

# Quantifying Double McCormick

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**Abstract:** When using the standard McCormick inequalities twice to convexify trilinear monomials, as is often the practice in modeling and software, there is a choice of which variables to group first. For the important case in which the domain is a nonnegative box, we calculate the volume of the resulting relaxation, as a function of the bounds defining the box. In this manner, we precisely quantify the strength of the different possible relaxations defined by all three groupings, in addition to the trilinear hull itself. As a by product, we characterize the best double McCormick relaxation.

We wish to emphasize that, in the context of spatial branch-and-bound for factorable formulations, our results do not only apply to variables in the input formulation. Our results apply to monomials that involve auxiliary variables as well. So, our results apply to the product of any three (possibly complicated) expressions in a formulation.

*Key words:* global optimization, mixed-integer nonlinear programming, spatial branch-and-bound, convexification, bilinear, trilinear, McCormick inequalities

*MSC2000 subject classification:* 90C26

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**1. Introduction.** Spatial branch-and-bound (sBB) (see [1, 19, 24]) for so-called factorable mathematical-optimization formulations (see [14]) is the workhorse general-purpose algorithm in the area of global optimization. It works by using additional variables to reformulate every function of the formulation as a labeled directed graph (DAG). Root nodes can be very complicated functions, and leaves are variables that appear in the input formulation, each labeled with its interval domain. Intermediate nodes are labeled with auxiliary variables together with operators from a small dictionary of basic functions of few (often one or two) variables. Also, we have a method for convexifying the graph of each dictionary function. sBB algorithms work by composing convex relaxations of the dictionary functions, according to the DAG, to get relaxations of the root functions. Bounds on the leaves propagate to other nodes and conversely. Branching (subdividing the domain interval of a variable) creates subproblems, which are treated. Objective bounds for subproblems are appropriately combined to achieve a global-optimization algorithm.

Much of the research on sBB has focused on developing tight convexifications for basic functions of few variables (many references can be found in [4]). Other research has focused on how bounds can be efficiently propagated and how branching can be judiciously be carried out (see [3], for example). From the viewpoint of good convexifications, much less attention as been payed to how the DAGs are created, but this can have a strong impact on the quality of the resulting convex relaxation of the input formulation; see [11, 12, 21, 26] for some key papers with other viewpoints concerning constructing DAGS.

For basic multilinear monomials  $f(x_1, \dots, x_n) := x_1 \cdots x_n$ , with  $x_i \in [a_i, b_i]$ , there is already a lot of flexibility which can have a significant impact on the overall convexification of the graph of  $f(x_1, \dots, x_n) := x_1 \cdots x_n$  on the box domain  $[a_1, b_1] \times \cdots \times [a_n, b_n]$ . For  $n = 2$ , we have the classic McCormick inequalities (see [14]), which is simply the tetrahedron that is the convex hull of

the points  $(f, x_1, x_2) := (a_1 a_2, a_1, a_2), (a_1 b_2, a_1, b_2), (b_1 a_2, b_1, a_2), (b_1 b_2, b_1, b_2)$ . The inequalities can be derived from the four inequalities

$$\begin{aligned} (x_1 - a_1)(x_2 - a_2) &\geq 0, \quad (x_1 - a_1)(b_2 - x_2) \geq 0, \\ (b_1 - x_1)(x_2 - a_2) &\geq 0, \quad (b_1 - x_1)(b_2 - x_2) \geq 0, \end{aligned}$$

by multiplying out and then replacing all occurrences of  $x_1 x_2$  by the variable  $f$ .

For general  $n$ , there are  $2^n$  points to consider (i.e., all choices of each variable at a bound), and the inequality descriptions in the space of  $(f, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$  get rather complicated (even for  $n = 3$ ; see [16, 15]). It is common practice, both in modeling and software, to repeatedly use the McCormick inequalities when  $n > 2$ . Already the trilinear case,  $n = 3$ , is an interesting one for analysis. Here, we have three choices, which can be thought of as  $f = (x_1 x_2) x_3$ ,  $f = (x_1 x_3) x_2$  and  $f = (x_2 x_3) x_1$ . Because the domain of each variable is its own interval  $[a_i, b_i]$ , the grouping can affect the quality of the convexification. In what follows, we analytically quantify the quality of these different convexification possibilities, in addition to the trilinear hull itself.

We do note that in some global-optimization software (e.g., BARON and ANTIGONE) the complicated inequality description of the trilinear hull is explicitly used. In other global-optimization software (e.g., COUENNE and SCIP) and as a technique at the formulation level, repeated McCormick is used for the trilinear case. It is by no means clear that either approach should be followed all of the time, because there is often a solution-time tradeoff in using more complicated but stronger convexifications, and this effect can be especially pronounced in the case of nonlinear optimization where solutions may not be on the boundary (see [9], for example). By quantifying the quality of different convexifications, we offer some firm means for deciding between them and some explanation for differing behavior of sBB software under different scenarios. We can well imagine choosing a simpler double McCormick in situations when that convexification is only a bit larger than the trilinear hull. Conversely, our work draws attention to the cases for which the trilinear hull is much stronger than any double McCormick convexification.

Some might advocate an extended variable-formulation treatment of multilinear monomials (see [5], for example), writing  $f = \sum_{v \in \mathcal{V}} \lambda_v x^v$ , with  $\sum_{v \in \mathcal{V}} \lambda_v = 1$ ,  $\lambda_v \geq 0$ , for  $v \in \mathcal{V}$ , where  $\{x^v : v \in \mathcal{V}\}$  is the set of  $2^n$  points that arise by considering all combinations of bounds for the  $n$  variables. In some situations this kind of “extended-variable approach” might well be viable. Our research does not bear directly on this approach for general  $n$  — we only seek to compare the different ways in which McCormick-style relaxations can be applied for  $n = 3$ . In terms of relaxations, for  $n = 3$ , this extended-variable approach has the same power as using the complicated trilinear inequality description of [16, 15]. In so far as we quantify the error in using the various possible double McCormick relaxations relative to the true convex hull given by [16], our results are relevant to the extended-variable approach for  $n = 3$  as well.

*Our results are not just relevant to trilinear monomials in formulations.* With the sBB approach for factorable formulations, our results are relevant whenever three quantities are multiplied. That is, as an expression DAG is created and auxiliary variables are introduced, a trilinear monomial will arise whenever three quantities (which can be complicated functions themselves) are multiplied.

In what follows, we use  $(n + 1)$ -dimensional volume to compare different natural convexifications of graphs of functions of  $n$  variables on the box domain  $[a_1, b_1] \times \cdots \times [a_n, b_n]$ . We present a complete analytic analysis of the case of  $n = 3$ , for all choices of  $0 \leq a_i < b_i$ . It is perhaps surprising that this can be carried out, and probably less surprising that the analysis is quite complicated.

Volume as a measure for comparing relaxations was first proposed in [10]; also see [8] and [25]. In fact, the practical use of volume as a measure for comparing relaxations in the context of nonlinear mixed-integer optimization, foreshadowed by [10], was later validated computationally for a nonlinear version of the uncapacitated facility-location problem (see [9]). The emphasis in [10, 8, 25] was not on sBB nor on low-dimensional functions. Because those results pertained to

varying dimension and related asymptotics, exactly how volumes are compared and scaled was important (in particular, see [10] which defines the “idealized radial distance”). Because we now focus on low-dimensional polytopes, the exact manner of comparison and scaling is much less relevant. Using volume as a measure corresponds to a uniform distribution of the optimal solution across a relaxation. This is justified in the context of *nonlinear* optimization if we want a measure that is robust across all formulations. One can well find situations where the volume measure is misleading. It would not make sense for evaluating polyhedral relaxations of the integer points in a polytope, if we were only concerned with *linear* objectives — in such a case, solutions are concentrated on the boundary and there are better measures available (see [10]). But if we are interested in a mathematically-tractable measure that robustly makes sense in the context of global optimization, volume is quite natural.

Motivated by two well-studied applications (the *Molecular Distance Geometry Problem* and the *Hartree-Fock Problem*), [4] first proposed volume in the context of sBB and monomials, but they leapfrogged to the case of  $n = 4$  and took a mostly experimental approach. They demonstrated that there can be a significant difference in performance depending on grouping, and they offered some guidance based on computational experiments. At the time of that work, it appeared that developing precise formulae for volumes relevant to repeated McCormick was not tractable. With our present work on  $n = 3$ , it now seems possible that the case of  $n = 4$  could be carried out (see §8 for an idea concerning how our results for  $n = 3$  could already be applied in practice to the  $n = 4$  case).

There has been considerable research on multilinear monomials and generalizations in the context of global optimization, notably [18, 13, 2, 20, 7, 17]. Our work adds to that literature.

In §2, we define the polytopes that we work with. In §3, we present our main results and their consequences. In §§4–7, we present our proofs. In §8, we describe future directions for investigation. §9, an appendix, contains technical lemmas and calculations.

**2. Double McCormick.** When using the double McCormick technique to convexify trilinear monomials, a modeling/algorithms choice is involved: we must choose to which pair of variables we will apply the first iteration of McCormick. For the variables  $x_i \in [a_i, b_i]$ ,  $i = 1, 2, 3$ , let  $\mathcal{O}_i := a_i(b_j b_k) + b_i(a_j a_k)$ . Then we can label the variables such that  $\mathcal{O}_1 \leq \mathcal{O}_2 \leq \mathcal{O}_3$ . In this manner, we can assume that

$$a_1 b_2 b_3 + b_1 a_2 a_3 \leq b_1 a_2 b_3 + a_1 b_2 a_3 \leq b_1 b_2 a_3 + a_1 a_2 b_3. \quad (\Omega)$$

Given the trilinear monomial  $f := x_1 x_2 x_3$ , there are three choices of convexifications depending on the bilinear sub-monomial we convexify first. We could first group  $x_1$  and  $x_2$  and convexify  $w = x_1 x_2$ ; after this, we are left with the monomial  $f = w x_3$  which we can also convexify using McCormick. Alternatively, we could first group variables  $x_1$  and  $x_3$ , or variables  $x_2$  and  $x_3$ .

**2.1. Convexification.** To see how to perform these convexifications in general, we show the double McCormick convexification that first groups the variables  $x_i$  and  $x_j$ . Therefore we have  $f = x_i x_j x_k$  and we let  $w_{ij} = x_i x_j$  so  $f = w_{ij} x_k$ .

Convexifying  $w_{ij} = x_i x_j$  we obtain the inequalities:

$$\begin{aligned} w_{ij} - a_j x_i - a_i x_j + a_i a_j &\geq 0, \\ -w_{ij} + b_j x_i + a_i x_j - a_i b_j &\geq 0, \\ -w_{ij} + a_j x_i + b_i x_j - b_i a_j &\geq 0, \\ w_{ij} - b_j x_i - b_i x_j + b_i b_j &\geq 0. \end{aligned}$$

Convexifying  $f = w_{ij} x_k$  we obtain the inequalities:

$$f - a_k w_{ij} - a_i a_j x_k + a_i a_j a_k \geq 0,$$

$$\begin{aligned} -f + b_k w_{ij} + a_i a_j x_k - a_i a_j b_k &\geq 0, \\ -f + a_k w_{ij} + b_i b_j x_k - b_i b_j a_k &\geq 0, \\ f - b_k w_{ij} - b_i b_j x_k + b_i b_j b_k &\geq 0. \end{aligned}$$

Using Fourier-Motzkin elimination, we then eliminate the variable  $w_{ij}$  to obtain the following system in our original variables  $f, x_i, x_j$  and  $x_k$ .

$$\begin{aligned} x_i - a_i &\geq 0, & (1) \\ x_j - a_j &\geq 0, & (2) \\ f - a_j a_k x_i - a_i a_k x_j - a_i a_j x_k + 2a_i a_j a_k &\geq 0, & (3) \\ f - a_j b_k x_i - a_i b_k x_j - b_i b_j x_k + a_i a_j b_k + b_i b_j b_k &\geq 0, & (4) \\ -x_j + b_j &\geq 0, & (5) \\ -x_i + b_i &\geq 0, & (6) \\ f - b_j a_k x_i - b_i a_k x_j - a_i a_j x_k + a_i a_j a_k + b_i b_j a_k &\geq 0, & (7) \\ f - b_j b_k x_i - b_i b_k x_j - b_i b_j x_k + 2b_i b_j b_k &\geq 0, & (8) \\ -f + b_j b_k x_i + a_i b_k x_j + a_i a_j x_k - a_i a_j b_k - a_i b_j b_k &\geq 0, & (9) \\ -f + a_j b_k x_i + b_i b_k x_j + a_i a_j x_k - a_i a_j b_k - b_i a_j b_k &\geq 0, & (10) \\ -x_k + b_k &\geq 0, & (11) \\ -f + b_j a_k x_i + a_i a_k x_j + b_i b_j x_k - a_i b_j a_k - b_i b_j a_k &\geq 0, & (12) \\ -f + a_j a_k x_i + b_i a_k x_j + b_i b_j x_k - b_i a_j a_k - b_i b_j a_k &\geq 0, & (13) \\ x_k - a_k &\geq 0, & (14) \\ f - a_i a_j x_k &\geq 0, & (15) \\ -f + b_i b_j x_k &\geq 0. & (16) \end{aligned}$$

It is easy to see that the inequalities 15 and 16 are redundant: 15 is  $a_j a_k (1) + a_i a_k (2) + (3)$ , and 16 is  $b_j a_k (6) + a_i a_k (5) + (12)$ .

We use the following notation in what follows. For  $i = 1, 2, 3$ , *system i* is defined to be the system of inequalities obtained by first grouping the pair of variables  $x_j$  and  $x_k$ , with  $j$  and  $k$  different from  $i$ .  $\mathcal{P}_i$  is defined to be the solution set of this system.

**2.2. Hull.** As we noted earlier, tight polyhedral convexifications for trilinear monomials are known. From [16], for any labeling that satisfies  $\Omega$  (or even just the first inequality of  $\Omega$ ), this inequality system which we refer to as  $H$  is:

$$f - a_2 a_3 x_1 - a_1 a_3 x_2 - a_1 a_2 x_3 + 2a_1 a_2 a_3 \geq 0, \quad (17)$$

$$f - b_2 b_3 x_1 - b_1 b_3 x_2 - b_1 b_2 x_3 + 2b_1 b_2 b_3 \geq 0, \quad (18)$$

$$f - a_2 b_3 x_1 - a_1 b_3 x_2 - b_1 a_2 x_3 + a_1 a_2 b_3 + b_1 a_2 b_3 \geq 0, \quad (19)$$

$$f - b_2 a_3 x_1 - b_1 a_3 x_2 - a_1 b_2 x_3 + b_1 b_2 a_3 + a_1 b_2 a_3 \geq 0, \quad (20)$$

$$f - \frac{\eta_1}{b_1 - a_1} x_1 - b_1 a_3 x_2 - b_1 a_2 x_3 + \left( \frac{\eta_1 a_1}{b_1 - a_1} + b_1 b_2 a_3 + b_1 a_2 b_3 - a_1 b_2 b_3 \right) \geq 0, \quad (21)$$

$$f - \frac{\eta_2}{a_1 - b_1} x_1 - a_1 b_3 x_2 - a_1 b_2 x_3 + \left( \frac{\eta_2 b_1}{a_1 - b_1} + a_1 a_2 b_3 + a_1 b_2 a_3 - b_1 a_2 a_3 \right) \geq 0, \quad (22)$$

$$-f + a_2 a_3 x_1 + b_1 a_3 x_2 + b_1 b_2 x_3 - b_1 b_2 a_3 - b_1 a_2 a_3 \geq 0, \quad (23)$$

$$-f + b_2 a_3 x_1 + a_1 a_3 x_2 + b_1 b_2 x_3 - b_1 b_2 a_3 - a_1 b_2 a_3 \geq 0, \quad (24)$$

$$-f + a_2 a_3 x_1 + b_1 b_3 x_2 + b_1 a_2 x_3 - b_1 a_2 b_3 - b_1 a_2 a_3 \geq 0, \quad (25)$$

$$-f + b_2 b_3 x_1 + a_1 a_3 x_2 + a_1 b_2 x_3 - a_1 b_2 b_3 - a_1 b_2 a_3 \geq 0, \quad (26)$$

$$-f + a_2 b_3 x_1 + b_1 b_3 x_2 + a_1 a_2 x_3 - b_1 a_2 b_3 - a_1 a_2 b_3 \geq 0, \quad (27)$$

$$-f + b_2 b_3 x_1 + a_1 b_3 x_2 + a_1 a_2 x_3 - a_1 b_2 b_3 - a_1 a_2 b_3 \geq 0, \quad (28)$$

$$x_1 - a_1 \geq 0, \quad (29)$$

$$-x_1 + b_1 \geq 0, \quad (30)$$

$$x_2 - a_2 \geq 0, \quad (31)$$

$$-x_2 + b_2 \geq 0, \quad (32)$$

$$x_3 - a_3 \geq 0, \quad (33)$$

$$-x_3 + b_3 \geq 0, \quad (34)$$

where  $\eta_1 = b_1 b_2 a_3 - a_1 b_2 b_3 - b_1 a_2 a_3 + b_1 a_2 b_3$  and  $\eta_2 = a_1 a_2 b_3 - b_1 a_2 a_3 - a_1 b_2 b_3 + a_1 b_2 a_3$ .

