Inner Approximations of Completely Positive Reformulations of Mixed Binary Quadratic Programs: A Unified Analysis*

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April 7, 2016

Abstract

Every quadratic programming problem with a mix of continuous and binary variables can be equivalently reformulated as a completely positive optimization problem, i.e., a linear optimization problem over the convex but computationally intractable cone of completely positive matrices. In this paper, we focus on general inner approximations of the cone of completely positive matrices on instances of completely positive optimization problems that arise from the reformulation of mixed binary quadratic programming problems. We provide a characterization of the feasibility of such an inner approximation as well as the optimal value of a feasible inner approximation. For polyhedral inner approximations, our characterization implies that computing an optimal solution of the corresponding inner approximation reduces to an optimization problem over a finite set. Our characterization yields, as a byproduct, an upper bound on the gap between the optimal value of an inner approximation and that of the original instance. We discuss the implications of this error bound for standard and box-constrained quadratic programs.

*This work was supported in part by TÜBİTAK (The Scientific and Technological Research Council of Turkey) Grant 112M870.

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Key words: Completely positive cone, mixed binary quadratic optimization problems, inner approximations, polyhedral approximations.

AMS Subject Classifications: 90C25, 90C26, 90C20.

1 Introduction

The cone of completely positive matrices is given by

$$\mathcal{CP}^n := \text{conv}\left\{uu^T : u \in \mathbb{R}^n_+\right\},$$

where conv$\{\cdot\}$ denotes the convex hull and $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$. A completely positive optimization problem is a linear optimization problem over an affine subset of the cone of completely positive matrices:

$$\text{(CoP)} \quad \min\left\{\langle C, X\rangle : \langle A_i, X\rangle = f_i, \quad i = 1,\ldots,\ell, \quad X \in \mathcal{CP}^n\right\}.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the trace inner product defined as $\langle U, V\rangle := \sum_{i=1}^n \sum_{j=1}^n U_{ij}V_{ij}$ for any $U \in \mathcal{S}^n$ and $V \in \mathcal{S}^n$, where $\mathcal{S}^n$ denotes the set of $n \times n$ real symmetric matrices; $C \in \mathcal{S}^n$, $A_i \in \mathcal{S}^n$, $i = 1,\ldots,\ell$, and $f \in \mathbb{R}^\ell$ constitute the parameters; and $X \in \mathcal{S}^n$ denotes the decision variable.

Burer [9] showed that every quadratic programming problem with a mix of binary and continuous variables, henceforth referred to as a mixed binary quadratic program, can be equivalently reformulated as a completely positive optimization problem. On the one hand, this result implies that a rather large class of nonconvex optimization problems admits a compact (i.e., polynomial-sized) reformulation as a convex optimization problem. On the other hand, it follows from this result that a completely positive optimization problem is, in general, NP-hard. Indeed, deciding if a given matrix belongs to $\mathcal{CP}^n$ is a computationally intractable problem [14, 20].

Despite the fact that Burer's reformulation does not seem to be helpful from a computational complexity point of view, it provides a new theoretical and computational perspective on approximately solving mixed binary quadratic programs. For instance, the computationally intractable cone of completely positive matrices can be approximated from the inside (resp., from the outside) by a tractable convex cone. Therefore, for a given mixed binary quadratic program, one can obtain upper (resp., lower) bounds on its optimal value by utilizing inner (resp., outer) approximations of $\mathcal{CP}^n$ in the corresponding completely positive reformulation. Indeed, various tractable inner and outer approximations of $\mathcal{CP}^n$ have been proposed in the literature. For further details, the reader is referred to survey papers [3, 8, 10, 15].

In this paper, given an instance of (CoP) that arises from the reformulation of a mixed binary quadratic program, we focus on the approximation obtained by replacing the difficult conic constraint $X \in \mathcal{CP}^n$ in (CoP) by $X \in \mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{CP}^n$ is a convex cone that approximates the cone of completely positive matrices from the inside. By identifying a particular structure of such an instance of (CoP) (see Proposition 2.2), we make three main contributions in this paper. First, for any such instance of (CoP) and any convex cone $\mathcal{I} \subseteq \mathcal{CP}^n$, we establish necessary and sufficient conditions under which the corresponding inner approximation has a nonempty feasible region (see Proposition 3.1). Second, we give a characterization of the optimal value of a feasible inner approximation, which yields an upper bound on the optimal value of (CoP) (see Proposition 3.2). For the special case of polyhedral inner approximations, our characterization implies that computing the corresponding upper bound reduces to an optimization problem over a finite set. Finally, as a third contribution, we derive an upper bound on the approximation error, i.e., the difference between the optimal value of the inner approximation and that of the original mixed binary quadratic program (see Proposition 3.3). We discuss the implications of this error bound for standard and box-constrained quadratic programs.

This paper is organized as follows. We define our notation in Section 1.1. We review Burer's reformulation of mixed binary quadratic programs and present a useful decomposition result for this formulation in Section 2. Section 3 is devoted to the characterizations of the feasibility, optimal value, and the error bound for general inner approximations. We focus on inner approximations generated by a set of nonnegative rank one matrices in Section 4. For this particular family of inner approximations, we show that our results can simply be expressed in terms of the feasible region of the original mixed binary quadratic program, in contrast with our corresponding results for general inner approximations which are stated in terms of the feasible region of the completely positive formulation in Section 3. We also consider mixed binary quadratic programs with a bounded feasible region and discuss the implications of our results on standard and box-constrained quadratic programs in this section. Finally, Section 5 concludes the paper.

1.1 Notation

We use $\mathbb{R}^n, \mathbb{R}^n_+$, and $\mathcal{S}^n$ to denote the n-dimensional Euclidean space, the nonnegative orthant, and the space of $n \times n$ real symmetric matrices, respectively. We reserve calligraphic letters for subsets of $\mathcal{S}^n$. The set of completely positive matrices is denoted by $\mathcal{CP}^n$. The inner product on $\mathcal{S}^n$ is the trace inner product given by $\langle U, V \rangle := \sum_{i=1}^n \sum_{j=1}^n U_{ij} V_{ij}$ for any $U \in \mathcal{S}^n$ and $V \in \mathcal{S}^n$. The norm associated with the trace inner product is the Frobenius norm denoted by $\|U\|_F := \langle U, U \rangle^{1/2}$. The ℓ_p-norm on $\mathbb{R}^n$ is denoted by $\| \cdot \|_p$, where $p \in \{1, 2, \infty\}$. The set of nonnegative integers is denoted by $\mathbb{N}$. Similarly, we use $\mathbb{N}^n$ to denote the set of n-dimensional vectors all of whose components are given by nonnegative integers. For $u \in \mathbb{R}^n$, we denote its jth component by $u_j,\ j = 1, \ldots, n$. For $u \in \mathbb{R}^{n+1}$, we start the indexing from zero. Similarly, U_{ij} denotes the (i, j) entry of a matrix $U \in \mathcal{S}^n,\ i = 1, \ldots, n;\ j = 1, \ldots, n$, and we start both indices from zero for $U \in \mathcal{S}^{n+1}$. We adopt Matlab-like notation. For a vector u, we use $u_{1:k} \in \mathbb{R}^k$ to denote the subvector given by the components of u indexed by $1, \ldots, k$. We use similar notation for matrices, i.e., $U_{I,J}$ denotes the submatrix of U whose rows and columns are given by the rows and columns of U indexed by the sets I and J, respectively. For two vectors $u \in \mathbb{R}^{n_1}$ and $v \in \mathbb{R}^{n_2}$, we use $[u; v] \in \mathbb{R}^{n_1+n_2}$ to denote the vector obtained by the vertical concatenation of u and v. The unit vectors in $\mathbb{R}^n$ are denoted by $e_j,\ j = 1, \ldots, n$, with the convention that indexing starts from zero for $\mathbb{R}^{n+1}$. We reserve e to denote the vector of all ones whose dimension will always be clear from the context.

