A Polyhedral Study of the Integrated Minimum-Up/-Down Time and Ramping Polytope*

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Abstract

In this paper, we consider the polyhedral structure of the integrated minimum-up/-down time and ramping polytope, which has broad applications in power generation scheduling problems. The generalized polytope we studied includes minimum-up/-down time, generation ramp-up/-down rate, logical, and generation upper/lower bound constraints. We derive strong valid inequalities for this polytope by utilizing its specialized structures. These inequalities, plus trivial inequalities described in the original formulation, are sufficient to provide the convex hull descriptions for variant two-period and three-period polytopes corresponding to different minimum-up/-down time limits. In addition, we derive more generalized strong valid inequalities (including one, two, and three continuous variable cases respectively) in polynomial size to strengthen the multi-period polytopes, and further prove that these inequalities are facet-defining under certain mild conditions. Finally, extensive computational experiments are conducted to verify the effectiveness of our proposed strong valid inequalities by testing the applications of these inequalities to solve both the network-constrained and self-scheduling unit commitment problems, for which our derived approach outperforms the default CPLEX significantly.

Key words: strong valid inequalities; polyhedral study; unit commitment; convex hull

1 Introduction

In this paper, we study the polyhedral structure of the integrated minimum-up/-down time and ramping polytope, which has broad applications in power generation scheduling problems, including the fundamental optimization problem in power system operations, the unit commitment (UC)

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problem. The UC problem decides the unit commitment status (online/offline) and power generation amount at each time period for each unit over a finite discrete time horizon so as to satisfy the load with a minimum total cost, with the associated physical restrictions, including generation upper/lower limits, ramp-rate limits, and minimum-up/-down time limits, to be satisfied. Due to its importance for power system operations, different approaches have been explored to solve the problem, such as dynamic programming [8, 10], Lagrangian relaxation [7, 12], decomposition methods [18], genetic algorithms [6], unit decommitment [21], and simulated annealing [11]. Detailed reviews of these approaches to solve the UC problem can be found in [19]. Among these approaches, the Lagrangian relaxation approach has been broadly adopted in industry, due to its advantages of targeting large-scale instances, because this approach can decompose the unit commitment problem into a group of subproblems with each subproblem solved by a dynamic programming algorithm efficiently.

However, the Lagrangian relaxation approach cannot guarantee to provide an optimal or even a feasible solution at the termination, in particular, when there are transmission network constraints currently faced by most wholesale electricity markets in US. Mixed-integer linear programming (MILP) approaches can guarantee to obtain an optimal solution [13]. Thus, recently, optimization algorithm developments for UCs are switched from Lagrangian relaxation to MILP approaches. For instance, MILP approaches have been adopted by all wholesale electricity markets in US [2] and creates more than 500 million annual savings [1]. Strong MILP formulations are crucial to speed up the MILP approach [14]. There has been research progress on developing strong formulations for the unit commitment problem by exploring its special structure. In [9], alternating up/down inequalities are proposed to strengthen the minimum-up/-down time polytope of the unit commitment problem. In [17], the convex hull of the minimum-up/-down time polytope considering start-up costs is provided, in which additional start-up and shut-down variables are introduced to provide the integral formulation. Recently, new families of strong valid inequalities are proposed in [15, 5] to tighten the ramping polytope of the unit commitment problem.

In the above studies, the minimum-up/-down time and ramping polytopes are studied separately. In this paper, we extend and generalize these studies to the integrated polytope that includes both the minimum-up/-down time and ramping polytopes. More specifically, this integrated polytope includes minimum-up/-down time constraints, logical constraints, power generation upper/lower bound constraints, and generation ramp-up/-down rate constraints. By studying this integrated polytope, we can deriving strong valid inequalities that are strong enough to provide convex hull descriptions for three-period cases and help speed up the branch-and-cut algorithm significantly to solve general multiple period cases. In addition, the proposed strong valid inequalities can be applied to solve other variants of UC problems, such as stochastic network-constrained UC [4, 20, 16] and stochastic self-scheduling UC problems [22].

In the remainder of this section, we introduce the integrated polytope and review the existing convex hull result for the two-period case. To describe the integrated polytope corresponding to a generator, we let T be the number of time periods for the whole operational horizon, L (ℓ) be the minimum-up (-down) time limit of the generator, $\overline{C}$ ($\underline{C}$) be its generation upper (lower) bound when it is online, $\overline{V}$ be its start-up/shut-down ramp rate, and V be its ramp-up/-down rate in the stable generation region. In addition, we let (x, y, u) be the decision variables to represent the generator's status, in which continuous variable x represents the generation amount, binary variable y represents the generator's online/offline status (i.e., $y_t = 1$ means the generator is online at t and $y_t = 0$ otherwise), and binary variable u represents whether the generator starts up or not (i.e., $u_t = 1$ means the generator starts up at t and $u_t = 0$ otherwise). The corresponding integrated minimum-up/-down time and ramping polytope can be described as follows:

$$
\begin{aligned}
P := \Big\{(x, y, u) \in \mathbb{R}_+^T \times \mathbb{B}^T \times \mathbb{B}^{T-1} : \quad & \sum_{i=t-L+1}^{t} u_i \leq y_t, \ \forall t \in [L+1, T]_{\mathbb{Z}}, && \text{(1a)} \\
& \sum_{i=t-\ell+1}^{t} u_i \leq 1 - y_{t-\ell}, \ \forall t \in [\ell+1, T]_{\mathbb{Z}}, && \text{(1b)} \\
& y_t - y_{t-1} - u_t \leq 0, \ \forall t \in [2, T]_{\mathbb{Z}}, && \text{(1c)} \\
& -x_t + \underline{C} y_t \leq 0, \ \forall t \in [1, T]_{\mathbb{Z}}, && \text{(1d)} \\
& x_t - \overline{C} y_t \leq 0, \ \forall i \in [1, T]_{\mathbb{Z}}, && \text{(1e)} \\
& x_t - x_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1}), \ \forall t \in [2, T]_{\mathbb{Z}}, && \text{(1f)} \\
& x_{t-1} - x_t \leq V y_t + \overline{V}(1 - y_t), \ \forall i \in [2, T]_{\mathbb{Z}} \Big\}, && \text{(1g)}
\end{aligned}
$$

where constraints (1a) and (1b) describe the minimum-up and minimum-down time limits [9, 17], respectively (i.e., if the generator starts up at time $t - L + 1$, it should keep online in the following

L consecutive time periods until time t; if the generator shuts down at time $t - \ell + 1$, it should keep offline in the following ℓ consecutive time periods until time t), constraints (1c) describe the logical relationship between y and u, constraints (1d) and (1e) describe the generation lower and upper bounds, and constraints (1f) and (1g) describe the generation ramp-up and ramp-down rate limits. Note here that, in our polytope description, there is no start-up decision corresponding to the first-time period. In this way, the derived inequalities can be applied to each time period and can be used recursively. Meanwhile, considering the physical characteristics of a thermal generator, without loss of generality, we can assume $\underline{C} < \overline{V} < \underline{C} + V$ and $\overline{C} - \underline{C} - V \geq 0$. In addition, we assume $\overline{C} - \overline{V} - V \geq 0$ so that the generator can ramp up at least once after its start-up, which is also reasonable for most thermal generators. For notation convenience, we define ϵ as an arbitrarily small positive real number and $[a, b]_{\mathbb{Z}}$ as the set of integer numbers between integers a and b, i.e., $\{a, a+1, \cdots, b\}$ with $[a, b]_{\mathbb{Z}} = \emptyset$ if $a > b$. Finally, we let conv(P) represent the convex hull description of P.

Before describing the details of our derived strong formulations in the following sections, we report the convex hull description of the two-period polytope as follows (note here that the convex hull descriptions for the separated studies on the ramp-up and ramp-down only polytopes are provided in [5]):

Theorem 1 *For $T = 2$ and $L = \ell = 1$, conv(P) can be described as follows:*

$$Q_2 := \Big\{(x, y, u) \in \mathbb{R}^5 : \quad u_2 \geq 0, \ u_2 \geq y_2 - y_1, \tag{2a}$$

$$u_2 \leq y_2, \ y_1 + u_2 \leq 1, \tag{2b}$$

$$x_1 \geq \underline{C} y_1, \ x_2 \geq \underline{C} y_2, \tag{2c}$$

$$x_1 \leq \overline{V} y_1 + (\overline{C} - \overline{V})(y_2 - u_2), \tag{2d}$$

$$x_2 \leq \overline{C} y_2 - (\overline{C} - \overline{V}) u_2, \tag{2e}$$

$$x_2 - x_1 \leq (\underline{C} + V) y_2 - \underline{C} y_1 - (\underline{C} + V - \overline{V}) u_2, \tag{2f}$$

$$x_1 - x_2 \leq \overline{V} y_1 - (\overline{V} - V) y_2 - (\underline{C} + V - \overline{V}) u_2 \Big\}. \tag{2g}$$

Proof: See Appendix A. ■

Remark 1 *Since the start-up decision is not considered in the first-time period in Q_2, the strong* *valid inequalities in Q_2 (e.g.,* (2d) - (2g)*) can be applied to any two consecutive time periods.*

In the remaining part of this paper, we derive strong valid inequalities and the further convex hull descriptions for the three-period polytopes with variant minimum-up/-down time limits in Section 2. In Section 3, we extend our study to derive strong valid inequalities covering multiple time periods so as to further strengthen the general multi-period polytopes. In Section 4, we perform computational studies on its applications in network-constrained and self-scheduling unit commitment problems to verify the effectiveness of our proposed strong valid inequalities. Finally, we conclude our study in Section 5.

2 Strengthening Three-period Formulations

In this section, we perform the polyhedral study for the three-period formulation, i.e., $T = 3$ in P, and propose convex hull descriptions for variant cases with different minimum-up/-down time limits. We first study the case in which $L = \ell = 2$ in the original polytope, which is the most complicated one among the cases in which $L = \ell = 1$, $L = 1$ and $\ell = 2$, $L = 2$ and $\ell = 1$, and $L = \ell = 2$. Since the derived strong valid inequalities are different for $\overline{C} - \underline{C} - 2V \geq 0$ and $\overline{C} - \underline{C} - 2V < 0$ cases, we first study the case in which $\overline{C} - \underline{C} - 2V \geq 0$. Under this setting, the corresponding formulation can be described as follows:

$$
\begin{aligned}
P_3^2 := \Big\{ \quad & (x, y, u) \in \mathbb{R}_+^3 \times \mathbb{B}^3 \times \mathbb{B}^2 : \\
& u_2 + u_3 \leq y_3, && \text{(3a)} \\
& y_1 + u_2 + u_3 \leq 1, && \text{(3b)} \\
& u_2 \geq y_2 - y_1, \ u_3 \geq y_3 - y_2, && \text{(3c)} \\
& x_1 \geq \underline{C} y_1, \ x_2 \geq \underline{C} y_2, \ x_3 \geq \underline{C} y_3, && \text{(3d)} \\
& x_1 \leq \overline{C} y_1, \ x_2 \leq \overline{C} y_2, \ x_3 \leq \overline{C} y_3, && \text{(3e)} \\
& x_2 - x_1 \leq V y_1 + \overline{V}(1 - y_1), \ x_3 - x_2 \leq V y_2 + \overline{V}(1 - y_2), && \text{(3f)} \\
& x_1 - x_2 \leq V y_2 + \overline{V}(1 - y_2), \ x_2 - x_3 \leq V y_3 + \overline{V}(1 - y_3) \Big\}. && \text{(3g)}
\end{aligned}
$$

For P_3^2, we first provide the strong valid inequalities in the following proposition. Then we provide a linear programming description Q_3^2 and further prove that Q_3^2 provides the convex hull description for P_3^2.

Proposition 1 *For P_3^2, the following inequalities*

$$x_1 \leq \overline{V} y_1 + V(y_2 - u_2) + (\overline{C} - \overline{V} - V)(y_3 - u_3 - u_2), \tag{4}$$

$$x_2 \leq \overline{V} y_2 + (\overline{C} - \overline{V})(y_3 - u_3 - u_2), \tag{5}$$

$$x_3 \leq \overline{C} y_3 - (\overline{C} - \overline{V})u_3 - (\overline{C} - \overline{V} - V)u_2, \tag{6}$$

$$x_2 - x_1 \leq \overline{V} y_2 - \underline{C} y_1 + (\underline{C} + V - \overline{V})(y_3 - u_3 - u_2), \tag{7}$$

$$x_3 - x_2 \leq (\underline{C} + V)y_3 - \underline{C} y_2 - (\underline{C} + V - \overline{V})u_3, \tag{8}$$

$$x_1 - x_2 \leq \overline{V} y_1 - (\overline{V} - V)y_2 - (\underline{C} + V - \overline{V})u_2, \tag{9}$$

$$x_2 - x_3 \leq \overline{V} y_2 - \underline{C} y_3 + (\underline{C} + V - \overline{V})(y_3 - u_3 - u_2), \tag{10}$$

$$x_3 - x_1 \leq (\underline{C} + 2V)y_3 - \underline{C} y_1 - (\underline{C} + 2V - \overline{V})u_3 - (\underline{C} + V - \overline{V})u_2, \tag{11}$$

$$x_1 - x_3 \leq \overline{V} y_1 - \underline{C} y_3 + V(y_2 - u_2) + (\underline{C} + V - \overline{V})(y_3 - u_3 - u_2), \tag{12}$$

$$x_1 - x_2 + x_3 \leq \overline{V} y_1 - (\overline{V} - V)y_2 + \overline{V} y_3 + (\overline{C} - \overline{V})(y_3 - u_3 - u_2), \tag{13}$$

are valid for conv(P_3^2).

Proof: To prove the validity of (4), we discuss the following two possible cases in terms of the value of y_1:

1) If $y_1 = 0$, then $x_1 = 0$ due to (3e). It follows that (4) is valid since $y_2 - u_2 \geq 0$ due to (3c) and (3a) and $y_3 - u_3 - u_2 \geq 0$ due to (3a).

2) If $y_1 = 1$, then $u_2 = u_3 = 0$ due to (3b). We consider the following three possible cases based on when the generator shuts down:

   (1) If the generator shuts down at the second time period, i.e., $y_2 = 0$, then we have $y_3 = 0$ since minimum-down time limit $\ell = 2$. Inequality (4) converts to $x_1 \leq \overline{V}$, which is valid due to ramp-down constraints (3g).

   (2) If the generator shuts down at the third time period, i.e., $y_3 = 0$ and $y_2 = 1$, then inequality (4) converts to $x_1 \leq \overline{V} + V$, which is valid due to ramp-down constraints (3g).

   (3) If the generator does not shut down, i.e., $y_2 = y_3 = 1$, then inequality (4) converts to $x_1 \leq \overline{C}$, which is valid due to (3e).

We can use the similar argument as above for (4) to prove that inequalities (5) and (6) are valid.

To prove the validity of (7), we discuss the following two possible cases in terms of the value of y_2:

1) If $y_2 = 0$, then $x_2 = 0$ due to (3e). It follows that inequality (7) is valid since $x_1 \geq \underline{C}y_1$ due to (3d), $y_3 - u_3 - u_2 \geq 0$ due to (3a), and $\underline{C} + V > \overline{V}$.

2) If $y_2 = 1$, then $u_3 = 0$ due to constraints (3b) and (3c) (i.e., $y_2 \leq y_1 + u_2 \leq 1 - u_3$). We further discuss the following two possible cases in terms of the value of u_2:

   (1) If $u_2 = 1$, then $y_1 = 0$ due to (3b) and $y_3 - u_3 - u_2 = 0$ due to (3a). It follows that inequality (7) converts to $x_2 \leq \overline{V}$, which is valid due to ramp-up constraints (3f).

   (2) If $u_2 = 0$, then $y_1 = 1$ due to (3c) (i.e., $y_1 \geq y_2 - u_2$). If $y_3 = 1$, then (7) converts to $x_2 - x_1 \leq V$, which is valid due to ramp-up constraints (3f); if $y_3 = 0$, then (7) converts to $x_2 - x_1 \leq \overline{V} - \underline{C}$, which is valid since $x_2 \leq \overline{V}$ due to (3f) and $x_1 \geq \underline{C}$ due to (3d).

We can use the similar argument for (7) to prove that inequality (8) is valid.

To prove the validity of (9), we discuss the following four possible cases in terms of the values of y_1 and y_2:

1) If $y_1 = y_2 = 1$, then $u_2 = 0$ due to (3b). Inequality (9) converts to $x_1 - x_2 \leq V$, which is valid following ramp-down constraints (3g).

2) If $y_1 = 1$ and $y_2 = 0$, then $u_2 = 0$ due to (3b). Inequality (9) converts to $x_1 \leq \overline{V}$, which is valid following ramp-down constraints (3g).

3) If $y_1 = 0$ and $y_2 = 1$, then $u_2 = 1$ due to (3c). Inequality (9) converts to $x_2 \geq \underline{C}$, which is valid following (3d).

4) If $y_1 = y_2 = 0$, (9) is clearly valid.

We can use the similar argument for (9) to prove that inequality (10) is valid.

To prove the validity of (11), we discuss the following four possible cases in terms of the values of y_1 and y_3:

1) If $y_1 = y_3 = 1$, then $u_2 = u_3 = 0$ due to (3b). Inequality (11) converts to $x_3 - x_1 \le 2V$, which is valid following ramp-up constraints (3f).

2) If $y_1 = 1$ and $y_3 = 0$, then $u_2 = u_3 = 0$ due to (3b). Inequality (11) converts to $x_1 \le \underline{C}$, which is valid following (3d).

3) If $y_1 = 0$ and $y_3 = 1$, then $u_2 + u_3 = 1$ due to (3a) - (3c). If $u_2 = 1$, i.e., $u_3 = 0$, then (11) converts to $x_3 \le \overline{V} + V$, which is valid following ramp-up constraints (3f); if $u_3 = 1$, i.e., $u_2 = 0$, then (11) converts to $x_3 \le \overline{V}$, which is valid following ramp-up constraints (3f).

4) If $y_1 = y_3 = 0$, (11) is clearly valid.

We can use the similar argument for (11) to prove that inequality (12) is valid.

To prove the validity of (13), we discuss the following two possible cases in terms of the value of y_3:

1) If $y_3 = 0$, then $u_2 = u_3 = 0$ due to (3a). It follows that inequality (13) converts to $x_1 - x_2 \le \overline{V} y_1 - (\overline{V} - V) y_2$, which can be proved to be valid following inequality (9).

2) If $y_3 = 1$, then $u_2 + u_3 \le 1$ due to (3a). We further discuss the following three possible cases based on when the generator starts up:

   (1) If $u_2 = 0$ and $u_3 = 1$, then $y_1 = y_2 = 0$ due to (3b) and (3c). It follows that (13) converts to $x_3 \le \overline{V}$, which is valid due to ramp-up constraints (3f).

   (2) If $u_2 = 1$ and $u_3 = 0$, then $y_1 = 0$ due to (3b). It follows that (13) converts to $x_3 - x_2 \le V$, which is valid due to ramp-up constraints (3f).

   (3) If $u_2 = u_3 = 0$, then $y_1 = y_2 = 1$ due to (3c). It follows that (13) converts to $x_1 - x_2 + x_3 \le V + \overline{C}$, which is valid since $x_1 - x_2 \le V$ due to ramp-down constraints (3g) and $x_3 \le \overline{C}$ due to (3e).

In sum, this completes the proof. $\blacksquare$

Now, through utilizing inequalities (4) - (13), we introduce the linear programming description of $\text{conv}(P_3^2)$ by adding trivial inequalities as follows:

$$Q_3^2 := \Big\{ \quad (x, y, u) \in \mathbb{R}^8 : (3\text{a}) - (3\text{d}), (4) - (13),$$

$$u_2 \geq 0,\ u_3 \geq 0\Big\}. \tag{14}$$

Note here that the nonnegativity of x in Q_3^2 is guaranteed by (3a), (3c) - (3d), and (14). In the following, we show that Q_3^2 describes the convex hull of P_3^2, i.e., $Q_3^2 = \text{conv}(P_3^2)$. We first provide the following preliminary results.

Proposition 2 *Q_3^2 is full-dimensional.*

Proof: We prove that $\dim(Q_3^2) = 8$, because there are eight decision variables in Q_3^2. Thus, we need to generate nine affinely independent points in Q_3^2. Since $0 \in Q_3^2$, it is sufficient to generate other eight linearly independent points in Q_3^2 as shown in Table 1. ■

Table 1: Eight linearly independent points in Q_3^2

x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\underline{C}$	0	0	1	0	0	0	0
$\underline{C}+\epsilon$	0	0	1	0	0	0	0
$\underline{C}$	$\underline{C}$	0	1	1	0	0	0
$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	0	1	1	0	0	0
$\underline{C}$	$\underline{C}$	$\underline{C}$	1	1	1	0	0
$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	1	1	1	0	0
0	$\underline{C}$	$\underline{C}$	0	1	1	1	0
0	0	$\underline{C}$	0	0	1	0	1

Proposition 3 *Every inequality in Q_3^2 is facet-defining for conv(P_3^2).*

Proof: The facet-defining proofs for inequalities (3a) - (3d) and (14) are trivial and thus omitted here. For inequalities (4) - (13), we provide eight affinely independent points in $\text{conv}(P_3^2)$ that satisfy each inequality at equality. Since $0 \in \text{conv}(P_3^2)$, it is sufficient to generate other seven linearly independent points in P_3^2, as shown in Table 2 and Tables 11 - 14. Note here that for inequalities (11) and (12), we create points satisfying $\overline{C} - \underline{C} - 2V > 0$, because (11) and (12) are not facet-defining and are dominated by other inequalities if $\overline{C} - \underline{C} - 2V = 0$.

See Appendix B.1 for the details of Tables 11 - 14. ■

Proposition 4 *Every extreme point in Q_3^2 is integral at y and u.*

Proof: It is sufficient to prove that every point $z \in Q_3^2$ can be written as $z = \sum_{s\in S} \lambda_s z^s$ for some

Table 2: Linearly independent points for inequalities (4) and (5)

(4)								(5)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\overline{V}$	0	0	1	0	0	0	0	$\underline{C}$	0	0	1	0	0	0	0
$\overline{V}+V$	$\overline{V}$	0	1	1	0	0	0	$\underline{C}+\epsilon$	0	0	1	0	0	0	0
$\overline{C}$	$\overline{C}$	$\overline{C}$	1	1	1	0	0	$\overline{V}$	$\overline{V}$	0	1	1	0	0	0
0	$\underline{C}$	$\underline{C}$	0	1	1	1	0	$\overline{C}$	$\overline{C}$	$\overline{C}$	1	1	1	0	0
0	$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	0	1	1	1	0	0	$\overline{V}$	$\overline{V}$	0	1	1	1	0
0	0	$\underline{C}$	0	0	1	0	1	0	0	$\underline{C}$	0	0	1	0	1
0	0	$\underline{C}+\epsilon$	0	0	1	0	1	0	0	$\underline{C}+\epsilon$	0	0	1	0	1

$\lambda_s \geq 0$ and $\sum_{s\in S}\lambda_s = 1$, where $z^s \in Q_3^2, s \in S$ with y and u binary and S is the index set for the candidate points.

For a given point $z = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3) \in Q_3^2$, we let the candidate points $z^1, z^2, \cdots, z^6 \in Q_3^2$ in the forms such that $z^1 = (\hat{x}_1, 0, 0, 1, 0, 0, 0, 0)$, $z^2 = (\hat{x}_2, \hat{x}_3, 0, 1, 1, 0, 0, 0)$, $z^3 = (\hat{x}_4, \hat{x}_5, \hat{x}_6, 1, 1, 1, 0, 0)$, $z^4 = (0, \hat{x}_7, \hat{x}_8, 0, 1, 1, 1, 0)$, $z^5 = (0, 0, \hat{x}_9, 0, 0, 1, 0, 1)$, and $z^6 = (0, 0, 0, 0, 0, 0, 0, 0)$, where $\hat{x}_i, i = 1, \cdots, 9$ are to be decided later. Meanwhile, we let

$$\lambda_1 = \bar{y}_1 - \bar{y}_2 + \bar{u}_2,\ \lambda_2 = \bar{y}_2 - \bar{y}_3 + \bar{u}_3,\ \lambda_3 = \bar{y}_3 - \bar{u}_2 - \bar{u}_3, \tag{15a}$$

$$\lambda_4 = \bar{u}_2,\ \lambda_5 = \bar{u}_3,\ \text{and}\ \lambda_6 = 1 - \bar{y}_1 - \bar{u}_2 - \bar{u}_3. \tag{15b}$$

First of all, based on this construction, we can check that $\sum_{s=1}^{6}\lambda_s = 1$ and $\lambda_s \geq 0$ for $\forall s = 1, \cdots, 6$ due to (3a) - (3c) and (14). Meanwhile, it can be checked that $\bar{y}_i = y_i(z) = \sum_{s=1}^{6}\lambda_s y_i(z^s)$ for $i = 1, 2, 3$ and $\bar{u}_i = u_i(z) = \sum_{s=1}^{6}\lambda_s u_i(z^s)$ for $i = 2, 3$, where $y_i(z)$ represents the $\bar{y}_i$ component value in the given point z and $u_i(z)$ represents the $\bar{u}_i$ component value in the given point z.

Thus, in the remaining part of this proof, we only need to decide the values of $\hat{x}_i$ for $i = 1, \cdots, 9$ such that $\bar{x}_i = x_i(z) = \sum_{s=1}^{6}\lambda_s x_i(z^s)$ for $i = 1, 2, 3$, i.e.,

$$\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4,\ \bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7,\ \bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9. \tag{16}$$

To show (16), in the following, we prove that for any $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in its feasible region corresponding to a given $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3)$, we can always find a $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_9)$ in its feasible region, corresponding to the same given $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3)$. Now we describe the feasible regions for $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_9)$ and $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, respectively. First, since y and u in $z^1, \cdots, z^6$ are given, by substituting $z^1, \cdots, z^6$ into Q_3^2, the corresponding feasible region for $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_9)$ can be described as set $A = \{(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_9) \in \mathbb{R}^9 : \underline{C} \leq \hat{x}_1 \leq \overline{V},\ \underline{C} \leq \hat{x}_2 \leq \overline{V} + V,\ \underline{C} \leq \hat{x}_3 \leq \overline{V},\ -V \leq \hat{x}_3 - \hat{x}_2 \leq \overline{V} - \underline{C},\ \underline{C} \leq \hat{x}_4 \leq \overline{C},\ \underline{C} \leq \hat{x}_5 \leq \overline{C},\ \underline{C} \leq \hat{x}_6 \leq \overline{C},\ -V \leq \hat{x}_5 - \hat{x}_4 \leq V,\ -V \leq \hat{x}_6 - \hat{x}_5 \leq V,\ \underline{C} \leq \hat{x}_7 \leq \overline{V},\ \underline{C} \leq \hat{x}_8 \leq \overline{V} + V,\ \underline{C} - \overline{V} \leq \hat{x}_8 - \hat{x}_7 \leq V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. Second, corresponding to a given $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3)$, following the description of Q_3^2, the feasible region for $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ can be described as follows:

$$
\begin{aligned}
C = \Big\{(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 :\ & \bar{x}_1 \geq \underline{C}\bar{y}_1,\ \bar{x}_2 \geq \underline{C}\bar{y}_2,\ \bar{x}_3 \geq \underline{C}\bar{y}_3, && \text{(17a)}\\
& \bar{x}_1 \leq \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), && \text{(17b)}\\
& \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), && \text{(17c)}\\
& \bar{x}_3 \leq \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2, && \text{(17d)}\\
& \bar{x}_2 - \bar{x}_1 \leq \overline{V}\bar{y}_2 - \underline{C}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), && \text{(17e)}\\
& \bar{x}_3 - \bar{x}_2 \leq (\underline{C} + V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_3, && \text{(17f)}\\
& \bar{x}_1 - \bar{x}_2 \leq \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2, && \text{(17g)}\\
& \bar{x}_2 - \bar{x}_3 \leq \overline{V}\bar{y}_2 - \underline{C}\bar{y}_3 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), && \text{(17h)}\\
& \bar{x}_3 - \bar{x}_1 \leq (\underline{C} + 2V)\bar{y}_3 - \underline{C}\bar{y}_1 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2, && \text{(17i)}\\
& \bar{x}_1 - \bar{x}_3 \leq \overline{V}\bar{y}_1 - \underline{C}\bar{y}_3 + V(\bar{y}_2 - \bar{u}_2) + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), && \text{(17j)}\\
& \bar{x}_1 - \bar{x}_2 + \bar{x}_3 \leq \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 + \overline{V}\bar{y}_3 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)\Big\}. && \text{(17k)}
\end{aligned}
$$

Accordingly, we can set up the linear transformation F from $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_9) \in A$ to $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in C$ as follows:

$$
F = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & \lambda_3 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & \lambda_4 & \lambda_5 \end{pmatrix},
$$

where $\lambda_1, \lambda_2, \cdots, \lambda_5$ follow the definitions described in (15). Thus, in the following, we only need to prove that $F : A \to C$ is surjective.