We refer to the polytope defined as the feasible set of system  $H$  as  $\mathcal{P}_H$ . The extreme points of  $\mathcal{P}_H$  are the 8 points that correspond to the  $2^3 = 8$  choices of each  $x$ -variable at its upper or lower bound. We label these 8 points (all of the form  $[f = x_1 x_2 x_3, x_1, x_2, x_3]$ ) as follows:

$$\begin{aligned} v^1 &:= \begin{bmatrix} b_1 a_2 a_3 \\ b_1 \\ a_2 \\ a_3 \end{bmatrix}, v^2 := \begin{bmatrix} a_1 a_2 a_3 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, v^3 := \begin{bmatrix} a_1 a_2 b_3 \\ a_1 \\ a_2 \\ b_3 \end{bmatrix}, v^4 := \begin{bmatrix} a_1 b_2 a_3 \\ a_1 \\ b_2 \\ a_3 \end{bmatrix}, \\ v^5 &:= \begin{bmatrix} a_1 b_2 b_3 \\ a_1 \\ b_2 \\ b_3 \end{bmatrix}, v^6 := \begin{bmatrix} b_1 b_2 b_3 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}, v^7 := \begin{bmatrix} b_1 b_2 a_3 \\ b_1 \\ b_2 \\ a_3 \end{bmatrix}, v^8 := \begin{bmatrix} b_1 a_2 b_3 \\ b_1 \\ a_2 \\ b_3 \end{bmatrix}. \end{aligned}$$

Each alternative polyhedral convexification leads to a different system of inequalities and therefore a different polytope in  $\mathbb{R}^4$  — all three contain the convex hull of the solution set of our original trilinear monomial (on the box domain), i.e.  $\mathcal{P}_H$ .

To establish if one of these three convexifications is better than another, we need to be able to compare these polytopes in a quantifiable manner. We take the (4-dimensional) volume as our measure, with the idea that a smaller volume corresponds to a tighter convexification. Volume seems to be a good measure in the context of nonlinear global optimization, corresponding to a uniform prior as to where an optimum may be located.

For trilinear monomials with domain being a box (in the nonnegative orthant), we derive exact expressions for the (4-dimensional) volume for the convex hull of the set of solutions and also for each of the three possible double-McCormick convexifications. These volumes are in terms of six parameters (the upper and lower bounds on each of the three variables) and are rather complicated. By comparing the volume expressions, we are able to draw conclusions regarding the optimal way to perform double McCormick for trilinear monomials.

**3. Theorems.** First we will define the following twelve points in  $\mathbb{R}^4$ , where  $j := i + 1 \pmod{3}$  and  $k := i + 2 \pmod{3}$ :

$$\begin{aligned} v_1^9 &:= \begin{bmatrix} \theta_1^1 \\ \theta_1^2 \\ a_2 \\ b_3 \end{bmatrix}, v_1^{10} := \begin{bmatrix} \theta_1^3 \\ \theta_1^4 \\ b_2 \\ a_3 \end{bmatrix}, v_1^{11} := \begin{bmatrix} \theta_1^5 \\ \theta_1^6 \\ b_2 \\ a_3 \end{bmatrix}, v_1^{12} := \begin{bmatrix} \theta_1^7 \\ \theta_1^8 \\ b_3 \end{bmatrix}, v_2^9 := \begin{bmatrix} \theta_2^1 \\ b_1 \\ \theta_2^2 \\ a_3 \end{bmatrix}, v_2^{10} := \begin{bmatrix} a_1 \\ \theta_2^4 \\ b_3 \end{bmatrix}, \\ v_2^{11} &:= \begin{bmatrix} \theta_2^5 \\ a_1 \\ \theta_2^6 \\ b_3 \end{bmatrix}, v_2^{12} := \begin{bmatrix} \theta_2^7 \\ b_1 \\ \theta_2^8 \\ a_3 \end{bmatrix}, v_3^9 := \begin{bmatrix} \theta_3^3 \\ b_1 \\ a_2 \\ \theta_3^4 \end{bmatrix}, v_3^{10} := \begin{bmatrix} \theta_3^1 \\ a_1 \\ b_2 \\ \theta_3^2 \end{bmatrix}, v_3^{11} := \begin{bmatrix} \theta_3^7 \\ a_1 \\ b_2 \\ \theta_3^8 \end{bmatrix}, v_3^{12} := \begin{bmatrix} \theta_3^5 \\ b_1 \\ a_2 \\ \theta_3^6 \end{bmatrix}, \end{aligned}$$

where:

$$\begin{aligned} \theta_i^1 &= a_i a_j a_k + \frac{a_j(b_k - a_k)(b_i b_j b_k - a_i a_j a_k)}{b_j b_k - a_j a_k}, & \theta_i^2 &= a_i + \frac{a_j(b_i - a_i)(b_k - a_k)}{b_j b_k - a_j a_k}, \\ \theta_i^3 &= a_i a_j a_k + \frac{a_k(b_j - a_j)(b_i b_j b_k - a_i a_j a_k)}{b_j b_k - a_j a_k}, & \theta_i^4 &= a_i + \frac{a_k(b_j - a_j)(b_i - a_i)}{b_j b_k - a_j a_k}, \end{aligned}$$

$$\begin{aligned}\theta_i^5 &= \frac{b_j a_k (a_i b_j b_k - a_i a_j b_k - b_i a_j a_k + b_i a_j b_k)}{b_j b_k - a_j a_k}, & \theta_i^6 &= a_i + \frac{b_j (b_i - a_i) (b_k - a_k)}{b_j b_k - a_j a_k}, \\ \theta_i^7 &= \frac{a_j b_k (b_i b_j a_k - b_i a_j a_k - a_i b_j a_k + a_i b_j b_k)}{b_j b_k - a_j a_k}, & \theta_i^8 &= a_i + \frac{b_k (b_j - a_j) (b_i - a_i)}{b_j b_k - a_j a_k}.\end{aligned}$$

Next, we state our main results.

**THEOREM 3.1.**

$$\begin{aligned}Vol_{\mathcal{P}_H} &= (b_1 - a_1)(b_2 - a_2)(b_3 - a_3) \times \\ &\quad (b_1(5b_2 b_3 - a_2 b_3 - b_2 a_3 - 3a_2 a_3) + a_1(5a_2 a_3 - b_2 a_3 - a_2 b_3 - 3b_2 b_3)) / 24.\end{aligned}$$

**THEOREM 3.2.** *The set of extreme points of  $\mathcal{P}_1$  is  $\{v^1, \dots, v^8\} \cup \{v_1^9, \dots, v_1^{12}\}$ . Moreover,*

$$Vol_{\mathcal{P}_1} = Vol_{\mathcal{P}_H} + (b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^2 \times \frac{3(b_1 b_2 a_3 - a_1 b_2 a_3 + b_1 a_2 b_3 - a_1 a_2 b_3) + 2(a_1 b_2 b_3 - b_1 a_2 a_3)}{24(b_2 b_3 - a_2 a_3)}.$$

**THEOREM 3.3.** *The set of extreme points of  $\mathcal{P}_2$  is  $\{v^1, \dots, v^8\} \cup \{v_2^9, \dots, v_2^{12}\}$ . Moreover,*

$$Vol_{\mathcal{P}_2} = Vol_{\mathcal{P}_H} + \frac{(b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^2 (5(a_1 b_1 b_3 - a_1 b_1 a_3) + 3(b_1^2 a_3 - a_1^2 b_3))}{24(b_1 b_3 - a_1 a_3)}.$$

**THEOREM 3.4.** *The set of extreme points of  $\mathcal{P}_3$  is  $\{v^1, \dots, v^8\} \cup \{v_3^9, \dots, v_3^{12}\}$ . Moreover,*

$$Vol_{\mathcal{P}_3} = Vol_{\mathcal{P}_H} + \frac{(b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^2 (5(a_1 b_1 b_2 - a_1 b_1 a_2) + 3(b_1^2 a_2 - a_1^2 b_2))}{24(b_1 b_2 - a_1 a_2)}.$$

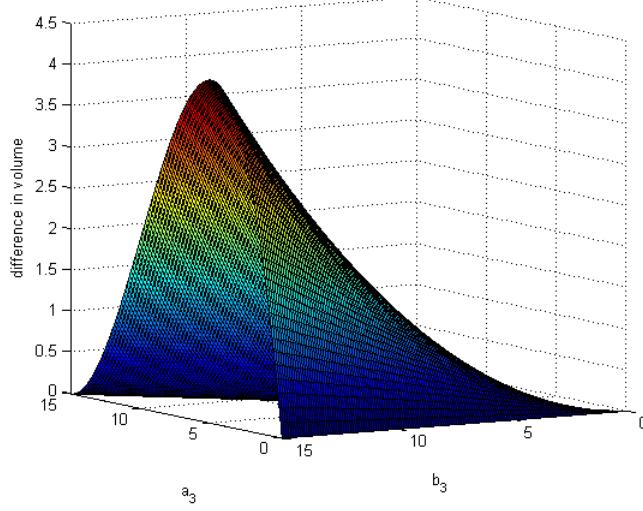
Our proofs in §§4–7 all assume that  $a_1, a_2, a_3 > 0$ . Next, we briefly explain why the theorems hold even when any of the  $a_i$  are zero. Taking the convex hull of a compact set is continuous (even 1-Lipschitz) in the Hausdorff metric (see [22, p. 51]). The volume functional is continuous (with respect to the Hausdorff metric) on the set  $K^n$  of convex bodies in  $\mathbb{R}^n$  (see [23, Theorem 1.8.20; p. 68]). If two sets of  $m$  points in  $\mathbb{R}^n$  are close as vectors in  $\mathbb{R}^{mn}$ , then they are also close in the Hausdorff metric. Therefore, the volume of the convex hull of a set of  $m$  points in  $\mathbb{R}^n$  is a continuous function of the coordinates of the points. Also, the coordinates of the extreme points of our polytopes are all continuous functions (of the six parameters) at  $a_i = 0$ . Finally, we note that the volume formulae that we derive are continuous functions (of the six parameters) at  $a_i = 0$ . Therefore, those formulae are also correct when some  $a_i = 0$ . We do note that we can also modify our constructions to handle these cases where some of the  $a_i$  are zero, but our continuity argument is much shorter.

**COROLLARY 3.1.** *For all values of the parameters  $a_1, b_1, a_2, b_2, a_3, b_3$ , meeting the conditions  $(\Omega)$ , we have:  $Vol_{\mathcal{P}_H} \leq Vol_{\mathcal{P}_3} \leq Vol_{\mathcal{P}_2} \leq Vol_{\mathcal{P}_1}$ .*

From this we can see that with the variables ordered according to their upper and lower bounds per  $(\Omega)$ , the smallest volume will always be obtained by using system 3 (i.e., first grouping variables  $x_1$  and  $x_2$ ). In addition, for different values of the upper and lower bounds, we can precisely quantify the difference in volume of the alternative convexifications.

Moreover, by substituting  $a_1 = a_2 = 0$  and  $b_1 = b_2 = 1$  into the conditions  $(\Omega)$ , we can easily see the following corollary relevant to mixed-integer nonlinear optimization.

FIGURE 1. Graph of difference in volume  $\left(\frac{3a_3(b_3-a_3)^2}{24b_3}\right)$  vs. parameter values



**COROLLARY 3.2.** *In the special case of two binary variables and one continuous variable, first grouping the two binary variables gives the convexification with the smallest volume.*

In this special case, we only have two parameters  $a_3$  and  $b_3$  and the volume formulae simplify considerably. In particular, for this special case,  $\mathcal{P}_3$  is equivalent to  $\mathcal{P}_H$ , and  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are equivalent. We compute the difference in volume between the two distinct choices of convexification and, in Figure 1, plot this expression as the parameters vary (satisfying  $0 \leq a_3 < b_3$ ). The following is easy to establish.

**COROLLARY 3.3.** *As  $a_3$  and  $b_3$  increase, the difference in volumes of  $\mathcal{P}_3$  and  $\mathcal{P}_2$  (or  $\mathcal{P}_1$ ) becomes arbitrarily large. Additionally, for a fixed  $b_3$ , the greatest difference in volume occurs when  $a_3 = b_3/3$ .*

Finally we note that in the special case in which  $a_1 = a_2 = a_3 = 0$ , each convexification reduces to the convex hull, which is a result of [20]. So in this case, *any* double-McCormick convexification has the power of the more-complicated inequality description of the convex hull. In fact, viewed this way, our results provide a quantified generalization of this result of [20]. We do wish to emphasize that because our results do not just apply to trilinear monomials on the formulation variables, but may well involve auxiliary variables, *the case of non-zero lower bounds is very relevant*.

**4. Proof of Thm. 3.1.** We will compute the volume of  $\mathcal{P}_H$  by constructing a triangulation. See Figure 2 for a diagram of the 8 extreme points of  $\mathcal{P}_H$ . Note that  $v^2$ , which has all of the variables at their lower bounds, is at the bottom of the “inner cube”, and  $v^6$ , which has all of the variables at their upper bounds, is at the top of the “outer cube”.

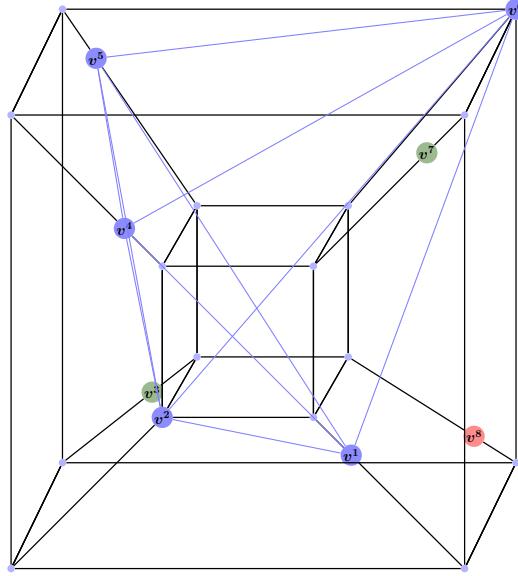
We use the fact that the volume of an  $n$ -simplex in  $\mathbb{R}^n$  with vertices  $(z^0, \dots, z^n)$  is:

$$|\det(z^1 - z^0 \ z^2 - z^0 \ \dots \ z^n - z^0)|/n!.$$

The volume of the 4-simplex with extreme points  $v^1, v^2, v^4, v^5$  and  $v^6$ , which we define as  $\mathcal{S} := \text{conv}\{v^1, v^2, v^4, v^5, v^6\}$ , is

$$(b_1 - a_1)^2(b_2 - a_2)(b_3 - a_3)(b_2b_3 - a_2a_3)/24.$$

FIGURE 2. Visual representation of simplex,  $\mathcal{S}$ , with extreme points  $v^1, v^2, v^4, v^5$  and  $v^6$ , plus the other 3 convex hull extreme points  $v^3, v^7$  and  $v^8$ .



