec et al. 2004). Note that the constraint $\sum_{j=1}^n \lambda_j = 1$ in \mathcal{R} ensures that all the possible portfolios are completely composed by the considered funds. This also means that our method is proposed under the assumption of variable returns to scale (Banker et al. 1984).

The performance of a target fund o , $o \in \{1, \dots, n\}$, can be evaluated by comparing its inputs and outputs with those corresponding inputs and outputs of all possible fund portfolios in \mathcal{R} . From Tables 1 and 2, we can see that transaction costs, as important factors affecting the efficiency of a fund (Murthi et al. 1997; Basso and Funari 2001), are not included in current DEA models with diversification. In this paper, we propose a new diversification DEA model by taking into account risk measures and transaction costs as inputs and return measures as outputs. We assume that only risk measures and return measures possess diversification since transaction costs do not show diversification in general. The directional distance function (DDF) is adopted to design this new model. An advantage of the DDF is that, by choosing a proper direction vector, the produced model can deal with negative inputs and outputs.

Let m be the number of risk measures, l the number of transaction costs, s the number of return measures. We denote the i th ($i = 1, \dots, m$) risk measure as $x_i(\cdot)$, the f th ($f = 1, \dots, l$) transaction cost as c_{fj} , the r th ($r = 1, \dots, s$) return measure as $y_r(\cdot)$. By referring to Branda (2013), we define the production possibility set (PPS) taking into account diversification as follows:

$$DT = \left\{ (x_i, c_f, y_r) \mid x_i \geq x_i \left(\sum_{j=1}^n \lambda_j R_j \right), i = 1, \dots, m; c_f \geq \sum_{j=1}^n \lambda_j c_{fj}, f = 1, \dots, l; \right.$$

$$y_r \leq y_r \left(\sum_{j=1}^n \lambda_j R_j \right), \quad r = 1, \dots, s; \quad \sum_{j=1}^n \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \dots, n\}.$$

Consider a target fund $o \in \{1, \dots, n\}$ with the input-output bundle $(x_i(R_o), c_{fo}, y_r(R_o))$ and a nonnegative and nonzero input-output direction vector $(\mu_i(R_o), \omega_{fo}, \nu_r(R_o))$. Based on the SBI measure (Fukuyama and Weber 2009), we introduce a generalized directional distance function (DDF) with respect to the PPS DT as

$$D(x_i(R_o), c_{fo}, y_r(R_o); \mu_i(R_o), \omega_{fo}) = \max_{\lambda_j, \beta_i^-, \tilde{\beta}_f^-, \beta_r^+} \frac{\frac{1}{m+l} \left(\sum_{i=1}^m \beta_i^- + \sum_{f=1}^l \tilde{\beta}_f^- \right) + \frac{1}{s} \sum_{r=1}^s \beta_r^+}{2} : \\ (x_i(R_o) - \beta_i^- \mu_i(R_o), c_{fo} - \tilde{\beta}_f^- \omega_{fo}, y_r(R_o) + \beta_r^+ \nu_r(R_o)) \in DT. \quad (2)$$

Unlike the conventional DDF (Chambers et al. 1998), the DDF defined in (2) considers the slacks and thus the relevant results can deal with strong Pareto-efficiency. From the definition of the PPS DT and the DDF in (2), the inefficiency of the fund o can be evaluated by solving the following model:

$$D^* = \max_{\lambda_j, \beta_i^-, \tilde{\beta}_f^-, \beta_r^+} \frac{\frac{1}{m+l} \left(\sum_{i=1}^m \beta_i^- + \sum_{f=1}^l \tilde{\beta}_f^- \right) + \frac{1}{s} \sum_{r=1}^s \beta_r^+}{2} \quad (3)$$

$$\text{s.t.} \quad x_i \left(\sum_{j=1}^n \lambda_j R_j \right) \leq x_i(R_o) - \beta_i^- \mu_i(R_o), \quad i = 1, \dots, m, \quad (4)$$

$$(M^S) \quad \sum_{j=1}^n \lambda_j c_{fj} \leq c_{fo} - \tilde{\beta}_f^- \omega_{fo}, \quad f = 1, \dots, l, \quad (5)$$

$$y_r \left(\sum_{j=1}^n \lambda_j R_j \right) \geq y_r(R_o) + \beta_r^+ \nu_r(R_o), \quad r = 1, \dots, s, \quad (6)$$

$$\sum_{j=1}^n \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \dots, n, \quad (7)$$

$$\beta_i^-, \tilde{\beta}_f^-, \beta_r^+ \geq 0, \quad \forall i, f, r. \quad (8)$$

Different from the DEA models in Branda (2015), the objective function of model (M^S) is linear. This property can simplify calculations of the multi-period network DEA model to be proposed. According to Fukuyama and Weber (2009), D^* is a shortage measure of the technical inefficiency for the target fund o . If $D^* = 0$, the fund o is efficient; otherwise, the fund o is inefficient. Let $\beta_i^{-*}, \tilde{\beta}_f^{-*}, \beta_r^{+*}$ denote the optimal variables of model (M^S) . The values of $\beta_i^{-*}, \tilde{\beta}_f^{-*}$ and β_r^{+*} indicate the necessary percentage change in order to catch up with the efficient frontier in terms of input reduction and output expansion along the direction $(\mu_i(R_o), \omega_{fo}, \nu_r(R_o))$. For example, when $\beta_i^{-*} = \tilde{\beta}_f^{-*} = 0.1, \beta_r^{+*} = 0.3, \forall i, f, r$, then by augmenting risks and transaction costs by 10% and meanwhile contracting outputs by 30% along the direction, the target fund o can join the efficient frontier.

The direction vector $(\mu_i(R_o), \omega_{fo}, \nu_r(R_o))$ should be nonnegative and nonzero. In order to deal with inputs and outputs that possibly take negative values, we

choose

$$\mu_i(R_o) = x_i(R_o) - \min_{R \in \mathcal{R}} x_i(R), \quad \omega_{f_o} = c_{f_o} - \min_j c_{f_j}, \quad \nu_r(R_o) = \max_{R \in \mathcal{R}} y_r(R) - y_r(R_o), \quad (9)$$

and if $\mu_i(R_o)$, ω_{f_o} or $\nu_r(R_o)$ is zero, the corresponding β_i^- , $\tilde{\beta}_f^-$ or β_r^+ is dropped from the objective function of model (M^S). The above choices of the directions are reasonable, see Portela et al. (2004) and Branda (2015). Moreover, we can show that $D^* \in [0, 1]$.