We use 0 to denote the real number zero, the vector of all zeroes as well as the matrix of all zeroes in the appropriate dimension, which will always be unambiguously determined from the context.

2 Burer's Reformulation and a Decomposition Result

A mixed binary quadratic program is given by

$$
\begin{array}{lll}
\text{(BQP)} & \min & x^T Q x + 2c^T x \\
& \text{s.t.} & \\
& & a_i^T x = b_i, \quad i = 1, \ldots, m, \\
& & x \geq 0, \\
& & x_j \in \{0, 1\}, \quad j \in B,
\end{array}
$$

where $Q \in \mathcal{S}^n$, $c \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n,\ i = 1, \ldots, m$, $b_i \in \mathbb{R},\ i = 1, \ldots, m$, and $B \subseteq \{1, \ldots, n\}$ constitute the data of the problem and $x \in \mathbb{R}^n$ denotes the decision variable. Note that (BQP) contains all (nonconvex) quadratic programs and all mixed binary linear and quadratic programs as special cases. Therefore, (BQP) is, in general, an NP-hard problem.

We denote the feasible region of (BQP) by F, i.e.,

$$
F := \{x \in \mathbb{R}^n : a_i^T x = b_i, \quad i = 1, \ldots, m, \quad x \geq 0, \quad x_j \in \{0, 1\}, \quad j \in B\}, \tag{1}
$$

and we assume that $F \neq \emptyset$. The set obtained from F by ignoring the binary constraints is denoted by L:

$$
L := \{x \in \mathbb{R}^n : a_i^T x = b_i, \quad i = 1, \ldots, m, \quad x \geq 0\}. \tag{2}
$$

The recession cone of L is given by

$$
L_\infty := \{d \in \mathbb{R}^n : a_i^T d = 0, \quad i = 1, \ldots, m, \quad d \geq 0\}. \tag{3}
$$

Burer [9] makes the following assumption, referred to as the *key assumption*:

$$
x \in L \Rightarrow 0 \leq x_j \leq 1, \quad j \in B. \tag{4}
$$

Note that the key assumption can always be enforced by adding the redundant constraints $x_j \leq 1$ for each $j \in B$, if necessary, and then introducing slack variables to convert the formulation into the form of (BQP). The key assumption implies that

$$d \in L_\infty \Rightarrow d_j = 0, \quad j \in B. \tag{5}$$

Under the key assumption (4), Burer [9] shows that (BQP) is equivalent to the following completely positive optimization problem:

$$\begin{array}{lllcl}
\text{(CP)} & \min & \langle Q, X \rangle + 2\, c^T x & & \\
& \text{s.t.} & a_i^T x & = & b_i, \quad i = 1, \ldots, m, \\
& & a_i^T X a_i & = & b_i^2, \quad i = 1, \ldots, m, \\
& & X_{jj} & = & x_j, \quad j \in B, \\
& & Y & = & \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}, \\
& & Y & \in & \mathcal{CP}^{n+1}.
\end{array}$$

Note that (CP) can be easily represented in the form of (CoP) by using the identity $a_i^T X a_i = \langle a_i a_i^T, X \rangle$ for $i = 1, \ldots, m$.

We remark that (CP) is an *exact* reformulation of (BQP) in the sense that the optimal values of the two problems are the same. Furthermore, the x-component of any optimal solution of (CP) lies in the convex hull of the set of optimal solutions of (BQP) [9]. Therefore, if (BQP) has a unique optimal solution, then the x-component of any optimal solution of (CP) is also an optimal solution of (BQP).

Similarly, let us denote the feasible region of (CP) by $\mathcal{F}$:

$$\mathcal{F} = \left\{ Y \in \mathcal{CP}^{n+1} : Y = \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}, \quad \begin{array}{ll} a_i^T x = b_i, & i = 1, \ldots, m, \\ a_i^T X a_i = b_i^2, & i = 1, \ldots, m, \\ X_{jj} = x_j, & j \in B \end{array} \right\}. \tag{6}$$

The recession cone of $\mathcal{F}$ is given by

$$\mathcal{L}_\infty = \left\{ \begin{bmatrix} 0 & 0^T \\ 0 & D \end{bmatrix} \in \mathcal{CP}^{n+1} : \begin{array}{ll} a_i^T D a_i = 0, & i = 1, \ldots, m, \\ D_{jj} = 0, & j \in B \end{array} \right\}. \tag{7}$$

By (1) and (6),

$$x \in F \Rightarrow \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \in \mathcal{F}. \tag{8}$$

Similarly, by (3), (5), and (7),

$$d \in L_\infty \Rightarrow \begin{bmatrix} 0 \\ d \end{bmatrix} \begin{bmatrix} 0 \\ d \end{bmatrix}^T \in \mathcal{L}_\infty. \tag{9}$$

The equivalence between (BQP) and (CP) relies on the following main result.

Proposition 2.1 (Burer [9]) *Given an instance of (BQP), let $Y \in \mathcal{CP}^{n+1}$ be a feasible solution of (CP) given by*

$$Y = \sum_{h \in H} \begin{bmatrix} \rho_h \\ r_h \end{bmatrix} \begin{bmatrix} \rho_h \\ r_h \end{bmatrix}^T,$$

where H is a finite set, $\rho_h \in \mathbb{R}_+$ and $r_h \in \mathbb{R}^n_+$ for each $h \in H$. Let

$$H_+ := \{h \in H : \rho_h > 0\}, \quad H_0 := \{h \in H : \rho_h = 0\}.$$

Then,

(i) $(1/\rho_h) r_h \in F$ *for each $h \in H_+$, and*

(ii) $r_h \in L_\infty$ *for each $h \in H_0$.*

Our next proposition presents a useful decomposition result for a given feasible solution of (CP).

Proposition 2.2 *Let $Y \in \mathcal{CP}^{n+1}$ be a feasible solution of (CP) and suppose that*

$$Y = \sum_{p \in P} V^p + \sum_{z \in Z} W^z,$$

where P and Z are finite sets; $V^p \in \mathcal{CP}^{n+1}$ and $V^p_{00} > 0$ for each $p \in P$; $W^z \in \mathcal{CP}^{n+1}$ and $W^z_{00} = 0$ for each $z \in Z$. Then,

(i) $P \neq \emptyset$,

(ii) $(1/V^p_{00})V^p \in \mathcal{F}$ *for each* $p \in P$*, and*

(iii) $W^z \in \mathcal{L}_\infty$ *for each* $z \in Z$*.*

Proof. Since $Y \in \mathcal{F}$, it follows that $Y_{00} = 1$, which implies that $P \neq \emptyset$ establishing (i).

Suppose that $V^p, p \in P$, and W^z, $z \in Z$, admit the following decompositions:

$$\begin{aligned} V^p &= \sum_{h \in H_0^p} \begin{bmatrix} 0 \\ v^{hp} \end{bmatrix} \begin{bmatrix} 0 \\ v^{hp} \end{bmatrix}^T + \sum_{h \in H_+^p} \begin{bmatrix} \sigma_{hp} \\ s^{hp} \end{bmatrix} \begin{bmatrix} \sigma_{hp} \\ s^{hp} \end{bmatrix}^T, \quad p \in P, \\ W^z &= \sum_{h \in H_0^z} \begin{bmatrix} 0 \\ w^{hz} \end{bmatrix} \begin{bmatrix} 0 \\ w^{hz} \end{bmatrix}^T, \quad z \in Z, \end{aligned}$$

where H_0^p and H_+^p are finite sets for each $p \in P$; H_0^z is a finite set for each $z \in Z$; $v^{hp} \in \mathbb{R}^n_+$ for each $p \in P$ and $h \in H_0^p$; $s^{hp} \in \mathbb{R}^n_+$ and $\sigma_{hp} > 0$ for each $p \in P$ and $h \in H_+^p$; and $w^{hz} \in \mathbb{R}^n_+$ for each $z \in Z$ and $h \in H_0^z$. Note that, for each $p \in P$, $H_+^p \neq \emptyset$ since $V^p_{00} > 0$.