A 4-simplex has 5 facets, each of which is a 3-simplex and is described by the hyperplane through a choice of 4 extreme points. To determine the facet-describing inequalities, we compute each hyperplane and then check the final point to obtain the direction of the inequality. The 5 facets of  $\mathcal{S}$  are described as follows:

$F^1$  (hyperplane through points  $v^1, v^2, v^4, v^6$ ):

$$\begin{aligned} & -f + a_2 a_3 x_1 + a_1 a_3 x_2 + \frac{(a_1 a_2 a_3 - a_1 b_2 a_3 - b_1 a_2 a_3 + b_1 b_2 b_3)}{(b_3 - a_3)} x_3 \\ & - \frac{(a_1 a_2 a_3 b_3 - a_1 b_2 a_3^2 - b_1 a_2 a_3^2 + b_1 b_2 a_3 b_3)}{(b_3 - a_3)} \geq 0 \end{aligned}$$

$F^2$  (hyperplane through points  $v^1, v^2, v^4, v^5$ ):

$$f - a_2 a_3 x_1 - a_1 a_3 x_2 - a_1 b_2 x_3 + a_1 a_2 a_3 + a_1 b_2 a_3 \geq 0$$

$F^3$  (hyperplane through points  $v^1, v^2, v^5, v^6$ ):

$$(b_3 - a_3) x_2 - (b_2 - a_2) x_3 + b_2 a_3 - a_2 b_3 \geq 0$$

$F^4$  (hyperplane through points  $v^1, v^4, v^5, v^6$ ):

$$\begin{aligned} & f - b_2 b_3 x_1 - \frac{(a_1 b_2 a_3 - a_1 b_2 b_3 - b_1 a_2 a_3 + b_1 b_2 b_3)}{(b_2 - a_2)} x_2 \\ & - a_1 b_2 x_3 + \frac{(-a_1 a_2 b_2 b_3 + a_1 b_2^2 a_3 - b_1 a_2 b_2 a_3 + b_1 b_2^2 b_3)}{(b_2 - a_2)} \geq 0 \end{aligned}$$

$F^5$  (hyperplane through points  $v^2, v^4, v^5, v^6$ ):

$$-f + b_2 b_3 x_1 + a_1 a_3 x_2 + a_1 b_2 x_3 - a_1 b_2 a_3 - a_1 b_2 b_3 \geq 0$$

If a hyperplane  $H$  intersects a polytope  $P$  on a facet  $F$ , then  $H^+$  (resp.,  $H^-$ ) denotes the half-space determined by  $H$  that contains (does not contain)  $P$ . If a point  $w$  is not in  $H$  but in  $H^+$  (resp.,  $H^-$ ), then  $w$  is *beneath* (*beyond*)  $F$  (see [6, p. 78]).

We will now compute the volume of  $\text{conv}(\mathcal{S} \cup \{v^8\})$ . To obtain the additional volume of this polytope compared with  $\mathcal{S}$  we will sum the volume of  $\text{conv}(\{v^8\} \cup F)$  for each facet,  $F$ , of  $\mathcal{S}$  such that  $v^8$  is beyond that facet. To do this we first check each of the 5 facets to determine if  $v^8$  is beneath or beyond that facet. To do this we substitute  $v^8$  into the relevant inequality, and if the result is negative then  $v^8$  lies beyond that facet.

It is easy to check that  $v^8$  satisfies  $F^1$  and  $F^5$  and violates  $F^3$ . Using Lemma 9.1 we can also check that  $v^8$  satisfies both  $F^2$  and  $F^4$ . From this we have that  $v^8$  is beyond one facet,  $F^3$ . Therefore we need to calculate the volume of  $\text{conv}(F^3 \cup \{v^8\}) = \text{conv}\{v^1, v^2, v^5, v^6, v^8\}$ , this a a 4-simplex with volume:

$$(b_1 - a_1)^2(b_2 - a_2)(b_3 - a_3)(b_2 b_3 - a_2 a_3)/24.$$

We now have a new polytope which is  $\text{conv}\{v^1, v^2, v^4, v^5, v^6, v^8\} = \text{conv}(\mathcal{S} \cup \{v^8\})$ . We refer to this polytope as  $\mathcal{Q}$ . The volume of  $\mathcal{Q}$  is given by the sum of the volumes of the two simplices we have computed thus far. The facets of  $\mathcal{Q}$  are the facets of the original simplex without  $F^3$ , along with the facets of the 4-simplex:  $\text{conv}(F^3 \cup v^8)$  (again not including  $F^3$  itself). A facet of  $\text{conv}(F^3 \cup \{v^8\})$  is supported by a hyperplane through a choice of 4 of the 5 extreme points (points  $v^1, v^2, v^5, v^6$  and  $v^8$ ). As before, to determine these facet inequalities we compute each hyperplane and then check the final point to obtain the direction of the inequality (note that we exclude the choice  $v^1, v^2, v^5, v^6$  because this corresponds to  $F^3$ ). The 4 facets are described below:

$F^6$  (plane through points  $v^1, v^2, v^5, v^8$ ):

$$\begin{aligned} f - a_2 a_3 x_1 - \frac{(-a_1 a_2 a_3 + a_1 b_2 b_3 + b_1 a_2 a_3 - b_1 a_2 b_3)}{(b_2 - a_2)} x_2 \\ - b_1 a_2 x_3 + \frac{(-a_1 a_2^2 a_3 + a_1 a_2 b_2 b_3 - b_1 a_2^2 b_3 + b_1 a_2 b_2 a_3)}{(b_2 - a_2)} \geq 0 \end{aligned}$$

$F^7$  (plane through points  $v^1, v^5, v^6, v^8$ ):

$$f - b_2 b_3 x_1 - b_1 b_3 x_2 - b_1 a_2 x_3 + b_1 a_2 b_3 + b_1 b_2 b_3 \geq 0$$

$F^8$  (plane through points  $v^2, v^5, v^6, v^8$ ):

$$\begin{aligned} -f + b_2 b_3 x_1 + b_1 b_3 x_2 + \frac{(-a_1 a_2 a_3 + a_1 b_2 b_3 + b_1 a_2 b_3 - b_1 b_2 b_3)}{(b_3 - a_3)} x_3 \\ - \frac{(-a_1 a_2 a_3 b_3 + a_1 b_2 b_3^2 + b_1 a_2 b_3^2 - b_1 b_2 a_3 b_3)}{(b_3 - a_3)} \geq 0 \end{aligned}$$

$F^9$  (plane through points  $v^1, v^2, v^6, v^8$ ):

$$-f + a_2 a_3 x_1 + b_1 b_3 x_2 + b_1 a_2 x_3 - b_1 a_2 a_3 - b_1 a_2 b_3 \geq 0$$

The facets of  $\mathcal{Q} = \text{conv}\{v^1, v^2, v^4, v^5, v^6, v^8\}$  are therefore  $F^1, F^2, F^4, F^5, F^6, F^7, F^8$  and  $F^9$ .

To obtain the entire volume of  $\mathcal{P}_H$  we need to consider two further extreme points:  $v^3$  and  $v^7$ . It would be convenient to add these points separately; i.e., compute the additional volume each produces when added to  $\mathcal{Q}$ , and sum the results. As the following lemma shows, this will give the correct volume if the intersection of the line segment between these points and  $\mathcal{Q}$  is not empty.

LEMMA 4.1. *Let  $P$  be a convex polytope and let  $w_1$  and  $w_2$  be points not in  $P$ . Let  $L(w_1, w_2)$  be the line segment between  $w_1$  and  $w_2$ . If  $L(w_1, w_2) \cap P \neq \emptyset$ , then  $\text{conv}(P, w_1) \cup \text{conv}(P, w_2)$  is convex. Moreover, in this case,  $\text{conv}(P, w_1, w_2) = \text{conv}(P, w_1) \cup \text{conv}(P, w_2)$ .*

*Proof.* First we show that  $\text{conv}(P, w_1) \cup \text{conv}(P, w_2)$  is convex. If we show that  $L(w_1, w_2)$  is completely contained in  $\text{conv}(P, w_1) \cup \text{conv}(P, w_2)$ , then we will be done. Choose  $z \in L(w_1, w_2) \cap P$ . Now consider  $L(w_1, z)$ . Because  $z \in P$ , this whole line segment must be in  $\text{conv}(P, w_1)$ . Similarly consider  $L(z, w_2)$ ; this whole line segment must be contained in  $\text{conv}(P, w_2)$ . Therefore the whole line segment  $L(w_1, w_2)$  must be contained in  $\text{conv}(P, w_1) \cup \text{conv}(P, w_2)$  and therefore this set is convex.

Next, we demonstrate that  $\text{conv}(P, w_1, w_2) = \text{conv}(P, w_1) \cup \text{conv}(P, w_2)$ . First, choose  $y \in \text{conv}(P, w_1) \cup \text{conv}(P, w_2)$ ; therefore  $y \in \text{conv}(P, w_1)$  or  $y \in \text{conv}(P, w_2)$  (or both); in either case it is clear that  $y \in \text{conv}(P, w_1, w_2)$ . In the other direction, choose  $y \in \text{conv}(P, w_1, w_2)$ ; therefore  $y$  can be written as a convex combination of the extreme points of  $P$  and  $w_1$  and  $w_2$ . Because  $\text{conv}(P, w_1) \cup \text{conv}(P, w_2)$  is convex, this means  $y \in \text{conv}(P, w_1) \cup \text{conv}(P, w_2)$ . Therefore the sets are equal as required.  $\square$

To show that the intersection of the line segment between  $v^3$  and  $v^7$  and  $\mathcal{Q}$  is non-empty, consider the midpoint

$$\left[ \frac{a_1 a_2 b_3 + b_1 b_2 a_3}{2} \quad \frac{a_1 + b_1}{2} \quad \frac{a_2 + b_2}{2} \quad \frac{b_3 + a_3}{2} \right]. \quad (35)$$

of this line segment.

We show that this point satisfies each of the inequalities of  $\mathcal{Q}$  by substituting into each inequality and checking the result. By showing that each resulting quantity is nonnegative, we can conclude that the midpoint intersects  $\mathcal{Q}$ . It is easy to see that the midpoint 35 satisfies  $F^1, F^5, F^8$  and  $F^9$ . Using Lemma 9.1, we can also check 35 satisfies  $F^2, F^4, F^6$  and  $F^7$ . Therefore  $\text{conv}(\mathcal{Q} \cup \{v^3\}) \cup \text{conv}(\mathcal{Q} \cup \{v^7\}) = \text{conv}(\mathcal{Q} \cup \{v^3\} \cup \{v^7\}) = \mathcal{P}_H$ .

**4.1. Computing the (additional) volume of  $\text{conv}(\mathcal{Q} \cup \{v^3\})$ .** We will now compute the additional volume of  $\text{conv}(\mathcal{Q} \cup \{v^3\})$  compared to the volume of  $\mathcal{Q}$ . To obtain this, we will sum the volumes of  $\text{conv}(\{v^3\} \cup F)$  for each facet,  $F$ , of  $\mathcal{Q}$  such that  $v^3$  is beyond that facet. We substitute  $v^3$  into each relevant inequality, and if the result is negative then  $v^3$  lies beyond that facet. It is easy to see that  $v^3$  satisfies  $F^5, F^7$  and  $F^9$  and violates  $F^2, F^6$  and  $F^8$ . It can then be checked that  $v^3$  satisfies  $F^1$  using Lemma 9.4 (with  $A = b_2, B = a_2, C = (b_1 b_3 - a_1 a_3), D = (2a_1 a_3 - a_1 b_3 - b_1 a_3)$ ). We can also check that  $v^3$  satisfies  $F^4$  using Lemma 9.4 (with  $A = b_2, B = a_2, C = (a_1 a_3 - 2a_1 b_3 + b_1 b_3), D = (a_1 b_3 - b_1 a_3)$ ) and Lemma 9.1.

From this we know that  $v^3$  is beyond  $F^2, F^6$  and  $F^8$ ; therefore we need to compute the volume of the convex hulls of  $v^3$  with each of these facets.

The polytope  $\text{conv}(F^2 \cup \{v^3\}) = \text{conv}\{v^1, v^2, v^4, v^5, v^3\}$  is a 4-simplex with volume:

$$a_1(b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^2/24.$$

The polytope  $\text{conv}(F^6 \cup \{v^3\}) = \text{conv}\{v^1, v^2, v^5, v^8, v^3\}$  is a 4-simplex with volume:

$$a_2(b_1 - a_1)^2(b_2 - a_2)(b_3 - a_3)^2/24.$$

The polytope  $\text{conv}(F^8 \cup \{v^3\}) = \text{conv}\{v^2, v^5, v^6, v^8, v^3\}$  is a 4-simplex with volume:

$$b_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)/24.$$

**4.2. Computing the (additional) volume of  $\text{conv}(\mathcal{Q} \cup \{v^7\})$ .** We will now compute the additional volume of  $\text{conv}(\mathcal{Q} \cup \{v^7\})$  compared to the volume of  $\mathcal{Q}$ . To obtain this, we will sum the volumes of  $\text{conv}(\{v^7\} \cup F)$  for each facet,  $F$ , of  $\mathcal{Q}$  such that  $v^7$  is beyond that facet. We substitute  $v^7$  into each relevant inequality, and if the result is negative then  $v^7$  lies beyond that facet. It is easy to see that  $v^7$  satisfies  $F^2, F^5$  and  $F^9$  and violates  $F^1, F^4$  and  $F^7$ . It can then be checked that  $v^7$  satisfies  $F^6$  using Lemma 9.4 (with  $A = b_2, B = a_2, C = (b_1 a_3 - a_1 b_3), D = (a_1 a_3 - 2b_1 a_3 + b_1 b_3)$ )

and Lemma 9.1. We can also check that  $v^7$  satisfies  $F^8$  using Lemma 9.4 (with  $A = b_2, B = a_2, C = (2b_1b_3 - a_1b_3 - b_1a_3), D = (a_1a_3 - b_1b_3)$ ).

From this we know that  $v^7$  is beyond  $F^1, F^4$  and  $F^7$ , therefore we need to compute the volume of the convex hulls of  $v^7$  with each of these facets.

The polytope  $\text{conv}(F^1 \cup \{v^7\}) = \text{conv}\{v^1, v^2, v^4, v^6, v^7\}$  is a 4-simplex with volume:

$$a_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)/24.$$

The polytope  $\text{conv}(F^4 \cup \{v^7\}) = \text{conv}\{v^1, v^4, v^5, v^6, v^7\}$  is a 4-simplex with volume:

$$b_2(b_1 - a_1)^2(b_2 - a_2)(b_3 - a_3)^2/24.$$

The polytope  $\text{conv}(F^7 \cup \{v^7\}) = \text{conv}\{v^1, v^5, v^6, v^8, v^7\}$  is a 4-simplex with volume:

$$b_1(b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^2/24.$$

To compute the volume of  $\mathcal{P}_H$ , we sum the volume of the appropriate eight simplices, and we obtain the volume of  $\mathcal{P}_H$  as stated in Theorem 3.1.  $\square$

**5. Proof of Thm. 3.4.** We will compute the volume of the convex hull of the 12 extreme points which we claim are exactly the extreme points of system 3. In computing the volume of this polytope, we will also prove that these are the correct extreme points and that we have therefore computed the volume of  $\mathcal{P}_3$ .