Since C is a closed and bounded polytope, any point can be expressed as a convex combination of the extreme points in C. Accordingly, we only need to show that for any extreme point $w^i \in C$ $(i = 1, \cdots, M)$, there exists a point $p^i \in A$ such that $Fp^i = w^i$, where M represents the number of extreme points in C (because for an arbitrary point $w \in C$, which can be represented as $w = \sum_{i=1}^M \mu_i w^i$ and $\sum_{i=1}^M \mu_i = 1$, there exists $p = \sum_{i=1}^M \mu_i p_i \in A$ such that $Fp = w$ due to the linearity of F and the convexity of A). Since it is difficult to enumerate all the extreme points in C, in the following proof we show the conclusion holds for any point in the faces of C, i.e., satisfying one of (17a) - (17k) at equality, which implies the conclusion holds for extreme points.

Satisfying $\bar{x}_1 \geq \underline{C}\bar{y}_1$ at equality. For this case, substituting $\bar{x}_1 = \underline{C}\bar{y}_1$ into (17b) - (17k), we obtain the feasible region of $(\bar{x}_2, \bar{x}_3)$ as $C' = \{(\bar{x}_2, \bar{x}_3) \in \mathbb{R}^2 : \underline{C}\bar{y}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \underline{C}\bar{y}_3 \leq \bar{x}_3 \leq (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2,\ \bar{x}_3 - \bar{x}_2 \leq (\underline{C} + V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_3\}$.

First, by letting $\hat{x}_1 = \hat{x}_2 = \hat{x}_4 = \underline{C}$, it is easy to check that $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. Then (16) holds for $\bar{x}_1$. Note here that once $(\hat{x}_1, \hat{x}_2, \hat{x}_4)$ fixed, the corresponding feasible region for $(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)$ can be described as set $A' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^6 : \underline{C} \leq \hat{x}_3 \leq \overline{V},\ \underline{C} \leq \hat{x}_5 \leq \underline{C} + V,\ \underline{C} \leq \hat{x}_6 \leq \overline{C},\ -V \leq \hat{x}_6 - \hat{x}_5 \leq V,\ \underline{C} \leq \hat{x}_7 \leq \overline{V},\ \underline{C} \leq \hat{x}_8 \leq \overline{V} + V,\ \underline{C} - \overline{V} \leq \hat{x}_8 - \hat{x}_7 \leq V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. In the following, we repeat the argument above to consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)$ from A'.

1) Satisfying $\bar{x}_2 \geq \underline{C}\bar{y}_2$ at equality. We obtain $\bar{x}_3 \in C'' = \{\bar{x}_3 \in \mathbb{R} : \underline{C}\bar{y}_3 \leq \bar{x}_3 \leq (\underline{C} + V)\bar{y}_3 - (\underline{C} + V - \overline{V})\bar{u}_3\}$ through substituting $\bar{x}_2 = \underline{C}\bar{y}_2$ into C'. By letting $\hat{x}_3 = \hat{x}_5 = \hat{x}_7 = \underline{C}$, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. Then (16) holds for $\bar{x}_2$. Thus, the corresponding feasible region for $(\hat{x}_6, \hat{x}_8, \hat{x}_9)$ can be described as set $A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \underline{C} \leq \hat{x}_6 \leq \underline{C} + V,\ \underline{C} \leq \hat{x}_8 \leq \underline{C} + V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. If $\bar{x}_3 \geq \underline{C}\bar{y}_3$ is satisfied at equality, we let $\hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 \leq (\underline{C} + V)\bar{y}_3 - (\underline{C} + V - \overline{V})\bar{u}_3$ is satisfied at equality, we let $\hat{x}_6 = \hat{x}_8 = \underline{C} + V$ and $\hat{x}_9 = \overline{V}$. It is easy to check that $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. Then (16) holds for $\bar{x}_3$.

2) Satisfying $\bar{x}_2 \leq \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\bar{x}_3 \in C'' = \{\bar{x}_3 \in \mathbb{R} : \underline{C}\bar{y}_3 \leq \bar{x}_3 \leq (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2\}$. By letting $\hat{x}_3 = \hat{x}_7 = \overline{V}$ and $\hat{x}_5 = \underline{C} + V$, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. Then (16) holds for $\bar{x}_2$. Thus, the corresponding feasible region for $(\hat{x}_6, \hat{x}_8, \hat{x}_9)$ can be described as set $A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \underline{C} \leq \hat{x}_6 \leq \underline{C} + 2V,\ \underline{C} \leq \hat{x}_8 \leq \overline{V} + V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. If $\bar{x}_3 \geq \underline{C}\bar{y}_3$ is satisfied at equality, we let $\hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 \leq (\underline{C}+2V)\bar{y}_3 - (\underline{C}+2V - \overline{V})\bar{u}_3 - (\underline{C}+V - \overline{V})\bar{u}_2$ is satisfied at equality, we let $\hat{x}_6 = \underline{C} + 2V$, $\hat{x}_8 = \overline{V} + V$ and $\hat{x}_9 = \overline{V}$. In this way, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. Then (16) holds for $\bar{x}_3$.

3) Satisfying $\bar{x}_3 \geq \underline{C}\bar{y}_3$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)\}$. By letting $\hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. Then

(16) holds for $\bar{x}_3$. Thus, the corresponding feasible region for $(\hat{x}_3, \hat{x}_5, \hat{x}_7)$ can be described as set $A'' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_7) \in \mathbb{R}^3 : \underline{C} \leq \hat{x}_3 \leq \overline{V},\ \underline{C} \leq \hat{x}_5 \leq \underline{C} + V,\ \underline{C} \leq \hat{x}_7 \leq \overline{V}\}$. If $\bar{x}_2 \geq \underline{C}\bar{y}_2$ is satisfied at equality, we let $\hat{x}_3 = \hat{x}_5 = \hat{x}_7 = \underline{C}$; if $\bar{x}_2 \leq \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ is satisfied at equality, we let $\hat{x}_3 = \hat{x}_7 = \overline{V}$ and $\hat{x}_5 = \underline{C} + V$. In this way, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. Then (16) holds for $\bar{x}_2$.

4) Satisfying $\bar{x}_3 \leq (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) + (\overline{V} - \underline{C})\bar{u}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)\}$. By letting $\hat{x}_6 = \underline{C} + 2V$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. Then (16) holds for $\bar{x}_3$. Thus, the corresponding feasible region for $(\hat{x}_3, \hat{x}_5, \hat{x}_7)$ can be described as set $A'' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_7) \in \mathbb{R}^3 : \underline{C} \leq \hat{x}_3 \leq \overline{V}, \hat{x}_5 = \underline{C} + V, \hat{x}_7 = \overline{V}\}$. If $\bar{x}_2 \geq \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) + (\overline{V} - \underline{C})\bar{u}_2$ is satisfied at equality, we let $\bar{x}_3 = \underline{C}$; if $\bar{x}_2 \leq \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ is satisfied at equality, we let $\bar{x}_3 = \overline{V}$. In this way, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. Then (16) holds for $\bar{x}_2$.

5) Satisfying $\bar{x}_3 - \bar{x}_2 \leq (\underline{C} + V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_3$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 \leq \bar{x}_2 \leq \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2\}$ through substituting $\bar{x}_3 = \bar{x}_2 + (\underline{C} + V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_3$ into set C'. By letting $\hat{x}_3 = \underline{C}$, $\hat{x}_9 = \overline{V}$, and $\hat{x}_6 - \hat{x}_5 = \hat{x}_8 - \hat{x}_7 = V$, we have $\bar{x}_3 - \bar{x}_2 = (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9) - (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7)$. If $\bar{x}_2 \geq \underline{C}\bar{y}_2$ is satisfied at equality, we let $\hat{x}_5 = \hat{x}_7 = \underline{C}$; if $\bar{x}_2 \leq \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2$ is satisfied at equality, we let $\hat{x}_5 = \underline{C} + V$, and $\hat{x}_7 = \overline{V}$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$ and thus $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. Then (16) holds for both $\bar{x}_2$ and $\bar{x}_3$.

Similar analyses hold for $\bar{x}_2 \geq \underline{C}\bar{y}_2$ and $\bar{x}_3 \geq \underline{C}\bar{y}_3$ due to the similar structure between $\bar{x}_1 \geq \underline{C}\bar{y}_1$, $\bar{x}_2 \geq \underline{C}\bar{y}_2$, and $\bar{x}_3 \geq \underline{C}\bar{y}_3$ and thus are omitted here. Furthermore, similar analyses also hold for inequalities (17b)-(17k), with the details included in Appendix B.2. ■

Theorem 2 *$Q_3^2 =$ conv(P_3^2).*

Proof: First, we have both P_3^2 and Q_3^2 bounded from their formulation representations. Since all the inequalities in Q_3^2 are valid and facet-defining for conv(P_3^2) based on Propositions 1 and 3, we have $Q_3^2 \supseteq$ conv(P_3^2). Meanwhile, we have that any extreme point in Q_3^2 is integral in y and u based on Proposition 4. Thus $Q_3^2 =$ conv(P_3^2). ■

For the case $L = \ell = 1$ and $\overline{C} - \underline{C} - 2V \geq 0$, we can obtain the convex hull representation of the original polytope (e.g., defined as P_3^1) described as follows:

Theorem 3 *For the case $L = \ell = 1$ and $\overline{C} - \underline{C} - 2V \geq 0$, the convex hull representation for the three-period problem is*

$$Q_3^1 = conv(P_3^1) = \Big\{ \quad (x, y, u) \in \mathbb{R}^8 : \text{(3c)} - \text{(3d)}, (14),$$

$$u_2 \leq y_2, \ u_3 \leq y_3, \tag{18a}$$

$$y_1 + u_2 \leq 1, \ y_2 + u_3 \leq 1, \tag{18b}$$

$$x_1 \leq \overline{V} y_1 + (\overline{C} - \overline{V})(y_2 - u_2), \tag{18c}$$

$$x_1 \leq \overline{V} y_1 + V(y_2 - u_2) + (\overline{C} - \overline{V} - V)(y_3 - u_3), \tag{18d}$$

$$x_2 \leq \overline{C} y_2 - (\overline{C} - \overline{V}) u_2, \tag{18e}$$

$$x_2 \leq \overline{V} y_2 + (\overline{C} - \overline{V})(y_3 - u_3), \tag{18f}$$

$$x_3 \leq \overline{C} y_3 - (\overline{C} - \overline{V}) u_3, \tag{18g}$$

$$x_3 \leq (\overline{V} + V) y_3 - V u_3 + (\overline{C} - \overline{V} - V)(y_2 - u_2), \tag{18h}$$

$$x_2 - x_1 \leq \overline{V} y_2 - \underline{C} y_1 + (\underline{C} + V - \overline{V})(y_3 - u_3), \tag{18i}$$

$$x_2 - x_1 \leq (\underline{C} + V) y_2 - \underline{C} y_1 - (\underline{C} + V - \overline{V}) u_2, \tag{18j}$$

$$x_3 - x_2 \leq (\underline{C} + V) y_3 - \underline{C} y_2 - (\underline{C} + V - \overline{V}) u_3, \tag{18k}$$

$$x_1 - x_2 \leq \overline{V} y_1 - (\overline{V} - V) y_2 - (\underline{C} + V - \overline{V}) u_2, \tag{18l}$$

$$x_2 - x_3 \leq \overline{V} y_2 - (\overline{V} - V) y_3 - (\underline{C} + V - \overline{V}) u_3, \tag{18m}$$

$$x_2 - x_3 \leq (\underline{C} + V) y_2 - \underline{C} y_3 - (\underline{C} + V - \overline{V}) u_2, \tag{18n}$$

$$x_3 - x_1 \leq (\underline{C} + 2V) y_3 - \underline{C} y_1 - (\underline{C} + 2V - \overline{V}) u_3, \tag{18o}$$

$$x_3 - x_1 \leq (\overline{V} + V) y_3 - V u_3 - \underline{C} y_1 + (\underline{C} + V - \overline{V})(y_2 - u_2), \tag{18p}$$

$$x_1 - x_3 \leq \overline{V} y_1 - \underline{C} y_3 + (\underline{C} + 2V - \overline{V})(y_2 - u_2), \tag{18q}$$

$$x_1 - x_3 \leq \overline{V} y_1 - \underline{C} y_3 + V(y_2 - u_2) + (\underline{C} + V - \overline{V})(y_3 - u_3) \Big\}. \tag{18r}$$

Proof: The proofs are similar with those for Theorem 2 and thus omitted here. ∎

Following the similar approach as described above, we can obtain the convex hull representations for other cases in terms of different minimum-up/-down times. The proofs are similar with those for Theorem 2, and thus we present the convex hull descriptions below without proofs.

Theorem 4 *For the case $L = 1$ and $\ell = 2$ and $\overline{C} - \underline{C} - 2V \geq 0$, the convex hull representation of the original polytope is the same as Q_3^1 except that* (18b) *is replaced by* (3b). *For the case $L = 2$ and $\ell = 1$ and $\overline{C} - \underline{C} - 2V \geq 0$, the convex hull representation of the original polytope is the same as Q_3^2 except that* (3b) *is replaced by* (18b).

Theorem 5 *For the case $L = \ell = 1$ and $\overline{C} - \underline{C} - 2V < 0$, the convex hull representation of the original polytope (e.g., denoted as $\hat{P}_3^1$) can be described as $\hat{Q}_3^1 = conv(\hat{P}_3^1) = \{(x, y, u) \in \mathbb{R}^8 :$ $(3c) - (3d), (14), (18a) - (18n)\}$. Similarly, for the case $L = 1$ and $\ell = 2$ and $\overline{C} - \underline{C} - 2V < 0$, the convex hull representation of the original polytope can be described the same as $\hat{Q}_3^1$ except that* (18b) *is replaced by* (3b).

Theorem 6 *For the case $L = 2$ and $\ell = 1$ and $\overline{C} - \underline{C} - 2V < 0$, the convex hull representation of the original polytope (e.g., denoted as $\hat{P}_3^{2,1}$) can be described as $\hat{Q}_3^{2,1} = conv(\hat{P}_3^{2,1}) = \{(x, y, u) \in \mathbb{R}^8 :$ $(3a), (3c) - (3d), (4) - (10), (13) - (14), (18b)\}$. Similarly, for the case $L = \ell = 2$ and $\overline{C} - \underline{C} - 2V < 0$, the convex hull representation of the original polytope can be described the same as $\hat{Q}_3^{2,1}$ except that* (18b) *is replaced by* (3b).

Note here that in our problem setting, we consider the most common reasonable cases for thermal generators in which $\underline{C} < \overline{V} < \underline{C} + V$, $\overline{C} - \underline{C} - V \geq 0$, and $\overline{C} - \overline{V} - V \geq 0$. We can also extend the study into the cases in which these assumptions are relaxed for the completeness of mathematical analysis and provide the convex hull descriptions for these cases in the rest of this section. For instance, we can consider the cases in which $\overline{V} \geq \underline{C} + V$, or $\overline{C} - \underline{C} - V < 0$, or $\overline{C} - \overline{V} - V < 0$, or the combinations of these. The derived convex hull results are very similar and we list them below for the readers' reference. Proofs for these results are omitted for description brevity, except that for Theorem 7 because it is the most complicated case among the rest convex hull results. First, we consider the case in which

$$\overline{V} \geq \underline{C} + V, \ \overline{C} - \underline{C} - V \geq 0, \text{ and } \overline{C} - \overline{V} - V \geq 0. \tag{19}$$

Note here that we have $\overline{C} - \underline{C} - 2V = \overline{C} - V - (\underline{C} + V) \geq \overline{V} - (\underline{C} + V) \geq 0$. Therefore, we do not need to consider the case in which $\overline{C} - \underline{C} - 2V < 0$ as described above. Since the two-period convex hull result is the same as that described in Theorem 1, we report the convex hull results for the three-period cases to show the slight difference.

Theorem 7 *For the case in which $L = \ell = 1$ and Condition* (19) *is satisfied, the convex hull representation of the original polytope (e.g., denoted as $\bar{P}_3^1$) can be described as*

$$\bar{Q}_3^1 = conv(\bar{P}_3^1) = \Big\{ \quad (x, y, u) \in \mathbb{R}^8 : (3c) - (3d), (14), (18a) - (18h), (18j) - (18m), (12),$$

$$x_3 - x_1 \leq (\underline{C} + 2V)y_3 - \underline{C}y_1 - (\underline{C} + 2V - \overline{V})u_3 - (\underline{C} + V - \overline{V})u_2, \quad (20a)$$

$$(\overline{C} - \underline{C} - 2V)(x_1 - \overline{V}y_1 - V(y_2 - u_2)) \leq (\overline{C} - \overline{V} - V)(x_3 - \underline{C}y_3), \quad (20b)$$

$$(\overline{C} - \underline{C} - 2V)(x_3 - (\overline{V} + V)y_3 + Vu_3) \leq (\overline{C} - \overline{V} - V)(x_1 - \underline{C}y_1)\Big\}. \quad (20c)$$

Similarly, for the case in which $L = 1, \ell = 2$ and Condition (19) *is satisfied, the convex hull representation of the original polytope is the same as $\bar{Q}_3^1$ except that* (18b) *is replaced by* (3b)*.*

Proof: Validity and facet-defining proofs are similar to those described in Propositions 1 and 3, which are omitted here. In this part, we only prove that all the extreme points of $\bar{Q}_3^1$ are integral in y and u. The details are shown in Appendix B.3. ■

Theorem 8 *For the case in which $L = \ell = 2$ and Condition* (19) *is satisfied, the convex hull representation of the original polytope (e.g., denoted as $\bar{P}_3^2$) can be described as $\bar{Q}_3^2$ =conv($\bar{P}_3^2$), which is the same as Q_3^2 except that* (7) *and* (10) *are replaced by* (18j) *and* (18m)*, plus the following one:*

$$x_1 - x_2 + x_3 \geq \underline{C}y_1 - (\underline{C} + V)y_2 + \underline{C}y_3. \quad (21)$$

Similarly, for the case in which $L = 2, \ell = 1$ and Condition (19) *is satisfied, the convex hull representation of the original polytope is the same as $\bar{Q}_3^2$ except that* (3b) *is replaced by* (18b)*.*

Finally, we consider the cases in which $\overline{C} - \underline{C} - V < 0$ and/or $\overline{C} - \overline{V} - V < 0$. Note here that $\overline{C} - \underline{C} - V < 0$ implies $\overline{C} - \overline{V} - V < 0$. We can summarize the remaining combinations in the following two theorems (note here that convex hulls corresponding to different values of ℓ with the rest being the same follow the same argument above by replacing (18b) with (3b) or replacing (3b) with (18b)). In general, these cases require less number of inequalities to describe the convex hulls.

Theorem 9 *For the cases in which $\overline{V} < \underline{C} + V$, $\overline{C} - \underline{C} - V < 0$, and $\overline{C} - \overline{V} - V < 0$, we have*

1. $L = \ell = 2$. *The convex hull description is* $\{(x, y, u) \in \mathbb{R}^8 : (3a) - (3d),\ (2d),\ (5),$ *and* $(18g)\}$,

2. $L = \ell = 1$. *The convex hull description is* $\{(x, y, u) \in \mathbb{R}^8 : (3c) - (3d),\ (18a) - (18c),\ (18e) - (18g)\}$.

For the cases in which $\overline{V} < \underline{C} + V,\ \overline{C} - \underline{C} - V > 0,$ *and* $\overline{C} - \overline{V} - V < 0$, *we have*

1. $L = \ell = 2$. *The convex hull description is* $\{(x, y, u) \in \mathbb{R}^8 : (3a) - (3d),\ (2d),\ (5),\ (18g),\ (7) - (10),$ *and* $(13)\}$,

2. $L = \ell = 1$. *The convex hull description is* $\{(x, y, u) \in \mathbb{R}^8 : (3c) - (3d),\ (18a) - (18c),\ (18e) - (18g),$ *and* $(18i) - (18n)\}$.

Theorem 10 *For the cases in which* $\overline{V} > \underline{C} + V,\ \overline{C} - \underline{C} - 2V \geq 0,$ *and* $\overline{C} - \overline{V} - V < 0$, *we have*

1. $L = \ell = 2$. *The convex hull description is*

$$\Big\{ (x, y, u) \in \mathbb{R}^8 : (3a) - (3d), (18c), (5), (18g), (18j) - (18m), (13), (21),$$

$$x_3 - x_1 \leq (\underline{C} + 2V)y_3 - \underline{C}y_1 - (\underline{C} + 2V - \overline{V})u_3 + (\overline{C} - \underline{C} - 2V)u_2, \tag{22a}$$

$$x_1 - x_3 \leq \overline{V}y_1 + (\overline{C} - \overline{V})y_2 - (\overline{C} - 2V)y_3 - (\underline{C} + 2V - \overline{V})u_2 + (\overline{C} - \underline{C} - 2V)u_3\Big\}, \tag{22b}$$

2. $L = \ell = 1$. *The convex hull description is* $\{(x, y, u) \in \mathbb{R}^8 : (3c) - (3d),\ (18a) - (18c),\ (18e) - (18g), (18j) - (18m), (22a),$ *and* $(22b)\}$.

For the cases in which $\overline{V} < \underline{C} + V,\ 2V \geq \overline{C} - \underline{C} > V,$ *and* $\overline{C} - \overline{V} - V < 0$, *we have the convex hull description for the* $L = \ell = 2$ *and* $L = \ell = 1$ *cases by removing* (22a) *and* (22b) *from the above expressions.*

Remark 2 *Since the start-up decision is not considered in the first-time period, the strong valid inequalities in this section can be applied to any three consecutive time periods.*

3 Strengthening Multi-period Formulations

First of all, the inequalities we derived in the previous sections can be applied to solve the general multi-period problems, because the start-up decision is not considered for the first-time period. These inequalities are polynomial in the order of $\mathcal{O}(T)$. In this section, we further strengthen the formulation for the general polytope P by exploring the inequalities covering multiple periods, mostly under the common reasonable setting. Strong valid inequalities containing one, two, and three continuous variables are derived in Sections 3.1, 3.2, and 3.3 respectively, through considering the effects of minimum-up/-down time, ramp rate, start-up decision, and capacity constraints. For notation brevity, we let $\sum_{t=a}^{b} x_t = \sum_{t=a}^{b} y_t = \sum_{t=a}^{b} u_t = 0$ if $b < a$. For each strong valid inequality, we provide the validity proof in this section and include the facet-defining proof in Appendix C.

3.1 Strong Valid Inequalities with Single Continuous Variable

First, it is obvious that based on capacity constraints (1e), the generation amount at each time period (e.g., x_t) is bounded from above by the generation upper bound (i.e., $\overline{C}$). Besides this, by additionally considering ramp-rate limits, minimum-up/-down time limits, and corresponding start-up or shut-down decision, tighter upper bounds can be obtained for the generation amount x_t as described in the following strong valid inequalities (23) - (27).

Proposition 5 *For $1 \leq k \leq \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1\}$, $t \in [k+1, T]_{\mathbb{Z}}$, the inequality*

$$x_t \leq \overline{C} y_t - \sum_{s=0}^{k-1} (\overline{C} - \overline{V} - sV) u_{t-s} \tag{23}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when $t = T$ and $k = \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor +$ $1\}$.

Proof: **(Validity)** We discuss the following two possible cases in terms of the value of y_t:

1) If $y_t = 0$, then we have $x_t = 0$ due to constraints (1e) and $u_{t-s} = 0$ for all $s \in [0, k-1]_{\mathbb{Z}}$ due to constraints (1a) since $k \leq L$. Thus, inequality (23) holds.

2) If $y_t = 1$, then we have $\sum_{s=0}^{k-1} u_{t-s} \leq 1$ due to constraints (1a) since $k \leq L$. We discuss the following two possible cases:

   - If $u_{t-s} = 0$ for all $s \in [0, k-1]_{\mathbb{Z}}$, then inequality (23) converts to $x_t \leq \overline{C}$, which is valid because of constraints (1e).

   - If $u_{t-s} = 0$ for some $s \in [0, k-1]_{\mathbb{Z}}$, then inequality (23) converts to $x_t \leq \overline{V} + sV$, which is valid because of ramp-up constraints (1f).

See Appendix C.1 for the facet-defining proof. ■

Proposition 6 *For $1 \leq k \leq \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 2\}$, $t \in [k, T-1]_{\mathbb{Z}}$, the inequality*

$$x_t \leq \overline{V} y_t + (\overline{C} - \overline{V})(y_{t+1} - u_{t+1}) - \sum_{s=1}^{k-1} (\overline{C} - \overline{V} - (s-1)V) u_{t-s+1} \tag{24}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when one of the following conditions is satisfied: (1) $L \leq 3$, $k = \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 2\}$ for all $t \in [k, T-1]_{\mathbb{Z}}$; (2) $L \geq 4$, $k = \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 2\}$ for $t = T-1$.

Proof: **(Validity)** We discuss the following four possible cases in terms of the values of y_t and y_{t+1}:

1) If $y_t = y_{t+1} = 1$, then we have $u_{t+1} = 0$ due to constraints (1b) and $\sum_{s=1}^{k-1} u_{t-s+1} \leq 1$ due to constraints (1a) since $k \leq L$. We further discuss the following two possible cases.

   - If $u_{t-s+1} = 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$, then (24) converts to $x_t \leq \overline{C}$, which is valid because of constraints (1e).
   - If $u_{t-s+1} = 1$ for some $s \in [1, k-1]_{\mathbb{Z}}$, then (24) converts to $x_t \leq \overline{V} + (s-1)V$, which is valid because of ramp-up constraints (1f).