Theorem 1 $D^* \in [0, 1]$.

Proof From the constraints of model (M^S) and (9), we have

$$\begin{aligned} 0 \leq \beta_i^- &\leq \frac{x_i(R_o) - x_i(\sum_{j=1}^n \lambda_j R_j)}{\mu_i(R_o)} = \frac{x_i(R_o) - x_i(\sum_{j=1}^n \lambda_j R_j)}{x_i(R_o) - \min_{R \in \mathcal{R}} x_i(R)} \leq 1, \quad i = 1, \dots, m, \\ 0 \leq \tilde{\beta}_f^- &\leq \frac{c_{f_o} - \sum_{j=1}^n \lambda_j c_{f_j}}{\omega_{f_o}} = \frac{c_{f_o} - \sum_{j=1}^n \lambda_j c_{f_j}}{c_{f_o} - \min_j c_{f_j}} \leq 1, \quad f = 1, \dots, l, \\ 0 \leq \beta_r^+ &\leq \frac{y_r(\sum_{j=1}^n \lambda_j R_j) - y_r(R_o)}{\nu_r(R_o)} = \frac{y_r(\sum_{j=1}^n \lambda_j R_j) - y_r(R_o)}{\max_{R \in \mathcal{R}} y_r(R) - y_r(R_o)} \leq 1, \quad r = 1, \dots, s, \end{aligned}$$

for all possible i, f, r satisfying $\mu_i(R_o), \omega_{f_o}, \nu_r(R_o) \neq 0$. Then, the optimal value of problem (M^S) satisfies $D^* \in [0, 1]$. \square

With Theorem 1, similar to most studies on the DDF-based DEA models (Ray 2008; Chen et al. 2013), we adopt $1 - D^*$, which lies in $[0, 1]$, to measure the efficiency of the target fund in this paper.

Note that an underlying assumption of the investment opportunity set \mathcal{R} , PPS DT and model (M^S) is that investors are interested in “limited diversification”, where all possible portfolios constructed by the linear combination of n funds are considered, as Branda (2013) did. It is not practical to consider “full diversification” in this paper, where the funds are compared with an artificial fund based on all financial assets that can be used to construct the funds. This is because that it is difficult to get the related data in real financial markets and the resulting models will be very large, which bring the enormous difficulty in problem solution.

3 Multi-period network DEA model with diversification

The proposed model (M^S) simply treats the whole investment process of funds as a “single” investment process, which consumes all the inputs and produces all the outputs in one move. Such a treatment is not suitable for assessing the efficiency of funds in medium-term or long-term investment problems since it cannot provide the information on operations or performance changes in internal investment periods. The network DEA technique takes into account the internal configuration of DMUs and decomposes the production process of each DMU into several sub-processes (Lozano et al. 2013). Inspired by the network DEA, we decompose the whole investment process into a number of successive periods. At each period,

funds consume exogenous inputs and intermediate products produced by the preceding period, and produce exogenous outputs and intermediate products for the next period to use.

Suppose there are n funds, and the return of the fund j ($j = 1, \dots, n$) is a random process defined on some probability space (Ω, \mathcal{F}, P) , with \mathcal{F} denoting the set of subsets of Ω , and P a probability measure assigning to any event B in \mathcal{F} its probability $P(B)$. We assume that there are $T + 1$ time points: $0, 1, 2, \dots, T$, and thus T consecutive investment periods. At each time point t , $\mathcal{F}_t \subseteq \mathcal{F}$ denotes the set of events corresponding to the information available until t . We define the space of random variables at period t , $t = 0, 1, \dots, T$, as $\mathcal{L}_t = \mathcal{L}_p(\Omega, \mathcal{F}_t, P)$, $p \in [1, +\infty]$. We suppose that the observed random process $R_j = (R_j^t)$, $R_j^t \in \mathcal{L}_t$, $t = 0, 1, 2, \dots, T$, is adapted to the filtration $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T)$, here $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, $t = 0, 1, \dots, T - 1$, $\mathcal{F}_T = \mathcal{F}$, and $\mathcal{F}_0 = \{0, \Omega\}$ is the trivial σ -algebra of the 0-th period. Similar to \mathcal{R} defined in (1), we consider the following investment opportunity sets for each period t and for the entire investment interval, respectively,

$$\mathcal{R}^t = \left\{ \sum_{j=1}^n \lambda_j^t R_j^t : \sum_{j=1}^n \lambda_j^t = 1, \lambda_j^t \geq 0, j = 1, \dots, n \right\}, \quad (10)$$

$$\mathcal{RT} = \left\{ \left(\sum_{j=1}^n \lambda_j^1 R_j^1, \dots, \sum_{j=1}^n \lambda_j^T R_j^T \right) : \sum_{j=1}^n \lambda_j^t R_j^t \in \mathcal{R}^t, t = 1, \dots, T \right\}, \quad (11)$$

where λ_j^t s are the combination weights at period t . For any portfolio $\sum_{j=1}^n \lambda_j^t R_j^t$ belonging to \mathcal{R}^t , let $x_i(\sum_{j=1}^n \lambda_j^t R_j^t)$ be the i th ($i = 1, \dots, m$) risk measure, $\sum_{j=1}^n \lambda_j^t c_{fj}^t$ be the f th ($f = 1, \dots, l$) transaction cost and $y_r(\sum_{j=1}^n \lambda_j^t R_j^t)$ be the r th ($r = 1, \dots, s$) return measure.

If investors do not cash their funds during the investment process, then the value of a mutual fund at the end of the preceding period is equal to the investment capital at the beginning of the current period. On the other hand, the value of a mutual fund at the end of each period represents a part of the wealth that an investor holds, which can be viewed as an income for the investor. Due to these reasons, we treat the net asset value (called NAV for short) of a fund at the end of each period as an intermediate variable. That is, for each fund, the NAV is not only an output for the preceding period, but also an input for the current period. Denote the NAV at the end of period t ($t = 1, \dots, T$) as z_j^t and the initial NAV as z_j^0 . The detailed decomposition of the investment process of each fund j is illustrated in Fig. 1.