The relevant points are the eight extreme points of  $\mathcal{P}_H$ , plus an additional four points. Because we have already computed the volume of  $\mathcal{P}_H$ , to compute the volume of  $\mathcal{P}_3$ , we need to compute the additional volume, compared with  $\mathcal{P}_H$ , added by these four extra extreme points. To show that this is indeed the volume of  $\mathcal{P}_3$ , we will keep track of which facets need to be deleted and added to the system of inequalities as we go. When this is complete we will have exactly system 3, and therefore we must also have the correct extreme points.

We begin with system  $H$  from §2.2. As discussed in §4, it would be convenient to add the four points to  $\mathcal{P}_H$  separately; i.e., compute the additional volume each produces when added to  $\mathcal{P}_H$ , and sum the results. To show that we can add two points separately and obtain the correct volume, we will show that the intersection of the line segment between these points and  $\mathcal{P}_H$  is non-empty (Lemma 4.1).

We will show that we can add  $v_3^9$  separately,  $v_3^{10}$  separately, and then  $v_3^{11}$  and  $v_3^{12}$  together by considering the midpoints of the line segments between the relevant points. We consider the line segments between points:  $v_3^9$  and  $v_3^{10}$ ;  $v_3^9$  and  $v_3^{11}$ ;  $v_3^9$  and  $v_3^{12}$ ;  $v_3^{10}$  and  $v_3^{11}$ ;  $v_3^{10}$  and  $v_3^{12}$ . We refer to the midpoint of the line between  $v_i^j$  and  $v_i^k$  as  $M(v_i^j, v_i^k)$ , and we will show that the midpoint of each line satisfies each of the inequalities of  $\mathcal{P}_H$  by substituting this point into each inequality and checking the result. See Table 1 for a summary of the resulting substitutions. The table notes whether nonnegativity of the resulting quantity follows immediately (after factoring), or by use of a technical lemma (after further explanation in the appendix), or after being rewritten in the way referenced in Figure 3. Because we have shown that each resulting quantity is nonnegative, we conclude that the midpoint intersects  $\mathcal{P}_H$ , and therefore we can add  $v_3^9$  separately,  $v_3^{10}$  separately, and then  $v_3^{11}$  and  $v_3^{12}$  together.

**5.1. Computing the (additional) volume of  $\text{conv}(\mathcal{P}_H \cup \{v_3^9\})$ .** We will now compute the additional volume of  $\text{conv}(\mathcal{P}_H \cup \{v_3^9\})$  compared to the volume of  $\mathcal{P}_H$ . To do this we will sum the volumes of  $\text{conv}(\{v_3^9\} \cup F)$  for each facet,  $F$ , of  $\mathcal{P}_H$  such that  $v_3^9$  is beyond that facet. We substitute  $v_3^9$  into each inequality of system  $H$ , and we can immediately see that it satisfies every inequality except 25.

TABLE 1

Ineq	$M(v_3^9, v_3^{10})$	$M(v_3^9, v_3^{11})$	$M(v_3^9, v_3^{12})$	$M(v_3^{10}, v_3^{11})$	$M(v_3^{10}, v_3^{12})$
17	immediate	immediate	immediate	immediate	immediate
18	immediate	immediate	immediate	immediate	immediate
19	immediate	by Lemma 9.1	immediate	by Lemma 9.1	by Lemma 9.1
20	immediate	by Lemma 9.1	by Lemma 9.1	immediate	by Lemma 9.1
21	immediate	by Lemma 9.1	immediate	by Lemma 9.1	by Lemma 9.1
22	immediate	by Lemma 9.1	by Lemma 9.1	immediate	by Lemma 9.1
23	immediate	immediate	immediate	immediate	immediate
24	immediate	immediate	immediate	immediate	immediate
25	see 36	see §9.2	immediate	see 37	see 38
26	see 39	see 40	see 37	immediate	See §9.3
27	immediate	immediate	immediate	immediate	immediate
28	immediate	immediate	immediate	immediate	immediate
29	immediate	immediate	immediate	immediate	immediate
30	immediate	immediate	immediate	immediate	immediate
31	immediate	immediate	immediate	immediate	immediate
32	immediate	immediate	immediate	immediate	immediate
33	immediate	immediate	immediate	immediate	immediate
34	immediate	immediate	immediate	immediate	immediate

FIGURE 3. For Table 1

$$\frac{(b_2 - a_2)(b_1 - a_1)(b_1 b_3(b_2 - a_2) + a_2 a_3(b_1 - a_1))}{2(b_1 b_2 - a_1 a_2)} \quad (36)$$

$$\frac{(b_2 b_3 - a_2 a_3)(b_1 - a_1) + (b_1 b_3 - a_1 a_3)(b_2 - a_2)}{2} \quad (37)$$

$$\frac{(b_2 - a_2)(b_1 - a_1)(b_1(b_2 b_3 - a_2 a_3) + a_2(b_1 b_3 - a_1 a_3))}{2(b_1 b_2 - a_1 a_2)} \quad (38)$$

$$\frac{(b_2 - a_2)(b_1 - a_1)(b_2 b_3(b_1 - a_1) + a_1 a_3(b_2 - a_2))}{2(b_1 b_2 - a_1 a_2)} \quad (39)$$

$$\frac{(b_2 - a_2)(b_1 - a_1)(b_2(b_1 b_3 - a_1 a_3) + a_1(b_2 b_3 - a_2 a_3))}{2(b_1 b_2 - a_1 a_2)} \quad (40)$$

From this we know that  $v_3^9$  is beyond only one facet. The extreme points that lie on this facet are points  $v^1, v^2, v^6$  and  $v^8$ . The polytope  $\text{conv}\{v^1, v^2, v^6, v^8, v_3^9\}$  is a 4-simplex with volume:

$$b_1 a_2 (b_1 - a_1)^2 (b_2 - a_2)^2 (b_3 - a_3)^2 / (24(b_1 b_2 - a_1 a_2)).$$

The facets of  $\text{conv}(\mathcal{P}_H \cup \{v_3^9\})$  are the facets of  $\mathcal{P}_H$  minus inequality 25. We can see this by computing the four additional facets that come from adding  $v_3^9$  and noting they are already contained in system  $H$ :

- The facet through points  $v^1, v^2, v^6$  and  $v_3^9$  is 23.
- The facet through points  $v^1, v^2, v^8$  and  $v_3^9$  is 31.
- The facet through points  $v^1, v^6, v^8$  and  $v_3^9$  is 30.
- The facet through points  $v^2, v^6, v^8$  and  $v_3^9$  is 27.

**5.2. Computing the (additional) volume of  $\text{conv}(\mathcal{P}_H \cup \{v_3^{10}\})$ .** We will now compute the additional volume of  $\text{conv}(\mathcal{P}_H \cup \{v_3^{10}\})$  compared to the volume of  $\mathcal{P}_H$ . To do this we will sum the volumes of  $\text{conv}(\{v_3^{10}\} \cup F)$  for each facet,  $F$ , of  $\mathcal{P}_H$  such that  $v_3^{10}$  is beyond that facet. We substitute  $v_3^{10}$  into each inequality of system  $H$ , and we can immediately see that every inequality is satisfied except 26.

From this we know that  $v_3^{10}$  is beyond only one facet. The extreme points that lie on this facet are points  $v^2, v^4, v^5$  and  $v^6$ . The polytope  $\text{conv}\{v^2, v^4, v^5, v^6, v_3^{10}\}$  is a 4-simplex with volume:

$$a_1 b_2 (b_1 - a_1)^2 (b_2 - a_2)^2 (b_3 - a_3)^2 / (24(b_1 b_2 - a_1 a_2)).$$

The facets of  $\text{conv}(\mathcal{P}_H \cup \{v_3^{10}\})$  are the facets of  $\mathcal{P}_H$  minus inequality 26. We can see this by computing the four additional facets that come from adding  $v_3^{10}$  and noting they are already contained in system  $H$ :

- The facet through points  $v^2, v^4, v^5$  and  $v_3^{10}$  is 29.
- The facet through points  $v^2, v^4, v^6$  and  $v_3^{10}$  is 24.
- The facet through points  $v^2, v^5, v^6$  and  $v_3^{10}$  is 28.
- The facet through points  $v^4, v^5, v^6$  and  $v_3^{10}$  is 32.

**5.3. Computing the (additional) volume of  $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\} \cup \{v_3^{12}\})$ .** We will now compute the additional volume of  $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\} \cup \{v_3^{12}\})$  compared to the volume of  $\mathcal{P}_H$ . Because the line segment between  $v_3^{11}$  and  $v_3^{12}$  lies entirely outside of  $\mathcal{P}_3$ , we will need to add them sequentially.

We will first compute the additional volume of  $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\})$  compared to the volume of  $\mathcal{P}_H$ . As we have done previously, we will sum the volumes of  $\text{conv}(\{v_3^{11}\} \cup F)$  for each facet,  $F$ , of  $\mathcal{P}_H$  such that  $v_3^{11}$  is beyond that facet. We substitute  $v_3^{11}$  into each relevant inequality, and if the result is negative then  $v_3^{11}$  lies beyond that facet. It is immediate that  $v_3^{11}$  violates inequalities 19–22 and satisfies inequalities 17–18, 23–24 and 26–34. To see that inequality 25 is also satisfied see §9.4.

Therefore we have that  $v_3^{11}$  is beyond four facets, and we need to compute the volume of the convex hulls of  $v_3^{11}$  with each of these facets.

The extreme points that lie on the first facet are points  $v^1, v^3, v^5$  and  $v^8$ . The polytope  $\text{conv}\{v^1, v^3, v^5, v^8, v_3^{11}\}$  is a 4-simplex with volume:

$$a_1 b_1 (b_1 - a_1) (b_2 - a_2)^3 (b_3 - a_3)^2 / (24(b_1 b_2 - a_1 a_2)).$$

The extreme points that lie on the second facet are points  $v^1, v^4, v^5$  and  $v^7$ . The polytope  $\text{conv}\{v^1, v^4, v^5, v^7, v_3^{11}\}$  is a 4-simplex with volume:

$$a_1 b_2 (b_1 - a_1)^2 (b_2 - a_2)^2 (b_3 - a_3)^2 / (24(b_1 b_2 - a_1 a_2)).$$

The extreme points that lie on the third facet are points  $v^1, v^5, v^7$  and  $v^8$ . The polytope  $\text{conv}\{v^1, v^5, v^7, v^8, v_3^{11}\}$  is a 4-simplex with volume:

$$a_1 b_1 (b_1 - a_1) (b_2 - a_2)^3 (b_3 - a_3)^2 / (24(b_1 b_2 - a_1 a_2)).$$

The extreme points that lie on the fourth facet are points  $v^1, v^3, v^4$  and  $v^5$ . The polytope  $\text{conv}\{v^1, v^3, v^4, v^5, v_3^{11}\}$  is a 4-simplex with volume:

$$a_1 b_2 (b_1 - a_1)^2 (b_2 - a_2)^2 (b_3 - a_3)^2 / (24(b_1 b_2 - a_1 a_2)).$$

We now have a new polytope which is  $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\})$ . We refer to this polytope as  $\mathcal{T}_3$  and will compute the facets of  $\mathcal{T}_3$ .

We begin with the facets of  $\mathcal{P}_H$  and delete the four facets that  $v_3^{11}$  violated (19–22). Let us call this system  $\mathcal{T}_3^-$ . Now consider the four simplices we dealt with when computing the additional volume produced with  $v_3^{11}$ . Each of these simplices has 5 facets; one of which corresponds to a deleted facet of  $\mathcal{P}_H$ .

The remaining 4 facets of the first simplex are described by the planes through the following sets of points:  $\{v^1, v^3, v^5, v_3^{11}\}$ ,  $\{v^1, v^3, v^8, v_3^{11}\}$ ,  $\{v^1, v^5, v^8, v_3^{11}\}$  and  $\{v^3, v^5, v^8, v_3^{11}\}$ .

The remaining 4 facets of the second simplex are described by the planes through the following sets of points:  $\{v^1, v^4, v^5, v_3^{11}\}$ ,  $\{v^1, v^4, v^7, v_3^{11}\}$ ,  $\{v^1, v^5, v^7, v_3^{11}\}$  and  $\{v^4, v^5, v^7, v_3^{11}\}$ .

The remaining 4 facets of the third simplex are described by the planes through the following sets of points:  $\{v^1, v^5, v^7, v_3^{11}\}$ ,  $\{v^1, v^5, v^8, v_3^{11}\}$ ,  $\{v^1, v^7, v^8, v_3^{11}\}$  and  $\{v^5, v^7, v^8, v_3^{11}\}$ .

The remaining 4 facets of the fourth simplex are described by the planes through the following sets of points:  $\{v^1, v^3, v^4, v_3^{11}\}$ ,  $\{v^1, v^3, v^5, v_3^{11}\}$ ,  $\{v^1, v^4, v^5, v_3^{11}\}$  and  $\{v^3, v^4, v^5, v_3^{11}\}$ .

Consider these sixteen facets and exclude the facets that are shared by more than one simplex. This leaves eight facets.

We can compute these eight facets to obtain the following:

- The facet through points  $v^1, v^3, v^8$  and  $v_3^{11}$  is

$$\begin{aligned} & \frac{1}{b_1 b_2 - a_1 a_2} (-a_1^2 a_2^2 b_3 + a_1^2 a_2 b_3 x_2 - a_1 b_1 a_2^2 a_3 + a_1 b_1 a_2^2 x_3 \\ & + a_1 a_2^2 b_3 x_1 + a_1 b_1 a_2 b_2 a_3 + a_1 b_1 a_2 a_3 x_2 - a_1 b_1 a_2 b_3 x_2 - a_1 b_1 b_2 a_3 x_2 \\ & + b_1^2 a_2 b_2 b_3 - b_1^2 a_2 b_2 x_3 - b_1 a_2 b_2 b_3 x_1 - a_1 a_2 f + b_1 b_2 f) \geq 0. \end{aligned} \quad (41)$$

- The facet through points  $v^3, v^5, v^8$  and  $v_3^{11}$  is

$$f - a_2 b_3 x_1 - a_1 b_3 x_2 - b_1 b_2 x_3 + a_1 a_2 b_3 + b_1 b_2 b_3 \geq 0. \quad (42)$$

- The facet through points  $v^1, v^4, v^7$  and  $v_3^{11}$  is

$$f - b_2 a_3 x_1 - b_1 a_3 x_2 - a_1 a_2 x_3 + a_1 a_2 a_3 + b_1 b_2 a_3 \geq 0. \quad (43)$$

- The facet through points  $v^4, v^5, v^7$  and  $v_3^{11}$  is 32.
- The facet through points  $v^1, v^7, v^8$  and  $v_3^{11}$  is

$$\begin{aligned} & \frac{1}{b_1 b_2 - a_1 a_2} (-a_1 b_1 a_2^2 a_3 + a_1 b_1 a_2^2 x_3 + a_1 b_1 a_2 a_3 x_2 - a_1 b_1 a_2 b_2 b_3 + a_1 a_2 b_2 b_3 x_1 + b_1 a_2 b_2 a_3 x_1 \\ & + b_1^2 a_2 b_2 b_3 - b_1^2 a_2 b_2 x_3 - b_1 a_2 b_2 b_3 x_1 + b_1^2 b_2^2 a_3 - b_1^2 b_2 a_3 x_2 - b_1 b_2^2 a_3 x_1 - a_1 a_2 f + b_1 b_2 f) \geq 0. \end{aligned} \quad (44)$$

- The facet through points  $v^5, v^7, v^8$  and  $v_3^{11}$  is 18.
- The facet through points  $v^1, v^3, v^4$  and  $v_3^{11}$  is 17.
- The facet through points  $v^3, v^4, v^5$  and  $v_3^{11}$  is 29.