2) If $y_t = 1$ and $y_{t+1} = 0$, then $u_{t-s+1} = 0$ for all $s \in [0, k-1]_{\mathbb{Z}}$ due to constraints (1a) since $k \leq L$. It follows that (24) converts to $x_t \leq \overline{V}$, which is valid because of ramp-down constraints (1g).

3) If $y_t = 0$ and $y_{t+1} = 1$, then we have $u_{t+1} = 1$ due to constraints (1c) and $u_{t-s+1} = 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$ due to constraints (1a) since $k \leq L$. It follows (24) is valid.

4) If $y_t = y_{t+1} = 0$, (24) is clearly valid.

See Appendix C.2 for the facet-defining proof. ■

Proposition 7 *For $k = \min\{L-1, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor\}$, $t \in [k+3, T]_{\mathbb{Z}}$, the inequality*

$$x_{t-1} \leq (\overline{C} - kV) y_{t-1} + kV(y_t - u_t) - \sum_{s=0}^{k} (\overline{C} - \overline{V} - sV) u_{t-s-1} \tag{25}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when one of the following conditions is satisfied: (1) $L \leq 3$, $L - 1 \leq \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor$ for all $t \in [k+3, T]_{\mathbb{Z}}$; (2) $L \geq 4$, $L - 1 \leq \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor$ for $t = T$.

Proof: **(Validity)** We discuss the following four possible cases in terms of the values of y_{t-1} and y_t:

1) If $y_{t-1} = y_t = 1$, we have $u_t = 0$ due to constraints (1b) and $\sum_{s=0}^{k} u_{t-s-1} \le 1$ due to constraints (1a) since $k \le L-1$. We further discuss the following two possible cases.

   - If $u_{t-s-1} = 0$ for all $s \in [0,k]_{\mathbb{Z}}$, then (25) converts to $x_t \le \overline{C}$, which is valid because of constraints (1e).
   - If $u_{t-s+1} = 1$ for some $s \in [0,k]_{\mathbb{Z}}$, then (25) converts to $x_t \le \overline{V} + sV$, which is valid because of ramp-up constraints (1f).

2) If $y_{t-1} = 1$ and $y_t = 0$, then $u_{t-s-1} = 0$ for all $s \in [0, L-2]_{\mathbb{Z}}$ and $\sum_{s=0}^{k} u_{t-s-1} \le 1$ due to constraints (1a) since $k \le L-1$. We further discuss the following two possible cases.

   - If $u_{t-s-1} = 0$ for all $s \in [0,k]_{\mathbb{Z}}$, then (25) converts to $x_t \le \overline{C} - kV$, which is valid since $x_t \le \overline{V}$ due to ramp-down constraints (1g) and $k \le \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor$.
   - If $k = L-1$ and $u_{t-k-1} = 1$, then (25) converts to $x_t \le \overline{V}$, which is valid because of ramp-down constraints (1g).

3) If $y_{t-1} = 0$ and $y_t = 1$, we have $u_t = 1$ due to constraints (1c) and $u_{t-s-1} = 0$ for all $s \in [0,k]_{\mathbb{Z}}$ due to constraints (1a) since $k \le L-1$. It follows (25) is valid.

4) If $y_t = y_{t+1} = 0$, (25) is clearly valid.

See Appendix C.3 for the facet-defining proof. ■

Proposition 8 *For each $k \in \{[2, T-2]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k-1)V > 0\}$, the inequality*

$$x_{t-k} \le \overline{V} y_{t-k} + V \sum_{s=1}^{k-1} (y_{t-s} - \sum_{i=s}^{\min\{k, s+L-1\}} u_{t-i}) + (\overline{C} - \overline{V} - (k-1)V)(y_t - \sum_{s=0}^{\min\{k, L-1\}} u_{t-s}) \quad (26)$$

is valid for conv(P) for each $t \in [\max\{\min\{k, k+L-2\}+2, \min\{k, L-1\}+2\}, T]_{\mathbb{Z}}$. Furthermore, it is facet-defining for conv(P) when one of the following conditions is satisfied: (1) $L \le 3$ and $t = T$; (2) $L \le 3$ and $k = \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1$ for all $t \in [\max\{\min\{k, k+L-2\}+2, \min\{k, L-1\}+2\}, T]_{\mathbb{Z}}$.

Proof: **(Validity)** We discuss the following two possible cases in terms of the value of y_{t-k}:

1) If $y_{t-k} = 0$, then $x_{t-k} = 0$ due to constraints (1e). It follows that inequality (26) is valid since $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$ and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$ due to minimum-up time constraints (1a).

2) If $y_{t-k} = 1$, then we consider the following two possible cases in terms of the value of u_{t-k}:

   (1) If $u_{t-k} = 1$, then we have $x_{t-k} \leq \overline{V}$ due to ramp-up constraints (1f). It follows that inequality (26) is valid since $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$ and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$ due to minimum-up time constraints (1a).

   (2) If $u_{t-k} = 0$, then it means that the generator starts up at a time period prior to time $t-k$. To show inequality (26) is valid, we consider the following two possible cases based on when this generator shuts down as follows.

   - If the generator shuts down at $t - \bar{s}$ for some $\bar{s} \in [1, k-1]_{\mathbb{Z}}$, i.e., $y_{t-\bar{s}} = 0$, then $u_{t-s} = 0$ for all $s \in [\bar{s}, \min\{k, k+L-2\}]_{\mathbb{Z}}$. It follows that inequality (26) converts to $x_{t-k} \leq \overline{V} + (k - \bar{s} - 1)V + V\sum_{s=1}^{\bar{s}-1}(y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i}) + (\overline{C} - \overline{V} - (k-1)V)(y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s})$, which is valid since $x_{t-k} \leq \overline{V} + (k - \bar{s} - 1)V$ due to ramp-down constraints (1g), $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, \bar{s}-1]_{\mathbb{Z}}$, and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$.
   - If the generator shuts down at $\bar{t}$ such that $\bar{t} \geq t$, then inequality (26) converts to $x_{t-k} \leq \overline{C}$, which is clearly valid due to constraints (1e).

See Appendix C.4 for the facet-defining proof. ∎

Proposition 9 *For each $k \in \{[2, T-1]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k-1)V > 0\}$, the inequality*

$$x_1 \leq \overline{V} y_1 + V\sum_{s=2}^{k}(y_s - \sum_{i=\max\{2,s-L+1\}}^{s} u_i) + (\overline{C} - \overline{V} - (k-1)V)(y_{k+1} - \sum_{i=\max\{2,k-L+2\}}^{k+1} u_i) \quad (27)$$

is valid and facet-defining for conv(P) for each $k \in [2, T-1]_{\mathbb{Z}}$.

Proof: The proofs are similar with that for Proposition 8 and thus omitted here. ∎

From Propositions 5 - 9, we can observe that the total number of derived inequalities is in the order of at most $\mathcal{O}(T^2)$.

Note here that there are also strong valid inequalities corresponding to other parameter settings, for instance, for those uncommon generators. Here we list one family of inequalities, under the condition $\overline{V} < \underline{C} + V,\ \overline{C} - \underline{C} - V < 0,$ and $\overline{C} - \overline{V} - V < 0$, for illustration purpose.

Proposition 10 *For each $t \in [2, T-1]_{\mathbb{Z}}$, the following inequality*

$$x_t \leq \overline{V} y_t + (\overline{C} - \overline{V}) \left(y_{t+1} - \sum_{s=0}^{\min\{L-1,1\}} u_{t+1-s} \right) \tag{28}$$

is valid and facet-defining for conv(P).

Proof: The proofs are similar and thus omitted here. ∎

3.2 Strong Valid Inequalities with Two Continuous Variables

We extend the study to derive strong valid inequalities to bound the difference of generation amounts at two different time periods, e.g., $x_t - x_{t-k}$, $x_{t-1} - x_{t-k-1}$, and $x_{t-k} - x_t$. These values are bounded from above and below by the combination of generation bound constraints (1d) - (1e) and ramp-rate constraints (1f) - (1g). Through additionally considering the minimum-up/-down time limits and start-up decisions, we derive the explicit formulas of upper bounds of $x_t - x_{t-k}$, $x_{t-1} - x_{t-k-1}$, and $x_{t-k} - x_t$ in inequalities (29) - (32).

Proposition 11 *For each $k \in [1, T-1]_{\mathbb{Z}}$ such that $\overline{C} - \underline{C} - kV > 0$, $t \in [k+1, T]_{\mathbb{Z}}$, the inequality*

$$x_t - x_{t-k} \leq (\underline{C} + kV) y_t - \underline{C} y_{t-k} - \sum_{s=0}^{\min\{k-1,L-1\}} (\underline{C} + (k-s)V - \overline{V}) u_{t-s} \tag{29}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when $t = T$.

Proof: **(Validity)** We discuss the following four possible cases in terms of the values of y_{t-k} and y_t:

1) If $y_{t-k} = y_t = 1$, then $\sum_{s=0}^{\min\{k-1,L-1\}} u_{t-s} \leq 1$ due to constraints (1a). We further discuss the following two possible cases.

   - If $u_{t-s} = 0$ for all $s \in [0, \min\{k-1, L-1\}]_{\mathbb{Z}}$, then (29) converts to $x_t - x_{t-k} \leq kV$, which is valid due to ramp-up constraints (1f).

- If $u_{t-s} = 1$ for some $s \in [0, \min\{k-1, L-1\}]_{\mathbb{Z}}$, then (29) converts to $x_t - x_{t-k} \leq sV - \underline{C}$, which is valid since $x_t \leq sV$ due to ramp-up constraints (1f) and $x_{t-k} \geq \underline{C}$ due to constraints (1d).

2) If $y_{t-k} = 1$ and $y_t = 0$, then $\sum_{s=0}^{\min\{k-1,L-1\}} u_{t-s} = 0$ due to constraints (1a). (29) converts to $x_{t-k} \geq \underline{C}$, which is valid due to constraints (1d).

3) If $y_{t-k} = 0$ and $y_t = 1$, then the generator should start up at time period $\bar{t} \in [t-k+1, t]_{\mathbb{Z}}$. Meanwhile, we have $\sum_{s=0}^{\min\{k-1,L-1\}} u_{t-s} \leq 1$ due to constraints (1a). We further discuss the following two possible cases.

   - If $u_{t-s} = 0$ for all $s \in [0, \min\{k-1, L-1\}]_{\mathbb{Z}}$, then it follows $\bar{t} \in [t-k+1, t-\min\{k-1, L-1\}-1]_{\mathbb{Z}}$, i.e., $t - \bar{t} \in [\min\{k-1, L-1\}+1, k-1]_{\mathbb{Z}}$. Meanwhile, (29) converts to $x_t \leq \underline{C} + kV$, which is valid since $x_t \leq \overline{V} + (t-\bar{t})V \leq \overline{V} + (k-1)V$ due to ramp-up constraints (1f) and $\overline{V} < \underline{C} + V$.
   - If $u_{t-s} = 1$ for some $s \in [0, \min\{k-1, L-1\}]_{\mathbb{Z}}$, then (29) converts to $x_t \leq sV$, which is valid since $x_t \leq sV$ due to ramp-up constraints (1f).

4) If $y_{t-k} = y_t = 0$, then (29) is clearly valid.

(Facet-defining) The proof is similar with that for Proposition 5 and thus omitted here. ■

Proposition 12 *For each* $k \in \{[1, T-2]_{\mathbb{Z}} : \overline{C} - \underline{C} - kV > 0\}$, $t \in [k+2, T]_{\mathbb{Z}}$, *the inequality*

$$x_{t-1} - x_{t-k-1} \leq \overline{V} y_{t-1} - \underline{C} y_{t-k-1} + (\underline{C} + kV - \overline{V})(y_t - u_t) - \sum_{s=1}^{\min\{k,L-1\}} (\underline{C} + (k-s+1)V - \overline{V}) u_{t-s} \quad (30)$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when one of the following conditions is satisfied: (1) $t = T$*; (2)* $\min\{k, L-1\} \leq 2$ *for all* $t \in [k+2, T]_{\mathbb{Z}}$.

Proof: **(Validity)** We discuss the following two possible cases in terms of the value of y_t:

1) If $y_t = 0$, then $u_{t-s} = 0$ for all $s \in [0, \min\{k, L-1\}]_{\mathbb{Z}}$ due to constraints (1a). Inequality (30) converts to $x_{t-1} - x_{t-k-1} \leq \overline{V} y_{t-1} - \underline{C} y_{t-k-1}$ since we have $x_{t-1} \leq \overline{V} y_{t-1}$ due to constraints (1g) and $x_{t-k-1} \geq \underline{C} y_{t-k-1}$ due to constraints (1d).

2) If $y_t = 1$, then $\sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \leq 1$ due to constraints (1a). We further discuss the following three possible cases.

   (1) If $u_{t-s} = 0$ for all $s \in [0, \min\{k, L-1\}]_{\mathbb{Z}}$, then $y_{t-1} = 1$ due to constraints (1c). Thus (30) converts to $x_{t-1} - x_{t-k-1} \leq \underline{C} + kV - \underline{C} y_{t-k-1}$. We further discuss the following two possible cases in terms of the value of y_{t-k-1}.

      - If $y_{t-k-1} = 1$, then (30) converts to $x_{t-1} - x_{t-k-1} \leq kV$, which is valid due to ramp-up constraints (1f).
      - If $y_{t-k-1} = 0$, then it follows the generator starts up at time $\bar{t} \in [t-k, \min\{k, L-1\}-1]_{\mathbb{Z}}$. Meanwhile, (30) converts to $x_{t-1} \leq \underline{C} + kV$, which is valid since $x_{t-1} \leq \overline{V} + (t-1-\bar{t})V < \underline{C} + V + (k-1)V = \underline{C} + kV$, where the first inequality is due to ramp-up constraints (1f) and the second inequality is due to $\overline{V} < \underline{C} + V$.

   (2) If $u_t = 1$, then $y_{t-1} = 0$ due to constraints (1b). It follows that inequality (30) converts to $x_{t-k-1} \geq \underline{C} y_{t-k-1}$, which is valid due to constraints (1d).

   (3) If $u_{t-s} = 1$ for some $s \in [1, \min\{k, L-1\}]_{\mathbb{Z}}$, then inequality (30) converts to $x_{t-1} - x_{t-k-1} \leq \overline{V} + (s-1)V - \underline{C} y_{t-k-1}$, which is valid since $x_{t-1} \leq \overline{V} + (s-1)V$ due to ramp-up constraints (1f) and $x_{t-k-1} \geq \underline{C} y_{t-k-1}$ due to constraints (1d).

See Appendix C.5 for the facet-defining proof. ■

Proposition 13 *For each* $k \in \{[2, T-1]_{\mathbb{Z}} : \overline{C} - \underline{C} - kV > 0\}$, $t \in [k + \min\{k, L-1\} + 1, T]_{\mathbb{Z}}$, *the inequality*

$$x_{t-k} - x_t \leq \overline{V} y_{t-k} - \underline{C} y_t + (\underline{C} + kV - \overline{V})(y_{t-k+1} - u_{t-k+1}) - \sum_{s=1}^{\min\{k,L-1\}} (\underline{C} + (k-s+1)V - \overline{V}) u_{t-k-s+1} \tag{31}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when $\min\{k, L-1\} \leq 2$ *for all* $t \in [k + \min\{k, L-1\} + 1, T]_{\mathbb{Z}}$.

Proof: As a symmetry of (30), inequality (31) can be proved to be valid and facet-defining similarly and thus the proofs are omitted here. ■

Proposition 14 *For each* $k \in \{[1, T-1]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k-1)V > 0\}$*, the inequality*

$$x_{t-k} - x_t \leq \overline{V} y_{t-k} - \underline{C} y_t + V \sum_{s=1}^{k-1} (y_{t-s} - \sum_{i=s}^{\min\{k, s+L-1\}} u_{t-i}) + (\underline{C} + V - \overline{V})(y_t - \sum_{s=0}^{\min\{k, L-1\}} u_{t-s}) \quad (32)$$

is valid for conv(P) for each $t \in [\max\{\min\{k, k+L-2\}+2, \min\{k, L-1\}+2\}, T]_{\mathbb{Z}}$*. Furthermore, it is facet-defining for conv(P) for each* $t \in [\max\{\min\{k, k+L-2\}+2, \min\{k, L-1\}+2\}, T]_{\mathbb{Z}}$ *when* $L \leq 3$*.*

Proof: **(Validity)** We discuss the following two possible cases in terms of the values of y_{t-k}:

1) If $y_{t-k} = 0$, then inequality (32) is valid since $x_t \geq \underline{C} y_t$ due to constraints (1d), $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$ and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$ due to constraints (1a), and $\underline{C} + V - \overline{V} > 0$.

2) If $y_{t-k} = 1$, then we only consider the case that $u_{t-k} = 0$ since we can easily verify that (32) is valid when $u_{t-k} = 1$ (following $x_{t-k} \leq \overline{V}$ and case 1) above). We further discuss the following cases in terms of the time period when the generator shuts down.

   (1) If the generator shuts down at $\bar{t}$ such that $\bar{t} \geq t$, then inequality (32) converts to $x_{t-k} - x_t \leq kV$, which is valid due to ramp-down constraints (1g).

   (2) If the generator shuts down at $t - \bar{s}$ such that $\bar{s} \in [1, k-1]_{\mathbb{Z}}$, i.e., $y_{t-\bar{s}} = 0$, then inequality (32) converts to $x_{t-k} - x_t \leq \overline{V} + (k-1-\bar{s})V - \underline{C} y_t + V \sum_{s=1}^{\bar{s}-1} (y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i}) + (\underline{C} + V - \overline{V})(y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s})$, which is clearly valid since $x_{t-k} \leq \overline{V} + (k-1-\bar{s})V$ due to ramp-down constraints (1g), $x_t \geq \underline{C} y_t$ due to constraints (1d), $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, \bar{s}-1]_{\mathbb{Z}}$ and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$ due to constraints (1a), and $\underline{C} + V - \overline{V} > 0$.

(Facet-defining) The proof is similar with that for Proposition 8 and thus omitted here. ∎

From Propositions 11 - 14, we can observe that these derived inequalities contain two continuous variables, and the total number of inequalities is in the order of $\mathcal{O}(T^2)$.

Similar to the one continuous variable case, there are also strong valid inequalities corresponding to other parameter settings. Here we list two families of inequalities, under the condition $\overline{V} > \underline{C} + V$, $\overline{C} - \underline{C} - V > 0$, and $\overline{C} - \overline{V} - V > 0$, for illustration purpose.

Proposition 15 *For each $t \in [2,T]$, the following inequalities*

$$
\begin{aligned}
x_t - x_{t-1} &\leq (\underline{C} + V)y_t - \underline{C}y_{t-1} - (\underline{C} + V - \overline{V})u_t, \\
x_{t-1} - x_t &\leq \overline{V}y_{t-1} - (\overline{V} - V)y_t - (\underline{C} + V - \overline{V})u_t,
\end{aligned}
$$

are valid and facet-defining for conv(P).

Proof: The proofs are similar and thus omitted here. ■

3.3 Strong Valid Inequalities with Three Continuous Variables

We continue to strengthen the integrated minimum-up/-down time and ramping polytope P by deriving strong valid inequalities containing three continuous variables, e.g., x_{t-3}, x_{t-2}, and x_{t-1}.

Proposition 16 *For each $t \in [\max\{L+2,4\},T]_{\mathbb{Z}}$, the inequality*

$$
\begin{aligned}
x_{t-3} - x_{t-2} + x_{t-1} \leq \overline{V}y_{t-3} - (\overline{V} - V)y_{t-2} + \overline{V}y_{t-1} + (\underline{C} + V - \overline{V})(y_t - u_t - y_{t-1}) \\
+ (\overline{C} - \overline{V})(y_{t-1} - u_{t-1} - u_{t-2}) - \sum_{s=0}^{L-3}(\overline{C} - \overline{V} - sV)u_{t-s-3} \qquad (33)
\end{aligned}
$$

is valid for conv(P) when $L \geq 2$. Furthermore, it is facet-defining for conv(P) for each $t \in [\max\{L+2,4\},T]_{\mathbb{Z}}$ when $L \leq 3$.

Proof: **(Validity)** We discuss the following two possible cases in terms of the value of y_t:

1) If $y_{t-1} = 0$, then $u_{t-s-1} = 0$ for all $s \in [0, L-1]_{\mathbb{Z}}$ due to constraints (1a) and $y_t = u_t$ due to constraints (1a) and (1c). It follows that inequality (33) converts to $x_{t-3} - x_{t-2} \leq \overline{V}y_{t-3} - (\overline{V} - V)y_{t-2}$, which can be easily verified to be valid through consider all the three possible cases, i.e., (1) $y_{t-3} = y_{t-2} = 1$, (2) $y_{t-3} = 1$ and $y_{t-2} = 0$, and (3) $y_{t-3} = y_{t-2} = 0$.

2) If $y_{t-1} = 1$, then $\sum_{s=0}^{L-1} u_{t-s-1} \leq 1$ due to constraints (1a). We further discuss the following four possible cases.

   (1) If $u_{t-s-1} = 0$ for all $s \in [0, L-1]_{\mathbb{Z}}$, then $y_{t-2} = 1$ due to constraints (1c) and $L \geq 2$. It follows that inequality (33) converts to $x_{t-3} - x_{t-2} + x_{t-1} \leq \overline{V}y_{t-3} - (\overline{V} - V) + \overline{C} + (\underline{C} + V - \overline{V})(y_t - 1)$, which can be easily verified to be valid through consider all the four possible cases, i.e., (1) $y_{t-3} = y_t = 1$, (2) $y_{t-3} = 1$ and $y_t = 0$, (3) $y_{t-3} = 0$ and $y_t = 1$, and (4) $y_{t-3} = y_t = 0$.

(2) If $u_{t-1} = 1$, then $u_{t-s-1} = 0$ for all $s \in [1, L-1]_{\mathbb{Z}}$. Meanwhile, we have $y_t = 1$ and $u_t = 0$ due to $L \geq 2$ and $y_{t-2} = 0$ due to constraints (1g). It follows that inequality (33) converts to $x_{t-3} + x_{t-1} \leq \overline{V} y_{t-3} + \overline{V}$, which is valid since $x_{t-3} \leq \overline{V} y_{t-3}$ and $x_{t-1} \leq \overline{V}$ due to constraints (1f) and (1g).

(3) If $u_{t-2} = 1$, then $u_{t-s-1} = 0$ for all $s \in [2, L-1]_{\mathbb{Z}}$ and $u_{t-1} = 0$. Meanwhile, we have $y_{t-3} = 0$ due to (1g) and $y_{t-1} = 1$ and $u_t = 0$ due to $L \geq 2$. It follows that inequality (33) converts to $x_{t-1} - x_{t-2} \leq V + (\underline{C} + V - \overline{V})(y_t - 1)$, which can be easily verified to be valid for either $y_t = 1$ or $y_t = 0$.

(4) If $u_{t-s-3} = 1$ for some $s \in [0, L-3]_{\mathbb{Z}}$ when $L \geq 3$, then $y_{t-3} = y_{t-2} = y_{t-1} = 1$ due to minimum-up time constraints (1a). It follows that inequality (33) converts to $x_{t-3} - x_{t-2} + x_{t-1} \leq \overline{V} + sV + V + (\underline{C} + V - \overline{V})(y_t - 1)$, which can be easily verified to be valid either $y_t = 1$ or $y_t = 0$ since $x_{t-3} \leq \overline{V} + sV$ and $x_{t-1} - x_{t-2} \leq V + (\underline{C} + V - \overline{V})(y_t - 1)$.

See Appendix C.6 for the facet-defining proof. ∎

Proposition 17 *For each $k \in \{[0, T-4]_{\mathbb{Z}} : \overline{C} - \overline{V} - kV > 0\}$, $t \in [\max\{1, L-2\}, T-k-3]_{\mathbb{Z}}$, the inequality*

$$x_t - x_{t+1} + x_{t+2} \leq \overline{V} y_t - (\overline{V} - V) y_{t+1} + \overline{V} y_{t+2} - \phi + V \sum_{s=1}^{k} (y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3}) \quad (34)$$

is valid and facet-defining for conv(P) when $L \geq 2$, where $\phi = 0$ if $L \geq 4$ or $t = 1$, and $\phi = (\underline{C} + V - \overline{V}) u_t$ otherwise.

Proof: **(Validity)** We prove the validity for the case that $\phi = (\underline{C} + V - \overline{V}) u_t$, i.e., $L \leq 3$ and $t \geq 2$, while other cases can be proved similarly. We discuss the following two possible cases in terms of the value of y_{t+2}:

1) If $y_{t+2} = 0$, to show inequality (34) is valid, we show $x_t - x_{t+1} \leq \overline{V} y_t - (\overline{V} - V) y_{t+1} - (\underline{C} + V - \overline{V}) u_t$. Then inequality (34) is valid since $y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2} \geq 0$ for all $s \in [1, k]_{\mathbb{Z}}$ and $y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3} \geq 0$ due to minimum-up constraints (1a). We discuss the following three possible cases.

(1) If $y_t = y_{t+1} = 1$ and $u_t = 0$, then (34) converts to $x_t - x_{t+1} \leq V$, which is valid due to ramp-down constraints (1g).

(2) If $y_t = 1$ and $y_{t+1} = u_t = 0$, then (34) converts to $x_t \leq \overline{V}$, which is valid due to ramp-down constraints (1g).

(3) If $y_t = y_{t+1} = u_t = 1$, then (34) converts to $x_t - x_{t+1} \leq \overline{V} - \underline{C}$, which is valid since $x_t \leq \overline{V}$ due to ramp-up constraints (1f) and $x_{t+1} \geq \underline{C}$ due to constraints (1d).

2) If $y_{t+2} = 1$, we discuss the following two possible cases in terms of the value of u_{t+2}:

(1) If $u_{t+2} = 1$, then we have $y_{t+1} = u_t = 0$ due to constraints (1a) - (1c). Thus, we have $x_t \leq \overline{V} y_t$ due to ramp-down constraints (1g) and $x_{t+2} \leq \overline{V}$ due to ramp-up constraints (1f). It follows that inequality (34) is valid since $y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2} \geq 0$ for all $s \in [1, k]_{\mathbb{Z}}$ and $y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3} \geq 0$ due to minimum-up constraints (1a).

(2) If $u_{t+2} = 0$, we discuss the following three possible cases.

- If $u_{t+1} = 1$, then $y_t = u_t = 0$ due to constraints (1b) - (1c) and $y_{t+1} = y_{t+2} = 1$ due to $L \geq 2$. Then (34) is clearly valid since $x_{t+2} - x_{t+1} \leq V$ due to ramp-up constrains (1f).
- If $u_t = 1$, then $y_{\bar{s}} = 1$ for all $\bar{s} \in [t, t + L - 1]_{\mathbb{Z}}$ and $u_{\hat{s}} = 0$ for all $\hat{s} \in [t - L + 4, t + L]_{\mathbb{Z}}$ since we consider $L \leq 3$. Inequality (34) converts to $x_t - x_{t+1} + x_{t+2} \leq \overline{V} - \underline{C} + \overline{V} + V \sum_{s=1}^{k} (y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3})$, which is valid since $x_t \leq \overline{V}$, $x_{t+1} \geq \underline{C}$, and $x_{t+2} \leq \overline{V} + V \sum_{s=1}^{k} (y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3})$ due to inequality (26).
- If $u_{\bar{t}} = 1$ for some $\bar{t} \leq t - 1$, then (34) converts to $x_t - x_{t+1} + x_{t+2} \leq V + \overline{V} + V \sum_{s=1}^{k} (y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3})$, which is valid since $x_t - x_{t+1} \leq V$, and $x_{t+2} \leq \overline{V} + V \sum_{s=1}^{k} (y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3})$ due to inequality (26).