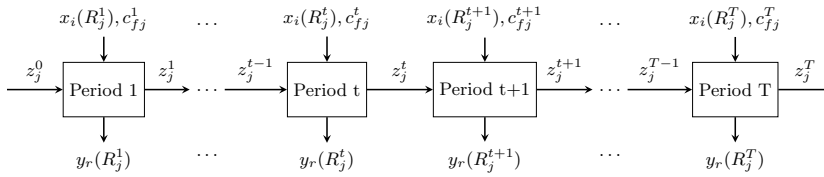


Fig. 1. The detailed decomposition of the investment process of fund j

As we can see from Fig. 1, z_j^0 and z_j^T should be treated as an input for period 1 and an output for period T , respectively, and all other z_j^t s, $1 \leq t \leq T - 1$, are intermediate variables for periods 1 to $T - 1$, respectively. Under this decomposition,

one cannot treat these T periods independently and apply model (M^S) separately for each period. The reason is: z_j^t ($t = 1, \dots, T-1$) is not only an output to the period t , but also an input to the period $t+1$. According to Liang et al. (2008), two adjacent periods can be treated as players in a cooperative game where both players “negotiate” on the expected intermediate value. The network DEA technique can deal with such an inter-relationship between two adjacent periods and can determine the relative efficiency for a set of DMUs in a dynamic manner. However, as we mentioned in Section 1, the current network DEA approaches cannot be directly applied to the investment process depicted in Fig. 1, since they do not consider the diversification which is necessary for financial applications.

Let ρ^t be the efficiency measure of the target fund o ($o \in \{1, \dots, n\}$) at the t th ($t = 1, \dots, T$) period. For each fund, the overall efficiency in the whole investment interval should be made up of the efficiencies at individual periods. In this paper, we compute the overall efficiency in the whole investment interval as a weighted average of the efficiency scores at periods 1 to T . That is, the overall efficiency is determined as $\sum_{t=1}^T w_t \rho^t$, where w_t s are weights satisfying $\sum_{t=1}^T w_t = 1$. For the purpose of practicality and implementability, we pre-specify w_t s. Otherwise, the corresponding model is nonlinear and time-consuming to solve. A reasonable choice of w_t should reflect the relative importance or contribution of the efficiency at period t to the overall efficiency. Inspired by Briec and Kerstens (2009), we adopt a retrospective benchmark: the present is more important than the past, so we attribute the largest weight to the efficiency score at the most recent period. Specifically, we introduce a time discount factor denoted as ξ ($0 < \xi < 1$), and set $w_t = \frac{(1-\xi)\xi^{T-t}}{1-\xi^T}$, $t = 1, \dots, T$. Therefore, we have the following definition for the overall efficiency.

Definition 1 For any target fund $o \in \{1, \dots, n\}$, its overall efficiency score during the whole investment process is defined as

$$E^O = \frac{1-\xi}{1-\xi^T} \sum_{t=1}^T \xi^{T-t} \rho^t.$$

Definition 1 gives a weighted inter-temporal efficiency measure, where the weights become smaller and smaller as one moves away from the present to the past.

Based on model (M^S) and the network DEA technique, we develop the following model for measuring the overall efficiency of the target fund.

$$E^* = \min \frac{1-\xi}{1-\xi^T} \sum_{t=1}^T \xi^{T-t} \rho^t \quad (12)$$

$$\text{s.t. } \rho^t = 1 - \frac{\frac{1}{m+l} \left(\sum_{i=1}^m \beta_i^{-t} + \sum_{f=1}^l \tilde{\beta}_f^{-t} \right) + \frac{1}{s} \sum_{r=1}^s \beta_r^{+t}}{2}, \quad t = 1, \dots, T-1, \quad (13)$$

$$\rho^T = 1 - \frac{\frac{1}{m+l} \left(\sum_{i=1}^m \beta_i^{-t} + \sum_{f=1}^l \tilde{\beta}_f^{-t} \right) + \frac{1}{s+1} \sum_{r=1}^{s+1} \beta_r^{+T}}{2}, \quad (14)$$

$$x_i \left(\sum_{j=1}^n \lambda_j^t R_j^t \right) \leq x_i(R_o^t) - \beta_i^{-t} \mu_i(R_o^t), \quad i = 1, \dots, m, \quad t = 1, \dots, T, \quad (15)$$

$$\sum_{j=1}^n \lambda_j^t c_{fj}^t \leq c_{fo}^t - \tilde{\beta}_f^{-t} \omega_{fo}^t, \quad f = 1, \dots, l, \quad t = 1, \dots, T, \quad (16)$$

$$(M^o) \quad y_r \left(\sum_{j=1}^n \lambda_j^t R_j^t \right) \geq y_r(R_o^t) + \beta_r^{+t} \nu_r(R_o^t), \quad r = 1, \dots, s, \quad t = 1, \dots, T, \quad (17)$$

$$\sum_{j=1}^n \lambda_j^1 z_j^0 = z_o^0, \quad (18)$$

$$\sum_{j=1}^n \lambda_j^t z_j^t = \sum_{j=1}^n \lambda_j^{t+1} z_j^t, \quad t = 1, \dots, T-1, \quad (19)$$

$$\sum_{j=1}^n \lambda_j^T z_j^T \geq z_o^T + \beta_{s+1}^{+T} \eta^T, \quad (20)$$

$$\sum_{j=1}^n \lambda_j^t = 1, \quad \lambda_j^t \geq 0, \quad j = 1, \dots, n, \quad t = 1, \dots, T, \quad (21)$$

$$\beta_i^{-t}, \tilde{\beta}_f^{-t}, \beta_r^{+t} \geq 0, \quad \forall i, f, r, \quad (22)$$

where $\beta_i^{-t}, \tilde{\beta}_f^{-t}, \beta_r^{+t}$ are the shortage measures of the input and output inefficiency of the target fund o ($o \in \{1, \dots, n\}$) at the t th ($t = 1, \dots, T$) period, and $(\mu_i(R_o^t), \omega_{fo}^t, \nu_r(R_o^t))$ is the nonnegative direction vector with respect to the input-output vector $(x_i(R_o^t), \omega_{fo}^t, y_r(R_o^t))$ for the target fund o at period t ($t = 1, \dots, T$), $\eta_z^T > 0$ is the direction variable with respect to z_o^T . Like the direction vector we adopted in Section 2, here

$$\mu_i(R_o^t) = x_i(R_o^t) - \min_{R \in \mathcal{R}^t} x_i(R), \quad \omega_{fo} = c_{fo}^t - \min_j c_{fj}^t, \quad (23)$$

$$\nu_r(R_o^t) = \max_{R \in \mathcal{R}^t} y_r(R) - y_r(R_o^t), \quad \forall t, \quad \eta^T = \max_j z_j^T - z_o^T. \quad (24)$$