There are four inequalities that are not already contained in system  $\mathcal{T}_3^-$ , we add these and in doing so obtain the system of inequalities that describes  $\mathcal{T}_3 = \text{conv}(\mathcal{P}_H \cup \{v_3^{11}\})$ .

We will now compute the additional volume of  $\text{conv}(\mathcal{T}_3 \cup \{v_3^{12}\})$  compared to the volume of  $\mathcal{T}_3$ . As we have done previously, we will sum the volumes of  $\text{conv}(\{v_3^{12}\} \cup F)$  for each facet,  $F$ , of  $\mathcal{T}_3$  such that  $v_3^{12}$  is beyond that facet. We substitute  $v_3^{12}$  into each relevant inequality (i.e., the system of inequalities that describes  $\mathcal{T}_3$ ) and if the result is negative then  $v_3^{12}$  lies beyond that facet. It is immediately clear that  $v_3^{12}$  satisfies inequalities 17-18, 23-25, 27-34 and 42-43. We can also see immediately that  $v_3^{12}$  violates inequalities 41 and 44. To see that inequality 26 is also satisfied see §9.4.

Therefore we see that  $v_3^{12}$  is beyond two facets and we need to compute the volume of the convex hull of  $v_3^{12}$  with each of these facets.

The extreme points that lie on the first facet are points  $v^1, v^3, v^8$  and  $v_3^{11}$ . The polytope  $\text{conv}\{v^1, v^3, v^8, v_3^{11}, v_3^{12}\}$  is a 4-simplex with volume:

$$b_1 a_2 (b_1 - a_1)^2 (b_2 - a_2)^2 (b_3 - a_3)^2 / (24(b_1 b_2 - a_1 a_2)).$$

The extreme points that lie on the second facet are points  $v^1, v^7, v^8$  and  $v_3^{11}$ . The polytope  $\text{conv}\{v^1, v^7, v^8, v_3^{11}, v_3^{12}\}$  is a 4-simplex with volume:

$$b_1 a_2 (b_1 - a_1)^2 (b_2 - a_2)^2 (b_3 - a_3)^2 / (24(b_1 b_2 - a_1 a_2)).$$

We now compute the additional facets; we take the four facets from adding each simplex and delete the facet that repeats. This leaves us with the following six facets to compute:

- The facet through points  $v^1, v^7, v^8$  and  $v_3^{12}$  is 30.
- The facet through points  $v^1, v^7, v_3^{11}$  and  $v_3^{12}$  is 43.
- The facet through points  $v^7, v^8, v_3^{11}$  and  $v_3^{12}$  is 18.
- The facet through points  $v^1, v^3, v^8$  and  $v_3^{12}$  is 31.
- The facet through points  $v^1, v^3, v_3^{11}$  and  $v_3^{12}$  is 17.
- The facet through points  $v^3, v^8, v_3^{11}$  and  $v_3^{12}$  is 42.

By adding and deleting the appropriate facets to and from system  $H$ , we see that we arrive at system 3.

Therefore, to compute the volume of  $\mathcal{P}_3$ , we sum the volume of  $\mathcal{P}_H$  with that of the appropriate eight simplices, and we obtain our result.  $\square$

**6. Proof of Thm. 3.2.** As with Theorem 3.4, we will compute the volume of the convex hull of the 12 extreme points which we claim are exactly the extreme points of system 1. In computing the volume of this polytope we will also prove that these are the correct extreme points and that we have therefore computed the volume of  $\mathcal{P}_1$ .

The relevant points are the eight extreme points of  $\mathcal{P}_H$ , plus an additional four points. Because we have already computed the volume of  $\mathcal{P}_H$ , to compute the volume of  $\mathcal{P}_1$  we need to compute the additional volume, compared with  $\mathcal{P}_H$ , added by these four extra extreme points. To show that this is indeed the volume of  $\mathcal{P}_1$  we will keep track of which facets need to be deleted and added to the system of inequalities as we go. When this is complete we will have exactly system 1 and therefore we must also have the correct extreme points.

We begin with system  $H$  which can be found in §2.2 and we will use the same principles as we used in the previous proof to compute the volume of  $\mathcal{P}_3$ .

First we argue that we can add  $v_1^9$  to  $\mathcal{P}_H$  separately,  $v_1^{10}$  to  $\mathcal{P}_H$  separately and then  $v_1^{11}$  and  $v_1^{12}$  together. To do this we will show that the midpoint of the line segment between  $v_1^9$  and all other additional points ( $v_1^{10}, v_1^{11}$  and  $v_1^{12}$ ) intersects  $\mathcal{P}_H$ . We will also show this is true for  $v_1^{10}$ .

As in the previous proof we refer to the midpoint of the line between  $v_i^j$  and  $v_i^k$  as  $M(v_i^j, v_i^k)$  and we show that the midpoint of each line satisfies each of the inequalities of  $\mathcal{P}_H$  by substituting this point into each inequality and checking the result. See Table 2 for a summary of the resulting substitutions. The table notes whether nonnegativity of the resulting quantity follows immediately (after factoring), or by using a technical lemma, after further explanation in the appendix or after being rewritten in the way referenced in Figure 4. Because we have shown that each resulting quantity is nonnegative, we know that the midpoint intersects  $\mathcal{P}_H$ , and therefore we can add  $v_1^9$  separately,  $v_1^{10}$  separately, and then  $v_1^{11}$  and  $v_1^{12}$  together.

TABLE 2

Ineq	$M(v_1^9, v_1^{10})$	$M(v_1^9, v_1^{11})$	$M(v_1^9, v_1^{12})$	$M(v_1^{10}, v_1^{11})$	$M(v_1^{10}, v_1^{12})$
17	immediate	immediate	immediate	immediate	immediate
18	immediate	immediate	immediate	immediate	immediate
19	see §9.5	see 45 and Lemma 9.1	immediate	see 46 and Lemma 9.1	see §9.11
20	see §9.6	see §9.7	see 46 and Lemma 9.1	immediate	see 47 and Lemma 9.1
21	see §9.17	see §9.8	by Lemma 9.1	by Lemma 9.1	see §9.12
22	see §9.18	see §9.9	by Lemma 9.1	by Lemma 9.1	see §9.13
23	immediate	immediate	immediate	immediate	immediate
24	see 48	see 49	see 50	immediate	see §9.14
25	immediate	immediate	immediate	immediate	immediate
26	immediate	immediate	immediate	immediate	immediate
27	see 51	See §9.10	immediate	see 50	see 52
28	immediate	immediate	immediate	immediate	immediate
29	immediate	immediate	immediate	immediate	immediate
30	immediate	immediate	immediate	immediate	immediate
31	immediate	immediate	immediate	immediate	immediate
32	immediate	immediate	immediate	immediate	immediate
33	immediate	immediate	immediate	immediate	immediate
34	immediate	immediate	immediate	immediate	immediate

FIGURE 4. For Table 2

$$\frac{(b_3 - a_3)(b_2 - a_2)(b_3(b_1a_2 - a_1b_2) + a_2(b_1a_3 - a_1b_3))}{2(b_2b_3 - a_2a_3)} \quad (45)$$

$$\frac{(b_1a_3 - a_1b_3)(b_2 - a_2) + (b_1a_2 - a_1b_2)(b_3 - a_3)}{2} \quad (46)$$

$$\frac{(b_3 - a_3)(b_2 - a_2)(b_2(b_1a_3 - a_1b_3) + a_3(b_1a_2 - a_1b_2))}{2(b_2b_3 - a_2a_3)} \quad (47)$$

$$\frac{(b_3 - a_3)(b_2 - a_2)(b_1b_2(b_3 - a_3) + a_1a_3(b_2 - a_2))}{2(b_2b_3 - a_2a_3)} \quad (48)$$

$$\frac{(b_3 - a_3)(b_2 - a_2)(b_2(b_1b_3 - a_1a_3) + a_3(b_1b_2 - a_1a_2))}{2(b_2b_3 - a_2a_3)} \quad (49)$$

$$\frac{(b_1b_3 - a_1a_3)(b_2 - a_2) + (b_1b_2 - a_1a_2)(b_3 - a_3)}{2} \quad (50)$$

$$\frac{(b_3 - a_3)(b_2 - a_2)(b_1b_3(b_2 - a_2) + a_1a_2(b_3 - a_3))}{2(b_2b_3 - a_2a_3)} \quad (51)$$

$$\frac{(b_3 - a_3)(b_2 - a_2)(b_3(b_1b_2 - a_1a_2) + a_2(b_1b_3 - a_1a_3))}{2(b_2b_3 - a_2a_3)} \quad (52)$$

**6.1. Computing the (additional) volume of  $\text{conv}(\mathcal{P}_H \cup \{v_1^9\})$ .** We will now compute the additional volume of  $\text{conv}(\mathcal{P}_H \cup \{v_1^9\})$  compared to the volume of  $\mathcal{P}_H$ . To do this we will sum the volumes of  $\text{conv}(\{v_1^9\} \cup F)$  for each facet,  $F$ , of  $\mathcal{P}_H$  such that  $v_1^9$  is beyond that facet. We substitute  $v_1^9$  into the 18 relevant inequalities (17-34) and can immediately see that it satisfies 17-19, 23-26 and 28-34. It is also immediate to see that inequality 27 is violated. To show that the remaining three inequalities are satisfied (20, 21 and 22) see §9.15, §9.19 and §9.20.

From this we know that  $v_1^9$  is beyond only one facet. The extreme points that lie on this facet are points  $v^2, v^3, v^6$  and  $v^8$ . The polytope  $\text{conv}\{v^2, v^3, v^6, v^8, v_1^9\}$  is a 4-simplex with volume:

$$a_2b_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)^2 / (24(b_2b_3 - a_2a_3)).$$

The facets of  $\text{conv}(\mathcal{P}_H \cup \{v_1^9\})$  are the facets of  $\mathcal{P}_H$  minus inequality 27. We can see this by computing the four additional facets that come from adding  $v_1^9$  and noting they are already contained in system  $H$ :

- The facet through points  $v^2, v^3, v^6$  and  $v_1^9$  is 28.
- The facet through points  $v^2, v^3, v^8$  and  $v_1^9$  is 31.
- The facet through points  $v^2, v^6, v^8$  and  $v_1^9$  is 25.
- The facet through points  $v^3, v^6, v^8$  and  $v_1^9$  is 34.

**6.2. Computing the (additional) volume of  $\text{conv}(\mathcal{P}_H \cup \{v_1^{10}\})$ .** We will now compute the additional volume of  $\text{conv}(\mathcal{P}_H \cup \{v_1^{10}\})$  compared to the volume of  $\mathcal{P}_H$ . To do this we will sum the volumes of  $\text{conv}(\{v_1^{10}\} \cup F)$  for each facet,  $F$ , of  $\mathcal{P}_H$  such that  $v_1^{10}$  is beyond that facet. We substitute  $v_1^{10}$  into the 18 relevant inequalities and can immediately see that it satisfies 17, 18, 20, 23 and 25-34. It is also immediate to see that inequality 24 is violated. To show that the remaining three inequalities are satisfied (19, 21 and 22) see §9.15, §9.21 and §9.22.

From this we know that  $v_1^{10}$  is beyond only one facet. The extreme points that lie on this facet are points  $v^2, v^4, v^6$  and  $v^7$ . The polytope  $\text{conv}\{v^2, v^4, v^6, v^7, v_3^{10}\}$  is a 4-simplex with volume:

$$b_2 a_3 (b_1 - a_1)^2 (b_2 - a_2)^2 (b_3 - a_3)^2 / (24(b_2 b_3 - a_2 a_3)).$$

The facets of  $\text{conv}(\mathcal{P}_H \cup \{v_1^{10}\})$  are the facets of  $\mathcal{P}_H$  minus inequality 24. We can see this by computing the four additional facets that come from adding  $v_1^{10}$  and noting they are already contained in system  $H$ :

- The facet through points  $v^2, v^4, v^6$  and  $v_1^{10}$  is 26.
- The facet through points  $v^2, v^4, v^7$  and  $v_1^{10}$  is 33.
- The facet through points  $v^2, v^6, v^7$  and  $v_1^{10}$  is 23.
- The facet through points  $v^4, v^6, v^7$  and  $v_1^{10}$  is 32.

**6.3. Computing the (additional) volume of  $\text{conv}(\mathcal{P}_H \cup \{v_1^{11}\} \cup \{v_1^{12}\})$ .** We will now compute the additional volume of  $\text{conv}(\mathcal{P}_H \cup \{v_1^{11}\} \cup \{v_1^{12}\})$  compared to the volume of  $\mathcal{P}_H$ . Because the line segment between  $v_1^{11}$  and  $v_1^{12}$  lies entirely outside of  $\mathcal{P}_1$ , we will need to add them sequentially.

We will first compute the additional volume of  $\text{conv}(\mathcal{P}_H \cup \{v_1^{11}\})$  compared to the volume of  $\mathcal{P}_H$ . As we have done previously, we will sum the volumes of  $\text{conv}(\{v_1^{11}\} \cup F)$  for each facet,  $F$ , of  $\mathcal{P}_H$  such that  $v_1^{11}$  is beyond that facet. We substitute  $v_1^{11}$  into each relevant inequality and if the result is negative then  $v_1^{11}$  lies beyond that facet. It is immediate that  $v_1^{11}$  satisfies inequalities 17, 18, 23-26 and 28-34. In §9.16 we show that 27 is also satisfied. It is also immediate that  $v_1^{11}$  violates the three facets described by 20-22. We compute the volume of the convex hulls of  $v_1^{11}$  with each of these facets.

The extreme points that lie on the first facet are points  $v^1, v^4, v^5$  and  $v^7$ . The polytope  $\text{conv}\{v^1, v^4, v^5, v^7, v_1^{11}\}$  is a 4-simplex with volume:

$$b_2 a_3 (b_1 - a_1)^2 (b_2 - a_2)^2 (b_3 - a_3)^2 / (24(b_2 b_3 - a_2 a_3)).$$

The extreme points that lie on the second facet are points  $v^1, v^5, v^7$  and  $v^8$ . The polytope  $\text{conv}\{v^1, v^5, v^7, v^8, v_1^{11}\}$  is a 4-simplex with volume:

$$b_1 a_3 (b_1 - a_1) (b_2 - a_2)^3 (b_3 - a_3)^2 / (24(b_2 b_3 - a_2 a_3)).$$

The extreme points that lie on the third facet are points  $v^1, v^3, v^4$  and  $v^5$ . The polytope  $\text{conv}\{v^1, v^3, v^4, v^5, v_1^{11}\}$  is a 4-simplex with volume:

$$a_1 b_2 (b_1 - a_1) (b_2 - a_2)^2 (b_3 - a_3)^3 / (24(b_2 b_3 - a_2 a_3)).$$

Unlike in system 3, we see immediately that there exists a fourth facet (described by 19) which, under certain circumstances,  $v_1^{11}$  is beyond. In particular, this is true if and only if  $a_1b_2b_3 - b_1a_2a_3 > 0$ . Therefore we continue with two cases.

