How to Reach his Desires: Variational Rationality and the Equilibrium Problem on Hadamard Manifolds

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Abstract In this paper we present a sufficient condition for the existence of a solution for an equilibrium problem on an Hadamard manifold and under suitable assumptions on the sectional curvature, we propose a framework for the convergence analysis of a proximal point algorithm to solve this equilibrium problem. Finally, we offer an application to the necessity for an agent to escape to a succession of temporary traps to be able to reach, at the end, his desires, using a recent "variational rationality" approach of human behavior.

Keywords. Proximal algorithms · Equilibrium problem · Hadamard manifold · Desires · Trap · Worthwhile changes.
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1 Introduction

The equilibrium problem EP has been widely studied and is a very active field of research. One of the motivations is that various problems may be formulated as an equilibrium problem, for instance, optimization problems, Nash equilibria problems, complementarity problems, fixed point problems and variational inequality problems. An extensive development can be found in Blum and Oettli [1], Bianchi and Schaible [2] and their references.

An important issue is under what conditions there exists a solution to EP. In the linear setting, several authors have provided results answering this question; see, for instance, Iusem and Sosa [3] and Iusem et al. [4]. As far as we know, Colao et al. [5] were the first to provide an existence result for equilibrium problems in a Riemannian context, more accurately, on Hadamard manifolds, in the case where EP is associated to a monotone bifunction which satisfies a certain coercivity condition. Following the ideas presented in [4], in this paper we have extended the existence result in [5] (for EP) by considering pseudomonotone bifunctions and a weaker sufficient condition than the coercivity assumption used there. In [5] the authors presented an iterative process, a Picard iteration, to approximate a solution of the equilibrium problem on an Hadamard manifold, which is associated to the proximal iteration studied, for example, by Moudafi [6], Konnov [7] and Iusem and Sosa [8], both in the linear setting. In this present paper, we introduce a new proximal algorithm for EP. The novelty here is the new regularization term which, unlike the classical model, does not come as a natural extension of the optimality condition of a minimization problem. It is known that, although natural, the script of the convergence analysis in the linear setting imposes a very restrictive condition about Hadamard manifolds. Our convergence analysis applies to genuinely Hadamard manifolds. Several researchers have studied the proximal point method on Hadamard manifolds for particular instances of the EP; see, for example, Ferreira and Oliveira [9], Li et al. [10] and Tang et al. [11]. In recent years, extensions to

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Riemannian manifolds of concepts and techniques which fit in Euclidean spaces are natural; see, for instance, [26, 12, 15, 24, 25] and the references therein. One reason for the success in the extension of techniques from the linear setting to the Riemannian context, is the possibility to transform nonconvex problems in convex problems by introducing a suitable Riemannian metric; see Rapcsák [14], Cruz Neto et al. [13], Bento and Melo [23] and Colao et al.[5].

The organization of our paper is as follows. In Section 2, we give some elementary facts on Riemannian manifolds and convexity needed for reading this paper. In Section 3, we present a sufficient condition for existence of a solution for the equilibrium problem on Hadamard manifolds under conditions similar to the linear case. In Section 4, the proximal point algorithm for equilibrium problems on Hadamard manifolds is presented and convergence analysis is derived. In Section 5, we give a behavioral application to the existence of temporary traps and the existence/reachability of desires, in the context of a recent and unifying approach of a lot of stability and change dynamics in Behavioral Sciences, the "Variational rationality" approach of worthwhile stays and changes human behavior (see, Soubeyran [18, 19]). This approach focuses the attention on four main concepts:

i) worthwhile single changes, where, for an isolated agent or several interrelated agents, their motivation to change from the current position to a new position is higher than some adaptive and satisficing worthwhile to change ratio, time their resistance to change. Motivation to change refers to the utility of advantages to change, while resistance to change refers to the disutility of inconveniences to change. Resistance to change includes inertia, frictions, obstacles, difficulties to change, costs to be able to change and inconveniences to change;

ii) worthwhile transitions, i.e., succession of worthwhile single stays and changes;

iii) traps, which can be stationary or variational. A trap is stationary when, starting from it, no feasible change is worthwhile. An equilibrium appears to be a very particular stationary trap, in a world with no resistance to change, when only motivation to change matters. In this case the agent has no motivation to change (no advantage to change, i.e only losses to change), and zero resistance to change. A trap is variational with respect to a subset of initial positions, when it is stationary, and, starting from any of these initial positions, agents can find a succession of worthwhile single changes and temporary stays which converge to this stationary trap. Then, a variational trap is rather easy to reach and difficult to leave in a worthwhile way. Furthermore, traps can be weak or strong, depending of large or strict inequalities;

iv) desires,which represent, both to have what you want and to want what you have. The idea is that to be able to reach your desires, you must escape to several temporary traps.

This last section, devoted to applications, focuses the attention on a succession of worthwhile changes and stays, moving from a weak stationary trap to a new one, given that the agent can change, each step, his satisficing worthwhile to change ratio. The algorithm given in Section 4 represents a nice instance of such a worthwhile stability and change dynamic. The result of this paper shows that this dynamic converges to an equilibrium which represents a desired situation or desire. This worthwhile stability and change dynamic is a very important benchmark case of the more general stability and change dynamic (see [18, 19] and a revised version, Soubeyran [20]), where a succession of worthwhile changes move from a position to a new one (which are not supposed to be stationary traps) and converges to an end point, which is shown to be a variational trap. Finally, Section 6 contains concluding discussions of the main results obtained in the paper.

2 Preliminary

In this paper, every manifold M is assumed to be Hadamard and finite dimensional. The notations, results, and concepts used throughout this paper can be found in Ferreira and Oliveira [9].