See Appendix C.7 for the facet-defining proof. ■

We can observe that the inequalities derived in Propositions 16 and 17 are polynomial in the order of $\mathcal{O}(T)$.

To summarize, all the derived strong valid inequalities in this section covering multiple periods are polynomial in the order of at most $\mathcal{O}(T^2)$. Therefore, we do not need to perform a separation procedure.

Finally, as described in the previous one and two continuous variable cases, there are also strong valid inequalities corresponding to other parameter settings for the three continuous variable case. Here we list one family of inequalities, under the condition $\overline{V} > \underline{C} + V$, $\overline{C} - \underline{C} - V > 0$, and $\overline{C} - \overline{V} - V > 0$, for illustration purpose.

Proposition 18 *For each $t \in [1, T-2]$, the following inequality*

$$x_t - x_{t+1} + x_{t+2} \geq \underline{C}y_t - (\underline{C} + V)y_{t+1} + \underline{C}y_{t+2} \tag{35}$$

is valid and facet-defining for conv(P).

Proof: The proofs are similar and thus omitted here. ■

4 Computational Experiments

In this section, we show the effectiveness of our proposed strong valid inequalities on solving both the network-constrained unit commitment (used by system operators) and self-scheduling unit commitment (used by market participants) problems. The experiments were performed on a computer node with two AMD Opteron 2378 Quad Core Processors at 2.4GHz. The addressable memory is 4GB and the time limit was set at one hour per run. CPLEX 12.3 under default settings was used to solve the problems.

4.1 Network-Constrained Unit Commitment Problem

For the network-constrained unit commitment problem, we first provide the mathematical formulation and then report the computational results for the power system data based on [3] and [15], and a modified IEEE 118-bus system based on the one given online at `http://motor.ece.iit.edu/data/SCUC_118/`, respectively.

For the mathematical formulation, we set the operational time interval to be 24 hours (i.e., $T = 24$) and let $\mathcal{G}$ and $\mathcal{B}$ represent the set of generators and buses respectively, with $|\mathcal{G}| = G$ and $|\mathcal{B}| = B$. Besides, we let $\mathcal{E}$ represent the set of transmission lines linking two buses. With superscripts g and b representing generator and bus index respectively, we introduce the notation for the whole system, with a part of them similar to those defined in Section 1. For each generator g, we let L^g (ℓ^g) be its minimum-up (-down) time limit, $\overline{C}^g$ ($\underline{C}^g$) be its generation upper (lower) bound, $\overline{V}^g$ be its start-up/shut-down ramp rate, V^g be its ramp-up/-down rate in stable generation, SU^g (SD^g) represent its start-up (shut-down) cost of generator g, (x_t^g, y_t^g, u_t^g) represent its status at each time period t for $t \in [1,T]_{\mathbb{Z}}$, and $f^g(x_t^g)$ represent its generation cost when its generation amount is x_t^g at t. In addition, we let d_t^b represent the load (demand) at bus b at time period t and r_t represent the system reserve factor at t. For each transmission line $(j,h) \in \mathcal{E}$, we let C_{jh} represent its capacity, and K_{jh}^b represent the line flow distribution factor for the flow on the transmission line (j,h) contributed by the net injection at bus b. Meanwhile, for notation convenience, we let $\mathcal{G}_b \subseteq \mathcal{G}$ represent the set of generators at bus b (e.g., $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$ for $i,j \in \mathcal{B}$ and $i \neq j$, $\bigcup_{b=1}^B \mathcal{G}_b = \mathcal{G}$) and $G_b = |\mathcal{G}_b|$. Accordingly, the network-constrained unit commitment problem can be described as follows:

$$\min_{x,y,u} \quad \sum_{g=1}^G \Big(\sum_{t=2}^T \big(\mathrm{SU}^g u_t^g + \mathrm{SD}^g (y_{t-1}^g - y_t^g + u_t^g) \big) + \sum_{t=1}^T f^g(x_t^g) \Big) \tag{36a}$$

$$\text{s.t.} \quad \sum_{i=t-L^g+1}^{t} u_i^g \le y_t^g, \quad \forall t \in [L^g+1, T]_{\mathbb{Z}}, \forall g \in [1,G]_{\mathbb{Z}}, \tag{36b}$$

$$\sum_{i=t-\ell^g+1}^{t} u_i^g \le 1 - y_{t-\ell^g}^g, \forall t \in [\ell^g+1, T]_{\mathbb{Z}}, \forall g \in [1,G]_{\mathbb{Z}}, \tag{36c}$$

$$-y_{t-1}^g + y_t^g - u_t^g \le 0, \quad \forall t \in [2,T]_{\mathbb{Z}}, \forall g \in [1,G]_{\mathbb{Z}}, \tag{36d}$$

$$\underline{C}^g y_t^g \le x_t^g \le \overline{C}^g y_t^g, \quad \forall t \in [1,T]_{\mathbb{Z}}, \forall g \in [1,G]_{\mathbb{Z}}, \tag{36e}$$

$$x_t^g - x_{t-1}^g \le V^g y_{t-1}^g + \overline{V}^g (1 - y_{t-1}^g), \quad \forall t \in [2,T]_{\mathbb{Z}}, \forall g \in [1,G]_{\mathbb{Z}}, \tag{36f}$$

$$x_{t-1}^g - x_t^g \le V^g y_t^g + \overline{V}^g (1 - y_t^g), \quad \forall t \in [2,T]_{\mathbb{Z}}, \forall g \in [1,G]_{\mathbb{Z}}, \tag{36g}$$

$$\sum_{g=1}^G x_t^g = \sum_{b=1}^B d_t^b, \quad \forall t \in [1,T]_{\mathbb{Z}}, \tag{36h}$$

$$\sum_{g=1}^G \overline{C}_g y_t^g \ge r_t \sum_{b=1}^B d_t^b, \quad \forall t \in [1,T]_{\mathbb{Z}}, \tag{36i}$$

$$-C_{jh} \le \sum_{b=1}^B K_{jh}^b \Big(\sum_{g=1}^{G_b} x_t^g - d_t^b \Big) \le C_{jh}, \quad \forall t \in [1,T]_{\mathbb{Z}}, \forall (j,h) \in \mathcal{E}, \tag{36j}$$

$$y_t^g \in \{0,1\}, \ \forall t \in [1,T]_{\mathbb{Z}}; \ u_t^g \in \{0,1\}, \ \forall t \in [2,T]_{\mathbb{Z}}, \forall g \in [1,G]_{\mathbb{Z}}, \tag{36k}$$

where the objective is to minimize the total cost, including start-up cost, shut-down cost, and the generation cost that is represented by $f^g(x_t^g)$, which is typically a nondecreasing quadratic function, i.e., $f^g(x_t^g) = a^g(x_t^g)^2 + b^g x_t^g + c^g$. Constraints (36b) (resp. (36c)) describe the minimum-up (resp. minimum-down) time restrictions and constraints (36d) describe the relationship between y and u. Constraints (36e) describe the generation upper and lower bounds for generator g if it is online at time period t. Constraints (36f) (resp. (36g)) describe the maximum generation increment (resp. decrement) between two consecutive time periods (i.e., ramp-rate restrictions). Constraints (36h) enforce the load balance at each time period t. Constraints (36i) describe the system reserve requirements. Finally, constraints (36j) represent the capacity limit of each transmission line (j, h) (see, e.g., [23]). Note here that the generation cost function $f^g(\cdot)$ can be approximated by a piecewise linear function [3]. With this approximation, the formulation above can be reformulated as an MILP formulation.

4.1.1 Power System Data Based on [3] and [15]

In this experiment, there are eight types of generators (see Table 3), and twenty instances with each containing different combinations of each type of generators (see Table 4). The system load setting is reported in Table 5. Constraints (36i) and (36j) are not included in this experiment since the system reserve and transmission data are not provided in [3] and [15].

Table 3: Generator Data

Generators	$\underline{C}$ (MW)	$\overline{C}$ (MW)	L/ℓ (h)	V (MW/h)	$\overline{V}$ (MW/h)	SU (\$/h)	a (\$/MW^{2}h)	b (\$/MWh)	c (\$/h)
1	150	455	8	91	180	2000	0.00048	16.19	1000
2	150	455	8	91	180	2000	0.00031	17.26	970
3	20	130	5	26	35	500	0.002	16.6	700
4	20	130	5	26	35	500	0.00211	16.5	680
5	25	162	6	32.4	40	700	0.00398	19.7	450
6	20	80	3	16	28	150	0.00712	22.26	370
7	25	85	3	17	33	200	0.00079	27.74	480
8	10	55	1	11	15	60	0.00413	25.92	660

For each instance, we compare four formulations (i.e., "MILP", "Strong", "Strong-1", and "Strong-2") and report the results in Table 6, where "MILP" represents the original MILP formulation given in (36), "Strong" represents the original MILP formulation plus our proposed strong valid inequalities in Sections 1 - 3 (i.e., (2d) - (2g), (4) - (13), (18c) - (18r), and (23) - (34)) as con-

Table 4: Problem Instances [15]

Instances	Generators								# of Generators
	1	2	3	4	5	6	7	8	
1	12	11	0	0	1	4	0	0	28
2	13	15	2	0	4	0	0	1	35
3	15	13	2	6	3	1	1	3	44
4	15	11	0	1	4	5	6	3	45
5	15	13	3	7	5	3	2	1	49
6	10	10	2	5	7	5	6	5	50
7	17	16	1	3	1	7	2	4	51
8	17	10	6	5	2	1	3	7	51
9	12	17	4	7	5	2	0	5	52
10	13	12	5	7	2	5	4	6	54
11	46	45	8	0	5	0	12	16	132
12	40	54	14	8	3	15	9	13	156
13	50	41	19	11	4	4	12	15	156
14	51	58	17	19	16	1	2	1	165
15	43	46	17	15	13	15	6	12	167
16	50	59	8	15	1	18	4	17	172
17	53	50	17	15	16	5	14	12	182
18	45	57	19	7	19	19	5	11	182
19	58	50	15	7	16	18	7	12	183
20	55	48	18	5	18	17	15	11	187

Table 5: System Load (% of Total Capacity) [15]

Time	1	2	3	4	5	6	7	8	9	10	11	12
Load	71%	65%	62%	60%	58%	58%	60%	64%	73%	80%	82%	83%
Time	13	14	15	16	17	18	19	20	21	22	23	24
Load	82%	80%	79%	79%	83%	91%	90%	88%	85%	84%	79%	74%

straints in the formulation, "Strong-1" represents the original MILP formulation plus inequalities (2d) - (2g) as constraints and inequalities in Sections 2 and 3 (i.e., (4) - (13), (18c) - (18r), and (23) - (34)) as user cuts, and "Strong-2" represents the original MILP formulation plus all the strong valid inequalities as user cuts.

In Table 6, the column labelled "Integer OBJ. ($)" provides the best objective value corresponding to the best integer solution obtained from all four different formulations, i.e., "MILP", "Strong", "Strong-1", and "Strong-2", within the time limit. The column labelled "IGap (%)" provides the root-node integrality gaps of "MILP" and "Strong", respectively. The integrality gap is defined as $(Z_{\text{MILP}} - Z_{\text{LP}})/Z_{\text{MILP}}$, where Z_{LP} is the objective value of the LP relaxation and Z_{MILP} is the objective value of the best integer solution, i.e., the value in the column labelled "Integer

Table 6: Computational Performance for the Data Based on [3] and [15]

Inst	Integer OBJ. ($)	IGap (%)		Percent-age (%)	CPU Time(s) (TGap (10^{-4}))				# of Nodes				# of User	
		MILP	Strong		MILP	Strong	Strong-1	Strong-2	MILP	Strong	Strong-1	Strong-2	Strong-1	Strong-2
1	3794100	0.76	0.12	84.94	*** (6.97)	2132.38	*** (1.18)	*** (3.22)	204521	61127	361175	258106	28	315
2	4770702	0.78	0.14	82.56	*** (14.7)	*** (2.54)	*** (4.96)	*** (5.42)	131085	64282	188069	135795	171	654
3	5080033	0.82	0.06	92.62	*** (1.93)	*** (1.15)	1060.64	*** (1.49)	155292	74126	56715	231671	262	884
4	4755459	0.78	0.05	93.05	1921.24	1640.69	510.39	723.99	90231	83913	64685	63963	90	756
5	5354093	0.91	0.04	95.63	*** (1.39)	*** (1.28)	2107.45	*** (1.34)	161350	61669	110622	424892	295	1539
6	4383414	1.09	0.04	95.94	*** (1.4)	*** (1.11)	*** (1.32)	1361.15	546367	149716	944040	324041	209	1581
7	5784804	0.75	0.08	88.7	*** (3.33)	*** (1.82)	*** (2.52)	*** (2.64)	145248	127640	499276	139909	165	1391
8	5136903	0.96	0.04	95.76	*** (1.3)	707.3	590.98	436.67	167748	19879	41011	21481	164	1251
9	5584115	0.91	0.05	95.01	*** (2.2)	*** (1.85)	*** (1.36)	669.01	174427	49344	166697	29233	313	1800
10	5046209	1.15	0.06	94.52	*** (1.93)	*** (1.5)	*** (1.69)	*** (1.48)	140382	119148	383328	544832	252	2129
11	15681132	0.72	0.07	89.92	*** (9.99)	*** (2.25)	*** (3.53)	*** (4.9)	28211	4368	38212	25296	646	3066
12	17079158	0.78	0.04	95.17	*** (8.17)	*** (1.14)	2315.29	*** (1.72)	33515	10457	12621	22343	447	3768
13	16758002	0.85	0.03	96.07	*** (6.57)	*** (1.08)	*** (1.11)	*** (1.73)	41118	10847	26170	27864	660	3656
14	19976963	0.8	0.04	95.01	*** (6.76)	*** (1.33)	*** (1.33)	*** (1.95)	42719	2537	13296	15090	1262	3983
15	17242043	0.93	0.03	97.29	*** (1.9)	1652.43	*** (1.07)	870.97	29106	1654	33767	5577	820	5692
16	19342401	0.74	0.04	94.03	*** (8.15)	3356.66	2084.96	*** (1.1)	60224	5235	11787	22108	867	4038
17	19534390	0.87	0.02	97.35	*** (2.24)	1445.89	2482.16	908.61	13924	769	11988	2152	981	5399
18	19455610	0.85	0.03	96.74	*** (2.24)	*** (1.08)	1958.65	2224.8	16340	3452	12177	12809	661	4829
19	19963596	0.81	0.03	96.3	*** (5.08)	*** (1.13)	*** (1.25)	*** (1.49)	24223	5595	14013	17027	814	4501
20	19571381	0.86	0.03	96.86	*** (1.93)	3595.03	2248.15	666.37	16862	5406	11336	1447	483	6314

OBJ. ($)". We can observe that, our proposed strong valid inequalities tighten the LP relaxation dramatically, with the integrality gap reduction (from "MILP" to "Strong") reported in the column labelled "Percentage (%)". In the column labelled "CPU Time(s) (TGap (10^{-4}))", we report the computational time that CPLEX takes to solve the problem for each approach. For the cases in which CPLEX cannot solve the problem to optimality (i.e., reach the default 0.01% optimality gap) within one hour time limit, we provide the label "***" and accordingly report the terminating gap labelled "TGap (10^{-4})", which indicates the relative gap between the objective value corresponding to the best integer solution and the best lower bound when the time limit is reached. We can observe that all "Strong", "Strong-1", and "Strong-2" approaches perform much better than the original model "MILP". Almost all instances cannot be solved to optimality by "MILP" within one hour limit (except instance 4), while most instances can be solved by at least one of "Strong", "Strong-1", and "Strong-2" approaches with our proposed strong valid inequalities added. The number of explored branch-and-bound nodes is reported in the column labelled "# of Nodes". The final column labelled "# of User" reports the number of user cuts added to solve the problem for

“Strong-1” and “Strong-2”.

4.1.2 Modified IEEE 118-Bus System

For this experiment, there are 54 generators, 118 buses, 186 transmission lines, and 91 load buses in the modified IEEE 118-bus system. We generate 15 instances, each with different load profile. Corresponding to each nominal load d_t^n given in the IEEE 118-bus system, we randomly generate a load $\bar{d}_t^n \in [1.8d_t^n, 2.2d_t^n]$. This random generation process is conducted fifteen times corresponding to each (n,t) to generate the 15 instances. In this experiment, both constraints (36i) and (36j) are included with the system reserve factor r_t set at 5% for each time period $t \in [1,T]_{\mathbb{Z}}$.

Table 7: Computational Performance for the IEEE 118-Bus System

Inst	Integer OBJ. ($)	IGap (%)		Percent -age (%)	CPU Time(s) (TGap (10^{-4}))		# of Nodes		# of User
		MILP	Strong-1		MILP	Strong-1	MILP	Strong-1	
1	3358217	1.54	0.09	94.42	*** (1.39)	1432.02	180121	85936	100
2	3356847	1.37	0.05	96.65	*** (1.43)	2371.36	229259	342774	222
3	3367104	1.61	0.06	96.29	*** (3)	*** (1.8)	159795	136426	340
4	3362632	1.64	0.06	96.26	*** (1.96)	*** (1.37)	272480	238904	225
5	3349280	1.47	0.09	93.97	*** (2.23)	*** (1.47)	150695	373875	299
6	3364177	1.45	0.07	95.28	*** (1.28)	848.11	152427	69191	257
7	3353272	1.58	0.08	95.19	*** (2.29)	*** (1.51)	180557	594986	182
8	3348885	1.27	0.04	97.12	758.44	289.94	54354	28080	215
9	3354399	1.5	0.06	96.02	*** (3.27)	*** (1.9)	127050	102107	199
10	3352652	1.53	0.06	96.21	*** (1.91)	*** (1.38)	191125	187788	280
11	3357921	1.54	0.06	95.85	*** (1.31)	665.88	166568	58687	249
12	3359379	1.55	0.05	96.57	1074.87	405.07	94365	29781	262
13	3359624	1.57	0.07	95.78	*** (1.23)	1162.33	166052	66590	236
14	3362072	1.57	0.06	96.07	671.6	480.58	36746	19262	271
15	3351562	1.51	0.1	93.61	*** (2.12)	2615.75	142626	98899	294

For each instance, we compare the computational performance between “MILP” and “Strong-1”, as defined in Section 4.1.1 and shown in Table 7. The labels in Table 7 are similar to those in Table 6. For “Strong-1”, in the column labelled “IGap (%)”, we report the integrality gap when all the strong valid inequalities are added as constraints. We continue to observe that the strong valid inequalities tighten the LP relaxation significantly, with about 95% reduction between the integrality gaps of the original MILP model and the model with the strong valid inequalities added. “Strong-1” also performs much better in terms of the computational time and terminating gap reported in the column labelled “CPU Time(s) (TGap (10^{-4}))”. The number of explored branch-and-bound nodes is also reduced for most instances as indicated in the column labelled "# of Nodes". The final column reports the number of user cuts added in the formulation "Strong-1".

4.2 Self-Scheduling Unit Commitment Problem

For the self-scheduling unit commitment problem in which a single generator is considered, we first provide the mathematical formulation and then report the computational results for the eight single generators described in Table 3.

For the mathematical formulation, besides the notation defined in Section 1, we let p_t represent the electricity price at time period t, $f(x_t)$ represent the generation cost corresponding to the generation amount of x_t at t, and SU (SD) represent the start-up (shut-down) cost. Accordingly, the self-scheduling unit commitment problem can be described as follows:

$$\max_{x,y,u} \quad \sum_{t=1}^{T}\big(p_t x_t - f(x_t)\big) - \sum_{t=2}^{T}\big(\mathrm{SU} u_t + \mathrm{SD}(y_{t-1} - y_t + u_t)\big) \tag{37a}$$

$$\text{s.t.} \quad \sum_{i=t-L+1}^{t} u_i \le y_t, \quad \forall t \in [L+1, T]_{\mathbb{Z}}, \tag{37b}$$

$$\sum_{i=t-\ell+1}^{t} u_i \le 1 - y_{t-\ell}, \ \forall t \in [\ell+1, T]_{\mathbb{Z}}, \tag{37c}$$

$$-y_{t-1} + y_t - u_t \le 0, \quad \forall t \in [2, T]_{\mathbb{Z}}, \tag{37d}$$

$$\underline{C} y_t \le x_t \le \overline{C} y_t, \quad \forall t \in [1, T]_{\mathbb{Z}}, \tag{37e}$$

$$x_t - x_{t-1} \le V y_{t-1} + \overline{V}(1 - y_{t-1}), \quad \forall t \in [2, T]_{\mathbb{Z}}, \tag{37f}$$

$$x_{t-1} - x_t \le V y_t + \overline{V}(1 - y_t), \quad \forall t \in [2, T]_{\mathbb{Z}}, \tag{37g}$$

$$y_t \in \{0,1\}, \ \forall t \in [1, T]_{\mathbb{Z}}; \ u_t \in \{0,1\}, \ \forall t \in [2, T]_{\mathbb{Z}}, \tag{37h}$$

where the objective is to maximize the total profit, i.e., the total revenue from selling electricity minus the total cost from producing electricity. The generation cost function $f(x_t) = a(x_t)^2 + bx_t + c$ can be approximated by a piecewise linear function and accordingly the above formulation can be reformulated as an MILP formulation. Constraints (37b) (resp. (37c)) represent the minimum-up (resp. minimum-down) time restrictions, constraints (37d) represent the relationship between y and u, constraints (37e) represent the generation upper and lower bounds, and constraints (37f) (resp. (37g)) represent the ramp-up (resp. ramp-down) rate limits.

For each generator in Table 3, we test three instances with the price p_t, $\forall t \in [1, T]_{\mathbb{Z}}$ with

$T = 10000$, randomly generated and report the average result over these three instances. For generators 1 and 2, we randomly generate $p_t \in [0, 35]$; for generators 3 and 4, we randomly generate $p_t \in [0, 41]$; for generator 5, we randomly generate $p_t \in [0, 44]$; for generator 6, we randomly generate $p_t \in [0, 48]$; for generator 7, we randomly generate $p_t \in [0, 60]$; for generator 8, we randomly generate $p_t \in [0, 67]$. These price ranges are selected based on the generator data in Table 3. We compare two formulations for each generator: "MILP" and "Strong" that are similarly defined in Section 4.1.1, i.e, "MILP" represents the original MILP formulation described in (37), "Strong" represents the original MILP formulation plus our proposed strong valid inequalities in Sections 1 - 3 (i.e., (2d) - (2g), (4) - (13), (18c) - (18r), and (23) - (34)) as constraints.

Table 8: Computational Performance for Eight Single Generators

Generator	IGap (%)		Percent -age (%)	CPU Time(s) (TGap (%))		# of Nodes	
	MILP	Strong		MILP	Strong	MILP	Strong
1	30.96	0.07	99.76	1612.55	84.4	11396	0
2	39.07	0.09	99.78	1557.55	66.11	11716	0
3	56.77	0.16	99.71	*** (0.91) [3]	82.16	49493	0
4	53.31	0.15	99.71	*** (0.82) [3]	104.69	44752	0
5	32.2	0.26	99.2	*** (0.1) [3]	201.97	47635	55
6	57.69	0.58	98.99	*** (0.15) [3]	109.45	58441	0
7	50.18	0.19	99.62	1387.44	88.96	11640	0
8	81.63	6.54	91.99	*** (3.49) [3]	403.65	36612	591

We report the computational results in Table 8. The integrality gaps of two formulations are reported in the column labelled "IGap (%)", in which the integrality gap is defined as $(Z_{\text{LP}} - Z_{\text{MILP}})/Z_{\text{LP}}$, where Z_{LP} is the objective value of the LP relaxation and Z_{MILP} is the objective value corresponding to the best integer solution we obtained from these two formulations within the time limit. Since the self-scheduling unit commitment problem is a maximization problem, the integrality gap definition is different from that defined for the network-constrained unit commitment problem in Section 4.1. We can observe that the strong valid inequalities tighten the LP relaxation dramatically, as the gap reduction between these two formulations is reported in the column labelled "Percentage (%)". In the column labelled "CPU Time(s) (TGap (%))", we report the computational time that CPLEX needs to solve each instance. For the case in which CPLEX cannot solve it to optimality (i.e., 0.01%) within one hour time limit, we use "***" to indicate it and report the terminating gap ("TGap (%)"). The number in the square bracket indicates the number of instances not solved to default optimality when the one hour time limit is reached. The column labelled "# of Nodes" reports explored branch-and-bound nodes for each formulation. From the table, we can observe significant advantages of applying our derived strong valid inequalities as cutting planes. For most cases, the "Strong" formulation can be solved at the root node without getting into the branch-and-bound procedure.

5 Conclusion

In this paper, we performed the polyhedral study of the integrated minimum-up/-down time and ramping polytope for the unit commitment problem. We derived strong valid inequalities to strengthen the original MILP formulation. In particular, our derived valid inequalities are strong enough to provide the convex hull description for the polytope up to three time periods with variant minimum-up/-down time limits. To the best of our knowledge, this is the first study that provides the convex hull description for the three-period cases. In addition, our derived strong valid inequalities for the general multi-period case cover one, two, and three continuous variables, respectively. All these inequalities are facet-defining under certain conditions. Furthermore, these inequalities are in polynomial size in the order of at most $\mathcal{O}(T^2)$. Thus, the separation procedure is not needed. Finally, the computational results showed the high efficiency of our proposed strong valid inequalities by solving both the network-constrained unit commitment problem for a system operator and the self-scheduling unit commitment problem for a market participant under various data settings.

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Appendix A Proof for Two-period Convex Hull

To begin with, we denote P_2 as the original polytope P with $T = 2$ and $L = \ell = 1$. To show $Q_2 = \text{conv}(P_2)$, we show that 1) Q_2 is full-dimensional; 2) inequalities (2a) - (2g) are valid for conv(P_2); 3) (2a) - (2g) are facet-defining for conv(P_2); and 4) every extreme point in Q_2 is integral in y and u.

Claim 1 *Q_2 is full-dimensional.*

Proof: We prove that $\dim(Q_2) = 5$, because there are five decision variables. We generate six affinely independent points in Q_2. Since $0 \in Q_2$, we only need to generate the following five linearly independent points in Q_2: 1) $(x_1, x_2, y_1, y_2, u_2) = (\underline{C}, 0, 1, 0, 0)$; 2) $(x_1, x_2, y_1, y_2, u_2) = (\underline{C} + \epsilon, 0, 1, 0, 0)$; 3) $(x_1, x_2, y_1, y_2, u_2) = (0, \underline{C}, 0, 1, 1)$; 4) $(x_1, x_2, y_1, y_2, u_2) = (0, \underline{C} + \epsilon, 0, 1, 1)$; 5) $(x_1, x_2, y_1, y_2, u_2) = (\underline{C}, \underline{C} + V, 1, 1, 0)$. ■

Claim 2 *Inequalities* (2a) - (2g) *are valid for conv(P_2).*

Proof: Since inequalities (2a) - (2c) are from the original polytope P_2, here we only need to provide the validity proofs for (2d) - (2g).