**6.3.1. Case 1:  $a_1b_2b_3 - b_1a_2a_3 > 0$**  In this case there exists a fourth facet (described by 19) such that  $v_1^{11}$  is beyond this facet. The extreme points that lie on this fourth facet are points  $v^1, v^3, v^5$  and  $v^8$ . The polytope  $\text{conv}\{v^1, v^3, v^5, v^8, v_1^{11}\}$  is a 4-simplex with volume:

$$(a_1b_2b_3 - b_1a_2a_3)(b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^2/(24(b_2b_3 - a_2a_3)).$$

We now have a new polytope which is  $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\})$  (in case 1). We refer to this polytope as  $\mathcal{T}_1^1$  and will compute the facets of  $\mathcal{T}_1^1$ .

We begin with the facets of  $\mathcal{P}_H$  and delete the four facets that  $v_1^{11}$  lies beyond (19-22). Let us call this system  $\mathcal{T}_1^{1-}$ . Now consider the four simplices we dealt with when computing the additional volume produced with  $v_1^{11}$ . Each of these simplices has 5 facets; one of which corresponds to a deleted facet of  $\mathcal{P}_H$ .

The remaining 4 facets of the first simplex are described by the planes through the following sets of points:  $\{v^1, v^4, v^5, v_1^{11}\}$ ,  $\{v^1, v^4, v^7, v_1^{11}\}$ ,  $\{v^1, v^5, v^7, v_1^{11}\}$  and  $\{v^4, v^5, v^7, v_1^{11}\}$ .

The remaining 4 facets of the second simplex are described by the planes through the following sets of points:  $\{v^1, v^5, v^7, v_1^{11}\}$ ,  $\{v^1, v^5, v^8, v_1^{11}\}$ ,  $\{v^1, v^7, v^8, v_1^{11}\}$  and  $\{v^5, v^7, v^8, v_1^{11}\}$ .

The remaining 4 facets of the third simplex are described by the planes through the following sets of points:  $\{v^1, v^3, v^4, v_1^{11}\}$ ,  $\{v^1, v^3, v^5, v_1^{11}\}$ ,  $\{v^1, v^4, v^5, v_1^{11}\}$  and  $\{v^3, v^4, v^5, v_1^{11}\}$ .

The remaining 4 facets of the fourth simplex are described by the planes through the following sets of points:  $\{v^1, v^3, v^5, v_1^{11}\}$ ,  $\{v^1, v^3, v^8, v_1^{11}\}$ ,  $\{v^1, v^5, v^8, v_1^{11}\}$  and  $\{v^3, v^5, v^8, v_1^{11}\}$ .

Consider these sixteen facets and exclude the facets that are shared by more than one simplex. This leaves eight facets.

We can compute these eight facets to obtain the following:

- The facet through points  $v^1, v^3, v^8$  and  $v_1^{11}$  is

$$\begin{aligned} & \frac{1}{b_2b_3 - a_2a_3} (-a_1a_2^2a_3b_3 + a_1a_2b_2a_3b_3 + a_1a_2a_3b_3x_2 - a_1b_2a_3b_3x_2 - b_1a_2^2a_3^2 + b_1a_2^2a_3x_3 \\ & + a_2^2a_3b_3x_1 + b_1a_2a_3^2x_2 - b_1a_2a_3b_3x_2 + b_1a_2b_2b_3^2 - b_1a_2b_2b_3x_3 - a_2b_2b_3^2x_1 - a_2a_3f + b_2b_3f) \geq 0. \end{aligned} \quad (53)$$

- The facet through points  $v^3, v^5, v^8$  and  $v_1^{11}$  is

$$\begin{aligned} & \frac{1}{b_2b_3 - a_2a_3} (-a_1a_2^2a_3b_3 + a_1a_2a_3b_3x_2 + a_1a_2b_2b_3x_3 + a_1b_2^2b_3^2 \\ & - a_1b_2^2b_3x_3 - a_1b_2b_3^2x_2 + a_2^2a_3b_3x_1 - b_1a_2b_2a_3b_3 + b_1a_2b_2a_3x_3 \\ & + b_1a_2b_2b_3^2 - b_1a_2b_2b_3x_3 - a_2b_2b_3^2x_1 - a_2a_3f + b_2b_3f) \geq 0. \end{aligned} \quad (54)$$

- The facet through points  $v^1, v^4, v^7$  and  $v_1^{11}$  is 33.
- The facet through points  $v^4, v^5, v^7$  and  $v_1^{11}$  is 32.
- The facet through points  $v^1, v^7, v^8$  and  $v_1^{11}$  is

$$f - b_2b_3x_1 - b_1a_3x_2 - b_1a_2x_3 + b_1a_2a_3 + b_1b_2b_3 \geq 0. \quad (55)$$

- The facet through points  $v^5, v^7, v^8$  and  $v_1^{11}$  is 18.
- The facet through points  $v^1, v^3, v^4$  and  $v_1^{11}$  is 17.
- The facet through points  $v^3, v^4, v^5$  and  $v_1^{11}$  is

$$f - a_2a_3x_1 - a_1b_3x_2 - a_1b_2x_3 + a_1a_2a_3 + a_1b_2b_3 \geq 0. \quad (56)$$

There are four inequalities that are not already contained in system  $\mathcal{T}_1^{1-}$ , we add these and in doing so obtain the system of inequalities that describes  $\mathcal{T}_1^1 = \text{conv}(\mathcal{P}_H \cup \{v_1^{11}\})$  (in case 1).

We will now compute the additional volume of  $\text{conv}(\mathcal{T}_1^1 \cup \{v_1^{12}\})$  compared to the volume of  $\mathcal{T}_1^1$ . As we have done previously, we will sum the volumes of  $\text{conv}(\{v_1^{12}\} \cup F)$  for each facet,  $F$ , of  $\mathcal{T}_1^1$  such that  $v_1^{12}$  is beyond that facet. We substitute  $v_1^{12}$  into each relevant inequality (i.e., the system that describes  $\mathcal{T}_1^1$ ) and if the result is negative then  $v_1^{12}$  lies beyond that facet. It is immediately clear that  $v_1^{12}$  satisfies inequalities 17, 18, 23, 25-34 and 55-56. We can also see immediately that inequalities 53 and 54 are violated. To see that inequality 24 is also satisfied see appendix §9.16.

Therefore we see that  $v_1^{12}$  is beyond two facets and we need to compute the volume of the convex hull of  $v_1^{12}$  with each of these facets.

The extreme points that lie on the first facet are points  $v^1, v^3, v^8$  and  $v_1^{11}$ . The polytope  $\text{conv}\{v^1, v^3, v^8, v_1^{11}, v_1^{12}\}$  is a 4-simplex with volume:

$$a_2b_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)^2/(24(b_2b_3 - a_2a_3)).$$

The extreme points that lie on the second facet are points  $v^3, v^5, v^8$  and  $v_1^{11}$ . The polytope  $\text{conv}\{v^3, v^5, v^8, v_1^{11}, v_1^{12}\}$  is a 4-simplex with volume:

$$a_2b_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)^2/(24(b_2b_3 - a_2a_3)).$$

We now compute the additional facets; we take the four facets from adding each simplex and delete the facet that repeats. This leaves us with the following six facet defining inequalities to compute:

- The facet through points  $v^1, v^3, v^8$  and  $v_1^{12}$  is 31.
- The facet through points  $v^1, v^3, v_1^{11}$  and  $v_1^{12}$  is 17.
- The facet through points  $v^1, v^8, v_1^{11}$  and  $v_1^{12}$  is 55.
- The facet through points  $v^3, v^5, v^8$  and  $v_1^{12}$  is 34.
- The facet through points  $v^3, v^5, v_1^{11}$  and  $v_1^{12}$  is 56.
- The facet through points  $v^5, v^8, v_1^{11}$  and  $v_1^{12}$  is 18.

By adding and deleting the appropriate facets from system  $H$  we see that we arrive at system 1.

Therefore, to compute the volume of  $\mathcal{P}_1$ , we sum the volume of  $\mathcal{P}_H$  with that of the appropriate eight simplices, and we obtain our result for case 1.

**6.3.2. Case 2:  $a_1b_2b_3 - b_1a_2a_3 \leq 0$**  In this case it is immediate to see that  $v_1^{11}$  satisfies 19 and therefore lies beyond no further facets. This means we now have a new polytope which is  $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\})$  (in case 2). We refer to this polytope as  $\mathcal{T}_1^2$  and will compute the facets of  $\mathcal{T}_1^2$ .

We begin with the facets of  $\mathcal{P}_H$  and delete the three facets that  $v_1^{11}$  lies beyond (20-22). Let us call this system  $\mathcal{T}_1^{2-}$ . Now consider the four simplices we dealt with when computing the additional volume produced with  $v_1^{11}$ . Each of these simplices has 5 facets; one of which corresponds to a deleted facet of  $\mathcal{P}_H$ .

The remaining 4 facets of the first simplex are described by the planes through the following sets of points:  $\{v^1, v^4, v^5, v_1^{11}\}$ ,  $\{v^1, v^4, v^7, v_1^{11}\}$ ,  $\{v^1, v^5, v^7, v_1^{11}\}$  and  $\{v^4, v^5, v^7, v_1^{11}\}$ .

The remaining 4 facets of the second simplex are described by the planes through the following sets of points:  $\{v^1, v^5, v^7, v_1^{11}\}$ ,  $\{v^1, v^5, v^8, v_1^{11}\}$ ,  $\{v^1, v^7, v^8, v_1^{11}\}$  and  $\{v^5, v^7, v^8, v_1^{11}\}$ .

The remaining 4 facets of the third simplex are described by the planes through the following sets of points:  $\{v^1, v^3, v^4, v_1^{11}\}$ ,  $\{v^1, v^3, v^5, v_1^{11}\}$ ,  $\{v^1, v^4, v^5, v_1^{11}\}$  and  $\{v^3, v^4, v^5, v_1^{11}\}$ .

Consider these twelve facets and exclude the facets that are shared by more than one simplex. This leaves eight facets.

We can compute these eight facets to obtain the following:

- The facet through points  $v^1, v^4, v^7$  and  $v_1^{11}$  is 33.
- The facet through points  $v^4, v^5, v^7$  and  $v_1^{11}$  is 32.
- The facet through points  $v^1, v^5, v^8$  and  $v_1^{11}$  is

$$\frac{1}{b_2(b_1 - a_1)} (-a_1 b_1 a_2^2 a_3 + a_1 b_1 a_2 a_3 x_2 + a_1 b_1 a_2 b_2 x_3 - a_1 b_1 b_2^2 b_3 + a_1 b_2^2 b_3 x_1 + b_1 a_2^2 a_3 x_1 + b_1^2 a_2 b_2 a_3 - b_1^2 a_2 a_3 x_2 - b_1 a_2 b_2 a_3 x_1 + b_1^2 a_2 b_2 b_3 - b_1^2 a_2 b_2 x_3 - b_1 a_2 b_2 b_3 x_1 - a_1 b_2 f + b_1 b_2 f) \geq 0. \quad (57)$$

- The facet through points  $v^1, v^7, v^8$  and  $v_1^{11}$  is 55.
- The facet through points  $v^5, v^7, v^8$  and  $v_1^{11}$  is 18.
- The facet through points  $v^1, v^3, v^4$  and  $v_1^{11}$  is 17.
- The facet through points  $v^1, v^3, v^5$  and  $v_1^{11}$  is

$$\frac{1}{a_3(b_1 - a_1)} (-a_1^2 a_2 a_3 b_3 - a_1^2 b_2 a_3 b_3 + a_1^2 a_3 b_3 x_2 + a_1^2 b_2 b_3 x_3 + a_1 b_1 a_2 a_3^2 + a_1 a_2 a_3 b_3 x_1 - a_1 b_1 a_3 b_3 x_2 + a_1 b_2 a_3 b_3 x_1 + a_1 b_1 b_2 b_3^2 - a_1 b_1 b_2 b_3 x_3 - a_1 b_2 b_3^2 x_1 - b_1 a_2 a_3^2 x_1 - a_1 a_3 f + b_1 a_3 f) \geq 0. \quad (58)$$

- The facet through points  $v^3, v^4, v^5$  and  $v_1^{11}$  is 56.

There are four inequalities that are not already contained in system  $\mathcal{T}_1^{2-}$ ; we add these and in doing this we obtain the system of inequalities that describes  $T_1^2 = \text{conv}(\mathcal{P}_H \cup \{v_1^{11}\})$  (in case 2).

We will now compute the additional volume of  $\text{conv}(\mathcal{T}_1^2 \cup \{v_1^{12}\})$  compared to the volume of  $\mathcal{T}_1^2$ . As we have done previously, we will sum the volumes of  $\text{conv}(\{v_1^{12}\} \cup F)$  for each facet,  $F$ , of  $\mathcal{T}_1^2$  such that  $v_1^{12}$  is beyond that facet. We substitute  $v_1^{12}$  into each relevant inequality (i.e., the system of inequalities that describes  $T_1^2$  in case 2) and if the result is negative then  $v_1^{12}$  lies beyond that facet. It is immediately clear that  $v_1^{12}$  satisfies inequalities 17, 18, 23, 25-34, 55 and 56. We can also see immediately that  $v_1^{12}$  violates inequalities 19, 57 and 58. To see that 24 is also satisfied see §9.16.

Therefore we know that  $v_1^{12}$  is beyond three facets and we need to compute the volume of the convex hull of  $v_1^{12}$  with each of these facets.

The extreme points that lie on the first facet are points  $v^1, v^3, v^5$  and  $v^8$ . The polytope  $\text{conv}\{v^1, v^3, v^5, v^8, v_1^{12}\}$  is a 4-simplex with volume:

$$a_2 b_3 (b_1 - a_1)^2 (b_2 - a_2)^2 (b_3 - a_3)^2 / (24(b_2 b_3 - a_2 a_3)).$$

The extreme points that lie on the second facet are points  $v^1, v^5, v^8$  and  $v_1^{11}$ . The polytope  $\text{conv}\{v^1, v^5, v^8, v_1^{11}, v_1^{12}\}$  is a 4-simplex with volume:

$$(b_1 a_2 (b_1 - a_1) (b_2 - a_2)^2 (b_3 - a_3)^3 / (24(b_2 b_3 - a_2 a_3))).$$

The extreme points that lie on the third and final facet are points  $v^1, v^3, v^5$  and  $v_1^{11}$ . The polytope  $\text{conv}\{v^1, v^3, v^5, v_1^{11}, v_1^{12}\}$  is a 4-simplex with volume:

$$a_1 b_3 (b_1 - a_1) (b_2 - a_2)^3 (b_3 - a_3)^2 / (24(b_2 b_3 - a_2 a_3)).$$

We now compute the additional facets; we take the four facets from adding each simplex and delete the three facets that are repeated. This leaves us with the following six facet defining inequalities to compute:

- The facet through points  $v^1, v^3, v^8$  and  $v_1^{12}$  is 31.
- The facet through points  $v^3, v^5, v^8$  and  $v_1^{12}$  is 34.
- The facet through points  $v^1, v^8, v_1^{11}$  and  $v_1^{12}$  is 55.
- The facet through points  $v^5, v^8, v_1^{11}$  and  $v_1^{12}$  is 18.
- The facet through points  $v^1, v^3, v_1^{11}$  and  $v_1^{12}$  is 17.
- The facet through points  $v^3, v^5, v_1^{11}$  and  $v_1^{12}$  is 56.

By adding and deleting the appropriate facets from system  $H$  we see that we also arrive at system 1 in case 2.