By Proposition 2.1, we obtain

$$v^{hp} \in L_\infty, \quad p \in P, \quad h \in H_0^p, \quad \text{and} \quad w^{hz} \in L_\infty, \quad z \in Z, \quad h \in H_0^z, \tag{10}$$

where L_∞ is given by (3), and

$$\left(\frac{1}{\sigma_{hp}} \right) s^{hp} \in F, \quad p \in P, \quad h \in H_+^p, \tag{11}$$

where F is defined as in (1).

Since $\mathcal{L}_\infty$ is a convex cone, it follows from (9) and (10) that $W^z \in \mathcal{L}_\infty$ for each $z \in Z$, which establishes (iii).

Let

$$V^{p,0} := \sum_{h \in H_0^p} \begin{bmatrix} 0 \\ v^{hp} \end{bmatrix} \begin{bmatrix} 0 \\ v^{hp} \end{bmatrix}^T, \quad V^{p,+} := \sum_{h \in H_+^p} \begin{bmatrix} \sigma_{hp} \\ s^{hp} \end{bmatrix} \begin{bmatrix} \sigma_{hp} \\ s^{hp} \end{bmatrix}^T, \quad p \in P,$$

so that $V^p = V^{p,+} + V^{p,0}$ for each $p \in P$. Similarly, by (9) and (10), $V^{p,0} \in \mathcal{L}_\infty$ for each $p \in P$.

Let us now define

$$\hat{V}^p := \left(\frac{1}{V^p_{00}}\right) V^p = \left(\frac{1}{V^{p,+}_{00}}\right)(V^{p,+} + V^{p,0}), \quad p \in P.$$

We need to show that $\hat{V}^p \in \mathcal{F}$ for each $p \in P$. Let us fix $p \in P$. By (8) and (11), we have

$$\begin{bmatrix} 1 \\ \left(\frac{1}{\sigma_{hp}}\right) s^{hp} \end{bmatrix} \begin{bmatrix} 1 \\ \left(\frac{1}{\sigma_{hp}}\right) s^{hp} \end{bmatrix}^T \in \mathcal{F}, \quad h \in H^p_+. \tag{12}$$

Note that

$$\hat{V}^p = \left(\frac{1}{\sum_{h \in H^p_+} \sigma^2_{hp}}\right) \left(\sum_{h \in H^p_+} \sigma^2_{hp} \begin{bmatrix} 1 \\ \left(\frac{1}{\sigma_{hp}}\right) s^{hp} \end{bmatrix} \begin{bmatrix} 1 \\ \left(\frac{1}{\sigma_{hp}}\right) s^{hp} \end{bmatrix}^T + V^{p,0} \right).$$

By (12), the first term above is given by a convex combination of the elements of $\mathcal{F}$ and the second term, which is a positive multiple of $V^{p,0}$, belongs to $\mathcal{L}_\infty$. It follows that $\hat{V}^p \in \mathcal{F}$ as desired, thereby establishing (ii). □

3 General Inner Approximations

In this section, we consider replacing the difficult conic constraint $Y \in \mathcal{CP}^{n+1}$ in (CP) by $Y \in \mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{CP}^{n+1}$ is a closed and convex cone. First, we discuss the representation of such a cone $\mathcal{I}$.

Let $\mathcal{M} \subseteq \mathcal{CP}^{n+1}$ be a nonempty set such that $0 \notin \mathcal{M}$. We denote by $\text{cone}(\mathcal{M}) \subset \mathcal{S}^{n+1}$ the conic hull of $\mathcal{M}$, i.e., it is the smallest convex cone (with respect to inclusion) that contains $\mathcal{M} \cup \{0\}$. By [22, Theorem 4.24],

$$\text{cone}(\mathcal{M}) = \text{cone}\left(\bigcup_{M \in \mathcal{M}} \text{cone}(\{M\}) \right).$$

Since $0 \notin \mathcal{M}$, we may, by scaling, assume that $e^T M e = \langle M, ee^T \rangle = 1$ for each $M \in \mathcal{M}$ using the relation above. Therefore, without loss of generality, we may assume that $\mathcal{M}$ is a bounded set. If $\mathcal{M}$ is also closed, then $\text{cone}(\mathcal{M})$ is a closed and convex cone (see, e.g., [22, Theorem 4.37]). Otherwise, since $\mathcal{M} \subseteq \mathcal{CP}^{n+1}$ and since $\mathcal{CP}^{n+1}$ is closed, we obtain $\text{cl}(\mathcal{M}) \subseteq \mathcal{CP}^{n+1}$, where $\text{cl}(\cdot)$ denotes the closure. It follows that $\text{cone}(\text{cl}(\mathcal{M})) \subseteq \mathcal{CP}^{n+1}$. We therefore consider the following cone:

$$\mathcal{I} = \mathcal{I}(\mathcal{M}) := \text{cone}(\text{cl}(\mathcal{M})) = \text{cl}(\text{cone}(\mathcal{M})), \tag{13}$$

where the last equality follows from [22, Theorem 4.37]. Note that $\mathcal{I}(\mathcal{M})$ therefore yields a closed and convex inner approximation of $\mathcal{CP}^{n+1}$.

Conversely, any closed and convex cone $\mathcal{I} \subseteq \mathcal{CP}^{n+1}$ can be represented by (13) for an appropriate choice of $\mathcal{M}$. Indeed, by defining

$$\mathcal{M} := \mathcal{I} \cap \{Y \in \mathcal{S}^{n+1} : \langle Y, ee^T \rangle = 1\},$$

which is a closed and bounded set, we obtain $\mathcal{I} = \mathcal{I}(\mathcal{M}) = \text{cone}(\mathcal{M})$. In fact, there exist inner approximation hierarchies for the cone of completely positive matrices, i.e., there are sequences of tractable nested cones that provide increasingly better inner approximations with the property that such sequences are, in some sense, exact in the limit. The reader is referred to [23] and [18] for inner approximation hierarchies consisting of polyhedral and spectrahedral cones (i.e., cones given by the intersection of a linear subspace and the set of positive semidefinite matrices), respectively. For instance, each inner approximation in either of these hierarchies can be represented in the form of (13) for an appropriate choice of $\mathcal{M}$ (see, e.g., [23, Theorem 2.2] and [18, Corollary 2.4]).

Since $\mathcal{I}(\mathcal{M}) \subset \mathcal{S}^{n+1}$, by Carathéodory's theorem for conic hulls, we have

$$\begin{aligned} Y \in \mathcal{I}(\mathcal{M}) \quad \Leftrightarrow \quad \exists\ M^k \in \text{cl}(\mathcal{M}), \quad k = 1, \ldots, \kappa, \quad \kappa \leq \frac{(n+1)(n+2)}{2}, \\ Y = \sum_{k=1}^{\kappa} \lambda_k M^k, \quad \lambda_k \geq 0, \quad k = 1, \ldots, \kappa, \end{aligned} \tag{14}$$

i.e., any element of $\mathcal{I}(\mathcal{M})$ can be represented by a nonnegative combination of a finite number of matrices $M^k \in \text{cl}(\mathcal{M})$. We will refer to (14) as a finite conic representation of $Y \in \mathcal{I}(\mathcal{M})$.

We consider replacing the conic constraint $Y \in \mathcal{CP}^{n+1}$ in (CP) by $Y \in \mathcal{I}(\mathcal{M})$. The next proposition provides a characterization of the feasibility of the corresponding inner approximation of (CP). Furthermore, for any feasible solution of the resulting inner approximation, we present a useful representation result that will subsequently be used to characterize the corresponding optimal solution.