For the validity of inequality (2d), we discuss the following three cases: 1) if $y_1 = 0$, then $x_1 = 0$ due to (1d). (2d) converts to $x_1 \leq (\overline{C} - \overline{V})(y_2 - u_2)$, which is valid due to (1a); 2) if $y_1 = y_2 = 1$, then (2d) converts to $x_1 \leq \overline{C}$, which is valid due to (1e); 3) if $y_1 = y_2 = 0$, then it is clear that (2d) is valid.

For the validity of inequality (2e), we discuss the following two cases: 1) if $u_2 = 0$, then (2e) converts to $x_2 \leq \overline{C} y_2$, which is valid due to (1e); 2) if $u_2 = 1$, then (2e) converts to $x_2 \leq \overline{V}$, which is valid due to ramp-up constraints (1f).

For the validity of inequality (2f), we discuss the following three cases: 1) if $y_2 = 0$, then (2f) converts to $x_1 \geq \underline{C} y_1$, which is valid due to (1d); 2) if $y_1 = y_2 = 1$, then (2f) converts to $x_2 - x_1 \leq V$, which is valid due to ramp-up constraints (1f); 3) if $y_1 = 0$ and $y_2 = 1$, then (2f) converts to $x_2 \leq \overline{V}$, which is valid due to (1f).

For the validity of inequality (2g), we discuss the following four cases: 1) if $y_1 = y_2 = 1$, then (2g) converts to $x_1 - x_2 \leq V$, which is valid due to ramp-down constraints (1g); 2) if $y_1 = 1$ and

$y_2 = 0$, then (2g) converts to $x_1 \leq \overline{V}$, which is valid due to ramp-down constraints (1g); 3) if $y_1 = 0$ and $y_2 = 1$, then (2g) converts to $x_2 \leq \overline{V}$, which is valid due to (1f); 4) if $y_1 = y_2 = 0$, then it is clear that (2g) is valid. ■

Claim 3 *Inequalities* (2a) - (2g) *are facet-defining for conv(P_2).*

Proof: The facet-defining proofs for inequalities (2a) - (2c) are trivial and thus omitted here. For inequalities (2d) - (2g), we provide five affinely independent points in conv(P_2) that satisfy each inequality at equality. Since $0 \in \text{conv}(P_2)$, it is sufficient to generate other four linearly independent points in P_2, as shown in Tables 9 and 10.

Table 9: Linearly independent points for inequalities (2d) and (2e)

(2d)					(2e)				
x_1	x_2	y_1	y_2	u_2	x_1	x_2	y_1	y_2	u_2
$\overline{V}$	0	1	0	0	$\underline{C}$	0	1	0	0
$\overline{C}$	$\overline{C}$	1	1	0	$\underline{C} + \epsilon$	0	1	0	0
$\overline{C}$	$\overline{C} - \epsilon$	1	1	0	$\overline{C}$	$\overline{C}$	1	1	0
0	$\underline{C}$	0	1	1	0	$\overline{V}$	0	1	1

Table 10: Linearly independent points for inequalities (2f) and (2g)

(2f)					(2g)				
x_1	x_2	y_1	y_2	u_2	x_1	x_2	y_1	y_2	u_2
$\underline{C}$	0	1	0	0	$\overline{V}$	0	1	0	0
$\underline{C}$	$\underline{C} + V$	1	1	0	$\underline{C} + V$	$\underline{C}$	1	1	0
$\underline{C} + \epsilon$	$\underline{C} + V + \epsilon$	1	1	0	$\underline{C} + V + \epsilon$	$\underline{C} + \epsilon$	1	1	0
0	$\overline{V}$	0	1	1	0	$\overline{V}$	0	1	1

■

Claim 4 *Every extreme point in Q_2 is integral in y and u.*

Proof: It is sufficient to prove that every point $z = (x_1, x_2, y_1, y_2, u_2) \in Q_2$ can be written as $z = \sum_{s \in S} \lambda_s z^s$ for some $\lambda_s \geq 0$ and $\sum_{s \in S} \lambda_s = 1$, where $z^s \in Q_2, s \in S$ with y and u binary.

For a given $z = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{u}_2) \in Q_2$, we can pick $z^1, \cdots, z^4 \in Q_2$ such that $z^1 = (\hat{x}_1, 0, 1, 0, 0)$, $z^2 = (0, \hat{x}_2, 0, 1, 1)$, $z^3 = (\hat{x}_3, \hat{x}_4, 1, 1, 0)$, and $z^4 = (0, 0, 0, 0, 0)$. In addition, we let $\lambda_1 = \bar{y}_1 - \bar{y}_2 + \bar{u}_2$, $\lambda_2 = \bar{u}_2$, $\lambda_3 = \bar{y}_2 - \bar{u}_2$, and $\lambda_4 = 1 - \bar{y}_1 - \bar{u}_2$. Firs of all, it is clear that $\sum_{s=1}^{4} \lambda_s = 1$ and $\lambda_s \geq 0$ for $\forall s = 1, \cdots, 4$ due to (2a) - (2b).

Next, it is also obvious that $\bar{y}_i = y_i(z) = \sum_{s=1}^4 \lambda_s y_i(z^s)$ for $i = 1, 2$ and $\bar{u}_2 = u_2(z) = \sum_{s=1}^4 \lambda_s u_2(z^s)$. In the following, we decide the values of $\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4$ and show $\bar{x}_i = x_i(z) = \sum_{s=1}^4 \lambda_s x_i(z^s)$ for $i = 1, 2$, i.e., $\bar{x}_1 = \lambda_1 \hat{x}_1 + \lambda_3 \hat{x}_3$ and $\bar{x}_2 = \lambda_2 \hat{x}_2 + \lambda_3 \hat{x}_4$. Note that y and u are given in $z^1, \ldots, z^4$, the corresponding feasible region for $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)$ can be described as set $A = \{(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \in \mathbb{R}^4 : \underline{C} \leq \hat{x}_1 \leq \overline{V},\ \underline{C} \leq \hat{x}_2 \leq \overline{V},\ \underline{C} \leq \hat{x}_i \leq \overline{C} (i = 3, 4),\ -V \leq \hat{x}_3 - \hat{x}_4 \leq V\}$.

To show $\bar{x}_i = \sum_{s=1}^4 \lambda_s x_i(z^s)$ for $i = 1, 2$, equivalently we prove that fixing $(\bar{y}_1, \bar{y}_2, \bar{u}_2) \in B = \{(\bar{y}_1, \bar{y}_2, \bar{u}_2) \in [0, 1]^3 : \text{(2a)} - \text{(2b)}\}$, for $\forall (\bar{x}_1, \bar{x}_2)$ belonging to the set

$$C = \Big\{ (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 : \quad \bar{x}_1 \geq \underline{C} \bar{y}_1, \tag{38a}$$

$$\bar{x}_2 \geq \underline{C} \bar{y}_2, \tag{38b}$$

$$\bar{x}_1 \leq \overline{V} \bar{y}_1 + (\overline{C} - \overline{V})(\bar{y}_2 - \bar{u}_2), \tag{38c}$$

$$\bar{x}_2 \leq \overline{C} \bar{y}_2 - (\overline{C} - \overline{V}) \bar{u}_2, \tag{38d}$$

$$\bar{x}_2 - \bar{x}_1 \leq (\underline{C} + V) \bar{y}_2 - \underline{C} \bar{y}_1 - (\underline{C} + V - \overline{V}) \bar{u}_2, \tag{38e}$$

$$\bar{x}_1 - \bar{x}_2 \leq \overline{V} \bar{y}_1 - (\overline{V} - V) \bar{y}_2 - (\underline{C} + V - \overline{V}) \bar{u}_2 \Big\}, \tag{38f}$$

there exists $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \in A$ such that

$$\bar{x}_1 = \lambda_1 \hat{x}_1 + \lambda_3 \hat{x}_3, \quad \bar{x}_2 = \lambda_2 \hat{x}_2 + \lambda_3 \hat{x}_4, \tag{39}$$

i.e., the linear transformation $F : A \to C$ is surjective, where

$$F = \begin{pmatrix} \bar{y}_1 - \bar{y}_2 + \bar{u}_2 & 0 & \bar{y}_2 - \bar{u}_2 & 0 \\ 0 & \bar{u}_2 & 0 & \bar{y}_2 - \bar{u}_2 \end{pmatrix}.$$

Since C is a closed and bounded polytope, any point can be expressed as a convex combination of the extreme points in C. Accordingly, we only need to show that for any extreme point $w^i \in C$ $(i = 1, \cdots, M)$, there exists a point $p^i \in A$ such that $Fp^i = w^i$, where M represents the number of extreme points in C (because for an arbitrary point $w \in C$, which can be rewritten as $w = \sum_{i=1}^M \mu_i w^i$ and $\sum_{i=1}^M \mu_i = 1$, there exists $p = \sum_{i=1}^M \mu_i p_i \in A$ such that $Fp = w$ due to the linearity of F and the convexity of A). Since it is difficult to enumerate all the extreme points in C, in the following proof we show the conclusion holds for any point in the faces of C, i.e., satisfying one of (38a) - (38f) at equality, which implies the conclusion holds for extreme points.

Satisfying (38a) **at equality.** For this case, substituting $\bar{x}_1 = \underline{C} \bar{y}_1$ into (38b) - (38f), we obtain $\underline{C} \bar{y}_2 \leq \bar{x}_2 \leq (\underline{C} + V) \bar{y}_2 - (\underline{C} + V - \overline{V}) \bar{u}_2$ because $\underline{C} \bar{y}_2 \geq (\underline{C} - \overline{V}) \bar{y}_1 + (\overline{V} - V) \bar{y}_2 + (\underline{C} + V - \overline{V}) \bar{u}_2$

and $(\underline{C}+V)\bar{y}_2 - (\underline{C}+V-\overline{V})\bar{u}_2 \le \overline{C}\bar{y}_2 - (\overline{C}-V)\bar{u}_2$.

First, by letting $\hat{x}_1 = \hat{x}_3 = \underline{C}$, it is easy to check that $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_3\hat{x}_3$, following (39). Note here that once $\hat{x}_1$ and $\hat{x}_3$ are fixed at $\underline{C}$, the corresponding feasible region for $\hat{x}_2$ and $\hat{x}_4$ can be described as $A' = \{(\hat{x}_2, \hat{x}_4) \in \mathbb{R}^2 : \underline{C} \le \hat{x}_2 \le \overline{V},\ \underline{C} \le \hat{x}_4 \le \underline{C}+V\}$. Now we decide the values of $\hat{x}_2$ and $\hat{x}_4$ in the following ways: 1) if $\bar{x}_2 \le (\underline{C}+V)\bar{y}_2 - V\bar{u}_2$, let $\hat{x}_2 = \underline{C}$, $\hat{x}_4 = (\bar{x}_2 - \underline{C}\bar{u}_2)/(\bar{y}_2 - \bar{u}_2)$ if $\bar{y}_2 - \bar{u}_2 > 0$ or $\hat{x}_4$ free otherwise; 2) if $\bar{x}_2 > (\underline{C}+V)\bar{y}_2 - V\bar{u}_2$, let $\hat{x}_4 = \underline{C}+V$, $\hat{x}_2 = (\bar{x}_2 - (\underline{C}+V^+)(\bar{y}_2 - \bar{u}_2))/\bar{u}_2$ if $\bar{u}_2 > 0$ or $\hat{x}_2$ free otherwise. Note here that in both cases $(\hat{x}_2, \hat{x}_4) \in A'$ and $\bar{x}_2 = \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$, following (39).

Similar analyses hold for (38b) - (38d) as follows.

Satisfying (38b) **at equality.** Through substituting $\bar{x}_2 = \underline{C}\bar{y}_2$ into (38b) - (38f), we obtain $\underline{C}\bar{y}_1 \le \bar{x}_1 \le \overline{V}\bar{y}_1 + (\underline{C}+V-\overline{V})(\bar{y}_2 - \bar{u}_2)$. By letting $\hat{x}_2 = \hat{x}_4 = \underline{C}$, we have $\bar{x}_2 = \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$, following (39). Once $\hat{x}_2$ and $\hat{x}_4$ are fixed at $\underline{C}$, the corresponding feasible region for $\hat{x}_1$ and $\hat{x}_3$ can be described as $A' = \{(\hat{x}_1, \hat{x}_3) \in \mathbb{R}^2 : \underline{C} \le \hat{x}_1 \le \overline{V},\ \underline{C} \le \hat{x}_3 \le \underline{C}+V\}$. Now we decide the values of $\hat{x}_1$ and $\hat{x}_3$ in the following ways: 1) if $\bar{x}_1 \le \underline{C}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2)$, let $\hat{x}_1 = \underline{C}$, $\hat{x}_3 = (\bar{x}_1 - \underline{C}(\bar{y}_1 - \bar{y}_2 + \bar{u}_2))/(\bar{y}_2 - \bar{u}_2)$ if $\bar{y}_2 - \bar{u}_2 > 0$ or $\hat{x}_3$ free otherwise; 2) if $\bar{x}_1 > \underline{C}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2)$, let $\hat{x}_3 = \underline{C}+V$, $\hat{x}_1 = (\bar{x}_1 - (\underline{C}+V)(\bar{y}_2 - \bar{u}_2))/(\bar{y}_1 - \bar{y}_2 + \bar{u}_2)$ if $\bar{y}_1 - \bar{y}_2 + \bar{u}_2 > 0$ or $\hat{x}_1$ free otherwise. For both cases, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_3\hat{x}_3$, following (39).

Satisfying (38c) **at equality.** Through substituting $\bar{x}_1 = \overline{V}\bar{y}_1 + (\overline{C}-\overline{V})(\bar{y}_2 - \bar{u}_2)$ into (38b) - (38f), we obtain $(\overline{C}-V)\bar{y}_2 - (\overline{C}-\overline{V}-V)\bar{u}_2 \le \bar{x}_2 \le \overline{C}\bar{y}_2 - (\overline{C}-\overline{V})\bar{u}_2$. By letting $\hat{x}_1 = \overline{V}$ and $\hat{x}_3 = \overline{C}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_3\hat{x}_3$, following (39). Once $\hat{x}_1$ and $\hat{x}_3$ are fixed, the corresponding feasible region for $\hat{x}_2$ and $\hat{x}_4$ can be described as $A' = \{(\hat{x}_2, \hat{x}_4) \in \mathbb{R}^2 : \underline{C} \le \hat{x}_2 \le \overline{V},\ \overline{C}-V \le \hat{x}_4 \le \overline{C}\}$. Now we decide the values of $\hat{x}_2$ and $\hat{x}_4$ in the following ways: 1) if $\bar{x}_2 \le \overline{C}\bar{y}_2 - (\overline{C}-\underline{C})\bar{u}_2$, let $\hat{x}_2 = \underline{C}$, $\hat{x}_4 = (\bar{x}_2 - \underline{C}\bar{u}_2)/(\bar{y}_2 - \bar{u}_2)$ if $\bar{y}_2 - \bar{u}_2 > 0$ or $\hat{x}_4$ free otherwise; 2) if $\bar{x}_2 > \overline{C}\bar{y}_2 - (\overline{C}-\underline{C})\bar{u}_2$, let $\hat{x}_4 = \overline{C}$, $\hat{x}_2 = (\bar{x}_2 - \overline{C}(\bar{y}_2 - \bar{u}_2))/\bar{u}_2$ if $\bar{u}_2 > 0$ or $\hat{x}_2$ free otherwise. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$, following (39).

Satisfying (38d) **at equality.** Through substituting $\bar{x}_2 \le \overline{C}\bar{y}_2 - (\overline{C}-\overline{V})\bar{u}_2$ into (38b) - (38f), we obtain $\underline{C}\bar{y}_1 + (\overline{C}-\underline{C}-V)(\bar{y}_2 - \bar{u}_2) \le \bar{x}_1 \le \overline{V}\bar{y}_1 + (\overline{C}-\overline{V})(\bar{y}_2 - \bar{u}_2)$. By letting $\hat{x}_2 = \overline{V}, \hat{x}_4 = \overline{C}$, we have $\bar{x}_2 = \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$, following (39). Once $\hat{x}_2$ and $\hat{x}_4$ are fixed, the corresponding feasible region for $\hat{x}_1$ and $\hat{x}_3$ can be described as $A' = \{(\hat{x}_1, \hat{x}_3) \in \mathbb{R}^2 : \underline{C} \le \hat{x}_1 \le \overline{V},\ \overline{C}-V \le \hat{x}_3 \le \overline{C}\}$. Now we decide the values of $\hat{x}_1$ and $\hat{x}_3$ in the following ways: 1) if $\bar{x}_1 \leq \overline{C}(\bar{y}_2 - \bar{u}_2) + \underline{C}(\bar{y}_1 - \bar{y}_2 + \bar{u}_2)$, let $\hat{x}_1 = \underline{C}$, $\hat{x}_3 = (\bar{x}_1 - \underline{C}(\bar{y}_1 - \bar{y}_2 + \bar{u}_2))/(\bar{y}_2 - \bar{u}_2)$ if $\bar{y}_2 - \bar{u}_2 > 0$ or $\hat{x}_3$ free otherwise; 2) if $\bar{x}_1 > \overline{C}(\bar{y}_2 - \bar{u}_2) + \underline{C}(\bar{y}_1 - \bar{y}_2 + \bar{u}_2)$, let $\hat{x}_3 = \overline{C}$, $\hat{x}_1 = (\bar{x}_1 - \overline{C}(\bar{y}_2 - \bar{u}_2))/(\bar{y}_1 - \bar{y}_2 + \bar{u}_2)$ if $\bar{y}_1 - \bar{y}_2 + \bar{u}_2 > 0$ or $\hat{x}_1$ free otherwise. For both cases, we have $\bar{x}_1 = \lambda_1 \hat{x}_1 + \lambda_3 \hat{x}_3$, following (39).

Satisfying (38e) **at equality.** For this case, substituting $\bar{x}_2 = \bar{x}_1 + (\underline{C} + V)\bar{y}_2 - \underline{C}\bar{y}_1 - (\underline{C} + V - \overline{V})\bar{u}_2$ to (38a) - (38d), we obtain $\underline{C}\bar{y}_1 \leq \bar{x}_1 \leq \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_2 - \bar{u}_2)$ because $\underline{C}\bar{y}_1 \geq \underline{C}\bar{y}_1 - V\bar{y}_2 + (\underline{C} + V - \overline{V})\bar{u}_2$ and $\underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_2 - \bar{u}_2) \leq \overline{V}\bar{y}_1 - (\overline{C} - \overline{V})(\bar{y}_2 - \bar{u}_2)$.

First, by letting $\hat{x}_1 = \underline{C}, \hat{x}_2 = \overline{V}, \hat{x}_4 - \hat{x}_3 = V$, it is easy to check that $\bar{x}_2 - \bar{x}_1 = (\lambda_2 \hat{x}_2 + \lambda_3 \hat{x}_4) - (\lambda_1 \hat{x}_1 + \lambda_3 \hat{x}_3)$. Then, the feasible region for $\hat{x}_3$ can be described as $A' = \{\hat{x}_3 \in \mathbb{R} : \underline{C} \leq \hat{x}_3 \leq \overline{C} - V\}$. Now we let $\hat{x}_3 = (\bar{x}_1 - \underline{C}(\bar{y}_1 - \bar{y}_2 + \bar{u}_2))/(\bar{y}_2 - \bar{u}_2)$ if $\bar{y}_2 - \bar{u}_2 > 0$ or $\hat{x}_4$ free otherwise. Note that $\hat{x}_3 \in A'$ and (39) holds.

Similar analysis holds for (38f) as follows.

Satisfying (38f) **at equality.** Through substituting $\bar{x}_1 = \bar{x}_2 + \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2$ to (38a) - (38d), we obtain $\underline{C}\bar{y}_2 \leq \bar{x}_2 \leq (\overline{C} - V)\bar{y}_2 - (\overline{C} - \underline{C} - V)\bar{u}_2$ By letting $\hat{x}_1 = \overline{V}, \hat{x}_2 = \underline{C}, \hat{x}_3 - \hat{x}_4 = V$, it is easy to check that $\bar{x}_1 - \bar{x}_2 = (\lambda_1 x_1 + \lambda_3 x_3) - (\lambda_2 x_2 + \lambda_3 x_4)$. Then, the feasible region for $\hat{x}_4$ can be described as $A' = \{x_4 \in \mathbb{R} : \underline{C} \leq \hat{x}_4 \leq \overline{C} - V\}$. Now we let $\hat{x}_4 = (\bar{x}_2 - \underline{C}\bar{u}_2)/(\bar{y}_2 - \bar{u}_2)$ if $\bar{y}_2 - \bar{u}_2 > 0$ or $\hat{x}_4$ free otherwise. Note that $\hat{x}_4 \in A'$ and (39) holds.

Therefore, we have $z = \sum_{s=1}^{4} \lambda_s z^s$ with $\sum_{s=1}^{4} \lambda_s = 1$ and $\lambda_s \geq 0$ for $\forall s = 1, \cdots, 4$ and hence the claim holds. ∎

The above four claims immediately give us the two-period convex hull results.

Theorem 1 $Q_2 =$ *conv*(P_2)

Proof: Basically, we have both P_2 and Q_2 are bounded from their formulation representations. Since all the inequalities in Q_2 are valid and facet-defining for conv(P_2), we have $Q_2 \supseteq$ conv(P_2). Furthermore, we have that any extreme point in Q_2 is integral in y and u. Thus $Q_2 =$ conv(P_2). ∎

Appendix B Proofs for Three-period Formulations

B.1 Proof for Proposition 3

Proof: For inequalities (6) - (13), we provide seven linearly independent points in conv(P_3^2) that satisfy each inequality at equality in Tables 11 - 14.

Table 11: Linearly independent points for inequalities (6) and (7)

(6)								(7)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\underline{C}$	0	0	1	0	0	0	0	$\underline{C}$	0	0	1	0	0	0	0
$\underline{C}+\epsilon$	0	0	1	0	0	0	0	$\underline{C}$	$\overline{V}$	0	1	1	0	0	0
$\underline{C}$	$\underline{C}$	0	1	1	0	0	0	$\underline{C}$	$\underline{C}+V$	$\underline{C}$	1	1	1	0	0
$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	0	1	1	0	0	0	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	1	1	1	0	0
$\overline{C}$	$\overline{C}$	$\overline{C}$	1	1	1	0	0	0	$\overline{V}$	$\overline{V}$	0	1	1	1	0
0	$\overline{V}$	$\overline{V}+V$	0	1	1	1	0	0	0	$\underline{C}$	0	0	1	0	1
0	0	$\overline{V}$	0	0	1	0	1	0	0	$\underline{C}+\epsilon$	0	0	1	0	1

Table 12: Linearly independent points for inequalities (8) and (9)

(8)								(9)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\underline{C}$	0	0	1	0	0	0	0	$\overline{V}$	0	0	1	0	0	0	0
$\underline{C}+\epsilon$	0	0	1	0	0	0	0	$\underline{C}+V$	$\underline{C}$	0	1	1	0	0	0
$\underline{C}$	$\underline{C}$	0	1	1	0	0	0	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	0	1	1	0	0	0
$\underline{C}$	$\underline{C}$	$\underline{C}+V$	1	1	1	0	0	$\underline{C}+V$	$\underline{C}$	$\underline{C}$	1	1	1	0	0
$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	1	1	1	0	0	0	$\underline{C}$	$\underline{C}$	0	1	1	1	0
0	$\underline{C}$	$\underline{C}+V$	0	1	1	1	0	0	0	$\underline{C}$	0	0	1	0	1
0	0	$\overline{V}$	0	0	1	0	1	0	0	$\underline{C}+\epsilon$	0	0	1	0	1

Table 13: Linearly independent points for inequalities (10) and (11)

(10)								(11)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\underline{C}$	0	0	1	0	0	0	0	$\underline{C}$	0	0	1	0	0	0	0
$\underline{C}+\epsilon$	0	0	1	0	0	0	0	$\underline{C}$	$\underline{C}$	0	1	1	0	0	0
$\overline{V}$	$\overline{V}$	0	1	1	0	0	0	$\underline{C}$	$\underline{C}+\epsilon$	0	1	1	0	0	0
$\underline{C}+V$	$\underline{C}+V$	$\underline{C}$	1	1	1	0	0	$\underline{C}$	$\underline{C}+V$	$\underline{C}+2V$	1	1	1	0	0
$\underline{C}+V+\epsilon$	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	1	1	1	0	0	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	$\underline{C}+2V+\epsilon$	1	1	1	0	0
0	$\overline{V}$	$\underline{C}$	0	1	1	1	0	0	$\overline{V}$	$\overline{V}+V$	0	1	1	1	0
0	0	$\underline{C}$	0	0	1	0	1	0	0	$\overline{V}$	0	0	1	0	1

■

Table 14: Linearly independent points for inequalities (12) and (13)

(12)								(13)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\overline{V}$	0	0	1	0	0	0	0	$\overline{V}$	0	0	1	0	0	0	0
$\overline{V}+V$	$\overline{V}$	0	1	1	0	0	0	$\underline{C}+V$	$\underline{C}$	0	1	1	0	0	0
$\underline{C}+2V$	$\underline{C}+V$	$\underline{C}$	1	1	1	0	0	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	0	1	1	0	0	0
$\underline{C}+2V+\epsilon$	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	1	1	1	0	0	$\overline{C}$	$\overline{C}-V$	$\overline{C}$	1	1	1	0	0
0	$\underline{C}$	$\underline{C}$	0	1	1	1	0	0	$\underline{C}$	$\underline{C}+V$	0	1	1	1	0
0	$\underline{C}+\epsilon$	$\underline{C}$	0	1	1	1	0	0	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	0	1	1	1	0
0	0	$\underline{C}$	0	0	1	0	1	0	0	$\overline{V}$	0	0	1	0	1

B.2 Proof for Proposition 4

Proof: **Satisfying** (17b) **at equality**. For this case, substituting $\bar{x}_1 = \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ into (17e) - (17k), we obtain the feasible region of $(\bar{x}_2, \bar{x}_3)$ as $C' = \{(\bar{x}_2, \bar{x}_3) \in \mathbb{R}^2 : \overline{V}\bar{y}_2 - (\overline{C} - \underline{C} - V)\bar{u}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) \le \bar{x}_2 \le \overline{V}\bar{y}_2 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \le \bar{x}_3 \le \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2,\ \bar{x}_3 - \bar{x}_2 \le (\overline{V} + V)\bar{y}_3 - \overline{V}\bar{y}_2 - V\bar{u}_3,\ \bar{x}_2 - \bar{x}_3 \le \overline{V}\bar{y}_2 - \underline{C}\bar{y}_3 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)\}$.

First, by letting $\hat{x}_1 = \overline{V}$, $\hat{x}_2 = \overline{V} + V$, and $\hat{x}_4 = \overline{C}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. Then (16) holds for $\bar{x}_1$. Then the corresponding feasible region for $(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)$ can be described as set $A' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^6 : \hat{x}_3 = \overline{V},\ \overline{C} - V \le \hat{x}_5 \le \overline{C},\ \underline{C} \le \hat{x}_6 \le \overline{C},\ -V \le \hat{x}_6 - \hat{x}_5 \le V,\ \underline{C} \le \hat{x}_7 \le \overline{V},\ \underline{C} \le \hat{x}_8 \le \overline{V} + V,\ \underline{C} - \overline{V} \le \hat{x}_8 - \hat{x}_7 \le V,\ \underline{C} \le \hat{x}_9 \le \overline{V}\}$. We consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)$ from A' as follows.