Therefore, to compute the volume of  $\mathcal{P}_1$ , we sum the volume of  $\mathcal{P}_H$  with that of the appropriate eight simplicies, and we obtain our result for case 2.  $\square$

**7. Proof of Thm. 3.3.** A mapping from the proof of Theorem 3.4 allows us to claim Theorem 3.3 immediately.  $\square$

**8. Possible extensions.** Our results quantify the tradeoff between different convexifications of trilinear monomials.

Of course it would be nice to use our results to develop guidelines for attacking trilinear monomials within an sBB code. In doing so, it should prove important to develop guidelines for how our results could be heuristically applied to formulations having many trilinear monomials overlapping on the same variables. We can imagine scoring each possible relaxation according to its volume, and then aggregating the scores to decide on what to do for each trilinear monomial.

It would be natural and certainly difficult to extend our work to multilinear monomials having  $n > 3$ . In particular, advances for the important case of  $n = 4$  could have immediate impact; see [4]. In particular, [4] found, via experiments, that composing a trilinear and bilinear convexification in the manner suggested by  $(x_i x_j) x_k x_l$  was a good strategy. They further observed sensitivity to the bounds on the variables, but they reached no clear conclusion on how to factor in that aspect. Restricting to this type of convexification, we could apply our results by substituting  $w \in [a_i b_i, a_j b_j]$  to arrive at the trilinear monomial  $w x_k x_l$ , which can then be analyzed and relaxed according to our methodology. Of course, for a general quadrilinear monomial, there are six choices of which pair of variables will be treated as  $\{x_i, x_j\}$ , so we can analyze all six possibilities and take the best overall.

Also, there is the possibility of extending our results on trilinear monomials to (i) box domains that are not necessarily nonnegative, (ii) domains other than boxes, and (iii) other low-dimensional functions. We do wish to mention that we are currently working on using our viewpoint to get some mathematical insight into the selection of a branching point in the context of sBB.

We hope that our work is just a first step in using volume to better understand and mathematically quantify the tradeoffs involved in developing sBB strategies for factorable formulations.

**9. Appendix.** Throughout the proofs, we have repeatedly claimed that certain quantities are nonnegative for any choice of  $a_1, a_2, a_3, b_1, b_2, b_3$ , such that,  $0 < a_i < b_i$ , for all  $i$  and

$$a_1 b_2 b_3 + b_1 a_2 a_3 \leq b_1 a_2 b_3 + a_1 b_2 a_3 \leq b_1 b_2 a_3 + a_1 a_2 b_3.$$

In this appendix, we provide proofs for the cases that are not immediate. As will become apparent, we will need to demonstrate that many different 6-variable polynomials are nonnegative on the relevant parameter space. Generally, such demonstrations can be tricky global-optimization problems, and in many cases sum-of-squares proofs are not available; rather, we often make somewhat ad hoc arguments. Still, we can place some efficiency on all of this by establish some technical lemmas.

**9.1.** We begin with the following lemmas that will be helpful in establishing the nonnegativity of certain quantities:

LEMMA 9.1. *For all choices of parameters that meet our assumptions we have:  $b_1a_2 - a_1b_2 \geq 0$ ,  $b_1a_3 - a_1b_3 \geq 0$  and  $b_2a_3 - a_2b_3 \geq 0$ .*

*Proof.*  $(b_3 - a_3)(b_1a_2 - a_1b_2) = b_1a_2b_3 + a_1b_2a_3 - a_1b_2b_3 - b_1a_2a_3 \geq 0$  by our original assumptions  $\Omega$ . This implies  $b_1a_2 - a_1b_2 \geq 0$ , because  $b_3 - a_3 > 0$ .  $b_1a_3 - a_1b_3 \geq 0$  and  $b_2a_3 - a_2b_3 \geq 0$  follow from  $\Omega$  in a similar way.  $\square$

LEMMA 9.2. *Let  $A, B, C, D, E, F \in \mathbb{R}$  with  $A \geq B \geq C \geq 0$  and  $D \geq 0$ ,  $E \geq 0$ ,  $F \leq 0$ . Also let,  $D + E + F = 0$ . Then  $AD + BE + CF \geq 0$ .*

*Proof.*  $AD + BE + CF = AD + BE - C(D + E) \geq BD + BE - CD - CE = (B - C)(D + E) \geq 0$ .  $\square$

LEMMA 9.3. *Let  $A, B, C, D, E, F \in \mathbb{R}$  with  $A \geq B \geq C \geq 0$  and  $D \geq 0$ ,  $E \leq 0$ ,  $F \leq 0$ . Also let,  $D + E + F = 0$ . Then  $AD + BE + CF \geq 0$ .*

*Proof.*  $AD + BE + CF = -A(E + F) + BE + CF = E(B - A) + F(C - A) \geq 0$ .  $\square$

LEMMA 9.4. *Let  $A, B, C, D \in \mathbb{R}$  with  $A \geq B \geq 0$ ,  $C + D \geq 0$ ,  $C \geq 0$ . Then  $AC + BD \geq 0$ .*

*Proof.*  $AC + BD \geq B(C + D) \geq 0$ .  $\square$

**9.2.** Substituting the midpoint of the line segment between  $v_3^9$  and  $v_3^{11}$  into inequality 25 of the convex hull we obtain:

$$\begin{aligned} & (-a_1^2a_2^2a_3 + a_1^2a_2b_2b_3 + a_1a_2^2a_3b_1 + 2a_1a_2a_3b_1b_2 - 2a_1a_2b_1b_2b_3 \\ & -a_1a_3b_1b_2^2 - a_2^2a_3b_1^2 + a_2^2b_1^2b_3 - a_2b_1^2b_2b_3 + b_1^2b_2^2b_3) / 2(b_1b_2 - a_1a_2), \end{aligned}$$

the numerator of which can be rewritten as

$$b_1b_2((b_1b_3 - a_1a_3)(b_2 - a_2)) + b_1a_2((b_1a_2 - a_1b_2)(b_3 - a_3)) + a_1a_2((b_2b_3 - a_2a_3)(a_1 - b_1)),$$

and is nonnegative by Lemmas 9.1 and 9.2.

**9.3.** Substituting the midpoint of the line segment between  $v_3^{10}$  and  $v_3^{12}$  into inequality 26 of the convex hull we obtain:

$$\begin{aligned} & (-a_1^2a_2^2a_3 + a_1^2a_2a_3b_2 - a_1^2a_3b_2^2 + a_1^2b_2^2b_3 + a_1a_2^2b_1b_3 + 2a_1a_2a_3b_1b_2 \\ & - 2a_1a_2b_1b_2b_3 - a_1b_1b_2^2b_3 - a_2a_3b_1^2b_2 + b_1^2b_2^2b_3) / 2(b_1b_2 - a_1a_2), \end{aligned}$$

the numerator of which can be rewritten as

$$b_1b_2((b_2b_3 - a_2a_3)(b_1 - a_1)) + a_1b_2((b_1a_2 - a_1b_2)(a_3 - b_3)) + a_1a_2((b_1b_3 - a_1a_3)(a_2 - b_2)),$$

and is nonnegative by Lemmas 9.1 and 9.3.

**9.4.** Substituting point  $v_3^{11}$  into inequality 25 of the convex hull or substituting  $v_3^{12}$  into inequality 26 we obtain:

$$\begin{aligned} & (-a_1^2 a_2^2 a_3 + a_1^2 a_2 b_2 b_3 + a_1 a_2^2 b_1 b_3 + 3a_1 a_2 a_3 b_1 b_2 - 3a_1 a_2 b_1 b_2 b_3 \\ & \quad - a_1 a_3 b_1 b_2^2 - a_2 a_3 b_1^2 b_2 + b_1^2 b_2^2 b_3) / (b_1 b_2 - a_1 a_2). \end{aligned}$$

The numerator can be rewritten as

$$\begin{aligned} & b_3 \left( a_1 a_2 (a_1 b_2 - b_1 b_2) + b_1 a_2 (a_1 a_2 - a_1 b_2) + b_1 b_2 (b_1 b_2 - a_1 a_2) \right) \\ & + a_3 \left( a_1 a_2 (b_1 b_2 - a_1 a_2) + b_1 a_2 (a_1 b_2 - b_1 b_2) + b_1 b_2 (a_1 a_2 - a_1 b_2) \right) =: b_3 Y + a_3 Z. \end{aligned}$$

Then we can see  $Y + Z = (b_2 - a_2)(b_1 - a_1)(b_1 b_2 - a_1 a_2)$ , which is nonnegative. Furthermore, by Lemma 9.3 we have  $Y \geq 0$ . Therefore, by Lemma 9.4 the numerator is nonnegative.

**9.5.** Substituting the midpoint of the line segment between  $v_1^9$  and  $v_1^{10}$  into inequality 19 of the convex hull we obtain:

$$\begin{aligned} & (-2a_1 a_2^2 a_3 b_3 + a_1 a_2^2 b_3^2 - a_1 a_2 a_3^2 b_2 + 4a_1 a_2 a_3 b_2 b_3 - a_1 a_2 b_2 b_3^2 - a_1 b_2^2 b_3^2 + a_2^2 a_3^2 b_1 \\ & \quad + a_2^2 a_3 b_1 b_3 - a_2^2 b_1 b_3^2 - 4a_2 a_3 b_1 b_2 b_3 + 2a_2 b_1 b_2 b_3^2 + a_3 b_1 b_2^2 b_3) / 2(b_2 b_3 - a_2 a_3), \end{aligned}$$

the numerator of which can be rewritten as

$$\begin{aligned} & b_2 b_3 \left( b_2 (b_1 a_3 - a_1 b_3) + a_2 (b_1 b_3 - 2b_1 a_3 + a_1 a_3) \right) + a_2 b_3 \left( (b_1 - a_1)(b_2 - a_2)(b_3 - a_3) \right) \\ & + a_2 a_3 \left( b_2 (2a_1 b_3 - b_1 b_3 - a_1 a_3) + a_2 (b_1 a_3 - a_1 b_3) \right) =: b_2 b_3 X + a_2 b_3 Y + a_2 a_3 Z. \end{aligned}$$

Now, we write  $X =: b_2 V + a_2 W$  and see that  $V + W = (b_1 - a_1)(b_3 - a_3) \geq 0$  and, by Lemma 9.1,  $V \geq 0$ . Therefore  $X \geq 0$  by Lemma 9.4. Because  $X + Y + Z = 0$ , by Lemma 9.2 we have that the numerator is nonnegative.

**9.6.** Substituting the midpoint of the line segment between  $v_1^9$  and  $v_1^{10}$  into inequality 20 of the convex hull, we obtain:

$$\begin{aligned} & (-a_1 a_2^2 a_3 b_3 - 2a_1 a_2 a_3^2 b_2 + 4a_1 a_2 a_3 b_2 b_3 + a_1 a_3^2 b_2^2 - a_1 a_3 b_2^2 b_3 - a_1 b_2^2 b_3^2 + a_2^2 a_3^2 b_1 \\ & \quad + a_2 a_3^2 b_1 b_2 - 4a_2 a_3 b_1 b_2 b_3 + a_2 b_1 b_2 b_3^2 - a_3^2 b_1 b_2^2 + 2a_3 b_1 b_2^2 b_3) / 2(b_2 b_3 - a_2 a_3), \end{aligned}$$

the numerator of which can be rewritten as

$$\begin{aligned} & b_2 b_3 \left( b_3 (b_1 a_2 - a_1 b_2) + a_3 (b_1 b_2 - 2b_1 a_2 + a_1 a_2) \right) + b_2 a_3 \left( (b_1 - a_1)(b_2 - a_2)(b_3 - a_3) \right) \\ & + a_2 a_3 \left( b_3 (2a_1 b_2 - b_1 b_2 - a_1 a_2) + a_3 (b_1 a_2 - a_1 b_2) \right) =: b_2 b_3 X + b_2 a_3 Y + a_2 a_3 Z. \end{aligned}$$

Now, we write  $X =: b_3 V + a_3 W$  and see that  $V + W = (b_1 - a_1)(b_2 - a_2) \geq 0$  and, by Lemma 9.1  $V \geq 0$ . Therefore  $X \geq 0$  by Lemma 9.4. Because  $X + Y + Z = 0$ , by Lemma 9.2 we have that the numerator is nonnegative.

**9.7.** Substituting the midpoint of the line segment between  $v_1^9$  and  $v_1^{11}$  into inequality 20 of the convex hull, we obtain:

$$\begin{aligned} & (-a_1 a_2^2 a_3 b_3 + 2a_1 a_2 a_3 b_2 b_3 - a_1 a_2^2 b_2^2 + a_1 a_3 b_2^2 b_3 - a_1 b_2^2 b_3^2 + a_2^2 a_3^2 b_1 - a_2 a_3^2 b_1 b_2 \\ & \quad - 2a_2 a_3 b_1 b_2 b_3 + a_2 b_1 b_2 b_3^2 + a_3^2 b_1 b_2^2) / 2(b_2 b_3 - a_2 a_3), \end{aligned}$$

the numerator of which can be rewritten as

$$b_2 b_3 \left( (b_1 a_2 - a_1 b_2)(b_3 - a_3) \right) + b_2 a_3 \left( (b_2 a_3 - a_2 b_3)(b_1 - a_1) \right) + a_2 a_3 \left( (b_1 a_3 - a_1 b_3)(a_2 - b_2) \right),$$

which is nonnegative by Lemmas 9.2 and 9.1.

**9.8.** Substituting the midpoint of the line segment between  $v_1^9$  and  $v_1^{11}$  into inequality 21 of the convex hull we obtain:

$$(b_2 - a_2) (a_1 a_2 a_3 b_3 - a_1 b_2 b_3^2 - a_2 a_3^2 b_1 - a_2 a_3 b_1 b_3 + a_2 b_1 b_3^2 + a_3^2 b_1 b_2) / 2(b_2 b_3 - a_2 a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_3(b_3(b_1 a_2 - a_1 b_2)) + a_3(a_1 a_2 b_3 - b_1 a_2 a_3 + b_1 b_2 a_3 - b_1 a_2 b_3) =: b_3 Y + a_3 Z,$$

now  $Y + Z = (b_1 a_3 - a_1 b_3)(b_2 - a_2) \geq 0$  (Lemma 9.1), and  $Y \geq 0$  (Lemma 9.1), therefore by Lemma 9.4 we have that  $b_3 Y + a_3 Z$  is nonnegative.

**9.9.** Substituting the midpoint of the line segment between  $v_1^9$  and  $v_1^{11}$  into inequality 22 of the convex hull we obtain:

$$(b_3 - a_3) (a_1 a_2^2 b_3 - a_1 a_2 b_2 b_3 + a_1 a_3 b_2^2 - a_1 b_2^2 b_3 - a_2^2 a_3 b_1 + a_2 b_1 b_2 b_3) / 2(b_2 b_3 - a_2 a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_2(b_3(b_1 a_2 - a_1 b_2) + a_1(b_2 a_3 - a_2 b_3)) + a_2(a_2(a_1 b_3 - b_1 a_3)) =: b_2 Y + a_2 Z,$$

where  $Y + Z = (b_1 a_2 - a_1 b_2)(b_3 - a_3) \geq 0$  and  $Y \geq 0$  (both Lemma 9.1), therefore by Lemma 9.4 we have that this term is nonnegative.