Proposition 3.1 *Given an instance of (CP), a nonempty set $\mathcal{M} \subseteq \mathcal{CP}^{n+1}$ such that $0 \notin \mathcal{M}$, suppose that the conic constraint $Y \in \mathcal{CP}^{n+1}$ in (CP) is replaced by $Y \in \mathcal{I}(\mathcal{M})$, where $\mathcal{I}(\mathcal{M})$ is defined as in (13). Let*

$$\mathcal{M}_0 = \{M \in cl(\mathcal{M}) : M_{00} = 0\}, \quad \mathcal{M}_+ = \{M \in cl(\mathcal{M}) : M_{00} > 0\}. \tag{15}$$

(i) We have $\mathcal{F} \cap \mathcal{I}(\mathcal{M}) \neq \emptyset$ (i.e., the inner approximation of (CP) has a nonempty feasible region) if and only if $\mathcal{M}_{++} \neq \emptyset$, where

$$\mathcal{M}_{++} := \{M \in \mathcal{M}_+ : \mathcal{F} \cap cone(\{M\}) \neq \emptyset\} = \{M \in \mathcal{M}_+ : (1/M_{00})\, M \in \mathcal{F}\}. \tag{16}$$

(ii) Suppose that $\mathcal{F} \cap \mathcal{I}(\mathcal{M}) \neq \emptyset$ and let $Y \in \mathcal{F} \cap \mathcal{I}(\mathcal{M})$. For any finite conic representation of Y given by (14), let

$$K_0 := \{k \in \{1, \ldots, \kappa\} : M^k \in \mathcal{M}_0, \quad \lambda_k > 0\}, \tag{17}$$

$$K_+ := \{k \in \{1, \ldots, \kappa\} : M^k \in \mathcal{M}_+, \quad \lambda_k > 0\}. \tag{18}$$

Then, $K_+ \neq \emptyset$. Furthermore, $M^k \in \mathcal{L}_\infty$ for each $k \in K_0$ and $M^k \in \mathcal{M}_{++}$ for each $k \in K_+$.

Proof. Let us first consider part (i). If $\mathcal{M}_{++} \neq \emptyset$, then $(1/M_{00})M \in \mathcal{F} \cap \mathcal{I}(\mathcal{M})$ for any $M \in \mathcal{M}_{++}$, which implies that $\mathcal{F} \cap \mathcal{I}(\mathcal{M}) \neq \emptyset$. Note that the reverse implication of part (i) directly follows from part (ii). Therefore, we will proceed with the proof of part (ii).

Suppose that $\mathcal{F} \cap \mathcal{I}(\mathcal{M}) \neq \emptyset$ and let $Y \in \mathcal{F} \cap \mathcal{I}(\mathcal{M})$. For any finite conic representation of Y given by (14), let us decompose $Y = Y^+ + Y^0$, where

$$Y^+ := \sum_{k \in K_+} \lambda_k M^k, \quad Y^0 := \sum_{k \in K_0} \lambda_k M^k.$$

Note that $M^k \in \mathcal{CP}^{n+1}$ for each $k \in \{1, \ldots, K\}$. Since $Y \in \mathcal{F}$, we have $Y^+ \neq 0$ by Proposition 2.2. Therefore, $K_+ \neq \emptyset$. Furthermore, for each $k \in K_+$, we have $(1/\lambda_k M^k_{00})\lambda_k M^k =$

$(1/M^k_{00})M^k \in \mathcal{F}$, i.e., $M^k \in \mathcal{M}_{++}$. Therefore, $\mathcal{M}_{++} \neq \emptyset$, which establishes the reverse implication of (i). Finally, $\lambda_k M^k \in \mathcal{L}_\infty$ for each $k \in K_0$ by Proposition 2.2, i.e., $M^k \in \mathcal{L}_\infty$ for each $k \in K_0$, which concludes the proof of part (ii). □

By Proposition 3.1, the feasibility of an arbitrary inner approximation of (CP) depends only on whether there exists at least one element $M \in \mathrm{cl}(\mathcal{M})$ such that $\mathcal{I}(\{M\}) = \mathrm{cone}(\{M\}) = \{\lambda M : \lambda \geq 0\}$, i.e., the cone generated by M, intersects the feasible region $\mathcal{F}$ of (CP). The next example illustrates that such a characterization does not necessarily hold true for the feasible region of an arbitrary completely positive optimization problem.

Example 3.1 *Consider the feasible region of a general completely positive optimization problem (CoP) given by*

$$\mathcal{F} := \left\{ X \in \mathcal{CP}^2 : \left\langle \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, X \right\rangle = 1, \quad \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X \right\rangle = 2 \right\}.$$

One can easily verify that

$$\mathcal{F} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \mu \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} : \mu \geq 0 \right\}.$$

Let

$$\mathcal{M} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right\} \subset \mathcal{CP}^2.$$

We have

$$\mathcal{F} \cap \mathcal{I}(\mathcal{M}) = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right\},$$

whereas the cone generated by either matrix in $\mathcal{M}$ does not intersect $\mathcal{F}$.

It follows from this example that our decomposition result for the completely positive reformulation of mixed binary quadratic programs given by Proposition 2.2 plays a central role for establishing our general characterization result presented in Proposition 3.1.

Our next result gives a useful characterization of the upper bound obtained by replacing the conic constraint $Y \in \mathcal{CP}^{n+1}$ in (CP) by $Y \in \mathcal{I}(\mathcal{M})$.

Proposition 3.2 *Given an instance of (CP), a nonempty set $\mathcal{M} \subseteq \mathcal{CP}^{n+1}$ such that $0 \notin \mathcal{M}$, suppose that the conic constraint $Y \in \mathcal{CP}^{n+1}$ in (CP) is replaced by $Y \in \mathcal{I}(\mathcal{M})$, where $\mathcal{I}(\mathcal{M})$ is defined as in (13). Let $\mathcal{M}_0$ and $\mathcal{M}_+$ be defined as in (15) and $\mathcal{M}_{++}$ as in (16). Let $\nu(\mathcal{M})$ denote the optimal value of the corresponding inner approximation of (CP).*

(i) If $\mathcal{M}_{++} = \emptyset$, then the inner approximation of (CP) is infeasible and $\nu(\mathcal{M}) := +\infty$.

(ii) If $\mathcal{M}_{++} \neq \emptyset$ and there exists $M \in \mathcal{M}_0$ such that $M \in \mathcal{L}_\infty$ and $\langle Q, M_{1:n,1:n}\rangle < 0$, then the inner approximation of (CP) is unbounded below and $\nu(\mathcal{M}) = -\infty$.

(iii) If $\mathcal{M}_{++} \neq \emptyset$ and $\inf\limits_{M \in \mathcal{M}_0 : M \in \mathcal{L}_\infty} \langle Q, M_{1:n,1:n}\rangle \geq 0$, then

$$\nu(\mathcal{M}) = \inf_{M \in \mathcal{M}_{++}} \left(\frac{1}{M_{00}}\right) \left(\langle Q, M_{1:n,1:n}\rangle + 2c^T M_{1:n,1}\right).$$

Proof. Note that (i) follows directly from part (i) of Proposition 3.1.

Considering (ii), let $M^+ \in \mathcal{M}_{++}$ and suppose that there exists $M^0 \in \mathcal{M}_0$ such that $M^0 \in \mathcal{L}_\infty$ and $\langle Q, M^0_{1:n,1:n}\rangle < 0$. Then, $Y^\mu := (1/M^+_{00})M^+ + \mu M^0 \in \mathcal{F} \cap \mathcal{I}(\mathcal{M})$ for each $\mu \geq 0$ by Proposition 3.1. Clearly,

$$\begin{aligned}\langle Q, Y^\mu_{1:n,1:n}\rangle + 2c^T Y^\mu_{1:n,1} &= \left(\frac{1}{M^+_{00}}\right)\left(\langle Q, M^+_{1:n,1:n}\rangle + 2c^T M^+_{1:n,1}\right) \\ &\quad + \mu\langle Q, M^0_{1:n,1:n}\rangle \to -\infty\end{aligned}$$

as $\mu \to +\infty$, which establishes (ii).