1) Satisfying $\bar{x}_2 \ge \overline{V}\bar{y}_2 - (\overline{C} - \underline{C} - V)\bar{u}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3)$ at equality. We obtain $\underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \le \bar{x}_3 \le \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \underline{C} - V)\bar{u}_2$. By letting $\hat{x}_3 = \overline{V}$, $\hat{x}_5 = \overline{C} - V$, and $\hat{x}_7 = \underline{C}$, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. Then (16) holds for $\bar{x}_2$. As a result, we have $(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \overline{C} - 2V \le \hat{x}_6 \le \overline{C},\ \underline{C} \le \hat{x}_8 \le \underline{C} + V,\ \underline{C} \le \hat{x}_9 \le \overline{V}\}$. If $\bar{x}_3 = \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_6 = \overline{C} - 2V$ and $\hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 = \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \underline{C} - V)\bar{u}_2$, we let $\hat{x}_6 = \overline{C}$, $\hat{x}_8 = \underline{C} + V$, and $\hat{x}_9 = \overline{V}$. In this way, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. Then (16) holds for $\bar{x}_3$.

2) Satisfying $\bar{x}_2 \le \overline{V}\bar{y}_2 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \le \bar{x}_3 \le \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$. By letting $\hat{x}_3 = \hat{x}_7 = \overline{V}$ and $\hat{x}_5 = \overline{C}$, we have $\bar{x}_2 = \lambda_2\hat{x}_3 +$

$\lambda_3\hat{x}_5+\lambda_4\hat{x}_7$. Then (16) holds for $\bar{x}_2$. As a result, we have $(\hat{x}_6,\hat{x}_8,\hat{x}_9)\in A''=\{(\hat{x}_6,\hat{x}_8,\hat{x}_9)\in\mathbb{R}^3:\overline{C}-V\le\hat{x}_6\le\overline{C},\ \underline{C}\le\hat{x}_8\le\overline{V}+V,\ \underline{C}\le\hat{x}_9\le\overline{V}\}$. If $\bar{x}_3=\underline{C}\bar{y}_3+(\overline{C}-\underline{C}-V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)$, we let $\hat{x}_6=\overline{C}-V$ and $\hat{x}_8=\hat{x}_9=\underline{C}$; if $\bar{x}_3=\overline{C}\bar{y}_3-(\overline{C}-\overline{V})\bar{u}_3-(\overline{C}-\overline{V}-V)\bar{u}_2$, we let $\hat{x}_6=\overline{C}$, $\hat{x}_8=\overline{V}+V$, and $\hat{x}_9=\overline{V}$. In this way, we have $\bar{x}_3=\lambda_3\hat{x}_6+\lambda_4\hat{x}_8+\lambda_5\hat{x}_9$. Then (16) holds for $\bar{x}_3$.

3) Satisfying $\bar{x}_3\ge\underline{C}\bar{y}_3+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)$ at equality. We obtain $\overline{V}\bar{y}_2-(\overline{C}-\underline{C}-V)\bar{u}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3)\le\bar{x}_2\le\overline{V}\bar{y}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)$. By letting $\hat{x}_6=\overline{C}-2V$ and $\hat{x}_8=\hat{x}_9=\underline{C}$, we have $\bar{x}_3=\lambda_3\hat{x}_6+\lambda_4\hat{x}_8+\lambda_5\hat{x}_9$. Then (16) holds for $\bar{x}_3$. As a result, we have $(\hat{x}_3,\hat{x}_5,\hat{x}_7)\in A''=\{(\hat{x}_3,\hat{x}_5,\hat{x}_7)\in\mathbb{R}^3:\hat{x}_3=\overline{V},\ \hat{x}_5=\overline{C}-V,\ \underline{C}\le\hat{x}_7\le\overline{V}\}$. If $\bar{x}_2=\overline{V}\bar{y}_2-(\overline{C}-\underline{C}-V)\bar{u}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3)$, we let $\hat{x}_7=\underline{C}$; if $\bar{x}_2\le\overline{V}\bar{y}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)$, we let $\hat{x}_7=\overline{V}$. In this way, we have $\bar{x}_2=\lambda_2\hat{x}_3+\lambda_3\hat{x}_5+\lambda_4\hat{x}_7$. Then (16) holds for $\bar{x}_2$.

4) Satisfying $\bar{x}_3\le\overline{C}\bar{y}_3-(\overline{C}-\overline{V})\bar{u}_3-(\overline{C}-\overline{V}-V)\bar{u}_2$ at equality. We obtain $\overline{V}\bar{y}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)\le\bar{x}_2\le\overline{V}\bar{y}_2+(\overline{C}-\overline{V})(\bar{y}_3-\bar{u}_3-\bar{u}_2)$. By letting $\hat{x}_6=\overline{C}$, $\hat{x}_8=\overline{V}+V$, and $\hat{x}_9=\overline{V}$, we have $\bar{x}_3=\lambda_3\hat{x}_6+\lambda_4\hat{x}_8+\lambda_5\hat{x}_9$. Then (16) holds for $\bar{x}_3$. As a result, we have $(\hat{x}_3,\hat{x}_5,\hat{x}_7)\in A''=\{(\hat{x}_3,\hat{x}_5,\hat{x}_7)\in\mathbb{R}^3:\hat{x}_3=\overline{V},\ \overline{C}-V\le\hat{x}_5\le\overline{C},\ \hat{x}_7=\overline{V}\}$. If $\bar{x}_2=\overline{V}\bar{y}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)$, we let $\hat{x}_5=\overline{C}-V$; if $\bar{x}_2=\overline{V}\bar{y}_2+(\overline{C}-\overline{V})(\bar{y}_3-\bar{u}_3-\bar{u}_2)$, we let $\hat{x}_5=\overline{C}$. In this way, we have $\bar{x}_2=\lambda_2\hat{x}_3+\lambda_3\hat{x}_5+\lambda_4\hat{x}_7$. Then (16) holds for $\bar{x}_2$.

5) Satisfying $\bar{x}_3-\bar{x}_2\le(\overline{V}+V)\bar{y}_3-\overline{V}\bar{y}_2-V\bar{u}_3$ at equality. We obtain $\overline{V}\bar{y}_2-(\overline{C}-\underline{C}-V)\bar{u}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3)\le\bar{x}_2\le\overline{V}\bar{y}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)$ through substituting $\bar{x}_3=\bar{x}_2+(\overline{V}+V)\bar{y}_3-\overline{V}\bar{y}_2-V\bar{u}_3$ into set C'. By letting $\hat{x}_3=\hat{x}_9=\overline{V}$ and $\hat{x}_6-\hat{x}_5=\hat{x}_8-\hat{x}_7=V$, we have $\bar{x}_3-\bar{x}_2=(\lambda_3\hat{x}_6+\lambda_4\hat{x}_8+\lambda_5\hat{x}_9)-(\lambda_2\hat{x}_3+\lambda_3\hat{x}_5+\lambda_4\hat{x}_7)$. If $\bar{x}_2=\overline{V}\bar{y}_2-(\overline{C}-\underline{C}-V)\bar{u}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3)$, we let $\hat{x}_5=\overline{C}-V$ and $\hat{x}_7=\underline{C}$; if $\bar{x}_2\le\overline{V}\bar{y}_2+(\overline{C}-\overline{V}-V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)$, we let $\hat{x}_5=\overline{C}-V$ and $\hat{x}_7=\overline{V}$. For both cases, we have $\bar{x}_2=\lambda_2\hat{x}_3+\lambda_3\hat{x}_5+\lambda_4\hat{x}_7$ and thus $\bar{x}_3=\lambda_3\hat{x}_6+\lambda_4\hat{x}_8+\lambda_5\hat{x}_9$. Then (16) holds for both $\bar{x}_2$ and $\bar{x}_3$.

6) Satisfying $\bar{x}_2-\bar{x}_3\le\overline{V}\bar{y}_2-\underline{C}\bar{y}_3+(\underline{C}+V-\overline{V})(\bar{y}_3-\bar{u}_3-\bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_3+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)\le\bar{x}_3\le\underline{C}\bar{y}_3+(\overline{C}-\underline{C}-V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)$. By letting $\hat{x}_3=\overline{V}$, $\hat{x}_5-\hat{x}_6=V$, $\hat{x}_8-\hat{x}_7=\underline{C}-\overline{V}$, and $\hat{x}_9=\underline{C}$, we have $\bar{x}_2-\bar{x}_3=(\lambda_2\hat{x}_3+\lambda_3\hat{x}_5+\lambda_4\hat{x}_7)-(\lambda_3\hat{x}_6+\lambda_4\hat{x}_8+\lambda_5\hat{x}_9)$. If $\bar{x}_3\ge\underline{C}\bar{y}_3+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\bar{u}_2)$ is satisfied at equality, we let $\hat{x}_6=\overline{C}-2V$ and $\hat{x}_8=\underline{C}$;

if $\bar{x}_3 \leq \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ is satisfied at equality, we let $\hat{x}_6 = \overline{C} - V$ and $\hat{x}_8 = \underline{C}$. For both cases, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$ and thus $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. Then (16) holds for both $\bar{x}_2$ and $\bar{x}_3$.

Similar analyses hold for (17c) and (17d) due to the similar structure between (17b), (17c), and (17d) and thus are omitted here.

Satisfying (17e) **at equality**. For this case, substituting $\bar{x}_2 = \bar{x}_1 + \overline{V}\bar{y}_2 - \underline{C}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ into (17a) - (17k), we obtain the feasible region of $(\bar{x}_1, \bar{x}_3)$ as $C' = \{(\bar{x}_1, \bar{x}_3) \in \mathbb{R}^2 : \underline{C}\bar{y}_1 \leq \bar{x}_1 \leq \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \underline{C}\bar{y}_3 \leq \bar{x}_3 \leq \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2,\ \bar{x}_3 - \bar{x}_1 \leq (\underline{C} + 2V)\bar{y}_3 - \underline{C}\bar{y}_1 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2,\ \bar{x}_1 - \bar{x}_3 \leq \underline{C}\bar{y}_1 - \underline{C}\bar{y}_3\}$.

First, by letting $\hat{x}_1 = \underline{C}$, $\hat{x}_3 - \hat{x}_2 = \overline{V} - \underline{C}$, $\hat{x}_5 - \hat{x}_4 = V$, and $\hat{x}_7 = \overline{V}$, we have $\bar{x}_2 - \bar{x}_1 = (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. Since $\underline{C} \leq \hat{x}_3 \leq \overline{V}$, it follows that $\hat{x}_2 = \underline{C}$ and $\hat{x}_3 = \overline{V}$. Then the corresponding feasible region for $(\hat{x}_4, \hat{x}_6, \hat{x}_8, \hat{x}_9)$ can be described as set $A' = \{(\hat{x}_4, \hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^6 : \underline{C} \leq \hat{x}_4 \leq \overline{C} - V,\ \underline{C} \leq \hat{x}_6 \leq \overline{C},\ 0 \leq \hat{x}_6 - \hat{x}_4 \leq 2V,\ \underline{C} \leq \hat{x}_8 \leq \overline{V} + V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. Next, we only need to show $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$ and $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. Then (16) will hold. We consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_4, \hat{x}_6, \hat{x}_8, \hat{x}_9)$ from A' as follows.

1) Satisfying $\bar{x}_1 \geq \underline{C}\bar{y}_1$ at equality. We obtain $\underline{C}\bar{y}_3 \leq \bar{x}_3 \leq (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$. By letting $\hat{x}_4 = \underline{C}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. As a result, we have $(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \underline{C} \leq \hat{x}_6 \leq \underline{C} + 2V,\ \underline{C} \leq \hat{x}_8 \leq \overline{V} + V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. If $\bar{x}_3 = \underline{C}\bar{y}_3$, we let $\hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 = (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$, we let $\hat{x}_6 = \underline{C} + 2V$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$. For both cases, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$.

2) Satisfying $\bar{x}_1 \leq \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_3 \leq \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$. By letting $\hat{x}_4 = \overline{C} - V$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. As a result, we have $(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \overline{C} - V \leq \hat{x}_6 \leq \overline{C},\ \underline{C} \leq \hat{x}_8 \leq \overline{V} + V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. If $\bar{x}_3 = \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_6 = \overline{C} - V$ and $\hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 = \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$, we let $\hat{x}_6 = \overline{C}$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$. For both cases, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$.

3) Satisfying $\bar{x}_3 \geq \underline{C}\bar{y}_3$ at equality. We obtain $\bar{x}_1 = \underline{C}\bar{y}_1$ since $\bar{x}_1 - \bar{x}_3 \leq \underline{C}\bar{y}_1 - \underline{C}\bar{y}_3$. By letting

$\hat{x}_4 = \hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$ and $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$.

4) Satisfying $\bar{x}_3 \le \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \le \bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_6 = \overline{C}$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. Thus it follows that $\overline{C} - 2V \le \hat{x}_4 \le \overline{C} - V$. If $\bar{x}_1 = \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_4 = \overline{C} - 2V$; if $\bar{x}_1 = \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_4 = \overline{C} - V$. For both cases, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$.

5) Satisfying $\bar{x}_3 - \bar{x}_1 \le (\underline{C} + 2V)\bar{y}_3 - \underline{C}\bar{y}_1 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_1 \le \bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_6 - \hat{x}_4 = 2V$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_3 - \bar{x}_1 = (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. If $\bar{x}_1 = \underline{C}\bar{y}_1$, we let $\hat{x}_4 = \underline{C}$ and thus $\hat{x}_6 = \underline{C} + 2V$; if $\bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_4 = \overline{C} - 2V$. For both cases, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$ and thus $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$.

6) Satisfying $\bar{x}_1 - \bar{x}_3 \le \underline{C}\bar{y}_1 - \underline{C}\bar{y}_3$ at equality. We obtain $\underline{C}\bar{y}_3 \le \bar{x}_3 \le \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_4 = \hat{x}_6$, $\hat{x}_8 = \hat{x}_9 = \underline{C}$, we have $\bar{x}_1 - \bar{x}_3 = (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4) - (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9)$. If $\bar{x}_3 = \underline{C}\bar{y}_3$, we let $\hat{x}_6 = \underline{C}$; if $\bar{x}_3 \le \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_6 = \overline{C} - V$. For both cases, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$ and thus $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$.

Similar analyses hold for (17f) - (17h) due to the similar structure between (17e) and (17f) - (17h) and thus are omitted here.

Satisfying (17i) **at equality**. For this case, substituting $\bar{x}_3 = \bar{x}_1 + (\underline{C} + 2V)\bar{y}_3 - \underline{C}\bar{y}_1 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$ into (17a) - (17k), we obtain the feasible region of $(\bar{x}_1, \bar{x}_2)$ as $C' = \{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 : \underline{C}\bar{y}_1 \le \bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \bar{x}_2 - \bar{x}_1 \le \overline{V}\bar{y}_2 - \underline{C}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \bar{x}_1 - \bar{x}_2 \le \underline{C}\bar{y}_1 - \underline{C}\bar{y}_2 - V\bar{y}_3 + V\bar{u}_3 + (\underline{C} + V - \overline{V})\bar{u}_2\}$.

First, by letting $\hat{x}_1 = \hat{x}_2 = \underline{C}$, $\hat{x}_6 - \hat{x}_4 = 2V$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_3 - \bar{x}_1 = (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. Since $\underline{C} \le \hat{x}_7 \le \overline{V}$ and $\underline{C} - \overline{V} \le \hat{x}_8 - \hat{x}_7 \le V$, we have $\hat{x}_7 = \overline{V}$. Then the corresponding feasible region for $(\hat{x}_3, \hat{x}_4, \hat{x}_5)$ can be described as set $A' = \{(\hat{x}_3, \hat{x}_4, \hat{x}_5) \in \mathbb{R}^3 : \underline{C} \le \hat{x}_3 \le \overline{V},\ \underline{C} \le \hat{x}_4 \le \overline{C} - 2V,\ \underline{C} \le \hat{x}_5 \le \overline{C} - V, \hat{x}_5 - \hat{x}_4 = V\}$. Next, we only need to show $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$ and $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. Then (16) will hold. We consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_3, \hat{x}_4, \hat{x}_5)$ from A' as follows.

1) Satisfying $\bar{x}_1 \geq \underline{C}\bar{y}_1$ at equality. We obtain $\underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_4 = \underline{C}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. As a result, we have $\underline{C} \leq \hat{x}_3 \leq \overline{V}$ and $\hat{x}_5 = \underline{C} + V$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2$, we let $\hat{x}_3 = \underline{C}$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_3 = \overline{V}$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$.

2) Satisfying $\bar{x}_1 \leq \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_4 = \overline{C} - 2V$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. As a result, we have $\underline{C} \leq \hat{x}_3 \leq \overline{V}$ and $\hat{x}_5 = \overline{C} - V$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2$, we let $\hat{x}_3 = \underline{C}$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_3 = \overline{V}$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$.

3) Satisfying $\bar{x}_2 - \bar{x}_1 \leq \overline{V}\bar{y}_2 - \underline{C}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_3 = \overline{V}$, we have $\bar{x}_2 - \bar{x}_1 = (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. As result, we have $\underline{C} + V \leq \hat{x}_5 \leq \overline{C} - V$. If $\bar{x}_2 = \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_5 = \underline{C} + V$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_5 = \overline{C} - V$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$ and thus $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$.

4) Satisfying $\bar{x}_1 - \bar{x}_2 \leq \underline{C}\bar{y}_1 - \underline{C}\bar{y}_2 - V\bar{y}_3 + V\bar{u}_3 + (\underline{C} + V - \overline{V})\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2 \leq \bar{x}_2 \leq \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) + (\overline{V} - \underline{C})\bar{u}_2$. By letting $\hat{x}_3 = \underline{C}$, we have $\bar{x}_1 - \bar{x}_2 = (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4) - (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7)$. As result, we have $\underline{C} + V \leq \hat{x}_5 \leq \overline{C} - V$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2$, we let $\hat{x}_5 = \underline{C} + V$; if $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) + (\overline{V} - \underline{C})\bar{u}_2$, we let $\hat{x}_5 = \overline{C} - V$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$ and thus $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$.

Similar analyses hold for (17j) due to the similar structure between (17i) and (17j) and thus are omitted here.

Satisfying (17k) **at equality**. For this case, substituting $\bar{x}_3 = \bar{x}_2 - \bar{x}_1 + \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 + \overline{V}\bar{y}_3 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ into (17a) - (17k), we obtain the feasible region of $(\bar{x}_1, \bar{x}_2)$ as $C' = \{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 : \overline{V}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_2 - \bar{y}_3 + \bar{u}_3) + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_1 \leq \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \bar{x}_2 - \bar{x}_1 \leq (\overline{V} - V)\bar{y}_2 - \overline{V}\bar{y}_1 + V\bar{u}_2,\ \bar{x}_1 - \bar{x}_2 \leq \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2\}$.

First, by letting $\hat{x}_1 = \overline{V}$, $\hat{x}_2 - \hat{x}_3 = V$, $\hat{x}_4 = \hat{x}_6 = \overline{C}$, $\hat{x}_5 = \overline{C} - V$, $\hat{x}_8 - \hat{x}_7 = V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_1 - \bar{x}_2 + \bar{x}_3 = (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4) - (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7) + (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9)$. Then the corresponding feasible region for $(\hat{x}_3, \hat{x}_7)$ can be described as set $A' = \{(\hat{x}_3, \hat{x}_7) \in \mathbb{R}^2 : \underline{C} \leq \hat{x}_3 \leq \overline{V},\ \underline{C} \leq \hat{x}_7 \leq \overline{V}\}$. Next, we only need to show $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$ and $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. Then (16) will accordingly hold. We consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_3, \hat{x}_7)$ from A' as follows.

1) Satisfying $\bar{x}_1 \geq \overline{V}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_2 - \bar{y}_3 + \bar{u}_3) + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_2 \leq \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2$. By letting $\hat{x}_3 = \underline{C}$ and thus $\hat{x}_2 = \underline{C} + V$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_7 = \underline{C}$ and thus $\hat{x}_8 = \underline{C} + V$; if $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2$, we let $\hat{x}_7 = \overline{V}$ and thus $\hat{x}_8 = \overline{V} + V$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$.

2) Satisfying $\bar{x}_1 \leq \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \underline{C} - V)\bar{u}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_3 = \overline{V}$ and thus $\hat{x}_2 = \overline{V} + V$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. If $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \underline{C} - V)\bar{u}_2$, we let $\hat{x}_7 = \underline{C}$ and thus $\hat{x}_8 = \underline{C} + V$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_7 = \overline{V}$ and thus $\hat{x}_8 = \overline{V} + V$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$.

3) Satisfying $\bar{x}_2 - \bar{x}_1 \leq (\overline{V} - V)\bar{y}_2 - \overline{V}\bar{y}_1 + V\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_7 = \overline{V}$ and thus $\hat{x}_8 = \overline{V} + V$, we have $\bar{x}_2 - \bar{x}_1 = (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2$, we let $\hat{x}_3 = \underline{C}$ and thus $\hat{x}_2 = \underline{C} + V$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_3 = \overline{V}$ and thus $\hat{x}_2 = \overline{V} + V$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$ and thus $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$.

4) Satisfying $\bar{x}_1 - \bar{x}_2 \leq \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \underline{C} - V)\bar{u}_2$. By letting $\hat{x}_7 = \underline{C}$ and thus $\hat{x}_8 = \underline{C} + V$, we have $\bar{x}_1 - \bar{x}_2 = (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4) - (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7)$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_3 = \underline{C}$ and thus $\hat{x}_2 = \underline{C} + V$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \underline{C} - V)\bar{u}_2$, we let $\hat{x}_3 = \overline{V}$ and thus $\hat{x}_2 = \overline{V} + V$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$ and thus $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$.

Thus, the whole claim holds, and we have proved the conclusion. $\blacksquare$

B.3 Proof for Theorem 7

Proof: Here we only provide the proof for the case in which $\overline{C} - \overline{V} - V > 0$ and $\overline{C} - \underline{C} - 2V > 0$ since the cases in which $\overline{C} - \overline{V} - V = 0$ or $\overline{C} - \underline{C} - 2V = 0$ can be proved similarly. Similar to the proof for Proposition 4, we prove that every point $z \in \bar{Q}_3^1$ can be written as $z = \sum_{s\in S} \lambda_s z^s$ for some $\lambda_s \geq 0$ and $\sum_{s\in S} \lambda_s = 1$, where $z^s \in \bar{Q}_3^1, s \in S$ with y and u binary and S is the index set for the candidate points.

For a given point $z = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3) \in \bar{Q}_3^1$, we let the candidate points $z^1, z^2, \cdots, z^8 \in \bar{Q}_3^1$ in the forms such that $z^1 = (\hat{x}_1, \hat{x}_2, \hat{x}_3, 1, 1, 1, 0, 0)$, $z^2 = (\hat{x}_4, \hat{x}_5, 0, 1, 1, 0, 0, 0)$, $z^3 = (\hat{x}_6, 0, \hat{x}_7, 1, 0, 1, 0, 1)$, $z^4 = (\hat{x}_8, 0, 0, 1, 0, 0, 0, 0)$, $z^5 = (0, \hat{x}_9, \hat{x}_{10}, 0, 1, 1, 1, 0)$, $z^6 = (0, \hat{x}_{11}, 0, 0, 1, 0, 1, 0)$, $z^7 = (0, 0, \hat{x}_{12}, 0, 0, 1, 0, 1)$, and $z^8 = (0, 0, 0, 0, 0, 0, 0, 0)$, where $\hat{x}_i, i = 1, \cdots, 12$ are to be decided later. Meanwhile, we let

$$\lambda_1 = \bar{y}_3 - \bar{u}_3 - \lambda_5,\ \lambda_2 = \bar{y}_2 - \bar{u}_2 - \bar{y}_3 + \bar{u}_3 + \lambda_5, \tag{40a}$$

$$\max\{0, \bar{y}_1 + \bar{u}_2 + \bar{u}_3 - 1\} \leq \lambda_3 \leq \min\{\bar{u}_3, \bar{y}_1 - \bar{y}_2 + \bar{u}_2\}, \tag{40b}$$

$$\lambda_4 = \bar{y}_1 - \bar{y}_2 + \bar{u}_2 - \lambda_3,\ \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\} \leq \lambda_5 \leq \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}, \tag{40c}$$

$$\lambda_6 = \bar{u}_2 - \lambda_5,\ \lambda_7 = \bar{u}_3 - \lambda_3 \text{ and } \lambda_8 = 1 - \bar{y}_1 - \bar{u}_2 - \bar{u}_3 + \lambda_3. \tag{40d}$$

First of all, based on this construction, we can check that λ_3 and λ_5 exist and $\sum_{s=1}^8 \lambda_s = 1$ and $\lambda_s \geq 0$ for $\forall s = 1, \cdots, 8$ due to (3c), (14), (18a), and (18b). Meanwhile, it can be checked that $\bar{y}_i = y_i(z) = \sum_{s=1}^8 \lambda_s y_i(z^s)$ for $i = 1, 2, 3$ and $\bar{u}_i = u_i(z) = \sum_{s=1}^8 \lambda_s u_i(z^s)$ for $i = 2, 3$, where $y_i(z)$ represents the $\bar{y}_i$ component value in the given point z and $u_i(z)$ represents the $\bar{u}_i$ component value in the given point z.