A set $\Omega \subset M$ is said to be *convex* iff any geodesic segment with end points in Ω is contained in Ω, that is, iff $\gamma : [a, b] \to M$ is a geodesic such that $x = \gamma(a) \in \Omega$ and $y = \gamma(b) \in \Omega$, then $\gamma((1-t)a + tb) \in \Omega$ for all $t \in [0, 1]$. We recall that a function f is called *concave* if $-f$ is convex. Furthermore, if f is both convex and concave then f is said to be linear *affine*. Given $\mathcal{B} \subset M$, we denote by $\mathrm{conv}(\mathcal{B})$ the convex *hull* of $\mathcal{B}$, that is, the smallest convex subset of M containing $\mathcal{B}$. Let $\Omega \subset M$ be a convex set. A function $f : \Omega \to \mathbb{R}$ is said to be *convex* iff for any geodesic segment $\gamma : [a, b] \to \Omega$ the composition $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex. Take $p \in \Omega$. A vector $s \in T_pM$ is said to be a *subgradient* of f at p iff

$$f(q) \geq f(p) + \langle s, \exp_p^{-1} q \rangle, \qquad q \in \Omega.$$

The set of all subgradients of f at p, denoted by $\partial f(p)$, is called the *subdifferential* of f at p. It is known that if f is convex and M is an Hadamard manifold, then $\partial f(p)$ is a nonempty set, for each $p \in \Omega$; see Udriste [22, Theorem 4.5, page 74].

Let $\mathcal{B} \subset M$ be a non-empty, convex and closed set. The distance function associated with $\mathcal{B}$ is given by

$$M \ni x \longmapsto d_{\mathcal{B}}(x) := \inf\{d(y, x) : y \in \mathcal{B}\} \in \mathbb{R}_+.$$

It is well-known (see [9, Corollary 3.1]) that for each $x \in M$ there exists a unique element $\tilde{x} \in \mathcal{B}$ such that

$$\langle \exp_{\tilde{x}}^{-1} x, \exp_{\tilde{x}}^{-1} y \rangle \leq 0, \qquad y \in \mathcal{B}.$$

In this case, $\tilde{x}$ is the projection of x onto the set $\mathcal{B}$ which we will denote by $P_{\mathcal{B}}(x)$.

Remark 2.1. *It is important to mention that for every $y \in M$, $x \mapsto d(x, y)$ is a continuous and convex function; see* [21, Proposition 4.3, page 222].

3 Equilibrium Problem

In this section, following the ideas given in [4], we present a sufficient condition for the existence of solution of equilibrium problems on Hadamard manifolds. We chose to present a proof only for the main result. With the exception to the proof of Proposition 3.1, the proof of the other results can be extended, from those presented in linear environments (see [4, 3]), with minor adjustments to the nonlinear context of this paper.

From now on, $\Omega \subset M$ will denote a nonempty closed convex set, unless explicitly stated otherwise. Given a bifunction $F : \Omega \times \Omega \to \mathbb{R}$ satisfying the property $F(x, x) = 0$, for all $x \in \Omega$, the *equilibrium problem* in the Riemannian context (denoted by EP) consists in:

$$\text{Find } x^* \in \Omega : \quad F(x^*, y) \geq 0, \qquad y \in \Omega. \tag{1}$$

In this case, the bifunction F is called a *equilibrium bifunction.* As far as we know, this problem was considered firstly, in this context, in [5] where the authors pointed out important problems, which are retrieved from (1). Particularly, given $V \in \mathcal{X}(M)$, if

$$F(x, y) = \langle V(x), \exp_x^{-1} y \rangle, \qquad x, y \in \Omega,$$

(1) reduces to the variational inequality problem; see, for instance, [17].

Remark 3.1. *Although the Variational Inequality Theory provides us a toll for formulating a variety of equilibrium problems, Iusem and Sosa* [3, Proposition 2.6] *showed that the generalization given by EP formulation with respect to VIP (Variational Inequality Problem) is genuina, in the sense that there are EP problems which do not fit the format of VIP. We affirm that it is possible to guarantee the genuineness of the by EP formulation compared to VIP, by considering the important class of by Quasi-Convex Optimization Problems that appear, for instance, in many micro-economical models devoted to maximize utilities. Indeed, the absence of convexity allows us to obtain situations where this important class of problems can not be considered as a VIP in the sense that their possible representation in this format would lead us to a problem, whose solution set contains points that do not necessarily belong to the solution set of the original optimization problem. On the other hand even in the absence of convexity, this class of problems can be placed in the EP format.*

Definition 3.1. *Let $F : \Omega \times \Omega \to \mathbb{R}$ be a bifunction. F is said to be*

(1) *monotone iff $F(x, y) + F(y, x) \leq 0$, for all $(x, y) \in \Omega \times \Omega$;*

(2) *pseudomonotone iff, for each $(x, y) \in \Omega \times \Omega$, $F(x, y) \geq 0$ implies $F(y, x) \leq 0$.*

Remark 3.2.

i) Clearly, monotonicity implies pseudomonotonicity, but the converse does not hold even in a linear context, see, for instance, Iusem and Sosa [3]*;*

ii) If F is pseudomonotone, then for $\tilde{x}, \tilde{y} \in \Omega$, $F(\tilde{x}, \tilde{y}) > 0$ implies $F(\tilde{y}, \tilde{x}) < 0$. Indeed, let us suppose, for contradiction, that $F(\tilde{y}, \tilde{x}) = 0$ (in particular $F(\tilde{y}, \tilde{x}) \geq 0$). From the pseudomonotonicity of F it follows that $F(\tilde{x}, \tilde{y}) \leq 0$, which is an absurd, and the affirmation is proved.

Next result was presented by Colao et al. in [5] and is fundamental to establish our existence result for the EP.

Proposition 3.1. *Let $\mathcal{B} \subset M$ be a closed convex subset and $H : \mathcal{B} \to 2^{\mathcal{B}}$ be a mapping such that, for each $y \in \mathcal{B}$, $H(y)$ is closed. Suppose that*

(i) *there exists $y_0 \in \mathcal{B}$ such that $H(y_0)$ is compact;*

(ii) $\forall y_1, \ldots, y_m \in \mathcal{B}$, $\operatorname{conv}(\{y_1, \ldots, y_m\}) \subset \bigcup_{i=1}^{m} H(y_i)$.