Finally, suppose that $\mathcal{M}_{++} \neq \emptyset$ and $\inf\limits_{M \in \mathcal{M}_0 : M \in \mathcal{L}_\infty} \langle Q, M_{1:n,1:n}\rangle \geq 0$. By Proposition 3.1, $(1/M_{00})M \in \mathcal{F}$ for each $M \in \mathcal{M}_{++}$. Therefore,

$$\nu(\mathcal{M}) \leq \eta := \inf_{M \in \mathcal{M}_{++}} \left(\frac{1}{M_{00}}\right)\left(\langle Q, M_{1:n,1:n}\rangle + 2c^T M_{1:n,1}\right). \tag{19}$$

We will use a contradiction argument to establish the reverse inequality. Suppose, for a contradiction, that $\nu(\mathcal{M}) < \eta$. Then, there exists a sequence $\{Y^r : r \in \mathbb{N}\} \subseteq \mathcal{F} \cap \mathcal{I}(\mathcal{M})$ such that $\langle Q, Y^r_{1:n,1:n}\rangle + 2c^T Y^r_{1:n,1} \to \nu(\mathcal{M})$ as $r \to \infty$. Therefore, there exists $r_* \in \mathbb{N}$ such that $\langle Q, Y^{r_*}_{1:n,1:n}\rangle + 2c^T Y^{r_*}_{1:n,1} < \eta$. Consider any finite representation of Y^{r_*} given by (14):

$$Y^{r_*} = \sum_{k \in K_0} \lambda_k M^k + \sum_{k \in K_+} \lambda_k M^k,$$

where K_0 and K_+ are defined as in (17) and (18), respectively. By Proposition 3.1, $K_+ \neq \emptyset$ and $M^k \in \mathcal{M}_{++}$ for each $k \in K_+$. Furthermore, $M^k \in \mathcal{L}_\infty$ for each $k \in K_0$. Therefore,

$$\begin{aligned}
\eta &> \langle Q, Y^{r_*}_{1:n,1:n} \rangle + 2c^T Y^{r_*}_{1:n,1} \\
&= \sum_{k \in K_0} \lambda_k \langle Q, M^k_{1:n,1:n} \rangle + \sum_{k \in K_+} \lambda_k \left(\langle Q, M^k_{1:n,1:n} \rangle + 2c^T M^k_{1:n,1} \right) \\
&\geq \sum_{k \in K_+} \lambda_k M^k_{00} \left(\frac{1}{M^k_{00}} \right) \left(\langle Q, M^k_{1:n,1:n} \rangle + 2c^T M^k_{1:n,1} \right) \\
&\geq \min_{k \in K_+} \left(\frac{1}{M^k_{00}} \right) \left(\langle Q, M^k_{1:n,1:n} \rangle + 2c^T M^k_{1:n,1} \right) \\
&\geq \eta,
\end{aligned}$$

where we used the assumption that $\inf_{M \in \mathcal{M}_0 : M \in \mathcal{L}_\infty} \langle Q, M_{1:n,1:n} \rangle \geq 0$ in the third line, the identity

$$Y^{r_*}_{00} = \sum_{k \in K_+} \lambda_k M^k_{00} = 1$$

in the fourth line, and the definition of η in (19) in the last line. Therefore, (iii) follows from the contradiction that arises from the chain of inequalities above. □

3.1 Error Bound

In this section, we consider the gap between the upper bound that arises from an arbitrary inner approximation of an instance of (CP) and its optimal value. The next proposition presents the main result.

Proposition 3.3 *Given an instance of (CP), a nonempty set $\mathcal{M} \subseteq \mathcal{CP}^{n+1}$ such that $0 \notin \mathcal{M}$, suppose that the conic constraint $Y \in \mathcal{CP}^{n+1}$ in (CP) is replaced by $Y \in \mathcal{I}(\mathcal{M})$, where $\mathcal{I}(\mathcal{M})$ is defined as in (13). Let $\mathcal{M}_0$ and $\mathcal{M}_+$ be defined as in (15) and $\mathcal{M}_{++}$ as in (16). Let $\nu(\mathcal{M})$ and ν^* denote the optimal values of the corresponding inner approximation and the original instance of (CP), respectively. Suppose that ν^* is finite. Then,*

$$\nu(\mathcal{M}) - \nu^* \leq \left(\|Q\|_F^2 + 2\|c\|_2^2 \right)^{1/2} \delta_1(\mathcal{F}, \mathcal{M}), \tag{20}$$

where

$$\delta_1(\mathcal{F}, \mathcal{M}) := \sup_{Y \in \mathcal{F}} \inf_{M \in \mathcal{M}_{++}} \left\| Y - \left(\frac{1}{M_{00}} \right) M \right\|_F . \tag{21}$$

Proof. If $\mathcal{M}_{++} = \emptyset$, then $\delta_1(\mathcal{F}, \mathcal{M}) := +\infty$ by Proposition 3.2 and (20) is trivially satisfied. Suppose now that $\mathcal{M}_{++} \neq \emptyset$. Since ν^* is finite, there exists $x^* \in \mathbb{R}^n$ such that $\nu^* = (x^*)^T Q x^* + 2c^T x^*$ by the Frank-Wolfe theorem [16] (note that, in the presence of binary variables, ν^* is given by the minimum of the optimal values of a finite number of quadratic programs in smaller dimensions and the optimal solution is attained for at least one of those quadratic programs since ν^* is finite). Therefore, by Burer's equivalence result [9] and by (8),

$$Y^* = \begin{bmatrix} 1 \\ x^* \end{bmatrix} \begin{bmatrix} 1 \\ x^* \end{bmatrix}^T \in \mathcal{F}$$

is an optimal solution of (CP). By (21), there exists $M^* \in \mathcal{M}_{++}$ such that $\|Y^* - (1/M^*_{00})M^*\|_F \leq \delta_1(\mathcal{F}, \mathcal{M})$. By Proposition 3.2,

$$\begin{aligned}
\nu(\mathcal{M}) - \nu^* &= \inf_{M \in \mathcal{M}_{++}} \left(\frac{1}{M_{00}} \right) \left(\langle Q, M_{1:n,1:n} \rangle + 2c^T M_{1:n,1} \right) - \nu^* \\
&\leq \left(\frac{1}{M^*_{00}} \right) \left(\langle Q, M^*_{1:n,1:n} \rangle + 2c^T M^*_{1:n,1} \right) - \nu^* \\
&= \left\langle \begin{bmatrix} 0 & c^T \\ c & Q \end{bmatrix}, \left(\frac{1}{M^*_{00}} \right) M^* - Y^* \right\rangle \\
&\leq \left(\|Q\|_F^2 + 2\|c\|_2^2 \right)^{1/2} \delta_1(\mathcal{F}, \mathcal{M}).
\end{aligned}$$

□

We remark that the parameter $\delta_1(\mathcal{F}, \mathcal{M})$ given by (21) is completely determined by the feasible region $\mathcal{F}$ of the completely positive reformulation of the underlying mixed binary quadratic program and by the particular inner approximation of $\mathcal{CP}^{n+1}$.

3.2 Polyhedral Inner Approximations

We close this section by specializing our results to the case of polyhedral inner approximations. Since every polyhedral cone is generated by a finite number of matrices, such a cone admits a representation given by (13) in which $\mathcal{M} = \{M^1, \ldots, M^\kappa\} \subset \mathcal{CP}^{n+1}$ is a finite set. In this case, most of our earlier discussions simplify considerably. For instance, by Proposition 3.1, checking the feasibility of the corresponding inner approximation simply reduces to checking whether the cone generated by each matrix $M^k,\ k = 1, \ldots, \kappa$ intersects the feasible region of (CP). Similarly, it follows from Proposition 3.2 that computing the corresponding upper bound reduces to solving an optimization problem over a finite set. Indeed, since $\mathcal{M}_0$ and $\mathcal{M}_+$ given by (15) are both finite sets, one can compute in a finite number of steps whether the inner approximation of (CP) is unbounded below or has a finite optimal value. Finally, unless the feasible region $\mathcal{F}$ of (CP) is bounded, we clearly have $\delta_1(\mathcal{F}, \mathcal{M}) = +\infty$ for any finite set $\mathcal{M}$. Therefore, Proposition 3.3 does not yield a nontrivial error bound in this case. However, we remark that, for any fixed finite set $\mathcal{M}$, it is easy to construct an instance of (CP) for which ν^* is finite while $\nu(\mathcal{M}) - \nu^*$ is as large as possible. It follows that a general finite error bound cannot be established for polyhedral inner approximations unless additional assumptions are made about the feasible region of $\mathcal{F}$. We will address this issue in the next section.