**9.10.** Substituting the midpoint of the line segment between  $v_1^9$  and  $v_1^{11}$  into inequality 27 of the convex hull we obtain:

$$\begin{aligned} & (-a_1 a_2^2 a_3^2 + a_1 a_2^2 a_3 b_3 - a_1 a_2^2 b_3^2 + 2a_1 a_2 a_3 b_2 b_3 - a_1 a_3 b_2^2 b_3 + a_2^2 b_1 b_3^2 \\ & + a_2 a_3^2 b_1 b_2 - 2a_2 a_3 b_1 b_2 b_3 - a_2 b_1 b_2 b_3^2 + b_1 b_2^2 b_3^2) / 2(b_2 b_3 - a_2 a_3), \end{aligned}$$

the numerator of which can be rewritten as

$$b_2 b_3((b_1 b_3 - a_1 a_3)(b_2 - a_2)) + a_2 b_3((b_2 a_3 - a_2 b_3)(a_1 - b_1)) + a_2 a_3((b_1 b_2 - a_1 a_2)(a_3 - b_3)),$$

which is nonnegative by Lemmas 9.3 and 9.1.

**9.11.** Substituting the midpoint of the line segment between  $v_1^{10}$  and  $v_1^{12}$  into inequality 19 of the convex hull we obtain:

$$\begin{aligned} & (-a_1 a_2^2 b_3^2 - a_1 a_2 a_3^2 b_2 + 2a_1 a_2 a_3 b_2 b_3 + a_1 a_2 b_2 b_3^2 - a_1 b_2^2 b_3^2 + a_2^2 a_3^2 b_1 \\ & - a_2^2 a_3 b_1 b_3 + a_2^2 b_1 b_3^2 - 2a_2 a_3 b_1 b_2 b_3 + a_3 b_1 b_2 b_3^2) / 2(b_2 b_3 - a_2 a_3), \end{aligned}$$

the numerator of which can be rewritten as

$$b_2 b_3((b_1 a_3 - a_1 b_3)(b_2 - a_2)) + a_2 b_3((b_2 a_3 - a_2 b_3)(a_1 - b_1)) + a_2 a_3((b_1 a_2 - a_1 b_2)(a_3 - b_3)),$$

which is nonnegative by Lemma 9.3 and Lemma 9.1.

**9.12.** Substituting the midpoint of the line segment between  $v_1^{10}$  and  $v_1^{12}$  into inequality 21 of the convex hull we obtain:

$$(b_3 - a_3) (a_1 a_2 a_3 b_2 - a_1 b_2 b_3 - a_2^2 a_3 b_1 + a_2^2 b_1 b_3 - a_2 a_3 b_1 b_2 + a_3 b_1 b_2^2) / 2(b_2 b_3 - a_2 a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_2(b_2(b_1 a_3 - a_1 b_3)) + a_2(b_1(a_2 b_3 - b_2 a_3) + a_3(a_1 b_2 - b_1 a_2)) =: b_2 Y + a_2 Z,$$

where  $Y + Z = (b_1 a_2 - a_1 b_2)(b_3 - a_3) \geq 0$  and  $Y \geq 0$  (both by Lemma 9.1). Therefore by Lemma 9.4 we have that  $b_2 Y + a_2 Z$  is nonnegative.

**9.13.** Substituting the midpoint of the line segment between  $v_1^{10}$  and  $v_1^{12}$  into inequality 22 of the convex hull we obtain:

$$(b_2 - a_2) (a_1 a_2 b_3^2 + a_1 a_3^2 b_2 - a_1 a_3 b_2 b_3 - a_1 b_2 b_3^2 - a_2 a_3^2 b_1 + a_3 b_1 b_2 b_3) / 2(b_2 b_3 - a_2 a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_3 (a_1 (a_2 b_3 - b_2 a_3) + b_2 (b_1 a_3 - a_1 b_3)) + a_3 (a_3 (a_1 b_2 - b_1 a_2)) =: b_3 Y + a_3 Z,$$

where  $Y + Z = (b_1 a_3 - a_1 b_3)(b_2 - a_2) \geq 0$  and  $Z \leq 0 \implies Y \geq 0$  (both by Lemma 9.1). Therefore by Lemma 9.4 we have that  $b_3 Y + a_3 Z$  is nonnegative.

**9.14.** Substituting the midpoint of the line segment between  $v_1^{10}$  and  $v_1^{12}$  into inequality 24 of the convex hull we obtain:

$$(-a_1 a_2^2 a_3^2 + a_1 a_2 a_3^2 b_2 + 2a_1 a_2 a_3 b_2 b_3 - a_1 a_2 b_2 b_3^2 - a_1 a_3^2 b_2^2 + a_2^2 a_3 b_1 b_3 \\ - 2a_2 a_3 b_1 b_2 b_3 + a_3^2 b_1 b_2^2 - a_3 b_1 b_2^2 b_3 + b_1 b_2^2 b_3^2) / 2(b_2 b_3 - a_2 a_3),$$

the numerator of which simplifies to

$$b_2 b_3 ((b_1 b_2 - a_1 a_2)(b_3 - a_3)) + b_2 a_3 ((b_2 a_3 - a_2 b_3)(b_1 - a_1)) + a_2 a_3 ((b_1 b_3 - a_1 a_3)(a_2 - b_2)),$$

and is nonnegative by Lemmas 9.1 and 9.2.

**9.15.** Substituting point  $v_1^9$  into inequality 20 of the convex hull or substituting point  $v_1^{10}$  into inequality 19 we obtain:

$$(-a_1 a_2^2 a_3 b_3 - a_1 a_2 a_3^2 b_2 + 3a_1 a_2 a_3 b_2 b_3 - a_1 b_2^2 b_3^2 + a_2^2 a_3^2 b_1 \\ - 3a_2 a_3 b_1 b_2 b_3 + a_2 b_1 b_2 b_3^2 + a_3 b_1 b_2^2 b_3) / (b_2 b_3 - a_2 a_3),$$

the numerator of which can be rewritten as

$$b_3 (b_2 (b_1 a_3 - a_1 b_3) + a_2 (a_1 a_3 + b_1 b_3 - 2b_1 a_3)) \\ + a_3 (a_2 (b_2 (2a_1 b_3 - a_1 a_3 - b_1 b_3) + a_2 (b_1 a_3 - a_1 b_3))) =: b_3 Y + a_3 Z.$$

Now, we write  $Y =: b_2^2 V + a_2 b_2 W$  and see that  $V + W = (b_1 - a_1)(b_3 - a_3) \geq 0$  and, by Lemma 9.1  $V \geq 0$ . Therefore  $Y \geq 0$  by Lemma 9.4. Because  $Y + Z = (b_1 a_3 - a_1 b_3)(b_2 - a_2)^2 \geq 0$  (Lemma 9.1), by Lemma 9.4 we have that the numerator is nonnegative.

**9.16.** Substituting point  $v_1^{11}$  into inequality 27 of the convex hull or substituting point  $v_1^{12}$  into 24 we obtain:

$$(-a_1 a_2^2 a_3^2 + 3a_1 a_2 a_3 b_2 b_3 - a_1 a_2 b_2 b_3^2 - a_1 a_3 b_2^2 b_3 + a_2^2 a_3 b_1 b_3 \\ + a_2 a_3^2 b_1 b_2 - 3a_2 a_3 b_1 b_2 b_3 + b_1 b_2^2 b_3^2) / (b_2 b_3 - a_2 a_3),$$

the numerator of which can be rewritten as

$$b_1 (b_2 b_3 (b_2 b_3 - a_2 a_3) + b_2 a_3 (a_2 a_3 - a_2 b_3) + a_2 a_3 (a_2 b_3 - b_2 b_3)) \\ + a_1 (b_2 b_3 (a_2 a_3 - a_2 b_3) + b_2 a_3 (a_2 b_3 - b_2 b_3) + a_2 a_3 (b_2 b_3 - a_2 a_3)) =: b_1 Y + a_1 Z,$$

where:  $Y + Z = (b_2 b_3 - a_2 a_3)(b_2 - a_2)(b_3 - a_3) \geq 0$ , and by Lemma 9.3 we have  $Y \geq 0$ . Therefore by Lemma 9.4 we have that the numerator is nonnegative.

**9.17.** Substituting the midpoint of the line segment between  $v_1^9$  and  $v_1^{10}$  into inequality 21 of the convex hull we obtain

$$(2b_2b_3 - a_2b_3 - b_2a_3)(a_1a_2a_3 - a_1b_2b_3 - 2b_1a_2a_3 + b_1a_2b_3 + b_1b_2a_3) / 2(b_2b_3 - a_2a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_2(b_1a_3 - a_1b_3) + a_2(a_1a_3 - 2b_1a_3 + b_1b_3),$$

which is nonnegative by Lemmas 9.1 and 9.4.

**9.18.** Substituting the midpoint of the line segment between  $v_1^9$  and  $v_1^{10}$  into inequality 22 of the convex hull we obtain

$$(a_2b_3 + b_2a_3 - 2a_2a_3)(a_1a_2b_3 + a_1b_2a_3 - 2a_1b_2b_3 - b_1a_2a_3 + b_1b_2b_3) / 2(b_2b_3 - a_2a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_2(b_1b_3 - 2a_1b_3 + a_1a_3) + a_2(a_1b_3 - b_1a_3),$$

which is nonnegative by Lemmas 9.1 and 9.4.

**9.19.** Substituting point  $v_1^9$  into inequality 21 of the convex hull we obtain

$$b_3(b_2 - a_2) \left( b_2(b_1a_3 - a_1b_3) + a_2(a_1a_3 - 2b_1a_3 + b_1b_3) \right) / (b_2b_3 - a_2a_3),$$

which is nonnegative by Lemmas 9.1 and 9.4.

**9.20.** Substituting point  $v_1^9$  into inequality 22 of the convex hull we obtain

$$a_2(b_3 - a_3) \left( b_2(b_1b_3 - 2a_1b_3 + a_1a_3) + a_2(a_1b_3 - b_1a_3) \right) / (b_2b_3 - a_2a_3),$$

which is nonnegative by Lemmas 9.1 and 9.4.

**9.21.** Substituting point  $v_1^{10}$  into inequality 21 of the convex hull we obtain

$$b_2(b_3 - a_3) \left( b_2(b_1a_3 - a_1b_3) + a_2(a_1a_3 - 2b_1a_3 + b_1b_3) \right) / (b_2b_3 - a_2a_3),$$

which is nonnegative by Lemmas 9.1 and 9.4.

**9.22.** Substituting point  $v_1^{10}$  into inequality 22 of the convex hull we obtain

$$a_3(b_2 - a_2) \left( b_2(b_1b_3 - 2a_1b_3 + a_1a_3) + a_2(a_1b_3 - b_1a_3) \right) / (b_2b_3 - a_2a_3),$$

which is nonnegative by Lemmas 9.1 and 9.4.

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## References

- [1] Adjiman, C.S., S. Dallwig, C.A. Floudas, A. Neumaier. 1998. A global optimization method,  $\alpha$ BB, for general twice-differentiable constrained NLPs: I. Theoretical advances. *Computers & Chemical Engineering* **22**(9) 1137–1158.
- [2] Bao, X., A. Khajavirad, N.V. Sahinidis, M. Tawarmalani. 2015. Global optimization of nonconvex problems with multilinear intermediates. *Math. Prog. Comp.* **7**(1) 1–37.
- [3] Belotti, P., J. Lee, L. Liberti, F. Margot, A. Wächter. 2009. Branching and bounds tightening techniques for non-convex MINLP. *Opt. Meth. & Software* **24**(4-5) 597–634.
- [4] Cafieri, S., J. Lee, L. Liberti. 2010. On convex relaxations of quadrilinear terms. *J. Global Optim.* **47**(4) 661–685.
- [5] Costa, Alberto, Leo Liberti. 2012. Relaxations of multilinear convex envelopes: Dual is better than primal. Ralf Klasing, ed., *Experimental Algorithms, Lecture Notes in Computer Science*, vol. 7276. Springer Berlin Heidelberg, 87–98.
- [6] Grünbaum, B. 2003. *Convex polytopes*. 2nd ed. Springer-Verlag, New York.
- [7] Jach, M., D. Michaels, R. Weismantel. 2008. The convex envelope of  $(n-1)$ -convex functions. *SIAM Journal on Optimization* **19**(3) 1451–1466.
- [8] Ko, C.-W., J. Lee, E.. Steingrímsson. 1997. The volume of relaxed Boolean-quadric and cut polytopes. *Discrete Math.* **163**(1-3) 293–298.
- [9] Lee, J. 2007. Mixed integer nonlinear programming: Some modeling and solution issues. *IBM Journal of Research and Development* **51**(3/4) 489–497.
- [10] Lee, J., W. Morris. 1994. Geometric comparison of combinatorial polytopes. *Discrete Applied Mathematics* **55** 163–182.
- [11] Liberti, Leo, Sonia Cafieri, David Savourey. 2010. The reformulation-optimization software engine. Komei Fukuda, Jorisvander Hoeven, Michael Joswig, Nobuki Takayama, eds., *Mathematical Software — ICMS 2010, Lecture Notes in Computer Science*, vol. 6327. Springer Berlin Heidelberg, 303–314.
- [12] Liberti, Leo, Sonia Cafieri, Fabien Tarissan. 2009. Reformulations in mathematical programming: A computational approach. Ajith Abraham, Aboul-Ella Hassani, Patrick Siarry, Andries Engelbrecht, eds., *Foundations of Computational Intelligence Volume 3, Studies in Computational Intelligence*, vol. 203. Springer Berlin Heidelberg, 153–234.
- [13] Luedtke, J., M. Namazifar, J. Linderoth. 2012. Some results on the strength of relaxations of multilinear functions. *Mathematical Programming* **136** 325–351.
- [14] McCormick, G.P. 1976. Computability of global solutions to factorable nonconvex programs: Part i. convex underestimating problems. *Mathematical Programming* **10** 147–175.
- [15] Meyer, C.A., C.A. Floudas. 2004. Trilinear monomials with mixed sign domains: Facets of the convex and concave envelopes. *Journal of Global Optimization* **29** 125–155.
- [16] Meyer, C.A., C.A. Floudas. 2004. Trilinear monomials with positive or negative domains: Facets of the convex and concave envelopes. *Frontiers in Global Optimization* 327–352.
- [17] Meyer, C.A., C.A. Floudas. 2005. Convex envelopes for edge-concave functions. *Math. Program.* **103**(2, Ser. B) 207–224.
- [18] Rikun, A. 1997. A convex envelope formula for multilinear functions. *Journal of Global Optimization* **10** 425–437.
- [19] Ryoo, H.S., N.V. Sahinidis. 1996. A branch-and-reduce approach to global optimization. *Journal of Global Optimization* **8**(2) 107–138.
- [20] Ryoo, H.S., N.V. Sahinidis. 2001. Analysis of bounds for multilinear functions. *J. Global Optim.* **19**(4) 403–424.
- [21] Schichl, Hermann, Arnold Neumaier. 2005. Interval analysis on directed acyclic graphs for global optimization. *J. Global Optim.* **33**(4) 541–562.
- [22] Schneider, R. 1993. *Convex bodies: the Brunn-Minkowski theory, Encyclopedia of Mathematics and its Applications*, vol. 44. Cambridge University Press, Cambridge.

- [23] Schneider, R. 2014. *Convex bodies: the Brunn-Minkowski theory, Encyclopedia of Mathematics and its Applications*, vol. 151. expanded ed. Cambridge University Press, Cambridge.
- [24] Smith, E.M.B., C.C. Pantelides. 1999. A symbolic reformulation/spatial branch-and-bound algorithm for the global optimisation of nonconvex MINLPs. *Computers & Chemical Engineering* **23** 457–478.
- [25] Steingrímsson, E. 1994. A decomposition of 2-weak vertex-packing polytopes. *Discrete Comput. Geom.* **12**(4) 465–479.
- [26] Vu, Xuan-Ha, Hermann Schichl, Djamil Sam-Haroud. 2009. Interval propagation and search on directed acyclic graphs for numerical constraint solving. *Computational Optimization and Applications* **45**(4) 499–531.