Model (M^o) can simultaneously realize the reduction in inputs, the expansion in outputs and the balance among intermediate variables at individual periods. The effects of the objective function and constraints of model (M^o) are explained as follows:

(i) From constraints (13) and (14), the objective function can be rewritten as

$$\frac{1-\xi}{1-\xi^T} \sum_{t=1}^T \xi^{T-t} \rho^t = 1 - \frac{1-\xi}{1-\xi^T} \left[\sum_{t=1}^{T-1} \xi^{T-t} \frac{\frac{1}{m+l} \left(\sum_{i=1}^m \beta_i^{-t} + \sum_{f=1}^l \tilde{\beta}_f^{-t} \right) + \frac{1}{s} \sum_{r=1}^s \beta_r^{+t}}{2} \right. \\ \left. + \frac{\frac{1}{m+l} \left(\sum_{i=1}^m \beta_i^{-T} + \sum_{f=1}^l \tilde{\beta}_f^{-T} \right) + \frac{1}{s+1} \sum_{r=1}^{s+1} \beta_r^{+T}}{2} \right].$$

Hence, minimizing $\frac{1-\xi}{1-\xi^T} \sum_{t=1}^T \xi^{T-t} \rho^t$ leads to maximizing

$$\sum_{t=1}^{T-1} \xi^{T-t} \frac{\frac{1}{m+l} \left(\sum_{i=1}^m \beta_i^{-t} + \sum_{f=1}^l \tilde{\beta}_f^{-t} \right) + \frac{1}{s} \sum_{r=1}^s \beta_r^{+t}}{2} + \frac{\frac{1}{m+l} \left(\sum_{i=1}^{m+l} \beta_i^{-T} + \sum_{f=1}^l \tilde{\beta}_f^{-T} \right) + \frac{1}{s+1} \sum_{r=1}^{s+1} \beta_r^{+T}}{2}.$$

(ii) According to the efficiency measure defined in Section 2, we adopt constraints (13) and (14) to calculate the efficiency measure of the target fund o ($o \in \{1, \dots, n\}$) at the t th period. By using a similar proof as that of Theorem 1, we can easily prove that $\rho^t \in [0, 1]$, $t = 1, \dots, T$. Therefore, the optimal value of problem (M^o) also lies in $[0, 1]$.

- (iii) Constraints (15)-(17) are used to evaluate the efficiency of the target fund by simultaneously reducing inputs and expanding outputs along the direction $(\mu_i(R_o^t), \omega_{f_o}^t, \nu_r(R_o^t))$ at each period t , $t = 1, \dots, T$.
- (iv) It seems more reasonable to treat the NAV at the initial period ($t = 0$), i.e., z_j^0 , as a non-discretionary input for the period 1. We hence do not reduce it in the constraint (18).
- (v) Constraints in (19) ensure that the expected intermediate variable from period t ($t = 1, \dots, T-1$), which is treated as an output, is equal to the expected intermediate variable at period $t+1$, which is treated as an input.
- (vi) Constraint (20) is imposed since z_j^T is an output for the final period T .
- (vii) Constraints in (21) are used to constrain portfolio weights at each period t , which correspond to investment opportunity sets in (10) and (11).
- (viii) Constraints in (22) are introduced to guarantee the non-negativity of all β_i^{-t} s, $\tilde{\beta}_f^{-t}$ s and β_r^{+t} s.

Previous works about the multi-period fund performance evaluation (Briec and Kerstens 2009; Morey and Morey 1999) did not consider the linkage among individual periods and moreover, only the mean and the variance of returns were adopted as the single input and the single output, respectively. Instead, we adopt the sequential structure depicted in Fig. 1. Our approach considers linkages between any two adjacent periods, as well as multiple inputs and multiple outputs. As a consequence, our model is more generic.

Problem (M^O) might have multiple optimal solutions with respect to ρ^t , the efficiency scores of individual periods yielded by problem (M^O) may not be unique. Considering that the shortage measure of the inefficiency at the present period is more important than that at previous periods, we minimize ρ^t period-wisely in a backward way. That is, we minimize ρ^t ($1 \leq t \leq T$) from period T to period 1 in turn while maintaining the overall efficiency score during the entire investment process. For $\tau = T, \dots, 2$, we denote the uniquely determined optimal ρ^τ as $\rho^{\tau,*}$, which equals to the optimal value of the following programming problem:

$$\min \rho^\tau \quad (25)$$

$$\text{s.t. } \frac{1-\xi}{1-\xi^T} \sum_{t=1}^T \xi^{T-t} \rho^t = E^*, \quad (26)$$

$$\rho^t = \rho^{t,*}, \quad t = T, \dots, \tau + 1, \quad (27)$$

$$\rho^t = 1 - \frac{\frac{1}{m+l} \left(\sum_{i=1}^m \beta_i^{-t} + \sum_{f=1}^l \tilde{\beta}_f^{-t} \right) + \frac{1}{s} \sum_{r=1}^s \beta_r^{+t}}{2}, \quad t = 1, \dots, T-1, \quad (28)$$

$$\rho^T = 1 - \frac{\frac{1}{m+l} \left(\sum_{i=1}^m \beta_i^{-T} + \sum_{f=1}^l \tilde{\beta}_f^{-T} \right) + \frac{1}{s+1} \sum_{r=1}^{s+1} \beta_r^{+T}}{2}, \quad (29)$$

$$x_i \left(\sum_{j=1}^n \lambda_j^t R_j^t \right) \leq x_i(R_o^t) - \beta_i^{-t} \mu_i(R_o^t), \quad i = 1, \dots, m, \quad t = 1, \dots, T, \quad (30)$$

$$\sum_{j=1}^n \lambda_j^t c_{fj}^t \leq c_{f_o}^t - \tilde{\beta}_f^{-t} \omega_{f_o}^t, \quad f = 1, \dots, l, \quad t = 1, \dots, T, \quad (31)$$

$$(M^P) \quad y_r \left(\sum_{j=1}^n \lambda_j^t R_j^t \right) \geq y_r(R_o^t) + \beta_r^{+t} \nu_r(R_o^t), \quad r = 1, \dots, s, \quad t = 1, \dots, T, \quad (32)$$

$$\sum_{j=1}^n \lambda_j^1 z_j^0 = z_o^0, \quad (33)$$

$$\sum_{j=1}^n \lambda_j^t z_j^t = \sum_{j=1}^n \lambda_j^{t+1} z_j^t, \quad t = 1, \dots, T-1, \quad (34)$$