4 Inner Approximations Generated by Nonnegative Rank One Matrices

In this section, we focus on inner approximations of $\mathcal{CP}^{n+1}$ generated by nonnegative rank one matrices. Note that

$$\mathcal{CP}^{n+1} = \text{cone}\left(\{vv^T : v \in \Delta_{n+1}\}\right),$$

where $\Delta_{n+1} = \{x \in \mathbb{R}^{n+1}_+ : e^T x = 1\}$ denotes the unit simplex in $\mathbb{R}^{n+1}$. Since the extreme rays of $\mathcal{CP}^{n+1}$ are given by rank one matrices [2], we can define inner approximations generated by rank one matrices pp^T, where $p \in S$ and S is a nonempty subset of Δ_{n+1}, so that the extreme rays of the inner approximation coincide with those of $\mathcal{CP}^{n+1}$.

Given a nonempty subset $S \subseteq \Delta_{n+1}$, let us therefore define

$$\mathcal{M}_S := \{pp^T : p \in S\}. \tag{22}$$

Note that $\mathcal{M}_S \subseteq \mathcal{CP}^{n+1}$.

First, we present a simple characterization of $\mathrm{cl}(\mathcal{M}_S)$.

Lemma 4.1 *Let $S \subseteq \Delta_{n+1}$ be a nonempty set and let $\mathcal{M}_S$ be given by (22). Then,*

$$cl(\mathcal{M}_S) = \{pp^T : p \in cl(S)\}.$$

Proof. Let $p \in \mathrm{cl}(S)$. Then, there exists a sequence $\{p^r : r \in \mathbb{N}\} \subset S$ such that $p^r \to p$ as $r \to +\infty$. Therefore, $p_i^r \to p_i$ for each $i = 1, \ldots, n+1$ as $r \to +\infty$. It follows that $\|p^r(p^r) - pp^T\|_F \to 0$, i.e., $p^r(p^r) \to pp^T$ as $r \to +\infty$. Therefore, $\{pp^T : p \in \mathrm{cl}(S)\} \subseteq \mathrm{cl}(\mathcal{M}_S)$.

For the reverse inclusion, let $P \in \mathrm{cl}(\mathcal{M}_S)$. Note that $\mathrm{cl}(\mathcal{M}_S) \subseteq \mathcal{CP}^{n+1} \cap \{Y \in \mathcal{S}^{n+1} : \langle Y, ee^T \rangle = 1\}$. Therefore, $P \in \mathcal{CP}^{n+1} \backslash \{0\}$. Since $P \in \mathrm{cl}(\mathcal{M}_S)$, there exists a sequence $\{p^r(p^r)^T : r \in \mathbb{N}\} \subset \mathcal{M}_S$ such that $p^r(p^r)^T \to P$ as $r \to +\infty$. Note that, for each $r \in \mathbb{N}$, $p^r(p^r)^T$ has one positive eigenvalue given by $(p^r)^T p^r$ and all the remaining eigenvalues are equal to zero. By the continuity of eigenvalues and using $P \in \mathcal{CP}^{n+1} \backslash \{0\}$, there exists $p \in \mathbb{R}_+^{n+1} \backslash \{0\}$ such that $P = pp^T$. We have

$$\left\|p^r(p^r) - pp^T\right\|_F^2 = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \left(p_i^r p_j^r - p_i p_j\right)^2 \geq \sum_{i=1}^{n+1} \left((p_i^r)^2 - p_i^2\right)^2.$$

Since $\|p^r(p^r) - pp^T\|_F \to 0$, $p \in \mathbb{R}_+^{n+1}$ and $p^r \in \mathbb{R}_+^{n+1}$ for each $r \in \mathbb{N}$, it follows that $p_i^r \to p_i$ for each $i = 1, \ldots, n+1$ as $r \to +\infty$. Therefore, $p \in \mathrm{cl}(S)$, which concludes the proof. $\square$

Next, we consider replacing the difficult conic constraint $Y \in \mathcal{CP}^{n+1}$ in (CP) by $Y \in \mathcal{I}(\mathcal{M}_S)$, where

$$\mathcal{I}(\mathcal{M}_S) := \mathrm{cone}(\mathrm{cl}(\mathcal{M}_S)) = \mathrm{cl}(\mathrm{cone}(\mathcal{M}_S)). \tag{23}$$

The following result is a direct consequence of Proposition 3.2.

Corollary 4.1 *Given an instance of (BQP) and its completely positive reformulation (CP), suppose that the conic constraint $Y \in \mathcal{CP}^{n+1}$ in (CP) is replaced by $Y \in \mathcal{I}(\mathcal{M}_S)$, where $S \subseteq \Delta_{n+1}$ is nonempty and $\mathcal{M}_S$ and $\mathcal{I}(\mathcal{M}_S)$ are defined as in (22) and (23), respectively. Let*

$$\begin{aligned} S_0 &:= \{p \in cl(S) : p_0 = 0\}, \\ S_+ &:= \{p \in cl(S) : p_0 > 0\}, \\ S_{++} &:= \left\{p \in S_+ : \left(\frac{1}{p_0}\right) p_{1:n} \in F\right\}, \end{aligned} \tag{24}$$

where F defined as in (1) denotes the feasible region of (BQP). Let $\nu(\mathcal{M}_S)$ denote the optimal value of the corresponding inner approximation of (CP).

(i) If $S_{++} = \emptyset$, then the inner approximation of (CP) is infeasible and $\nu(\mathcal{M}_S) = +\infty$.

(ii) If $S_{++} \neq \emptyset$ and there exists $p \in S_0$ such that $p_{1:n} \in L_\infty$, where L_∞ is given by (3), and $p_{1:n}^T Q p_{1:n} < 0$, then the inner approximation of (CP) is unbounded below and $\nu(\mathcal{M}_S) = -\infty$.

(iii) If $S_{++} \neq \emptyset$ and $\inf\limits_{p \in S_0 : p_{1:n} \in L_\infty} p_{1:n}^T Q p_{1:n} \geq 0$, then

$$\nu(\mathcal{M}_S) = \inf_{p \in S_{++}} \left\{ \left(\frac{1}{p_0}\right)^2 p_{1:n}^T Q p_{1:n} + 2\left(\frac{1}{p_0}\right) c^T p_{1:n} \right\}.$$

Proof. Let us define $M^p := pp^T$ for each $p \in \text{cl}(S)$. Note that $M^p_{00} > 0$ if and only if $p \in S_+$. For $p \in S_+$, we have $(1/M^p_{00})M^p \in \mathcal{F}$ if and only if $a_i^T(1/p_0)p_{1:n} = b_i$ for each $i = 1, \ldots, m$, and $(1/M^p_{00})M^p_{jj} = (1/p_0^2)e_j^T p\, p^T e_j = (1/p_0^2)p_j^2 = (1/M^p_{00})M^p_{0j} = (1/p_0)p_j$ for each $j \in B$, which holds if and only if $(1/p_0)p_{1:n} \in F$ since $p \in \mathbb{R}^{n+1}_+$. For $p \in S_{++}$,

$$\frac{1}{M^p_{00}}\left(\langle Q, M^p_{1:n,1:n}\rangle + 2c^T M^p_{1:n,1}\right) = \left(\frac{1}{p_0}\right)^2 \left(p_{1:n}^T Q p_{1:n} + 2(1/p_0)c^T p_{1:n}\right).$$