Thus, in the remaining part of this proof, we only need to decide the values of $\hat{x}_i$ for $i = 1, \cdots, 12$, λ_3, and λ_5 such that $\bar{x}_i = x_i(z) = \sum_{s=1}^8 \lambda_s x_i(z^s)$ for $i = 1, 2, 3$, i.e.,

$$\bar{x}_1 = \lambda_1 \hat{x}_1 + \lambda_2 \hat{x}_4 + \lambda_3 \hat{x}_6 + \lambda_4 \hat{x}_8, \tag{41a}$$

$$\bar{x}_2 = \lambda_1 \hat{x}_2 + \lambda_2 \hat{x}_5 + \lambda_5 \hat{x}_9 + \lambda_6 \hat{x}_{11}, \tag{41b}$$

$$\bar{x}_3 = \lambda_1 \hat{x}_3 + \lambda_3 \hat{x}_7 + \lambda_5 \hat{x}_{10} + \lambda_7 \hat{x}_{12}. \tag{41c}$$

To show (41), in the following, we prove that for any $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in its feasible region corresponding to a given $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3)$, we can always find a $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{12})$ in its feasible region, corresponding to the same given $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3)$. Now we describe the feasible regions for $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{12})$ and $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, respectively. First, since y and u in $z^1, \cdots, z^8$ are given, by substituting $z^1, \cdots, z^8$ into $\bar{Q}_3^1$, the corresponding feasible region for $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{12})$ can be described as set $A = \{(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{12}) \in \mathbb{R}^{12} : \underline{C} \leq \hat{x}_i \leq \overline{C} (i = 1, 2, 3),\ -V \leq \hat{x}_i - \hat{x}_{i+1} \leq V (i = 1, 2),\ \underline{C} \leq \hat{x}_4 \leq \overline{V} + V,\ \underline{C} \leq \hat{x}_5 \leq \overline{V},\ -V \leq \hat{x}_4 - \hat{x}_5 \leq V,\ \underline{C} \leq \hat{x}_i \leq \overline{V} (i = 6, 7, 8, 11, 12),\ \underline{C} \leq \hat{x}_9 \leq \overline{V},\ \underline{C} \leq \hat{x}_{10} \leq \overline{V} + V,\ -V \leq \hat{x}_9 - \hat{x}_{10} \leq V\}$. Second, corresponding to a given $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3)$, following the description of $\bar{Q}_3^1$, the feasible region for $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ can be described as follows:

$$
\begin{aligned}
C = \Big\{ (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 :\ & \bar{x}_1 \geq \underline{C}\bar{y}_1,\ \bar{x}_2 \geq \underline{C}\bar{y}_2,\ \bar{x}_3 \geq \underline{C}\bar{y}_3, && \text{(42a)} \\
& \bar{x}_1 \leq \overline{V}\bar{y}_1 + (\overline{C} - \overline{V})(\bar{y}_2 - \bar{u}_2), && \text{(42b)} \\
& \bar{x}_1 \leq \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3), && \text{(42c)} \\
& \bar{x}_2 \leq \overline{C}\bar{y}_2 - (\overline{C} - \overline{V})\bar{u}_2, && \text{(42d)} \\
& \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3), && \text{(42e)} \\
& \bar{x}_3 \leq \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3, && \text{(42f)} \\
& \bar{x}_3 \leq (\overline{V} + V)\bar{y}_3 - V\bar{u}_3 + (\overline{C} - \overline{V} - V)(\bar{y}_2 - \bar{u}_2), && \text{(42g)} \\
& \bar{x}_2 - \bar{x}_1 \leq (\underline{C} + V)\bar{y}_2 - \underline{C}\bar{y}_1 - (\underline{C} + V - \overline{V})\bar{u}_2, && \text{(42h)} \\
& \bar{x}_3 - \bar{x}_2 \leq (\underline{C} + V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_3, && \text{(42i)} \\
& \bar{x}_1 - \bar{x}_2 \leq \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2, && \text{(42j)} \\
& \bar{x}_2 - \bar{x}_3 \leq \overline{V}\bar{y}_2 - (\overline{V} - V)\bar{y}_3 - (\underline{C} + V - \overline{V})\bar{u}_3, && \text{(42k)} \\
& \bar{x}_3 - \bar{x}_1 \leq (\underline{C} + 2V)\bar{y}_3 - \underline{C}\bar{y}_1 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2, && \text{(42l)} \\
& \bar{x}_1 - \bar{x}_3 \leq \overline{V}\bar{y}_1 - \underline{C}\bar{y}_3 + V(\bar{y}_2 - \bar{u}_2) + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), && \text{(42m)} \\
& \bar{x}_3 - \alpha\bar{x}_1 \leq (\overline{V} + V)\bar{y}_3 - V\bar{u}_3 - \alpha\underline{C}\bar{y}_1, && \text{(42n)} \\
& \bar{x}_1 - \alpha\bar{x}_3 \leq \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) - \alpha\underline{C}\bar{y}_3, \Big\}, && \text{(42o)}
\end{aligned}
$$

where $\alpha = (\overline{C} - \overline{V} - V)/(\overline{C} - \underline{C} - 2V)$ $(0 < \alpha \leq 1)$.

Accordingly, we can set up the linear transformation F from $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{12}) \in A$ to $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in C$ as follows:

$$
F = \begin{pmatrix}
\lambda_1 & 0 & 0 & \lambda_2 & 0 & \lambda_3 & 0 & \lambda_4 & 0 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & \lambda_5 & 0 & \lambda_{11} & 0 \\
0 & 0 & \lambda_1 & 0 & 0 & 0 & \lambda_3 & 0 & 0 & \lambda_5 & 0 & \lambda_{12}
\end{pmatrix},
$$

where $\lambda_1, \lambda_2, \cdots, \lambda_7$ follow the definitions described in (40). Thus, in the following, we only need to prove that $F : A \to C$ is surjective.

Since C is a closed and bounded polytope, any point can be expressed as a convex combination of the extreme points in C. Accordingly, we only need to show that for any extreme point $w^i \in C$ $(i = 1, \cdots, M)$, there exists a point $p^i \in A$ such that $Fp^i = w^i$, where M represents the number of extreme points in C (because for an arbitrary point $w \in C$, which can be represented as $w = \sum_{i=1}^{M} \mu_i w^i$ and $\sum_{i=1}^{M} \mu_i = 1$, there exists $p = \sum_{i=1}^{M} \mu_i p_i \in A$ such that $Fp = w$ due to the linearity of F and the convexity of A). Since it is difficult to enumerate all the extreme points in C, in the following proof we show the conclusion holds for any point in the faces of C, i.e., satisfying one of (42a) - (42o) at equality, which implies the conclusion holds for extreme points.

Satisfying $\bar{x}_1 \geq \underline{C}\bar{y}_1$ at equality. For this case, substituting $\bar{x}_1 = \underline{C}\bar{y}_1$ into (42b) - (42o), we obtain the feasible region of $(\bar{x}_2, \bar{x}_3)$ as $C' = \{(\bar{x}_2, \bar{x}_3) \in \mathbb{R}^2 : \underline{C}\bar{y}_2 \leq \bar{x}_2 \leq (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2,\ \underline{C}\bar{y}_3 \leq \bar{x}_3 \leq (\underline{C}+2V)\bar{y}_3 - (\underline{C}+2V-\overline{V})\bar{u}_3 + (\overline{V}-\underline{C}-V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}, \bar{x}_3 - \bar{x}_2 \leq (\underline{C}+V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C}+V-\overline{V})\bar{u}_3, \bar{x}_2 - \bar{x}_3 \leq (\underline{C}+V)\bar{y}_2 - \underline{C}\bar{y}_3 - (\underline{C}+V-\overline{V})\bar{u}_2 + (\underline{C}+V-\overline{V})\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}\}$.

First, by letting $\hat{x}_1 = \hat{x}_4 = \hat{x}_6 = \hat{x}_8 = \underline{C}$, it is easy to check that $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_4 + \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8$. Then (41a) holds. Note here that once $(\hat{x}_1, \hat{x}_4, \hat{x}_6, \hat{x}_8)$ fixed, the corresponding feasible region for $(\hat{x}_2, \hat{x}_3, \hat{x}_5, \hat{x}_7, \hat{x}_9, \hat{x}_{10}, \hat{x}_{11}, \hat{x}_{12})$ can be described as set $A' = \{(\hat{x}_2, \hat{x}_3, \hat{x}_5, \hat{x}_7, \hat{x}_9, \hat{x}_{10}, \hat{x}_{11}, \hat{x}_{12}) \in \mathbb{R}^8 : \underline{C} \leq \hat{x}_2 \leq \underline{C}+V,\ \underline{C} \leq \hat{x}_3 \leq \underline{C}+2V,\ -V \leq \hat{x}_2 - \hat{x}_3 \leq V,\ \underline{C} \leq \hat{x}_5 \leq \underline{C}+V,\ \underline{C} \leq \hat{x}_i \leq \overline{V}(i = 7, 11, 12),$ $\underline{C} \leq \hat{x}_9 \leq \overline{V},\ \underline{C} \leq \hat{x}_{10} \leq \overline{V} + V,\ -V \leq \hat{x}_9 - \hat{x}_{10} \leq V\}$. In the following, we repeat the argument above to consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_2, \hat{x}_3, \hat{x}_5, \hat{x}_7, \hat{x}_9, \hat{x}_{10}, \hat{x}_{11}, \hat{x}_{12})$ from A'. In addition, we decide the corresponding λ_3 and λ_5 when a particular value of λ_3 or λ_5 is required (it follows other λ's are also decided), otherwise we let λ_3 and λ_5 be free in their ranges as described in (40) respectively.

1) Satisfying $\bar{x}_2 \geq \underline{C}\bar{y}_2$ at equality. We obtain $\bar{x}_3 \in C'' = \{\bar{x}_3 \in \mathbb{R} : \underline{C}\bar{y}_3 \leq \bar{x}_3 \leq (\underline{C} + V)\bar{y}_3 - (\underline{C} + V - \overline{V})\bar{u}_3\}$ through substituting $\bar{x}_2 = \underline{C}\bar{y}_2$ into C'. By letting $\hat{x}_2 = \hat{x}_5 = \hat{x}_9 = \hat{x}_{11} = \underline{C}$, we have $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$. Then (41b) holds. Thus, the corresponding feasible region for $(\hat{x}_3, \hat{x}_7, \hat{x}_{10}, \hat{x}_{12})$ can be described as set $A'' = \{(\hat{x}_3, \hat{x}_7, \hat{x}_{10}, \hat{x}_{12}) \in \mathbb{R}^4 : \underline{C} \leq \hat{x}_i \leq \underline{C} + V(i = 3, 10),\ \underline{C} \leq \hat{x}_i \leq \overline{V}(i = 7, 12)\}$. If $\bar{x}_3 \geq \underline{C}\bar{y}_3$ is satisfied at equality, we let $\hat{x}_3 = \hat{x}_7 = \hat{x}_{10} = \hat{x}_{12} = \underline{C}$; if $\bar{x}_3 \leq (\underline{C} + V)\bar{y}_3 - (\underline{C} + V - \overline{V})\bar{u}_3$ is satisfied at equality, we let

$\hat{x}_3 = \hat{x}_{10} = \underline{C} + V$ and $\hat{x}_7 = \hat{x}_{12} = \overline{V}$. It is easy to check that $\bar{x}_3 = \lambda_1\hat{x}_3 + \lambda_3\hat{x}_7 + \lambda_5\hat{x}_{10} + \lambda_7\hat{x}_{12}$. Then (41c) holds.

2) Satisfying $\bar{x}_2 \leq (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2$ at equality. We obtain $\bar{x}_3 \in C'' = \{\bar{x}_3 \in \mathbb{R} : \underline{C}\bar{y}_3 + (\overline{V} - \underline{C} - V)\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\} \leq \bar{x}_3 \leq (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 + (\overline{V} - \underline{C} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}\}$. By letting $\hat{x}_2 = \hat{x}_5 = \underline{C} + V$ and $\hat{x}_9 = \hat{x}_{11} = \overline{V}$, we have $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$. Then (41b) holds. Thus, the corresponding feasible region for $(\hat{x}_3, \hat{x}_7, \hat{x}_{10}, \hat{x}_{12})$ can be described as set $A'' = \{(\hat{x}_3, \hat{x}_7, \hat{x}_{10}, \hat{x}_{12}) \in \mathbb{R}^4 : \underline{C} \leq \hat{x}_3 \leq \underline{C} + 2V,\ \underline{C} \leq \hat{x}_i \leq \overline{V}(i = 7, 12),\ \overline{V} - V \leq \hat{x}_{10} \leq \overline{V} + V\}$. If $\bar{x}_3 \geq \underline{C}\bar{y}_3 + (\overline{V} - \underline{C} - V)\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$ is satisfied at equality, we let $\hat{x}_3 = \hat{x}_7 = \hat{x}_{12} = \underline{C}$, $\hat{x}_{10} = \overline{V} - V$ and $\lambda_5 = \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$; if $\bar{x}_3 \leq (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 + (\overline{V} - \underline{C} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$ is satisfied at equality, we let $\hat{x}_3 = \underline{C} + 2V$, $\hat{x}_7 = \hat{x}_{12} = \overline{V}$, $\hat{x}_{10} = \overline{V} + V$, and $\lambda_5 = \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$. In this way, we have $\bar{x}_3 = \lambda_1\hat{x}_3 + \lambda_3\hat{x}_7 + \lambda_5\hat{x}_{10} + \lambda_7\hat{x}_{12}$. Then (41c) holds.

3) Satisfying $\bar{x}_3 \geq \underline{C}\bar{y}_3$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 \leq \bar{x}_2 \leq (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2 - (\overline{V} - \underline{C} - V)\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}\}$. By letting $\hat{x}_3 = \hat{x}_7 = \hat{x}_{10} = \hat{x}_{12} = \underline{C}$, we have $\bar{x}_3 = \lambda_1\hat{x}_3 + \lambda_3\hat{x}_7 + \lambda_5\hat{x}_{10} + \lambda_7\hat{x}_{12}$. Then (41c) holds. Thus, the corresponding feasible region for $(\hat{x}_2, \hat{x}_5, \hat{x}_9, \hat{x}_{11})$ can be described as set $A'' = \{(\hat{x}_2, \hat{x}_5, \hat{x}_9, \hat{x}_{11}) \in \mathbb{R}^4 : \underline{C} \leq \hat{x}_i \leq \underline{C} + V(i = 2, 5, 9),\ \underline{C} \leq \hat{x}_{11} \leq \overline{V}\}$. If $\bar{x}_2 \geq \underline{C}\bar{y}_2$ is satisfied at equality, we let $\hat{x}_2 = \hat{x}_5 = \hat{x}_9 = \hat{x}_{11} = \underline{C}$; if $\bar{x}_2 \leq (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2 - (\overline{V} - \underline{C} - V)\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$ is satisfied at equality, we let $\hat{x}_2 = \hat{x}_5 = \hat{x}_9 = \underline{C} + V$, $\hat{x}_{11} = \overline{V}$, and $\lambda_5 = \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$. In this way, we have $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$. Then (41b) holds.

4) Satisfying $\bar{x}_3 \leq (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 + (\overline{V} - \underline{C} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) + (\overline{V} - \underline{C} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\} \leq \bar{x}_2 \leq (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2\}$. By letting $\hat{x}_3 = \underline{C} + 2V$, $\hat{x}_7 = \hat{x}_{12} = \overline{V}$, $\hat{x}_{10} = \overline{V} + V$, and $\lambda_5 = \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$, we have $\bar{x}_3 = \lambda_1\hat{x}_3 + \lambda_3\hat{x}_7 + \lambda_5\hat{x}_{10} + \lambda_7\hat{x}_{12}$. Then (41c) holds. In addition, we further have $\hat{x}_2 = \underline{C} + V$ and $\hat{x}_9 = \overline{V}$ based on the definition of A'. Thus, the corresponding feasible region for $(\hat{x}_5, \hat{x}_{11})$ can be described as set $A'' = \{(\hat{x}_5, \hat{x}_{11}) \in \mathbb{R}^2 : \underline{C} \leq \hat{x}_5 \leq \underline{C} + V,\ \underline{C} \leq \hat{x}_{11} \leq \overline{V}\}$. If $\bar{x}_2 \geq \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) + (\overline{V} - \underline{C} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$ is satisfied at equality, we let $\bar{x}_5 = \bar{x}_{11} = \underline{C}$; if $\bar{x}_2 \leq (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2$ is satisfied at equality, we let $\hat{x}_5 = \underline{C} + V$ and $\hat{x}_{11} = \overline{V}$. In

this way, we have $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$. Then (41b) holds.

5) Satisfying $\bar{x}_3 - \bar{x}_2 \le (\underline{C}+V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C}+V-\overline{V})\bar{u}_3$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 \le \bar{x}_2 \le \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) + (\overline{V} - \underline{C} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}\}$ through substituting $\bar{x}_3 = \bar{x}_2 + (\underline{C}+V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C}+V-\overline{V})\bar{u}_3$ into set C'. By letting $\hat{x}_3 - \hat{x}_2 = \hat{x}_{10} - \hat{x}_9 = V$, $\hat{x}_7 = \hat{x}_{12} = \overline{V}$, and $\hat{x}_5 = \hat{x}_{11} = \underline{C}$, we have $\bar{x}_3 - \bar{x}_2 = (\lambda_1\hat{x}_3 + \lambda_3\hat{x}_7 + \lambda_5\hat{x}_{10} + \lambda_7\hat{x}_{12}) - (\lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11})$. If $\bar{x}_2 \ge \underline{C}\bar{y}_2$ is satisfied at equality, we let $\hat{x}_2 = \hat{x}_9 = \underline{C}$ (and then $\hat{x}_3 = \hat{x}_{10} = \underline{C} + V$); if $\bar{x}_2 \le \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) + (\overline{V} - \underline{C} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$ is satisfied at equality, we let $\hat{x}_2 = \underline{C} + V$ (and then $\hat{x}_3 = \underline{C} + 2V$), $\hat{x}_9 = \overline{V}$ (thus $\hat{x}_{10} = \overline{V} + V$), and $\lambda_5 = \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$. For both cases, we have $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$ and thus $\bar{x}_3 = \lambda_1\hat{x}_3 + \lambda_3\hat{x}_7 + \lambda_5\hat{x}_{10} + \lambda_7\hat{x}_{12}$. Then both (41b) and (41c) hold.

6) Satisfying $\bar{x}_2 - \bar{x}_3 \le (\underline{C}+V)\bar{y}_2 - \underline{C}\bar{y}_3 - (\underline{C}+V-\overline{V})\bar{u}_2 + (\underline{C}+V-\overline{V})\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$ at equality. We obtain $\bar{x}_3 \in C'' = \{\bar{x}_3 \in \mathbb{R} : \underline{C}\bar{y}_3 \le \bar{x}_3 \le \underline{C}\bar{y}_3 + (\overline{V} - \underline{C} - V)\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}\}$ through substituting $\bar{x}_2 = \bar{x}_3 + (\underline{C}+V)\bar{y}_2 - \underline{C}\bar{y}_3 - (\underline{C}+V-\overline{V})\bar{u}_2 + (\underline{C}+V-\overline{V})\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$ into set C'. By letting $\hat{x}_2 - \hat{x}_3 = \hat{x}_9 - \hat{x}_{10} = V$, $\hat{x}_7 = \hat{x}_{12} = \underline{C}$, $\hat{x}_5 = \underline{C}$, $\hat{x}_{11} = \overline{V}$, and $\lambda_5 = \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$, we have $\bar{x}_2 - \bar{x}_3 = (\lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}) - (\lambda_1\hat{x}_3 + \lambda_3\hat{x}_7 + \lambda_5\hat{x}_{10} + \lambda_7\hat{x}_{12})$. In addition, we further have $\hat{x}_2 = \underline{C} + V$ and $\hat{x}_3 = \underline{C}$ based on the definition of A'. If $\bar{x}_3 \ge \underline{C}\bar{y}_3$ is satisfied at equality, we let $\hat{x}_{10} = \underline{C}$ (and then $\hat{x}_9 = \underline{C} + V$); if $\bar{x}_3 \le \underline{C}\bar{y}_3 + (\overline{V} - \underline{C} - V)\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$ is satisfied at equality, we let $\hat{x}_{10} = \overline{V} - V$ (and then $\hat{x}_9 = \overline{V}$). For both cases, we have $\bar{x}_3 = \lambda_1\hat{x}_3 + \lambda_3\hat{x}_7 + \lambda_5\hat{x}_{10} + \lambda_7\hat{x}_{12}$ and thus $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$. Then both (41b) and (41c) hold.

Similar analyses hold for $\bar{x}_2 \ge \underline{C}\bar{y}_2$ and $\bar{x}_3 \ge \underline{C}\bar{y}_3$ due to the similar structure and also hold for inequalities (42b) - (42m) following the proofs for Proposition 4, and thus are omitted here.

Satisfying (42n) **at equality**. For this case, substituting $\bar{x}_3 = \alpha\bar{x}_1 + (\overline{V}+V)\bar{y}_3 - V\bar{u}_3 - \alpha\underline{C}\bar{y}_1$ into (42a) - (42o), we obtain the feasible region of $(\bar{x}_1, \bar{x}_2)$ as $C' = \{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 : \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}) \le \bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\})$, $\underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\} \le \bar{x}_2 \le (\underline{C}+V)\bar{y}_2 - (\underline{C}+V-\overline{V})\bar{u}_2 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\})$, $\bar{x}_2 - \bar{x}_1 \le (\underline{C}+V)\bar{y}_2 - \underline{C}\bar{y}_1 - (\underline{C}+V-\overline{V})\bar{u}_2$, $\bar{x}_1 - \bar{x}_2 \le \underline{C}(\bar{y}_1 - \bar{y}_2) - V(\bar{y}_3 - \bar{u}_3) - (\overline{V} - \underline{C} - V)\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$, $\bar{x}_2 - \alpha\bar{x}_1 \ge \underline{C}\bar{y}_2 + (\overline{V} - \underline{C})(\bar{y}_3 - \bar{u}_3) - \alpha\underline{C}\bar{y}_1\}$.

First, by letting $\hat{x}_3 = \overline{C}$, $\hat{x}_1 = \overline{C} - 2V$, $\hat{x}_4 = \hat{x}_6 = \hat{x}_8 = \underline{C}$, $\hat{x}_7 = \hat{x}_{12} = \overline{V}$, and $\hat{x}_{10} = \overline{V} + V$, we have $\bar{x}_3 - \alpha\bar{x}_1 = (\lambda_1\hat{x}_3 + \lambda_3\hat{x}_7 + \lambda_5\hat{x}_{10} + \lambda_7\hat{x}_{12}) - \alpha(\lambda_1\hat{x}_1 + \lambda_2\hat{x}_4 + \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8)$. In addition, we further have $\hat{x}_2 = \overline{C} - V$ and $\hat{x}_9 = \overline{V}$ based on the definition of A. Then the corresponding feasible region for $(\hat{x}_5, \hat{x}_{11})$ can be described as set $A' = \{(\hat{x}_5, \hat{x}_{11}) \in \mathbb{R}^2 : \underline{C} \leq \hat{x}_5 \leq \underline{C} + V,\ \underline{C} \leq \hat{x}_{11} \leq \overline{V}\}$. Next, we only need to show $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_4 + \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8$ and $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$. Then (41) will hold. We consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_5, \hat{x}_{11})$ from A'. In addition, we decide the corresponding λ_3 and λ_5 when a particular value of λ_3 or λ_5 is required (it follows other λ's are also decided), otherwise we let λ_3 and λ_5 be free in their ranges as described in (40) respectively.

1) Satisfying $\bar{x}_1 \geq \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\})$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\} \leq \bar{x}_2 \leq (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\})\}$. By letting $\lambda_5 = \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_4 + \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$, we let $\hat{x}_5 = \hat{x}_{11} = \underline{C}$; if $\bar{x}_2 = (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\})$, we let $\hat{x}_5 = \underline{C} + V$ and $\hat{x}_{11} = \overline{V}$. For both cases, we have $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$.

2) Satisfying $\bar{x}_1 \leq \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\})$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\} \leq \bar{x}_2 \leq (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\})\}$. By letting $\lambda_5 = \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_4 + \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\}$, we let $\hat{x}_5 = \hat{x}_{11} = \underline{C}$; if $\bar{x}_2 = (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\})$, we let $\hat{x}_5 = \underline{C} + V$ and $\hat{x}_{11} = \overline{V}$. For both cases, we have $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$.

3) Satisfying $\bar{x}_2 \geq \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$ at equality. We obtain $\bar{x}_1 = \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\})$. By letting $\hat{x}_5 = \hat{x}_{11} = \underline{C}$ and $\lambda_5 = \min\{\bar{u}_2, \bar{y}_3 - \bar{u}_3\}$, we have $\bar{x}_2 = \lambda_1\hat{x}_2 + \lambda_2\hat{x}_5 + \lambda_5\hat{x}_9 + \lambda_6\hat{x}_{11}$. Furthermore, we also have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_4 + \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8$.

4) Satisfying $\bar{x}_2 \geq (\underline{C} + V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\})$ at equality. We obtain $\bar{x}_1 = \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \max\{0, \bar{y}_3 - \bar{u}_3 - \bar{y}_2 + \bar{u}_2\})$. By letting

$\hat{x}_5 = \underline{C}+V$, $\hat{x}_{11} = \overline{V}$, and $\lambda_5 = \max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\}$, we have $\bar{x}_2 = \lambda_1\hat{x}_2+\lambda_2\hat{x}_5+\lambda_5\hat{x}_9+\lambda_6\hat{x}_{11}$. Furthermore, we also have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_4 + \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8$.

5) Satisfying $\bar{x}_2-\bar{x}_1 \leq (\underline{C}+V)\bar{y}_2-\underline{C}\bar{y}_1-(\underline{C}+V-\overline{V})\bar{u}_2$ at equality. We obtain $\bar{x}_1 \in C'' = \{\bar{x}_1 \in \mathbb{R} : \underline{C}\bar{y}_1+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\min\{\bar{u}_2, \bar{y}_3-\bar{u}_3\}) \leq \bar{x}_1 \leq \underline{C}\bar{y}_1+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\})\}$. By letting $\hat{x}_5 = \underline{C}+V$ and $\hat{x}_{11} = \overline{V}$, we have $\bar{x}_2-\bar{x}_1 = (\lambda_1\hat{x}_2+\lambda_2\hat{x}_5+\lambda_5\hat{x}_9+\lambda_6\hat{x}_{11})-(\lambda_1\hat{x}_1+\lambda_2\hat{x}_4+\lambda_3\hat{x}_6+\lambda_4\hat{x}_8)$. If $\bar{x}_1 = \underline{C}\bar{y}_1+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\min\{\bar{u}_2, \bar{y}_3-\bar{u}_3\})$, we let $\lambda_5 = \min\{\bar{u}_2, \bar{y}_3-\bar{u}_3\}$; if $\bar{x}_1 = \underline{C}\bar{y}_1+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\})$, we let $\lambda_5 = \max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\}$. For both cases, we have $\bar{x}_1 = \lambda_1\hat{x}_1+\lambda_2\hat{x}_4+\lambda_3\hat{x}_6+\lambda_4\hat{x}_8$ and thus $\bar{x}_2 = \lambda_1\hat{x}_2+\lambda_2\hat{x}_5+\lambda_5\hat{x}_9+\lambda_6\hat{x}_{11}$.

6) Satisfying $\bar{x}_1-\bar{x}_2 \leq \underline{C}(\bar{y}_1-\bar{y}_2)-V(\bar{y}_3-\bar{u}_3)-(\overline{V}-\underline{C}-V)\max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\}$ at equality. We obtain $\bar{x}_2 = \underline{C}\bar{y}_2+(\overline{C}-\underline{C}-V)(\bar{y}_3-\bar{u}_3)-(\overline{C}-\overline{V}-V)\max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\}$. By letting $\hat{x}_5 = \hat{x}_{11} = \underline{C}$ and $\lambda_5 = \max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\}$, we have $\bar{x}_1-\bar{x}_2 = (\lambda_1\hat{x}_1+\lambda_2\hat{x}_4+\lambda_3\hat{x}_6+\lambda_4\hat{x}_8)-(\lambda_1\hat{x}_2+\lambda_2\hat{x}_5+\lambda_5\hat{x}_9+\lambda_6\hat{x}_{11})$. Furthermore, we also have $\bar{x}_1 = \lambda_1\hat{x}_1+\lambda_2\hat{x}_4+\lambda_3\hat{x}_6+\lambda_4\hat{x}_8$ and thus $\bar{x}_2 = \lambda_1\hat{x}_2+\lambda_2\hat{x}_5+\lambda_5\hat{x}_9+\lambda_6\hat{x}_{11}$.