$$\sum_{j=1}^n \lambda_j^T z_j^T \geq z_o^T + \beta_{s+1}^T \eta^T, \quad (35)$$

$$\sum_{j=1}^n \lambda_j^t = 1, \quad \lambda_j^t \geq 0, \quad j = 1, \dots, n, \quad t = 1, \dots, T, \quad (36)$$

$$\beta_i^{-t}, \tilde{\beta}_f^{-t}, \beta_r^{+t} \geq 0, \quad i, r, f, t, \quad (37)$$

where E^* is the optimal value of problem (M^o) . When $\tau = T$, the constraint set (27) should be removed from model (M^P) . For $\tau = T-1, \dots, 2$, the unique $\rho^{t,*}$, $t = T, \dots, \tau+1$, has been recursively determined by model (M^P) before it appears in (27). For $t = 2, \dots, T$, the efficiency at period t is measured by $\rho^{t,*}$. After $\rho^{T,*}, \dots, \rho^{2,*}$ have been uniquely determined by model (M^P) , the efficiency at period 1 is measured by

$$\rho^{1,*} = \xi^{1-T} \left(E^* \frac{1-\xi^T}{1-\xi} - \sum_{t=2}^T \xi^{T-t} \rho^{t,*} \right).$$

According to the definition of usual DEA-efficiency (Cooper et al. 2007), we introduce the following definitions to examine a fund's efficiency.

Definition 2 The target fund o is overall efficient if and only if $E^o = 1$; otherwise, it is overall inefficient.

Definition 3 For $t = 1, \dots, T$, the target fund o is efficient at period t if and only if $\beta_i^{-t,*} = \tilde{\beta}_f^{-t,*} = \beta_r^{+t,*} = 0, \forall i, f, r$.

It is clear that $E^o = 1$ if and only if $\rho^{t,*} = 0$ for $t = 1, \dots, T$. Then from Definitions 2 and 3, we have: *if the target fund is overall efficient, then it is efficient at each period; if the target fund is overall inefficient, there is at least one period at which the target fund is inefficient.* Hence, if a fund is overall inefficient, we can easily find periods at which the inefficiency happens. So, the proposed method can help the investor reveal the time when the inefficiency happens for inefficient funds.

Like conventional DEA models, our method can also provide information on how a badly performed fund can improve its performance. Let $\lambda_j^{t,*}$ denote the optimal solution variables of model (M^P) with $\tau = 2$. Denote the portfolio whose inputs, outputs and intermediate variables at period t ($t = 1, \dots, T$) are $x_i (\sum_{j=1}^n \lambda_j^{t,*} R_j^t)$, $\sum_{j=1}^n \lambda_j^{t,*} c_{fj}^t$, $\sum_{j=1}^n \lambda_j^{1,*} z_j^0$ (for period 1), $y_r (\sum_{j=1}^n \lambda_j^{t,*} R_j^t)$ and $\sum_{j=1}^n \lambda_j^{t,*} z_j^t$, respectively, as the portfolio o^* . We have the following conclusion.

Theorem 2 *The portfolio o^* is overall efficient.*

Proof Suppose that the portfolio o^* is not overall efficient. Then, there is at least one period at which the portfolio o^* is not efficient. Without loss of generality, we assume that the portfolio o^* is inefficient at period $t_0 \in \{2, \dots, T-1\}$. Then, there exist feasible weights $\lambda_j^{t_0}$ such that

$$\sum_{j=1}^n \lambda_j^{t_0-1,*} z_j^{t_0-1} = \sum_{j=1}^n \lambda_j^{t_0} z_j^{t_0-1}, \quad (38)$$

$$\sum_{j=1}^n \lambda_j^{t_0} z_j^{t_0} = \sum_{j=1}^n \lambda_j^{t_0+1,*} z_j^{t_0}, \quad (39)$$

hold, and meanwhile, at least one of the following inequalities holds:

$$x_i \left(\sum_{j=1}^n \lambda_j^{t_0} R_j^{t_0} \right) < x_i \left(\sum_{j=1}^n \lambda_j^{t_0,*} R_j^{t_0} \right), \quad (40)$$

$$\sum_{j=1}^n \lambda_j^{t_0} c_{fj}^{t_0} < \sum_{j=1}^n \lambda_j^{t_0,*} c_{fj}^{t_0}, \quad (41)$$

$$y_r \left(\sum_{j=1}^n \lambda_j^{t_0} R_j^{t_0} \right) > y_r \left(\sum_{j=1}^n \lambda_j^{t_0,*} R_j^{t_0} \right). \quad (42)$$

Otherwise, the portfolio o^* would be efficient at period t_0 . We can deduce from (38)-(42) that there exist some feasible $\beta_i^{-t_0}$, $\beta_f^{-t_0}$ or $\beta_r^{+t_0}$ such that $\rho^{t_0} < \rho^{t,*}$ holds. Then, we have

$$\frac{1-\xi}{1-\xi^T} \left(\sum_{\substack{t=1 \\ t \neq t_0}}^T \xi^{T-t} \rho^{t,*} + \xi^{T-t_0} \rho^{t_0} \right) < \frac{1-\xi}{1-\xi^T} \sum_{t=1}^T \xi^{T-t} \rho^{t,*} = E^*.$$

This contradicts with the optimality of E^* . Hence, the portfolio o^* is efficient at period t_0 . Analogous arguments can be used for the periods 1 and T and the case with multiple periods at which the portfolio o^* is not efficient. \square

According to Theorem 2, the investor who holds an overall inefficient fund o can compare the inputs, outputs and intermediate variables of the portfolio o^* with those of the target fund o at each period t , so that he/she can find those factors causing the overall inefficiency of the fund o .

Last but not least, we'd like to point out that, in addition to mutual funds, the proposed approach can also be applied to evaluate the performance of other financial assets, such as stock or bond portfolios. Although there is only one intermediate variable in models (M^O) and (M^P), they can be easily extended to situations with multiple intermediate variables by adding the corresponding constraints related with other intermediate variables. Moreover, our method can be adapted to the case of NIRS used by Lamb and Tee (2012a), where it is not necessary to invest all the budgets into considered funds.