Similarly, for $p \in P_0$, we have $M^p \in \mathcal{L}_\infty$ if and only if $p_0^2 = 0$, $a_i^T p_{1:n} p_{1:n}^T a_i = (a_i^T p_{1:n})^2 = 0$ for each $i = 1, \ldots, m$, and $M^p_{jj} = e_j^T p\, p^T e_j = p_j^2 = 0$ for each $j \in B$, which holds if and only if $p \in S_0$ and $p_{1:n} \in L_\infty$. Furthermore, for $p \in P_0$,

$$\langle Q, M^p_{1:n,1:n}\rangle = p_{1:n}^T Q p_{1:n}.$$

The assertions now follow from the observations above and Proposition 3.2. $\square$

An interesting and useful observation regarding Corollary 4.1 is that any inner approximation of (CP) given by a cone generated by nonnegative rank one matrices is equivalent to approximating the original problem (BQP) by evaluating the objective function only on a subset of the feasible solutions. Furthermore, if $S \subseteq \Delta_{n+1}$ is a nonempty finite set, then $\mathcal{I}(\mathcal{M}_S)$ is a polyhedral inner approximation of $\mathcal{CP}^{n+1}$ and the resulting inner approximation of (CP) reduces to a finite discretization of the feasible region of (BQP). On the other hand, if $S = \Delta_{n+1}$, then $\mathcal{I}(\mathcal{M}_S) = \mathcal{CP}^{n+1}$, i.e., the inner approximation of $\mathcal{CP}^{n+1}$ is exact. In this case, $x \in F$ if and only if $(1/p_0)p_{1:n} = x$, where $p := (1/(e^T x + 1))[1; x] \in \Delta_{n+1}$. We therefore recover Burer's reformulation.

Recall that Proposition 3.3 presents an error bound on the difference between the upper bound arising from an inner approximation of an instance of (CP) and its optimal value. Clearly, under the hypotheses of Proposition 3.3, the same error bound holds for inner approximations generated by nonnegative rank one matrices. By Corollary 4.1, such inner approximations are equivalent to evaluating the objective function only on a subset of the feasible solutions of (BQP). Therefore, an interesting question is whether a similar error bound can be established based on the maximum distance between any feasible solution of (BQP) and the subset of feasible solutions of (BQP) corresponding to the particular inner approximation as opposed to the corresponding maximum distance in the space of the feasible solutions of (CP) as in Proposition 3.3.

Note that the objective function of (CP) is linear in the space of $(n + 1) \times (n + 1)$ symmetric matrices and this observation allows us to bound the difference in the objective function as a function of the distance between two feasible solutions of (CP). On the other hand, the objective function of (BQP) is quadratic on $\mathbb{R}^n$. Therefore, we cannot find a global Lipschitz constant, unless we make further additional assumptions on (BQP). Henceforth, we will assume that the feasible region F of (BQP) is a bounded set, which allows us to use the following Lipschitz-type error bound.

Lemma 4.2 *Given an instance of (BQP) with a nonempty and bounded feasible region F defined as in (1), we have*

$$|f(y) - f(x)| \leq 2 \left(\max_{z \in L} \|Qz + c\|_1 \right) \|y - x\|_\infty, \quad x \in F, \quad y \in F, \tag{25}$$

where $f(x) := x^T Q x + 2c^T x$ and L given by (2) denotes the linear portion of F.

Proof. Note that F is bounded if and only if L is bounded due to (4). Let $x \in F$ and $y \in F$ be such that $x \neq y$. By the mean value theorem, there exists $\mu \in (0, 1)$ such that

$$f(y) - f(x) = \nabla f((1 - \mu)x + \mu y)^T (y - x).$$

Therefore,

$$|f(y) - f(x)| \leq \|\nabla f((1 - \mu)x + \mu y)\|_1 \|y - x\|_\infty.$$

Since $\nabla f((1-\mu)x+\mu y) = 2Q((1-\mu)x+\mu y))+c$ and L is a convex set, the assertion follows. □

Next, using the simple characterization of the inner approximation given by Corollary 4.1, we present an error bound on the gap between the corresponding upper bound and the optimal value of (CP) under the assumption that the feasible region F of (BQP) is nonempty and bounded.

Proposition 4.1 *Given an instance of (BQP) with a nonempty and bounded feasible region F, consider its completely positive reformulation (CP). Suppose that the conic constraint $Y \in \mathcal{CP}^{n+1}$ in (CP) is replaced by $Y \in \mathcal{I}(\mathcal{M}_S)$, where $S \subseteq \Delta_{n+1}$ is nonempty and $\mathcal{M}_S$ and $\mathcal{I}(\mathcal{M}_S)$ are defined as in (22) and (23), respectively. Let S_{++} be defined as in (24) and let $\nu(\mathcal{M}_S)$ and ν^* denote the optimal values of the corresponding inner approximation and the original instance of (CP), respectively. Then,*

$$\nu(\mathcal{M}_S) - \nu^* \leq 2 \left(\max_{z \in L} \|Qz + c\|_1 \right) \delta_2(F, S),$$

where L denotes the linear portion of F given by (2) and

$$\delta_2(F, S) := \sup_{x \in F} \inf_{p \in S_{++}} \left\| x - \left(\frac{1}{p_0} \right) p \right\|_\infty.$$

Proof. The proof follows from Lemma 4.2 and a similar argument as in the proof of Proposition 3.3. □

□

Once again, we remark that the parameter $\delta_2(F, S)$ is completely determined by the feasible region F of (BQP) and by the particular inner approximation of $\mathcal{CP}^{n+1}$.

4.1 Burer's Reduced Formulation

Given an instance of (BQP), suppose that

$$\exists\, y \in \mathbb{R}^m \text{ such that } g := \sum_{i=1}^m y_i a_i \in \mathbb{R}^n_+, \quad \sum_{i=1}^m y_i b_i = 1. \tag{26}$$

By (1) and (26), we obtain

$$x \in F \Rightarrow g^T x = 1.$$

Under the assumption (26), Burer [9] shows that (BQP) admits the following simpler completely positive formulation:

$$\begin{array}{llrcl}
(\mathrm{CP}') & \min & \langle Q, X\rangle + 2\, c^T X g & & \\
& \text{s.t.} & a_i^T X g & = & b_i, \quad i = 1, \ldots, m, \\
& & a_i^T X a_i & = & b_i^2, \quad i = 1, \ldots, m, \\
& & (Xg)_j & = & X_{jj}, \quad j \in B, \\
& & g^T X g & = & 1, \\
& & X & \in & \mathcal{CP}^n.
\end{array}$$

Note that, in contrast with (CP), the decision variable X lies in $\mathcal{S}^n$. Therefore, (CP′) yields an equivalent reformulation of (BQP) in a smaller dimension under the assumption (26).

Suppose now that (BQP) has a nonempty and bounded feasible region F. Therefore, there exists a positive $\beta \in \mathbb{R}_+$ such that $x \in F$ implies $e^T x \leq \beta$. Adding this redundant $(m+1)$st constraint to (BQP) and the corresponding slack variable to convert the inequality constraint to an equality constraint, it follows that the assumption (26) is satisfied by simply defining $y_i = 0$ for $i = 1, \ldots, m$ and $y_{m+1} = 1/\beta$, where $a_{m+1} := e \in \mathbb{R}^{n+1}$. Therefore, by adding additional variables if necessary, every instance of (BQP) with a nonempty and bounded feasible region F can be formulated as an instance of (CP$'$). Note that we obtain $g = (1/\beta)e$, which implies that $g \in \mathbb{R}^n_+$ can be chosen to be strictly positive under this assumption (after possibly redefining n to account for the slack variable).

In a similar fashion, the conic constraint $X \in \mathcal{CP}^n$ in (CP$'$) can be replaced by $X \in \mathcal{I}(\mathcal{M}_S)$, where $S \subseteq \Delta_n$ is nonempty. We first present a characterization of the feasibility of the corresponding inner approximation and the resulting upper bound.