Then,

$$\bigcap_{y \in \mathcal{B}} H(y) \neq \varnothing.$$

Proof. See [5]. □

Unless stated to the contrary, in the remainder of this paper we assume that $F : \Omega \times \Omega \to \mathbb{R}$ is an *equilibrium bifunction* satisfying the following assumptions:

$\mathcal{H}1$) For every $x \in \Omega$, $y \mapsto F(x, y)$ is convex and lower semicontinuous;

$\mathcal{H}2$) For every $y \in \Omega$, $x \mapsto F(x, y)$ is upper semicontinuous.

For each $y \in \Omega$, let us define:

$$L_F(y) := \{x \in \Omega : F(y, x) \leq 0\}.$$

From this set, we can consider the following *convex feasibility problem* (denoted by CFP):

$$\text{Find } x^* \in \bigcap_{y \in \Omega} L_F(y).$$

As far as we know, this problem was first studied, in the Riemannian context, by Bento and Melo in [23], in the particular case where the domain of F is $M \times \{1, \ldots, m\}$. In this case, $y \in \{1, \ldots, m\}$ and Ω is the whole M.

Next result establishes a relationship between CFP and EP.

Lemma 3.1. *The solution set of* CFP *is contained in the solution set of* EP*.*

Remark 3.3. *Note that, as it is in the Euclidean context, the equality between the two sets in the previous lemma in general does not happen, see* [3]*. However, in the particular case where F is pseudomonotone, the equality is immediately verified.*

Take $z_0 \in M$ fixed. For each $k \in \mathbb{N}$ consider the following set:

$$\Omega_k := \{x \in \Omega : \ d(x, z_0) \leq k\}.$$

Note that Ω_k is a nonempty set, for $k \in \mathbb{N}$ sufficiently large. For simplicity, we can suppose, without loss of generality, that Ω_k is a nonempty set for all $k \in \mathbb{N}$. Moreover, as Ω_k is contained in the closed ball $B(z_0, k) := \{x \in M : \ d(z_0, x) \leq k\}$, it is a bounded set. On the other hand, since $d(\cdot, z_0)$ is a continuous and convex function (this follows from Remark 2.1), Ω_k is a convex and closed set and, hence, compact; see Ropf-Rinow's Theorem. We denote, by Ω_k^0, the following set:

$$\Omega_k^0 := \{x \in \Omega : d(x, z_0) < k\}.$$

For each $y \in \Omega$, let us define:

$$L_F(k, y) := \{x \in \Omega_k : F(y, x) \leq 0\}.$$

Lemma 3.2. *Let $k \in \mathbb{N}$, $\bar{x} \in \bigcap_{y\in\Omega_k} L_F(k,y)$ and assume that there exists $\bar{y} \in \Omega_k^0$ such that $F(\bar{x},\bar{y}) \leq 0$. Then, $F(\bar{x},y) \geq 0$, for all $y \in \Omega$, i.e., $\bar{x}$ is a solution for* (1).

Assumption 3.1. *Given $k \in \mathbb{N}$, for all finite set $\{y_1,\dots,y_m\} \subset \Omega_k$, one has*

$$\operatorname{conv}(\{y_1,\dots,y_m\}) \subset \bigcup_{i=1}^{m} L_F(k,y_i).$$

Remark 3.4. *Note that, in the particular case where F is pseudomonotone, the property described by the previous assumption is naturally verified. Indeed, let $y_1,\dots,y_m \in \Omega_k$, take $\bar{y} \in \operatorname{conv}(\{y_1,\dots,y_n\})$ and let us suppose, for contradiction, that $\bar{y} \notin \bigcup_{i=1}^{m} L_F(k,y_i)$. Then,*

$$F(y_i,\bar{y}) > 0, \qquad i \in \{1,\dots,m\}. \tag{2}$$

Now, define the following set $B := \{x \in \Omega_k : \ F(\bar{y},x) < 0\}$. In the particular case where F is pseudomonotone, using (2) *and taking into account that B is convex (this follows from $\mathcal{H}1$), we conclude that $\bar{y} \in B$ (see item ii) of Remark 3.2). But this contradicts that $F(x,x) = 0$ and the affirmation is proved.*

Assumption 3.2. *Given $z_0 \in M$ fixed, consider a sequence $\{z^k\} \subset \Omega$ such that $\{d(z^k,z_0)\}$ converges to infinity as k goes to infinity. Then, there exists $x^* \in \Omega$ and $k_0 \in \mathbb{N}$ such that*

$$F(z^k,x^*) \leq 0, \qquad k \geq k_0.$$

It is worth noting that this last assumption has been presented by Iusem et al. [4], in a space with a linear structure. It is a sufficient condition for the existence of solutions of the equilibrium problem EP.

Next result (see [3] for similar results, in the linear setting) assures us that Assumption 3.2 is a weaker sufficient condition than the coercivity assumption used by Colao et al. [5], for the existence of solutions of EP.

Proposition 3.2. *Let $\mathcal{B} \subset M$ be a compact set and $y_0 \in \mathcal{B} \cap \Omega$ a point such that $F(x,y_0) < 0$, for all $x \in \Omega \setminus \mathcal{B}$. Then, F satisfies* Assumption 3.2.

The following is the main result of this section.