4 Empirical analysis

The effect of the proposed approach for the fund performance evaluation will be empirically examined in this section. The input-output data used here comes from the Chinese Security Investment Fund Research Database, which is compiled by the GTA Information Technology Company, Shenzhen, China. Considering that most of funds in China are open-ended, we only consider open-ended funds in what follows. We choose 40 (i.e., $n = 40$) open-ended funds that were set up early in China, the sample period spans from January 2004 to December 2013, comprising of 2382 daily observations. A calendar year is treated as one investment period. For each fund, the daily return rate with dividend re-invested is adopted to calculate all the risk and return measures.

4.1 Input-output factors

In order to select suitable risk measures, we first calculate the skewness and kurtosis values of return distributions of all funds. Most return distributions of chosen funds have the kurtosis value greater than 3, while the skewness value is obviously different from zero; all the return distributions show statistically significant leptokurtic and skewness, which stretch the tails. Therefore, it is important for us to select risk measures which can reflect return distributions with fat tails, so that we can properly evaluate the performance of these funds. Following Branda (2013), we choose the CVaR deviation as our risk measure. CVaR deviation is an important deviation measure derived from CVaR (Rockafellar et al. 2006b). Like CVaR, CVaR deviation is a quantile-based measure which is suitable for asymmetric return distributions with skewness and/or fat tails. Compared with CVaR, an advantage of CVaR deviation is that it is positive for any random returns (Branda 2013; Rockafellar et al. 2006a,b). This makes it appealing for including in DEA models. The CVaR deviation of R_j^t can be determined as (Branda 2013; Ogryczak and Ruszczyński 2002)

$$D_\alpha(R_j^t) = \min_{\zeta} \frac{1}{1-\alpha} E[\max\{(1-\alpha)(R_j^t - \zeta), \alpha(\zeta - R_j^t)\}],$$

where $\alpha \in (0, 1)$ is the confidence level. If R_j^t has discrete distribution γ_{jk}^t , $j = 1, \dots, n$, $k = 1, \dots, K_t$, with equal probabilities $p_k = 1/K_t$, where K_t is the number of sample points at period t , the CVaR deviation of a linear combination of funds at period t can be formulated as (Branda 2013)

$$D_\alpha(\sum_{j=1}^n \lambda_j^t R_j^t) = \min_{\zeta} \frac{1}{K_t} \sum_{k=1}^{K_t} \max\{(\sum_{j=1}^n \lambda_j^t \gamma_{jk}^t - \zeta), \frac{\alpha}{1-\alpha}(\zeta - \sum_{j=1}^n \lambda_j^t \gamma_{jk}^t)\}.$$

To measure the loss risk from those negative returns of a fund, we select the first-order lower partial moment $L(R) = E[\max(0, -R)]$ as another risk measure. Under the above discrete return distribution assumption, we have

$$L(R_j^t) = \frac{1}{K_t} \sum_{k=1}^{K_t} \max\{0, -\gamma_{jk}^t\}$$

for the loss risk of fund j at period t . In the following empirical analysis, we adopt $D_{\alpha_i}(\cdot)$, for $\alpha_i \in \{0.75, 0.95\}$, and $L(\cdot)$ as risk measures.

The expected return is an important return measure for any fund, thus we select it as the unique return measure. Like many other researchers, such as Basso and Funari (2001) and Murthi et al. (1997), we also consider transaction costs as inputs. Concretely, we take the annual expense ratio as the unique transaction cost indicator. Annual expense ratio refers to the annually total cost incurred by the fund in operating the portfolio, usually expressed as a percent of total assets under management. Although many other transaction cost indicators have been chosen in existing DEA studies on fund performance, we just consider the above one due to the following reasons: (i) Having too many inputs and outputs in a DEA model might diminish its discriminatory power (Galagedera 2013). (ii) Other available transaction cost indicators, such as the redemption fee and the subscription fee, are almost the same for sampled funds and keep unchanged for each sampled year. This is not benefit for diminishing the efficiencies of funds in each sampled year.

Based on the chosen input-output factors, we have $m = 3$, $l = 1$, $s = 1$. Similar to the DEA models with diversification proposed by Branda (2013) and Branda

(2015), models (M^O) and (M^P) including the above inputs and outputs can be written as the following equivalent linear programming models, respectively, under the assumption of discrete return distributions.

$$\begin{aligned}
E^* = \min & \frac{1-\xi}{1-\xi^T} \sum_{t=1}^T \xi^{T-t} \rho^t \\
\text{s.t. } & \rho^t = 1 - \frac{\frac{1}{4}(\sum_{i=1}^3 \beta_i^{-t} + \tilde{\beta}_1^{-t}) + \beta_1^{+t}}{2}, \quad t = 1, \dots, T-1, \\
& \rho^T = 1 - \frac{\frac{1}{4}(\sum_{i=1}^3 \beta_i^{-T} + \tilde{\beta}_1^{-T}) + \frac{1}{2} \sum_{r=1}^2 \beta_r^{+T}}{2}, \\
& \frac{1}{K_t} \sum_{k=1}^{K_t} u_{ki}^t \leq D_{\alpha_i}(R_o^t) - \beta_i^{-t} \mu_i(R_o^t), \quad i = 1, 2, \quad t = 1, \dots, T, \\
& u_{ki}^t \geq \sum_{j=1}^n \lambda_j^t \gamma_{jk}^t - \zeta_i^t, \quad k = 1, \dots, K_t, \quad i = 1, 2, \quad t = 1, \dots, T, \\
& u_{ki}^t \geq \frac{\alpha_i}{1-\alpha_i} (\zeta_i^t - \sum_{j=1}^n \lambda_j^t \gamma_{jk}^t), \quad k = 1, \dots, K_t, \quad i = 1, 2, \quad t = 1, \dots, T, \\
& \frac{1}{K_t} \sum_{k=1}^{K_t} \delta_k^t \leq L(R_o^t) - \beta_3^{-t} \mu_3(R_o^t), \quad t = 1, \dots, T, \\
& \delta_k^t \geq 0, \quad k = 1, \dots, K_t, \quad t = 1, \dots, T, \\
& \delta_k^t \geq -\sum_{j=1}^n \lambda_j^t \gamma_{jk}^t, \quad k = 1, \dots, K_t, \quad t = 1, \dots, T, \\
& \sum_{j=1}^n \lambda_j^t c_{1j}^t \leq c_{1o}^t - \tilde{\beta}_1^{-t} \omega_{1o}^t, \quad t = 1, \dots, T, \\
& \sum_{j=1}^n \lambda_j^t E(R_j^t) \geq E(R_o^t) + \beta_1^{+t} \nu_1(R_o^t), \quad t = 1, \dots, T, \\
& \sum_{j=1}^n \lambda_j^1 z_j^0 = z_o^0, \\
& \sum_{j=1}^n \lambda_j^t z_j^t = \sum_{j=1}^n \lambda_j^{t+1} z_j^t, \quad t = 1, \dots, T-1, \\
& \sum_{j=1}^n \lambda_j^T z_j^T \geq z_o^T + \eta^T \beta_2^{+T}, \\
& \sum_{j=1}^n \lambda_j^t = 1, \quad \lambda_j^t \geq 0, \quad j = 1, \dots, n, \quad t = 1, \dots, T, \\
& \beta_i^{-t}, \tilde{\beta}_1^{-t}, \beta_r^{+t} \geq 0, \quad \forall i, r, t.
\end{aligned}$$