7) Satisfying $\bar{x}_2-\alpha\bar{x}_1 \geq \underline{C}\bar{y}_2+(\overline{V}-\underline{C})(\bar{y}_3-\bar{u}_3)-\alpha\underline{C}\bar{y}_1$ at equality. We obtain $\bar{x}_1 \in C'' = \{\bar{x}_1 \in \mathbb{R} : \underline{C}\bar{y}_1+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\min\{\bar{u}_2, \bar{y}_3-\bar{u}_3\}) \leq \bar{x}_1 \leq \underline{C}\bar{y}_1+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\})\}$. By letting $\hat{x}_5 = \hat{x}_{11} = \underline{C}$, we have $\bar{x}_2-\alpha\bar{x}_1 = (\lambda_1\hat{x}_2+\lambda_2\hat{x}_5+\lambda_5\hat{x}_9+\lambda_6\hat{x}_{11})-\alpha(\lambda_1\hat{x}_1+\lambda_2\hat{x}_4+\lambda_3\hat{x}_6+\lambda_4\hat{x}_8)$. If $\bar{x}_1 = \underline{C}\bar{y}_1+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\min\{\bar{u}_2, \bar{y}_3-\bar{u}_3\})$, we let $\lambda_5 = \min\{\bar{u}_2, \bar{y}_3-\bar{u}_3\}$; if $\bar{x}_1 = \underline{C}\bar{y}_1+(\overline{C}-\underline{C}-2V)(\bar{y}_3-\bar{u}_3-\max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\})$, we let $\lambda_5 = \max\{0, \bar{y}_3-\bar{u}_3-\bar{y}_2+\bar{u}_2\}$. For both cases, we have $\bar{x}_1 = \lambda_1\hat{x}_1+\lambda_2\hat{x}_4+\lambda_3\hat{x}_6+\lambda_4\hat{x}_8$ and thus $\bar{x}_2 = \lambda_1\hat{x}_2+\lambda_2\hat{x}_5+\lambda_5\hat{x}_9+\lambda_6\hat{x}_{11}$.

Similar analysis holds for (42o) due to the similar structure and thus are omitted here. ∎

Appendix C Proofs for Multi-period Formulations

C.1 Proof for Proposition 5

Proof: **(Facet-defining)** We generate $3T-1$ affinely independent points in $\mathrm{conv}(P)$ that satisfy (23) at equality. Since $0 \in \mathrm{conv}(P)$, we generate another $3T-2$ linearly independent points in conv(P) in the following groups. In the following proofs, we use the superscript of (x, y, u), e.g., r in (x^r, y^r, u^r), to indicate the index of different points in conv(P).

First, we create T linearly independent points $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ $(r \in [1,T]_{\mathbb{Z}})$ such that $\bar{y}^r_s = 1$ for each $s \in [1,r]_{\mathbb{Z}}$ and $\bar{y}^r_s = 0$ otherwise. Thus we have $\bar{u}^r_s = 0$ for all $s \in [2,T]_{\mathbb{Z}}$ due to constraints (1a) - (1c). For the value of $\bar{x}^r$, we consider the following cases: 1) for each $r \in [1, T-1]_{\mathbb{Z}}$, we have $\bar{x}^r_s = \underline{C}$ for each $s \in [1,r]_{\mathbb{Z}}$ and $\bar{x}^r_s = 0$ otherwise; 2) for $r = T$, we have $\bar{x}^r_s = \overline{C}$ for each $s \in [1,T]_{\mathbb{Z}}$.

Second, we create $T-1$ linearly independent points $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ $(r \in [1, T-1]_{\mathbb{Z}})$ such that

$$\hat{x}^r_s = \begin{cases} \underline{C} + \epsilon, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \hat{y}^r_s = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \hat{u}^r_s = 0, \\ \forall s \end{array}.$$

Third, we create k linearly independent points $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ $(r \in [T-k+1, T]_{\mathbb{Z}})$ such that

$$\acute{x}^r_s = \begin{cases} \overline{V} + (s-r)V, & s \in [r,T]_{\mathbb{Z}} \\ 0, & s \in [1, r-1]_{\mathbb{Z}} \end{cases}, \quad \acute{y}^r_s = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & s \in [1,r-1]_{\mathbb{Z}} \end{cases}, \text{ and } \acute{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Fourth, for the remaining $T-k-1$ points, we consider $k = L$ and $\lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1$ respectively, since the condition requires $k = \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1\}$.

1) If $k = L$, then we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ for each $r \in [2, T-k]_{\mathbb{Z}}$, where

$$\acute{x}^r_s = \begin{cases} \underline{C}, & s \in [r, T-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}^r_s = \begin{cases} 1, & s \in [r, T-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \acute{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

2) If $k = \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1$, then we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ for each $r \in [2, T-k]_{\mathbb{Z}}$, where

$$\acute{x}^r_s = \begin{cases} \overline{V} + (s-r)V, & s \in [r, r+k-1]_{\mathbb{Z}} \\ \overline{C}, & s \in [r+k, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}^r_s = \begin{cases} 1, & s \in [r, T-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \acute{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=2}^T$ are linearly independent because they construct a lower-diagonal matrix. In addition, $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=1}^{T-1}$ are also linearly independent with them after Gaussian elimination between $(\bar{x}, \bar{y}, \bar{u})$ and $(\hat{x}, \hat{y}, \hat{u})$. Therefore the statement is proved.

■

C.2 Proof for Proposition 6

Proof: **(Facet-defining)** We provide the facet-defining proof for condition (1), as the proof for condition (2) is similar with that for Proposition 5 and thus omitted here.

We generate $3T-2$ linearly independent points in $\text{conv}(P)$ that satisfy (24) at equality in the following groups.

1) For each $r \in [1,t-1]_{\mathbb{Z}}$ (totally $t-1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

2) For each $r \in [1,t-1]_{\mathbb{Z}}$ (totally $t-1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}+\epsilon, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

3) For $r = t$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{V}, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

4) For each $r \in [t+1,T-1]_{\mathbb{Z}}$ (totally $T-t-1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [t-k+2,r]_{\mathbb{Z}}$ and $\bar{y}_s^r = 0$ otherwise. Thus $\bar{u}_s^r = 1$ for $s = t-k+2$ and $\bar{u}_s^r = 0$ otherwise due to constraints (1a) - (1c). Moreover, we let $\bar{x}_s^r = \max\{\underline{C}, \overline{V} + (s-(t-k+2))V\}$ for each $s \in [t-k+2,t]_{\mathbb{Z}}$, $\bar{x}_s^r = \max\{\underline{C}, \overline{V} + (k-3)V\}$ for each $s \in [t+1,r]_{\mathbb{Z}}$, and $\bar{x}_s^r = 0$ otherwise.

5) For $r = T$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [1,T]_{\mathbb{Z}}$, $\bar{u}_s^r = 0$ for each $s \in [2,T]_{\mathbb{Z}}$, and $\bar{x}_s^r = \overline{C}$ for each $s \in [1,T]_{\mathbb{Z}}$.

6) For each $r \in [2,t-k+1]_{\mathbb{Z}}$ (totally $t-k$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{V}, & s \in [r,t]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,t]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

7) For each $r \in [t-k+2,t]_{\mathbb{Z}}$ (totally $k-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{V} + (s-r)V, & s \in [r,t]_{\mathbb{Z}} \\ \max\{\underline{C}, \overline{V} + (t-r-1)V\}, & s \in [t+1,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

8) For each $r \in [t+1, T]_{\mathbb{Z}}$ (totally $T-t$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \hat{y}_s^r = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

9) For each $r \in [t+1, T]_{\mathbb{Z}}$ (totally $T-t$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \grave{y}_s^r = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \grave{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-1}$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t+1}^T$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-1}$ and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-1}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t+1}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t+1}^T$. ■

C.3 Proof for Proposition 7

Proof: **(Facet-defining)** We provide the facet-defining proof for condition (1), as the proof for condition (2) is similar with that for Proposition 5 and thus omitted here. Since $L - 1 \leq \lfloor \frac{\overline{C} - \overline{V}}{V} \rfloor$, we have $k = L - 1$.

We generate $3T - 2$ linearly independent points in conv(P) that satisfy (25) at equality in the following groups.

1) For each $r \in [1, t-2]_{\mathbb{Z}}$ (totally $t-2$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \ \acute{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{c} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

2) For each $r \in [1, t-2]_{\mathbb{Z}}$ (totally $t-2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \ \bar{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{c} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

3) For $r = t-1$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{V}, & s \in [t-k-1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \bar{y}_s^r = \begin{cases} 1, & s \in [t-k-1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \bar{u}_s^r = \begin{cases} 1, & s = t-k-1 \\ 0, & \text{o.w.} \end{cases}.$$

4) For each $r \in [t, T-1]_{\mathbb{Z}}$ (totally $T-t$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [t-k, r]_{\mathbb{Z}}$ and $\bar{y}_s^r = 0$ otherwise. Thus $\bar{u}_s^r = 1$ for $s = t-k$ and $\bar{u}_s^r = 0$ otherwise due to constraints (1a) - (1c). Moreover, we let $\bar{x}_s^r = \overline{V} + (s - (t-k))V$ for each $s \in [t-k, t-1]_{\mathbb{Z}}$, $\bar{x}_s^r = \max\{\underline{C}, \overline{V} + (k-2)V\}$ for each $s \in [t, r]_{\mathbb{Z}}$, and $\bar{x}_s^r = 0$ otherwise.

5) For $r = T$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [1, T]_{\mathbb{Z}}$, $\bar{u}_s^r = 0$ for each $s \in [2, T]_{\mathbb{Z}}$, and $\bar{x}_s^r = \overline{C}$ for each $s \in [1, T]_{\mathbb{Z}}$.

6) For each $r \in [2, t-k-2]_{\mathbb{Z}}$ (totally $t-k-3$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r, t-2]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r, t-2]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

7) For each $r \in [t-k-1, t-1]_{\mathbb{Z}}$ (totally $k+1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{V} + (s-r)V, & s \in [r, t]_{\mathbb{Z}} \\ \overline{V} + (t-r)V, & s \in [t+1, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

8) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

9) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \grave{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-2}$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-2}$ and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-2}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$. ■

C.4 Proof for Proposition 8

Proof: **(Facet-defining)** We provide the facet-defining proof for condition (2), as the proof for condition (1) is similar with that for Proposition 5 and thus omitted here.

We have $\overline{C} \leq \overline{V} + kV$ from condition (2) and generate $3T-2$ linearly independent points in $\text{conv}(P)$ that satisfy (26) at equality in the following groups.

1) For each $r \in [1, t-k-1]_{\mathbb{Z}}$ (totally $t-k-1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

2) For each $r \in [1, t-k-1]_{\mathbb{Z}}$ (totally $t-k-1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

3) For each $r \in [t-k, t-1]_{\mathbb{Z}}$ (totally k points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{V} + (r - (t-k))V, & s \in [1, t-k-1]_{\mathbb{Z}} \\ \overline{V} + (r-s)V, & s \in [t-k, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

4) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, & s \in [1, t-k]_{\mathbb{Z}} \\ \overline{V} + (t-s)V, & s \in [t-k+1, t-1]_{\mathbb{Z}} \\ \overline{V}, & s \in [t, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

5) For each $r \in [2, t-k]_{\mathbb{Z}}$ (totally $t-k-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that $\hat{y}_s^r = 1$ for each $s \in [r, r+L-1]_{\mathbb{Z}}$ and $\hat{y}_s^r = 0$ otherwise. Thus $\hat{u}_s^r = 1$ for each $s = r$ and $\hat{u}_s^r = 0$ otherwise due to constraints (1a) - (1c). Meanwhile, we let $\hat{x}_s^r = \overline{V}$ for each $s \in [r, r+L-1]_{\mathbb{Z}} \setminus \{t-k\}$ and $\hat{x}_s^r = 0$ for each $s \in [1, r-1]_{\mathbb{Z}} \cup [r+L, T]_{\mathbb{Z}}$. In addition, for the value of $\hat{x}_{t-k}^r$: 1) If $\hat{y}_{t-k}^r = 1$, we let $\hat{x}_{t-k}^r = \overline{V}$ if $\hat{y}_{t-k+1}^r = 0$ and $\hat{x}_{t-k}^r = \overline{V} + V$ otherwise; 2) If $\hat{y}_{t-k}^r = 0$, we let $\hat{x}_{t-k}^r = 0$.

6) For each $r \in [t-k+1, T]_{\mathbb{Z}}$ (totally $T-t+k$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r, \min\{r+L-1, T\}]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r, \min\{r+L-1, T\}]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

7) For each $r \in [t-k+1, T]_{\mathbb{Z}}$ (totally $T-t+k$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}_s^r = \begin{cases} \underline{C}, & s \in [r, \min\{r+L-1, T\}]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & s \in [r, \min\{r+L-1, T\}]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \grave{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-k-1}$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t-k+1}^T$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-k-1}$ and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-k-1}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t-k+1}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t-k+1}^T$. ■

C.5 Proof for Proposition 12

Proof: **(Facet-defining)** We provide the facet-defining proof for condition (2), as the proof for condition (1) is similar with that for Proposition 5 and thus omitted here.

We let $\kappa = \min\{k, L-1\}$ and generate $3T-2$ linearly independent points in $\text{conv}(P)$ that satisfy (30) at equality in the following groups.

1) For each $r \in [1, t-2]_{\mathbb{Z}}$ (totally $t-2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{matrix} \bar{u}_s^r = 0, \\ \forall s \end{matrix}.$$

2) For $r = t-1$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & s \in [1,r-1]_{\mathbb{Z}} \\ \overline{V}, & s = r \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{matrix} \bar{u}_s^r = 0, \\ \forall s \end{matrix}.$$

3) For each $r \in [t, T-1]_{\mathbb{Z}}$ (totally $T-t$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [t-\kappa, r]_{\mathbb{Z}}$ and $\bar{y}_s^r = 0$ otherwise. Thus $\bar{u}_s^r = 1$ for $s = t-\kappa$ and $\bar{u}_s^r = 0$ otherwise due to constraints (1a) - (1c). Moreover, we let $\bar{x}_s^r = \overline{V} + (s-(t-\kappa))V$ for each $s \in [t-\kappa, t-1]_{\mathbb{Z}}$, $\bar{x}_s^r = \max\{\underline{C}, \overline{V} + (\kappa-2)V\}$ for each $s \in [t, r]_{\mathbb{Z}}$, and $\bar{x}_s^r = 0$ otherwise.

4) For $r = T$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [1,T]_{\mathbb{Z}}$, $\bar{u}_s^r = 0$ for each $s \in [2,T]_{\mathbb{Z}}$, and $\bar{x}_s^r = \underline{C}$ for each $s \in [1, t-k-1]_{\mathbb{Z}}$, $\bar{x}_s^r = \underline{C} + (s-(t-k-1))V$ for each $s \in [1, t-1]_{\mathbb{Z}}$, and $\bar{x}_s^r = \underline{C} + kV$ for each $s \in [t, T]_{\mathbb{Z}}$.

5) We create $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P)$ (totally one point) such that $\dot{y}_s = 1$ for each $s \in [1,T]_{\mathbb{Z}}$, $\bar{u}_s = 0$ for each $s \in [2,T]_{\mathbb{Z}}$, and $\dot{x}_s = \underline{C} + \epsilon$ for each $s \in [1, t-k-1]_{\mathbb{Z}}$, $\bar{x}_s = \underline{C} + (s-(t-k-1))V + \epsilon$ for each $s \in [1, t-1]_{\mathbb{Z}}$, and $\bar{x}_s = \underline{C} + kV + \epsilon$ for each $s \in [t, T]_{\mathbb{Z}}$.

6) For each $r \in [1, t-2]_{\mathbb{Z}} \setminus \{t-k-1\}$ (totally $t-3$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [1,r]_{\mathbb{Z}} \setminus \{t-k-1\} \\ \underline{C}, & s \in [1,r]_{\mathbb{Z}} \cap \{t-k-1\} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{matrix} \acute{u}_s^r = 0, \\ \forall s \end{matrix}.$$

7) For each $r \in [t-\kappa, t-1]_{\mathbb{Z}}$ (totally κ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{V} + (s-r)V, & s \in [r, t-1]_{\mathbb{Z}} \\ \overline{V} + (t-1-r)V, & s \in [t,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

8) For each $r \in [t,T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

9) For each $r \in [t,T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \grave{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

10) For the remaining $\kappa - t - 2$ points, we consider $\kappa = L-1$ and k respectively since $\kappa = \min\{k, L-1\}$.

   - If $\kappa = L-1$, we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ for each $r \in [2, T-\kappa-1]_{\mathbb{Z}}$, where

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r,t-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,t-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

   - If $\kappa = k$, we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ for each $r \in [2, T-\kappa-1]_{\mathbb{Z}}$, where

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r,t-\kappa-1]_{\mathbb{Z}} \\ \underline{C} + (s-(t-\kappa-1))V, & \\ \quad s \in [t-\kappa, t-1]_{\mathbb{Z}} & \\ \underline{C} + \kappa V, & s \in [t,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t-k-1}^{t-2}$, $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$, and $(\dot{x}, \dot{y}, \dot{u})$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t-k-1}^{t-2}$, $(\dot{x}, \dot{y}, \dot{u})$, and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-2}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$. ■

C.6 Proof for Proposition 16

Proof: **(Facet-defining)** We only provide the facet-defining proof for the case when $L = 3$ since the case when $L = 2$ can be proved similarly and thus omitted here.

We generate generate $3T-2$ linearly independent points in conv(P) that satisfy (33) at equality in the following groups.

1) For each $r \in [1, t-4]_{\mathbb{Z}}$ (totally $t-4$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{matrix} \acute{u}_s^r = 0, \\ \forall s \end{matrix}.$$

2) For $r = t-2$ (totally one point), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}+V, & s \in [1, r-1]_{\mathbb{Z}} \\ \underline{C}, & s = r \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

3) For each $r \in [1, t-2]_{\mathbb{Z}}$ (totally $t-2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [1, r]_{\mathbb{Z}}$ and $\bar{y}_s^r = 0$ otherwise. Thus $\bar{u}_s^r = 0$ for each $s \in [2, T]_{\mathbb{Z}}$ due to constraints (1a) - (1c). For the value of $\bar{x}^r$: (1) for each $r \in [1, t-4]_{\mathbb{Z}}$, we let $\bar{x}_s^r = \underline{C} + \epsilon$ for each $s \in [1, r]_{\mathbb{Z}}$; (2) for $r = t-3$, we let $\bar{x}_s^r = \overline{V}$ for each $s \in [1, r]_{\mathbb{Z}}$; (3) for $r = t-2$, we let $\bar{x}_s^r = \underline{C} + V + \epsilon$ for each $s \in [1, r-1]_{\mathbb{Z}}$ and $\bar{x}_s^r = \underline{C} + \epsilon$ for $s = r$.

4) For $r = t-1$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [1, T]_{\mathbb{Z}}$ and thus $\bar{u}_s^r = 0$ for each $s \in [2, T]_{\mathbb{Z}}$ due to constraints (1a) - (1c). For the value of $\bar{x}^r$, we let $\bar{x}_s^r = \overline{C} - V$ for $s = t-2$ and $\bar{x}_s^r = \overline{C}$ otherwise.

5) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{V}, & s = t-3 \\ \underline{C}+V, & s = t-1 \\ \underline{C}, & s \in [r, T]_{\mathbb{Z}} \cup \{t-2\} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [t-3, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \bar{u}_s^r = \begin{cases} 1, & s = t-3 \\ 0, & \text{o.w.} \end{cases}.$$

6) For each $r \in [2, t-1]_{\mathbb{Z}}$ (totally $t-2$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that $\hat{y}_s^r = 1$ for each $s \in [r, r+L-1]_{\mathbb{Z}}$ (i.e., $s \in [r, r+2]_{\mathbb{Z}}$) and $\hat{y}_s^r = 0$ otherwise. Thus $\hat{u}_s^r = 1$ for $s = r$ due to constraints (1a) - (1c). For the value of $\hat{x}^r$: (1) for each $r \in [2, t-4]_{\mathbb{Z}} \cup \{t-2\}$, we let $\hat{x}_s^r = \underline{C}$ for each $s \in [r, r+2]_{\mathbb{Z}} \setminus \{t-3\}$ and $\hat{x}_s^r = \underline{C} + V$ for each $s \in [r, r+2]_{\mathbb{Z}} \cap \{t-3\}$; (2) for $r = t-3$, we let $\hat{x}_s^r = \overline{V}$ for each $s \in \{t-3, t-1\}$ and $\hat{x}_s^r = \underline{C}$ for each $s = t-2$; (3) for $r = t-1$, we let $\hat{x}_s^r = \overline{V}$ for each $s \in [r, r+L-1]_{\mathbb{Z}}$.

7) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

8) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}_s^r = \begin{cases} \underline{C}+\epsilon, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \grave{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

9) We create $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P)$ such that $\dot{y}_s = 1$ for each $s \in \{t-2, t-1, t\}$ and $\dot{y}_s = 0$ otherwise. Thus we have $\dot{u}_s = 1$ for $s = t-2$. Meanwhile, we let $\dot{x}_{t-2} = \dot{x}_t = \underline{C} + \epsilon$ and $\dot{x}_{t-1} = \underline{C} + V + \epsilon$.

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t-3}^{t-2}$, $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$, and $(\dot{x}, \dot{y}, \dot{u})$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t-3}^{t-2}$, $(\dot{x}, \dot{y}, \dot{u})$, and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-2}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$. ■

C.7 Proof for Proposition 17

Proof: **(Facet-defining)** We only provide the facet-defining proof for the case when $L \geq 4$ since other cases can be proved similarly and thus omitted here.

We generate $3T-2$ linearly independent points in conv(P) that satisfy (34) at equality in the following groups.

1) For each $r \in [1, t-1]_{\mathbb{Z}}$ (totally $t-1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that
$$\acute{x}^r_s = \begin{cases} \underline{C}, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}^r_s = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}^r_s = 0, \\ \forall s \end{array}.$$

2) For $r = t+1$ (totally one point), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that
$$\acute{x}^r_s = \begin{cases} \underline{C} + V, & s \in [1, r-1]_{\mathbb{Z}} \\ \underline{C}, & s = r \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}^r_s = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}^r_s = 0, \\ \forall s \end{array}.$$

3) For each $r \in [1, t+k+2]_{\mathbb{Z}}$ (totally $t+k+2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}^r_s = 1$ for each $s \in [1, r]_{\mathbb{Z}}$ and $\bar{y}^r_s = 0$ otherwise. Thus $\bar{u}^r_s = 0$ for each $s \in [2, T]_{\mathbb{Z}}$ due to constraints (1a) - (1c). For the value of $\bar{x}^r$: (1) for each $r \in [1, t-1]_{\mathbb{Z}}$, we let $\bar{x}^r_s = \underline{C} + \epsilon$ for each $s \in [1, r]_{\mathbb{Z}}$; (2) for $r = t$, we let $\bar{x}^r_s = \overline{V}$ for each $s \in [1, r]_{\mathbb{Z}}$; (3) for $r = t+1$, we let $\bar{x}^r_s = \underline{C} + V + \epsilon$ for each $s \in [1, r-1]_{\mathbb{Z}}$ and $\bar{x}^r_s = \underline{C} + \epsilon$ for $s = r$; (4) for $r = t+2$, we let $\bar{x}^r_s = \underline{C} + V$ for each $s \in [1, t]_{\mathbb{Z}}$, $\bar{x}^r_s = \underline{C}$ for $s = t+1$, and $\bar{x}^r_s = \overline{V}$ for $s = t+2$; (5) for each $r \in [t+3, t+k+2]_{\mathbb{Z}}$, we let $\bar{x}^r_s = \overline{V} + (r-s)V$ for each $s \in [t+2, r]_{\mathbb{Z}}$, $\bar{x}^r_s = \overline{V} + (r-t-3)V$ for $s = t+1$, and $\bar{x}^r_s = \overline{V} + (r-t-2)V$ for each $s \in [1, t]_{\mathbb{Z}}$.

4) For $r = t+k+3$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}^r_s = 1$ for each $s \in [1, T]_{\mathbb{Z}}$ and thus $\bar{u}^r_s = 0$ for each $s \in [2, T]_{\mathbb{Z}}$ due to constraints (1a) - (1c). For the value of $\bar{x}^r$, we let $\bar{x}^r_s = \overline{C} - V$ for $s = t+1$ and $\bar{x}^r_s = \overline{C}$ otherwise.

5) For each $r \in [t+k+4, T]_{\mathbb{Z}}$ (totally $T-t-k-3$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \operatorname{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}}$ and $\bar{y}_s^r = 0$ otherwise. Thus $\bar{u}_s^r = 1$ for $s = t+k-L+4$ due to constraints (1a) - (1c). For the value of $\bar{x}^r$, we consider the following cases:

   - If $t+k-L+4 \geq t+3$, we let $\bar{x}_s^r = \underline{C}$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}}$;
   - If $t+k-L+4 = t+2$, we let $\bar{x}_s^r = \overline{V}$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}}$;
   - If $t+k-L+4 = t+1$, we let $\bar{x}_s^r = \underline{C} + V$ for $s = t+2$ and $\bar{x}_s^r = \underline{C}$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}} \setminus \{t+2\}$;
   - If $t+k-L+4 \leq t$, we let $\bar{x}_s^r = \overline{V}$ for each $s \in [t+k-L+4, t]_{\mathbb{Z}}$, $\bar{x}_s^r = \underline{C} + V$ for $s = t+2$, and $\bar{x}_s^r = \underline{C}$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}} \setminus \{t+2\}$.

6) For each $r \in [2, T]_{\mathbb{Z}}$ (totally $T-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \operatorname{conv}(P)$ such that $\hat{u}_s^r = 1$ for $s = r$ and $\hat{u}_s^r = 0$ otherwise. For the values of $\hat{x}^r$ and $\hat{y}^r$: (1) for each $r \in [2, t-L+3]_{\mathbb{Z}}$, we let $\hat{y}_s^r = 1$ for $s \in [r, t+2]_{\mathbb{Z}}$ and $\hat{y}_s^r = 0$ otherwise; we let $\hat{x}_s^r = \underline{C} + V$ for each $s \in [r, t]_{\mathbb{Z}}$, $\hat{x}_s^r = \underline{C}$ for $s = t+1$, and $\hat{x}_s^r = \overline{V}$ for $s = t+2$; (2) for each $r \in [t-L+4, T]_{\mathbb{Z}}$, we let $\hat{y}_s^r = 1$ for $s \in [r, r+L-1]_{\mathbb{Z}}$ and $\hat{y}_s^r = 0$ otherwise; the value of $\hat{x}^r$ can be assigned similarly as above and thus omitted here.

7) For each $r \in [t+1, T]_{\mathbb{Z}} \setminus \{t+2\}$ (totally $T-t-1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \operatorname{conv}(P)$ such that $\grave{y}_s^r = 1$ for $s \in [r, r+L-1]_{\mathbb{Z}}$ and $\grave{y}_s^r = 0$ otherwise. Thus $\grave{u}_s^r = 1$ for $s = r$ due to constraints (1a) - (1c). We assign the value of $\grave{x}^r$ to make $(\grave{x}^r, \grave{y}^r, \grave{u}^r)$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ linearly independent for each $r \in [t+1, T]_{\mathbb{Z}} \setminus \{t+2\}$. It can be easily assigned following the similar rule above and thus omitted here.

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t}^{t+1}$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t+1, r\neq t+2}^T$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t}^{t+1}$ and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1, r\neq t}^{t+1}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t+1, r\neq t+2}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t+1, r\neq t+2}^T$. ■