Theorem 3.1. *Under* Assumptions 3.1 *and* 3.2, *EP admits a solution.*

Proof. Recall that Ω_k is a convex and compact set for each $k \in \mathbb{N}$. Now, given $k \in \mathbb{N}$ and $y \in \Omega$, note that $L_F(k,y)$ is a compact set. Indeed, this fact follows from the definition of $L_F(k,y)$ combined with assumption $\mathcal{H}1$ ($F(y,\cdot)$ is a lower semicontinuous function on Ω) and compactness of Ω_k. Now, since Assumption 3.1 holds true, using Proposition 3.1 with $\mathcal{B} = \Omega_k$ and $H(y) = L_F(k,y)$, we conclude that, for each $k \in \mathbb{N}$,

$$\bigcap_{y\in\Omega_k} L_F(k,y) \neq \emptyset.$$

For each k, choose $z^k \in \bigcap_{y\in\Omega_k} L_F(k,y)$ and take $z_0 \in M$ fixed. If there exists $k \in \mathbb{N}$ such that $d(z^k,z_0) < k$, then $z^k \in \Omega_k^0$ and, from Lemma 3.2, it follows that z^k solves EP. On the other hand, if $d(z^k,z_0) = k$, from Assumption 3.2, there exists, $x^* \in \Omega$ and $k_0 \in \mathbb{N}$ such that $F(z^k,x^*) \leq 0$, for all $k \geq k_0$. Taking $k' > k_0$ such that $d(x^*,z_0) < k'$, we have $F(z^{k'},x^*) \leq 0$ and $x^* \in \Omega_{k'}^0$. Therefore, using again Lemma 3.2, we conclude that $z^{k'}$ solves EP, and the proof is complete. □

Next example was inspired by [5, Example 3.4]. It illustrates the usefulness of the our previous result, in the sense that it applies to some situations not covered in the linear setting. For other papers that highlight such advantage, in regard to the linear setting, see [16, 23].

Example 3.1. *Let $\Omega = \{(x,y,z) : 0 \leq x \leq 1, y^2 - z^2 = -1, y \geq 0, z \geq 1\} \subset \mathbb{R} \times \mathbb{H}^1$ and consider the following bifunction $F : \Omega \times \Omega \to \mathbb{R}$, given by:*

$$F((x_1,y_1,z_1),(x_2,y_2,z_2)) := (2 - x_1)\left(\left(y_2^2 + z_2^2\right) - \left(y_1^2 + z_1^2\right)\right).$$

Note that Ω is indeed a not convex set in $\mathbb{R}^3$. So, an equilibrium problem defined on Ω cannot be solved by using the classical results known in the linear context. Let $(\mathbb{H}^n, \langle\ ,\ \rangle)$ be the Riemannian manifold, where

$$\mathbb{H}^n := \{x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0 \text{ and } \langle x, x\rangle = -1\} \quad \text{(hyperbolic n space)},$$

and $\langle\cdot,\cdot\rangle$ is the Riemannian metric $\langle x, y\rangle := x_1y_1 + x_2y_2 + \ldots + x_ny_n - x_{n+1}y_{n+1}$ (Lorentz metric). As noted in [5], *$(\mathbb{H}^n, \langle\cdot,\cdot\rangle)$ is a Hadamard manifold with sectional curvature -1 and, given initial conditions $x \in \mathbb{H}^n$, $v \in T_x\mathbb{H}^n$ ($\|v\| = 1$), the normalized geodesic $\gamma : \mathbb{R} \to \mathbb{H}^n$, is given by:*

$$\gamma(t) = (\cosh t)x + (\sinh t)v, \qquad t \in \mathbb{R}.$$

Hence, we obtain the following expression for the Riemannian distance d:

$$d(x, y) = \operatorname{arccosh}(-\langle x, y\rangle), \qquad x, y \in \mathbb{H}^n.$$

Again, as observed in [5], *Ω is a convex set which is immersed in the Hadamard manifold $M := \mathbb{R} \times \mathbb{H}^1$. Using the expression of the geodesic curves, it can be deduced that F is a convex function in the second variable. Moreover, from the definition of F, it is easy to see that $F(x, x) = 0$, assumptions $\mathcal{H}1$ and $\mathcal{H}2$ are satisfied, and F is a pseudomonotone bifunction. In particular, from Remark 3.4, it follows that* Assumption *3.1 holds. Now, take $w^0 \in M$ fixed and a sequence $\{w^k\} \subset \Omega$, $w^k := (x_k, y_k, z_k)$, such that $d(w^k, w^0) \to +\infty$. There exists $x^* := (1, 0, 1) \in \Omega$ such that $F(w^k, x^*) \leq 0$, for all $k \in \mathbb{N}$, i.e.,* Assumption 3.2 *holds. Therefore,* Theorem 3.1 *implies the existence of an equilibrium point for F.*

4 Proximal Point for Equilibrium Problem

In this section, we propose a new proximal point algorithm for equilibrium problems on Hadamard manifolds where the convergence result is obtained for monotone bifunctions. As far as we know, a proximal point algorithm for EP was first introduced in [5]. In comparison with the algorithm given in [5], the novelty here is the new regularization term, whose convexity does not impose any restrictive hypothesis about the Hadamard manifold.

Let us denote the equilibrium point set of F by EP(F,Ω) and, for $\lambda > 0$ and $z \in \Omega$ fixed, consider the regularized bifunction

$$F_{\lambda,z}(x, y) := F(x, y) + \lambda\left[d^2(y, z) - d^2(x, z)\right], \qquad x, y \in \Omega. \tag{3}$$

Now, we describe a proximal point algorithm to solve the equilibrium problem (1).

Algorithm 4.1. *Take $\{\lambda_k\}$ a bounded sequence of positive real numbers.*
Initialization. *Choose an initial point $x^0 \in \Omega$;*
Stopping criterion. *Given x^k, if $x^{k+1} = x^k$ STOP. Otherwise;*
Iterative Step. *Given x^k, take as the next iterate any $x^{k+1} \in \Omega$ such that:*

$$x^{k+1} \in \mathrm{EP}(F_k, \Omega), \quad F_k := F_{\lambda_k, x^k}. \tag{4}$$

Remark 4.1. *Note that if $x^{k+1} = x^k$, then $x^k \in EP(F,\Omega)$. It is worth noting that the iterative process* (4) *retrieves the proximal point method for minimization problems on Hadamard manifolds; see* [9].

Next results are useful to ensure the well-definition of Algorithm 4.1. In the remainder of the paper we assume that F is monotone, λ is a positive real number and $z \in \Omega$.