$$\begin{aligned}
\min & \rho^\tau \\
\text{s.t. } & \frac{1-\xi}{1-\xi^T} \sum_{t=1}^T \xi^{T-t} \rho^t = E^*, \\
& \rho^t = \rho^{t,*}, \quad t = T, \dots, \tau+1,
\end{aligned}$$

$$\begin{aligned}
\rho^t &= 1 - \frac{\frac{1}{4}(\sum_{i=1}^3 \beta_i^{-t} + \tilde{\beta}_1^{-t}) + \beta_1^{+t}}{2}, \quad t = 1, \dots, T-1, \\
\rho^T &= 1 - \frac{\frac{1}{4}(\sum_{i=1}^3 \beta_i^{-T} + \tilde{\beta}_1^{-T}) + \frac{1}{2} \sum_{r=1}^2 \beta_r^{+T}}{2}, \\
\frac{1}{K_t} \sum_{k=1}^{K_t} u_{ki}^t &\leq D_{\alpha_i}(R_o^t) - \beta_i^{-t} \mu_i(R_o^t), \quad i = 1, 2, \quad t = 1, \dots, T, \\
u_{ki}^t &\geq \sum_{j=1}^n \lambda_j^t \gamma_{jk}^t - \zeta_i^t, \quad k = 1, \dots, K_t, \quad i = 1, 2, \quad t = 1, \dots, T, \\
u_{ki}^t &\geq \frac{\alpha_i}{1 - \alpha_i} (\zeta_i^t - \sum_{j=1}^n \lambda_j^t \gamma_{jk}^t), \quad k = 1, \dots, K_t, \quad i = 1, 2, \quad t = 1, \dots, T, \\
\frac{1}{K_t} \sum_{k=1}^{K_t} \delta_k^t &\leq L(R_o^t) - \beta_3^{-t} \mu_3(R_o^t), \quad t = 1, \dots, T, \\
\delta_k^t &\geq 0, \quad k = 1, \dots, K_t, \quad t = 1, \dots, T, \\
\delta_k^t &\geq - \sum_{j=1}^n \lambda_j^t \gamma_{jk}^t, \quad k = 1, \dots, K_t, \quad t = 1, \dots, T, \\
\sum_{j=1}^n \lambda_j^t c_{1j}^t &\leq c_{1o}^t - \tilde{\beta}_1^{-t} \omega_{1o}^t, \quad t = 1, \dots, T, \\
\sum_{j=1}^n \lambda_j^t E(R_j^t) &\geq E(R_o^t) + \beta_1^{+t} \nu_1(R_o^t), \quad t = 1, \dots, T, \\
\sum_{j=1}^n \lambda_j^1 z_j^0 &= z_o^0, \\
\sum_{j=1}^n \lambda_j^t z_j^t &= \sum_{j=1}^n \lambda_j^{t+1} z_j^t, \quad t = 1, \dots, T-1, \\
\sum_{j=1}^n \lambda_j^T z_j^T &\geq z_o^T + \eta^T \beta_2^{+T}, \\
\sum_{j=1}^n \lambda_j^t &= 1, \quad \lambda_j^t \geq 0, \quad j = 1, \dots, n, \quad t = 1, \dots, T, \\
\beta_i^{-t}, \tilde{\beta}_1^{-t}, \beta_r^{+t} &\geq 0, \quad \forall i, r, t.
\end{aligned}$$

In the above two formulations, we have

$$\begin{aligned}
\mu_i(R_o^t) &= D_{\alpha_i}(R_o^t) - \min_{R \in \mathcal{R}^t} D_{\alpha_i}(R), \quad i = 1, 2, \quad \mu_3(R_o^t) = L(R_o^t) - \min_{R \in \mathcal{R}^t} L(R), \\
\omega_{1o} &= c_{1o}^t - \min_j c_{1j}^t, \quad \nu_1(R_o^t) = \max_{R \in \mathcal{R}^t} E(R) - E(R_o^t), \quad \forall t, \\
\eta^T &= \max_j z_j^T - z_o^T,
\end{aligned}$$

according to (23) and (24). We know from Branda (2013) that the above LPs are well-defined because

$$\frac{1}{K_t} \sum_{k=1}^{K_t} u_{ki}^t \geq D_{\alpha_i}(\sum_{j=1}^n \lambda_j^t R_j^t), \quad i = 1, 2, \quad t = 1, \dots, T,$$

and

$$\frac{1}{K_t} \sum_{k=1}^{K_t} \delta_k^t \geq L \left(\sum_{j=1}^n \lambda_j^t R_j^t \right), \quad t = 1, \dots, T.$$

4.2 Empirical results

We first use the proposed multi-period network model (M^O) to evaluate the overall efficiency of 40 chosen funds in three different sample intervals: 2011-2013, 2009-2013 and 2004-2013, respectively. As a comparison, we also treat the relevant investment process as a black-box and then compute the corresponding efficiency scores of 40 funds in these three sample intervals by solving model (M^S). When the investment process is treated as a black-box, we choose $D_{\alpha_i}(R_j)$, $L(R_j)$, $\frac{1}{T} \sum_{t=1}^T c_{1j}^t$ and z_j^0 as inputs (z_j^0 is also a non-discretionary input), $E(R_j)$ and z_j^T as outputs, where R_j is the return rate of the fund j in the corresponding sample interval. Table 3 lists the resulting efficiency scores and corresponding ranks, which are shown in brackets. To distinguish, we call the efficiency determined by model (M^S) the static efficiency. As we can see from Table 3, the static efficiency identifies more efficient funds than the overall efficiency in each sample interval. The overall efficiency in the sample interval 2004-2013 shows a quite strong discriminating power. Just three funds, ZSPH, JSZZ and GTJX, are identified as efficient in the whole period.