Lemma 4.1. *Let F be a monotone bifunction. Then,*

i) $F_{\lambda,z}$ is monotone;

ii) $F_{\lambda,z}$ satisfies assumption $\mathcal{H}1$.

Proof. Item i) follows immediately from the monotony of F and item ii) for considering that F and $d^2(\cdot,\cdot)$ satisfy $\mathcal{H}1$. □

Lemma 4.2. *Let* EP*(F,Ω) be a nonempty set. If F satisfies* Assumption 3.2*, then $F_{\lambda,z}$ also satisfies this assumption.*

Proof. First of all, given $z_0 \in M$, consider a sequence $\{z^k\} \subset \Omega$ such that $\{d(z^k, z_0)\}$ converges to infinity as k goes to infinity. Take $\tilde{x} \in \mathrm{EP}(F, \Omega)$. Using (3) with $x = z^k$ and $y = \tilde{x}$, we get

$$F_{\lambda,z}(z^k, \tilde{x}) = F(z^k, \tilde{x}) + \lambda \left[d^2(\tilde{x}, z) - d^2(z^k, z)\right]. \tag{5}$$

Since $\{d(z^k, z_0)\}$ converges to infinity as k goes to infinity, in particular, $\{d(z^k, z)\}$ also converges to infinity. Moreover, as $\{z^k\} \subset \Omega$ and $\tilde{x} \in \mathrm{EP}(F, \Omega)$, monotonicity of F implies that $F(z^k, \tilde{x}) \leq 0$. Hence, the desired result follows immediately from (5), which concludes the proof. □

Theorem 4.1. *Assume that* Assumption 3.2 *holds and* EP*(F,Ω) is a nonempty set. Then, there exists a $x^* \in \Omega$ such that*

$$F_{\lambda,z}(x^*, y) \geq 0, \qquad y \in \Omega.$$

Proof. From item i) of Lemma 4.1 it follows that $F_{\lambda,z}$ is monotone and, in particular, pseudomotone (this follows from Remark 3.2). Moreover, Remark 3.4 implies that $F_{\lambda,z}$ satisfies Assumption 3.1 and Lemma 4.1 (resp. Lemma 4.2) tells us that $F_{\lambda,z}$ satisfies $\mathcal{H}1$ (resp. Assumption 3.2). Hence, the desired result follows from Theorem 3.1 and the proof is concluded. □

Corollary 4.1. *Assume that* Assumption 3.2 *holds and* EP*(F,Ω) is a nonempty set. Then, Algorithm 4.1 is well-defined.*

Proof. It follows immediately from Theorem 4.1. □

In the remainder of this paper we assume that the assumption of the previous corollary hold and $\{x^k\}$ is a sequence generated from Algorithm 4.1. Taking into account that if Algorithm 4.1 terminates after a finite number of iterations, it terminates at an equilibrium point of F, from now on, we assume also that $\{x^k\}$ is an infinite sequence.

4.1 Convergence Analysis

In this section we present the convergence of the sequence $\{x^k\}$.

We claim that

$$\langle \exp^{-1}_{x_{k+1}} x^k, \exp^{-1}_{x_{k+1}} x^* \rangle \leq 0, \qquad x^* \in EP(F, \Omega). \tag{6}$$

Indeed, from the definition of the iterate x^{k+1} and F_k in (4) combined with (3), we obtain

$$F_k(x^{k+1}, y) = F(x^{k+1}, y) + \lambda_k \left[d^2(y, x^k) - d^2(x^{k+1}, x^k)\right] \geq 0, \qquad y \in \Omega.$$

Since $F_k(x^{k+1}, x^{k+1}) = 0$, last inequality tells us that $x^{k+1} = \arg\min_{y \in \Omega} F_k(x^{k+1}, y)$ or, equivalently, that there exists $w^{k+1} \in \partial F_k(x^{k+1}, x^{k+1}) = \partial F(x^{k+1}, x^{k+1}) - \lambda_k \exp^{-1}_{x^{k+1}} x^k$ such that

$$\langle w^{k+1}, \exp^{-1}_{x^{k+1}} y \rangle \geq 0, \qquad y \in \Omega.$$

Note that $w^{k+1} = u^{k+1} - \lambda_k \exp^{-1}_{x^{k+1}} x^k$, where $u^{k+1} \in \partial F(x^{k+1}, x^{k+1})$. So, last inequality yields

$$\langle u^{k+1}, \exp^{-1}_{x^{k+1}} y \rangle - \lambda_k \langle \exp^{-1}_{x^{k+1}} x^k, \exp^{-1}_{x^{k+1}} y \rangle \geq 0, \qquad y \in \Omega. \tag{7}$$

On the other hand, convexity of $F(x^{k+1}, \cdot)$ (this follows from $\mathcal{H}1$) implies that

$$F(x^{k+1}, y) \geq F(x^{k+1}, x^{k+1}) + \langle u^{k+1}, \exp^{-1}_{x^{k+1}} y \rangle = \langle u^{k+1}, \exp^{-1}_{x^{k+1}} y \rangle, \qquad y \in \Omega.$$

Take $\bar{x} \in \mathrm{EP}(F, \Omega)$. As $\mathrm{EP}(F, \Omega) \subset \Omega$, from the last inequality, we obtain

$$\langle u^{k+1}, \exp^{-1}_{x^{k+1}} \bar{x} \rangle \leq F(x^{k+1}, \bar{x}) \leq 0, \tag{8}$$

where last inequality is a consequence of the monotony of F, since $F(\bar{x}, x^{k+1}) \geq 0$. Therefore, because $\lambda_k > 0$ for all k, the result of the claim immediately follows by combining inequalities (7) and (8).

Definition 4.1. A sequence $\{y^k\}$ in the complete metric space (M, d) is said to be *Fejér convergent* to a nonempty set $\mathcal{S} \subset M$ iff for every $\bar{y} \in \mathcal{S}$,

$$d(y^{k+1}, \bar{y}) \le d(y^k, \bar{y}) \qquad k = 0, 1, \ldots.$$

The following result is well known and its proof is elementary.

Proposition 4.1. *Let $\{y^k\}$ be a sequence in the complete metric space (M, d). If $\{y^k\}$ is Fejér convergent to a nonempty set $\mathcal{S} \subset M$, then $\{y^k\}$ is bounded. If, furthermore, an accumulation point $\bar{y}$ of $\{y^k\}$ belongs to $\mathcal{S}$, then $\lim_{k\to\infty} y^k = \bar{y}$.*

Now, we present our main convergence result.

Theorem 4.2. *Assume that the sequence $\{\lambda_k\}$ converges to zero. Then, the sequence $\{x^k\}$ converges to a point in* EP*(F, Ω).*

Proof. Take $x^* \in \mathrm{EP}(F, \Omega)$. Using [21, Theorem 4.2, page 161] (law of cosines) with $x_i = x^*$, $x_{i+1} = x^k$ and $x_{i+2} = x^{k+1}$, we obtain

$$d^2(x^{k+1}, x^*) + d^2(x^{k+1}, x^k) - 2\langle \exp^{-1}_{x^{k+1}} x^k, \exp^{-1}_{x^{k+1}} x^* \rangle \le d^2(x^k, x^*), \qquad k = 01, \ldots. \tag{9}$$

Since $x^* \in \mathrm{EP}(F, \Omega)$, combining inequality (6) with the last inequality and taking into account that $d^2(x^{k+1}, x^k) > 0$, it follows that $\{x^k\}$ is Fejér convergent to the set $\mathrm{EP}(F, \Omega)$. Moreover, from (9) combined with inequality (6), we have that $\{d(x^{k+1}, x^k)\}$ converges to zero as k goes to infinity. Applying Proposition 4.1 with $y^k = x^k$, $k \in \mathbb{N}$, and $\mathcal{S} = \mathrm{EP}(F, \Omega)$, we have that $\{x^k\}$ is a bounded sequence. In particular, from Hopf-Rinow Theorem, there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ converging to some point $\hat{x}$. Note that $\{d(x^{k_j+1}, x^{k_j})\}$, $\{d(x^{k_j+1}, \hat{x})\}$ converge to zero and, in particular, $\{x^{k_j+1}\}$ goes to $\hat{x}$ as j goes to infinity. Hence, using simple upper limit properties applied to $F(x^{k_j+1}, y) + \lambda_{k_j}\left[d^2(y, x^{k_j}) - d^2(x^{k_j+1}, x^{k_j})\right]$, and that $\{\lambda_{k_j}\}$ goes to zero, we obtain

$$0 \le \limsup_{j\to+\infty} F(x^{k_j}, y) \le F(\hat{x}, y), \qquad y \in \Omega,$$

where the first (resp. second) inequality follows, respectively, from definition of x^{k_j} (resp. assumption $\mathcal{H}2$). Therefore, the desired result is proved. □

5 Application. The Problem of Traps and the Reachability of Desires in Behavioral Sciences

In this last section we want to show how the present paper offers, in a specific context, a nice solution to the behavioral problem of how an agent can escape to a succession of traps to be able to reach his desires, using the recent variational rationality modelization of desires and temporary/permanent traps all along stay and change worthwhile transitions; see [18, 19, 20].

This paper considers two related problems, in a given space of alternatives Ω (possible choices of actions, states, positions, situations,...) on an Hadamard manifold:

i) the standard static equilibrium problem given in (1);

ii) a new and dynamic regularized equilibrium problem given in (4) which, as observed in the very beginning of the convergence analysis, consists to find in the current period $k+1$, a choice $x^{k+1} \in \Omega$ such that

$$F_{\lambda_k, x^k}(x^{k+1}, y) = F(x^{k+1}, y) + \lambda_k I_{x^k}(x^{k+1}, y) \ge 0, \quad y \in \Omega,$$

where the specific regularized term is given by $I_{x^k}(x^{k+1}, y) := \left[d^2(x^k, y) - d^2(x^k, x^{k+1})\right]$, $\lambda_k > 0$ is a weight, x^{k+1} is the current choice, y is a current alternative choice, and $x^k \in \Omega$ is the last choice. Under specific assumptions, this paper shows, first, the existence of a static equilibrium problem (1). Then, it shows that the solution $x^k \in \Omega$ of the dynamic equilibrium problem converges to a static equilibrium $x^* \in \Omega$.

5.1 Desires. The Duality between Equilibrium Points and Strong Aspiration Points

Let us see, using a recent variational rationality formalization of acceptable stays and changes approach and avoidance dynamics which unifies a lot of them in different disciplines, see [18, 19, 20], why a dual view of an equilibrium problem is so important for applications in Behavioral Sciences. Consider an active agent who derives the utility $V(y) \in \mathbb{R}$ to perform an action $y \in \Omega$, where his highest utility level $\overline{V} = \sup\{V(y) : y \in \Omega\} < +\infty$ is finite, because of bounded needs. In Applied Mathematics and very often in Behavioral Sciences, agents are supposed to minimize a cost function, or to try to decrease their level of unsatisfaction $U(y) \in \mathbb{R}$. The (VR) approach defines unsatisfaction as unsatisfied needs, i.e, the difference $U(y) = \overline{V} - V(y) \geq 0$. We will consider such unsatisfaction levels, given the Mathematical audience.

In a static context, the agent will not regret to have done action x^* if there is a loss

$$F(x^*, y) = U(y) - U(x^*) \geq 0,$$

to change, moving from doing x^* to do any other action $y \in \Omega$ with a highest unsatisfaction level. This view supposes that the agent is, at the very beginning of the story, right where he wants to be, at the maximum of utility $V(x^*)$, i.e, at the lowest level of unsatisfaction $U(x^*)$. But this is not very realistic. In real life, agent are not, initially, exactly where they want to be. Usually, they start to be in an unsatisfactory position, at $y = x^0 \in \Omega$, where $U(y) > U(x^*)$. This means that, initially, an agent will find an advantage to change $A(y, x^*) = U(y) - U(x^*) > 0$ from $y = x^0$ to x^*. In this case, $A(y, x^*) = -F(x^*, y) > 0$.