Compared with model (M^S), an advantage of our multi-period network DEA approach is that model (M^O) can decompose the overall efficiency into period-wise efficiencies among intermediate periods. Table 4 shows the efficiency scores $\rho^{t,*}$ at each period (year) in the sample interval 2004-2013. In Table 4, funds whose efficiency scores are equal to 1 are efficient in the corresponding year. Note that a fund with a poor overall efficiency might be efficient in some periods. For instance, the fund HAAG has a poor overall efficiency among all funds in 2004-2013 with the rank 34 (see Table 3), but it achieves efficiency in 2007, 2008 and 2009. This illustrates the dynamic property of the proposed multi-period network DEA method since it reveals the change of performance within the sample interval. In addition, the efficiency scores of each fund at individual periods tell us the time point at which an inefficient performance occurs for that fund. For example, the fund YJCL is overall inefficient in 2004-2013, it is easy to see from Table 4 that the inefficient performance of YJCL occurs in 2004, 2011, 2012 and 2013, respectively.

From Tables 3 and 4, we see that the result got from model (M^S) sometimes misguides investors or fund managers as it ignores the performance variation at internal periods. For sample, the fund TDCZ is efficient under model (M^S) in the sample interval 2004-2013. However, it is easy to see from Table 4 that TDCZ is only efficient in two years of 2004 and 2005. TDCZ is inefficient at most of time in the sample interval. The similar misleading result also happens for the fund JSPH in the sample interval 2004-2013.

As we can see from the last row of Table 4, the average score of funds' efficiency scores keeps increasing from 2004 to 2007 as a whole (there is a single slight decline in 2006), follows by a drop in 2008 and an increase in 2009, and then remains volatile from 2010 to 2013. Such an evaluation result is consistent with the trend of Chinese security markets. From 2004 to 2007, Chinese security markets gradually recovered. In 2007, Chinese security markets experienced a

frenzied boom, the Shanghai composite index surged from about 2600 points at the beginning of 2007 to 6124 points. Like most of security markets in the world, Chinese security markets experienced a steep decline in 2008 due to the financial crisis. The Shanghai composite index fell to 1664.93 points from the highest point of 5522.78. In 2009, as the Chinese government offered a 4 trillion Yuan economic stimulus plan, the Shanghai composite index rose from 1820.81 to 3478.01 points. From 2010 to 2014, Chinese security markets experienced many large fluctuations. Many analysts believe that the Shanghai and Shenzhen main board stock market is a typical “monkey” market. From the above analyses, we know that our evaluation results capture the overall trend of Chinese security markets rather well.

Moreover, our multi-period network DEA model can provide expected inputs, outputs and intermediate variables. This information can help investors identify factors causing the fund’s inefficiency at each period. For example, the fund TDZQ is overall inefficient in the sample interval 2004-2013. By solving models (M^O) and (M^P) , we can find the necessary changes for TDZQ’s inputs, outputs and intermediate variables to reach the corresponding expected values at all periods. The detailed data are shown in Table 5.

Table 5 Expected changes to improve TDZQ’s overall inefficiency

	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
$D_{0.75}(R_o^t)$	-0.00%	-0.00%	-0.00%	-35.60%	-16.45%	-19.38%	-53.78%	-75.90%	-27.81%	-45.33%
$D_{0.95}(R_o^t)$	-0.00%	-0.00%	-0.00%	-34.28%	-10.17%	-27.66%	-49.83%	-69.74%	-20.60%	-55.30%
$L(R_o^t)$	-0.00%	-0.00%	-0.00%	-37.05%	-68.00%	-70.21%	-0.00%	-57.82%	-67.43%	-31.22%
c_o^t	-0.00%	-0.00%	-0.00%	-35.27%	-18.03%	-20.92%	-57.95%	-77.67%	-32.32%	-48.51%
$E(R_o^t)$	0.00%	0.00%	0.00%	0.00%	32.34%	0.00%	0.00%	75.94%	16.87%	0.00%
z_o^t	0.00%	0.00%	0.00%	-10.40%	-6.36%	-4.97%	-5.07%	0.70%	0.64%	0.34%

Positive (negative) values in rows 2-6 of Table 5 mean that the outputs (inputs) should be increased (decreased) by the corresponding percentages along the related direction in order to reach the expected values. Since only the direction of the NAV at the T th period is set, we provide the corresponding positive (negative) percentage relative to the original NAV in order to reach the expected NAV. See the last row of Table 5. We see from Table 5 that for the fund TDZQ: CVaR deviation and cost should be decreased in each year of 2007-2013; the first-order lower partial moment should be decreased in each year of 2007-2009 and 2011-2013; the expected return should be increased in 2008, 2011 and 2012; the intermediate variable z_o^t corresponding to the years from 2007 to 2010 should be decreased and those should be increased in the interval 2011 to 2012; and the output z_o^T at period T (in 2013) should be increased. These different modifications at different periods demonstrate an interesting feature of the intermediate variable: it is treated as both an input and an output. This feature of the intermediate variable also means the proposed method is inconsistent with stochastic dominance, since an intermediate variable cannot be simply expanded or reduced to maximize the performance of DMUs. In the proposed model, optimal intermediate variables are determined by coordinating all periods in such a way that the overall efficiency in the whole investment horizon is maximized!

5 Conclusions

When applying the DEA models with diversification to evaluate the performance of funds, current studies treat the investment process as a black-box and ignore the interrelationship among inner periods. In this paper, we overcome this disadvantage by treating the investment process as serially inter-related periods. Each period has its exogenous inputs and outputs and also has intermediate variables. We decompose the investment process into a series of successive periods. By combining the DEA model with diversification and the network DEA model, we proposed a multi-period network DEA model with diversification. The proposed model can reveal the detailed dynamics of funds' performance by decomposing the overall efficiency into inter-dependent performance measures at individual periods. It dynamically evaluates investment funds in terms of the input reduction, the output expansion and the intermediate variable equilibrium. Empirical results show that the proposed overall efficiency has a good discriminating power for ranking the efficiency of funds and properly reflects the impact of the financial crisis and the policy stimulus on fund performance.

We propose the network DEA model with diversification from the perspective of time, which can reveal the change of a fund's performance within the sample interval. The proposed model cannot tell us the poor performance of an inefficient fund is due to the bad operational management of the fund company or the bad portfolio management of the fund manager. How to solve this problem is left for future research.

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