Then, the (VR) approach completely reverses the logic of the equilibrium problem. It focuses first the attention on agents who aspire for better and considers advantages to change to a better situation, and only, at the end, on agents who fear for less, and consider losses to change to a worst situation! Moreover, unfortunately, most of the time, an agent does not know, at least at the very beginning, what can be the best action to do, $x^* \in \Omega$. To hope to know it, he cannot escape to enter, if possible, in an improving dynamic, where each period, his unsatisfaction moves from $U(x^k)$ in the last period to a lower level $U(x^{k+1}) \leq U(x^k)$ in the current period, enjoying the advantage to change $A(x^k, x^{k+1}) = U(x^k) - U(x^{k+1}) \geq 0$. Then, the agent will follow an improving transition of stays and changes, where, each period, the agent will stay, $x^{k+1} = x^k$, if he fails to find an improving action, or will change, $x^{k+1} \neq x^k$, in the opposite case of success.

This dual dynamic context considers both:

i) the equilibrium concept, relative to losses to change (static negative view), where an agent, being there, at x^*, will prefer to stay there, than to move away, because moving from x^* to any $y \in \Omega$ will generate the loss $F(x^*, y) = U(y) - U(x^*) \geq 0$;

ii) the dual concept of strong aspiration, relative to advantages to change (dynamic positive view), where the agent being at $y = x^0 \neq x^*$, will prefer to move to the aspiration point x^* rather than to stay at y, because moving from any y to x^* will generate the advantage to change

$$A(y, x^*) = U(y) - U(x^*) = F(x^*, y) \geq 0.$$

Then, the dual equilibrium problem is: find $x^* \in X$ such that:

$$A(y, x^*) \geq 0, \qquad y \in \Omega,$$

i.e., $A(x^*, y) = -A(y, x^*) \leq 0$ for all $y \in \Omega$.

This dual view helps to defines desires as an equilibrium (to want what you have) and a strong aspiration point (to hope to have what you want, starting from anywhere). Of course, in this very simple formulation of a non "reference dependent" utility function, we have $A(x, y) = -F(x, y)$, i.e., if there is an advantage to change from x to y, there is an opposite loss to move the other way. This is not the case if utility functions are reference dependent; see [20].

5.2 The Regularized Equilibrium Function as an Instance of a Worthwhile to Change Payoff

We are now in a good position to see what represents, in the context of the variational rationality approach, see [18, 19, 20], the regularized equilibrium function given in (3). The previous discussion on losses to change

$F(x,y)$ and their opposites, advantages to change $A(x,y)=-F(x,y)$ directs our attention to the opposite of the regularized equilibrium function,

$$\Delta_{\lambda,z}(x,y)=-F_{\lambda,z}(x,y)=A(x,y)-\lambda I_z(x,y),$$

where $I_z(x,y)=\left[d^2(z,y)-d^2(z,x)\right]$ denotes the regularization term in (3). More explicitly, in a dynamic context, we move the attention from $F_{\lambda_k,x^k}(x^{k+1},y)=F_{x^k}(x^{k+1},y)+\lambda_k I_{x^k}(x^{k+1},y)$ to its opposite,

$$\Delta_{\lambda_k,x^k}(x^{k+1},y)=-F_{\lambda_k,x^k}(x^{k+1},y)=A_{x^k}(x^{k+1},y)-\lambda_k I_{x^k}(x^{k+1},y).$$

It turns out that $\Delta_{\lambda,z}(x,y)$ is a specific, but interesting instance of a worthwhile to change payoff, a central concept of the variational rationality approach, where an agent balances, each current period $k+1$, advantages $A(x^{k+1},y)$ and inconveniences to change $I_z(x^{k+1},y)$, using the balancing weight λ_k. This means that an hypothetic change from x^{k+1} to y in the current period is not (strictly) worthwhile, i.e, $\Delta_{\lambda_k,x^k}(x^{k+1},y)\leq 0$ if advantages to change are low high enough (the weight λ being hight enough) with respect to inconveniences to change, i.e, $A_{x^k}(x^{k+1},y)-\lambda_k I_{x^k}(x^{k+1},y)\leq 0$. In the specific context of this paper, the (VR) approach (see [18, 19, 20]) defines inconveniences to change as follows. They represent, in the current period, the difference $I_{x^k}(x^{k+1},y):=C_{x^k}(x^k,y)-C_{x^k}(x^k,x^{k+1})$ between costs to be able to change $C_{x^k}(x^k,y)\in\mathbb{R}_+$ from the last position $z=x^k$ to the alternative position y, and costs to be able to change $C_{x^k}(x^k,x^{k+1})$ from the last position $z=x^k$ to a current position x^{k+1}. Because they refer to a lot of different situations, in different disciplines, where resistance to change and inertia matter, these costs are not easy to define in a precise way. For a carefull formulation of these costs to be able to change and several examples, see [20]. In the present paper, $C_{x^k}(x^k,y)=d^2(x^k,y)$, where $d(z,y)$ is the geodesic distance on an Hadamard manifold.

Given that $F_{\lambda_k,x^k}(x^{k+1},x^{k+1})=0$, the current condition $F_{\lambda_k,x^k}(x^{k+1},y)\geq 0$, for all $y\in\Omega$ means that, each current period $k+1$, the agent minimizes his current regularized equilibrium function, i.e,

$$x^{k+1}=\text{argmin}_{y\in\Omega}F_{\lambda_k,x^k}(x^{k+1},y).$$

Then, the current opposite condition $\Delta_{\lambda_k,x^k}(x^{k+1},y)\leq 0$, for all $y\in\Omega$, means that, each current period, the agent maximizes his current worthwhile to change balance, i.e., $x^{k+1}=\text{argmax}_{y\in\Omega}\Delta_{\lambda_k,x^k}(x^{k+1},y)$.

5.3 The Algorithm as an Instance of a Worthwhile Transition

In the current period $k+1$, the current equilibrium condition (4) can be written, taking the opposite,

$$\Delta_{\lambda_k,x^k}(x^{k+1},y)\leq 0,\qquad y\in\Omega.$$

Take $y=x^k,k\in\mathbb{N}$. Then, $\Delta_{\lambda_k,x^k}(x^k,x^{k+1})=-\,\Delta_{\lambda_k,x^k}(x^{k+1},x^k)=F_{\lambda_k,x^k}(x^{k+1},y)\geq 0$ shows that it is worthwhile to change, $x^{k+1}\in W_{\lambda_k,x^k}(x^k),k\in\mathbb{N}$, from x^k in the last period to x^{k+1} in the current period. This shows that Algorithm 4.1 defines a worthwhile stay and change transition, where, each period, advantages to change are higher enough with respect to inconveniences to change, i.e, $A_{x^k}(x^k,x^{k+1})\geq\lambda_k I_{x^k}(x^k,x^{k+1})$. Then, inconveniences to change $I_{x^k}(x^k,x^{k+1})=d^2(x^k,x^{k+1})$ and advantages to change $A_{x^k}(x^k,x^{k+1})$ are non negative all along the path of worthwhile stays and changes.

5.4 The Two Main Results Show How to Reach Desires, Escaping to a Succession of Temporary Variational Traps

The concept of trap appears in an informal way in a lot of different disciplines in Behavioral Sciences; for a short survey see [20]. The variational rationality approach gives a general and formal definition. A (permanent) variational trap is the end of a worthwhile stay and change transition $x^{k+1}\in W_{\lambda_k,x^k}(x^k),k\in\mathbb{N}$. More precisely, a (permanent) variational trap is, both,

i) an aspiration point, more or less easy to approach, i.e, x^k converges to x^* as k goes to $+\infty$, and reachs it, $x^*\in W_{\lambda_k,x^k}(x^k)$, $k\in\mathbb{N}$, following a worthwhile stay and change transition, $x^{k+1}\in W_{\lambda_k,x^k}(x^k)$, $k\in\mathbb{N}$;

ii) a stationary trap (a static equilibrium with inconveniences to change), difficult to leave, i.e, $W_{\lambda_*,x^*}(x^*)=\{x^*\}$, where $\lambda_*>0$ is the end worthwhile to change ratio.

Variational traps can be permanent or temporary. In the present paper, they are temporary, $W_{\lambda_k, x^k}(x^k) = \{x^k\}$, $k \in \mathbb{N}$, because changing the worthwhile to change ratio λ_k breaks the current variational trap.

The present paper shows:

A) the existence and reachability of desires (both, as equilibria and strong aspiration points) as the ends of a succession of temporary variational traps;

B) Fejér convergence of a succession of temporary variational traps to some desire (equilibrium). This means that costs to be able to reach an equilibrium decrease;

C) costs to be able to change which go to zero (see the proof of Theorem 4.2). Then, resistance to change disappears in the long run.

5.5 Behavioral Hypothesis

Let us give a behavioral interpretation of the main hypothesis of this paper.

Assumption 3.1 (monotonicity) implies that if there is a losss $F(x, y) \geq 0$ to move from x to y, then, there is a gain to go in the reverse way, because $F(y, x) \leq -F(x, y) \leq 0$, for all $x, y \in \Omega$. This is a very natural hypothesis. With regarding Assumption 3.2, it is a kind of coercivity assumption which says that if costs to be able to change $C(x^0, x^k) = d^2(x^0, x^k)$ go to infinity as k goes to infinity, then, there exists $x^* \in \Omega$ and $k_0 \in \mathbb{N}$ such that $F(x^k, x^*) \leq 0$, i.e., $A(x^k, x^*) \geq 0$ for all $k \geq k_0$. In behavioral term, this means that it exists a kind of weak aspiration point along a sequence when costs to be able to change go to infinity.

In this paper, in view of applications, there are two specific behavioral hypothesis, namely:

i) costs to be able to change $C(x, y) = d^2(x, y)$ are symmetric because $d(y, x) = d(x, y)$. This is a strong assumption that must be removed in future researches. An immediate way to remove it is to consider costs to be able to change $C(x, y) = q^2(x, y)$ where $q(x, y) \geq 0$ is a quasi distance such that $q(x, y) = 0$ iff $y = x$ and $q(x, z) \leq q(x, y) + q(y, z)$, for all $x, y, z \in \Omega$. Then, we can suppose that when there is an advantage to change from x to y, i.e, $A(x, y) \geq 0$, costs to be able to change from x to y are higher than the reverse, i.e, $C(x, y) \geq C(y, x)$. This is a natural hypothesis which supposes that it is more costly to improve than the reverse. In this case $d(x, y) = \max\{q(x, y), q(y, x)\}$ is a distance, which satisfies the symmetric axiom all along a worthwhile transition, which is improving, because advantages to change $A(x^k, x^{k+1}) \geq 0$ are non negative on such a transition (see above the paragraph "The algorithm as an instance of a worthwhile transition");

ii) losses and advantages to change functions $F(x, y) = -A(x, y)$ are not reference dependent. In the general case they depend on experience (in the Markov case, on the last action $x^k = z$).

Finally, the choice of an Hadamard manifold is useful in Behavioral Sciences, because it helps to deal with inevitable constraints (to save space, see, for example, Cruz Neto et al. [27]).

6 Conclusions

In this paper, we provided a sufficient condition to obtain the existence of solutions of EP and we present a proximal algorithm for EP on an Hadamard manifold. We gave an application in the context of a recent unifying approach of a lot of stability and change theories in Behavioral Sciences, the "Variational rationality approach of human behavior", to the desire problem in Psychology, which refers to how to escape to a succession of temporary traps to finally succeed to reach his desires.

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