Proposition 4.2 *Given an instance of (BQP) with a nonempty and bounded feasible region F given by (1), consider its completely positive reformulation (CP$'$). Suppose that the conic constraint $X \in \mathcal{CP}^n$ in (CP$'$) is replaced by $X \in \mathcal{I}(\mathcal{M}_S)$, where $S \subseteq \Delta_n$ is nonempty and $\mathcal{M}_S$ and $\mathcal{I}(\mathcal{M}_S)$ are defined as in (22) and (23), respectively. Let $\nu(\mathcal{M}_S)$ denote the optimal value of the corresponding inner approximation.*

(i) We have $\mathcal{F}' \cap \mathcal{I}(\mathcal{M}_S) \neq \emptyset$ (i.e., the inner approximation has a nonempty feasible region) if and only if $S' \neq \emptyset$, where $\mathcal{F}'$ denotes the feasible region of (CP$'$) and

$$S' := \left\{ p \in S : \left(\frac{1}{g^T p} \right) p \in F \right\}, \tag{27}$$

where $g \in \mathbb{R}^n_+$ is given by (26).

(ii)

$$\nu(\mathcal{M}_S) = \inf_{p \in S'} \left\{ \left(\frac{1}{g^T p} \right)^2 p^T Q p + 2 \left(\frac{1}{g^T p} \right) c^T p \right\}.$$

Proof. Under the assumption that F is nonempty and bounded, note that $g \in \mathbb{R}^n_+$ given by (26) can be chosen to be strictly positive by our discussion preceding the statement. Since $S \subseteq \Delta_n$, it follows that $g^T p > 0$ for each $p \in S$.

Let us consider (i). If $S' \neq \emptyset$, it is easy to verify that $(1/(g^T p)^2) p p^T \in \mathcal{F}' \cap \mathcal{I}(\mathcal{M}_S)$. Conversely, suppose that $X \in \mathcal{F}' \cap \mathcal{I}(\mathcal{M}_S)$. Since $\mathcal{I}(\mathcal{M}_S) \subset \mathcal{S}^n$, by Carathéodory's theorem for conic hulls, there exists $\bar{S} \subset S$ such that $|\bar{S}| \leq n(n+1)/2$ and

$$X = \sum_{p \in \bar{S}} \lambda_p p p^T.$$

Since $X \in \mathcal{F}'$, we obtain

$$\begin{aligned}
\sum_{p \in \bar{S}} \hat{\lambda}_p a_i^T \hat{p} &= b_i, \quad i = 1, \ldots, m, & (28)\\
\sum_{p \in \bar{S}} \hat{\lambda}_p \left(a_i^T \hat{p}\right)^2 &= b_i^2, \quad i = 1, \ldots, m, & (29)\\
\sum_{p \in \bar{S}} \hat{\lambda}_p \hat{p}_j &= \sum_{p \in \bar{S}} \hat{\lambda}_p \hat{p}_j^2, \quad j \in B, & (30)\\
\sum_{p \in \bar{S}} \hat{\lambda}_p &= 1, & (31)
\end{aligned}$$

where

$$\hat{\lambda}_p := \lambda_p (g^T p)^2, \quad \hat{p} := \left(\frac{1}{g^T p}\right) p, \quad p \in \bar{S}. \tag{32}$$

Note that $\hat{\lambda}_p \geq 0$ and $\hat{p} \in \mathbb{R}^n_+$ for each $p \in \bar{S}$. Let $a_i^T \hat{p} = b_i + \alpha_{ip}$ for each $p \in \bar{S}$ and $i = 1, \ldots, m$. By (28) and (31), we obtain $\sum_{p \in \bar{S}} \hat{\lambda}_p \alpha_{ip} = 0$ for each $i = 1, \ldots, m$. Together with (29),

$$\sum_{p \in \bar{S}} \hat{\lambda}_p \left(b_i^2 + 2\alpha_{ip} b_i + \alpha_{ip}^2\right) = b_i^2 + \sum_{p \in \bar{S}} \hat{\lambda}_p \alpha_{ip}^2 = b_i^2, \quad i = 1, \ldots, m,$$

which implies that $\alpha_{ip} = 0$ for each $i = 1, \ldots, m$ and for each $p \in \bar{S}$ such that $\hat{\lambda}_p > 0$. Therefore, $\hat{p} \in L$ for each $p \in \bar{S}$ such that $\hat{\lambda}_p > 0$, where L given by (2) denotes the linear portion of F. For each $j \in B$, we obtain that $0 \leq \hat{p}_j \leq 1$ by (4), which implies that $\hat{p}_j - \hat{p}_j^2 \geq 0$. Therefore, it follows from (30) that $\sum_{p \in \bar{S}} \hat{\lambda}_p (\hat{p}_j - \hat{p}_j^2) = 0$, which holds if and only if $\hat{p}_j - \hat{p}_j^2 = 0$, or equivalently, $\hat{p}_j \in \{0, 1\}$ for each $j \in B$ and each $p \in \bar{S}$ such that $\hat{\lambda}_p > 0$. We therefore obtain that $\hat{p} \in F$ for each $p \in \bar{S}$ such that $\hat{\lambda}_p > 0$, or equivalently, $S' \neq \emptyset$ by (32) since there should be at least one $p \in \bar{S}$ such that $\hat{\lambda}_p > 0$ by (31). This completes the proof of (i).

The proof of part (ii) is very similar to the proof of part (iii) of Proposition 3.2 and is therefore omitted. □

The next result presents an error bound on the difference between the corresponding upper bound and the optimal value of (CP′).

Proposition 4.3 *Given an instance of (BQP) with a nonempty and bounded feasible region F given by (1), consider its completely positive reformulation (CP'). Suppose that the conic constraint $X \in \mathcal{CP}^n$ in (CP') is replaced by $X \in \mathcal{I}(\mathcal{M}_S)$, where $S \subseteq \Delta_n$ is nonempty and $\mathcal{M}_S$ and $\mathcal{I}(\mathcal{M}_S)$ are defined as in (22) and (23), respectively. Let $\nu(\mathcal{M}_S)$ and ν^* denote the optimal values of the corresponding inner approximation and the original instance of (CP'), respectively. Then,*

$$\nu(\mathcal{M}_S) - \nu^* \leq 2 \left(\max_{z \in L} \|Qz + c\|_1 \right) \delta_3(F, S),$$

where L given by (2) denotes the linear portion of F and

$$\delta_3(F, S) := \sup_{x \in F} \inf_{p \in S'} \left\| x - \left(\frac{1}{g^T p} \right) p \right\|_\infty ,$$

where $g \in \mathbb{R}^n_+$ given by (26) is strictly positive and S' is defined as in (27).

Proof. The proof follows from Lemma 4.2, Proposition 4.2, and a similar argument as in the proof of Proposition 3.3. □

4.2 Implications of Error Bounds

In this section, we discuss the implications of the error bound of Proposition 4.3 on standard quadratic programs and box-constrained quadratic programs, which constitute two important classes of quadratic programs that can be formulated in the form of (BQP).

4.2.1 Standard Quadratic Programs

A standard quadratic program is given by

$$\text{(StQP)} \quad \nu^* = \min_{x \in \Delta_n} x^T Q x, \tag{33}$$

where $Q \in \mathcal{S}^n$. Standard quadratic programs have numerous applications (see, e.g., [7]). They are known to be NP-hard since the maximum clique problem in a graph can be reformulated as an instance of (StQP) [19].

Clearly, the feasible region of (StQP) is bounded since it is given by the unit simplex. Since there is a single equality constraint, it follows that (26) is satisfied by defining $y_1 = 1$ and we obtain $g = e \in \mathbb{R}^n_+$. Therefore, (StQP) admits the following completely positive formulation in the form of (CP$'$) (see also [5] for an alternative derivation):

$$\text{(CP1)} \quad \nu^* = \min \left\{ \langle Q, X \rangle : \langle E, X \rangle = 1, \quad X \in \mathcal{CP}^n \right\},$$

where $E = ee^T \in \mathcal{S}^n$ is the matrix of all ones.

We now consider the error bounds arising from a particular class of inner approximations of $\mathcal{CP}^n$ proposed by [4] in the context of (CP1) (see also [23] for generalization to an inner approximation hierarchy). This class of inner approximations consists of polyhedral cones and each polyhedral cone is defined as follows. Let us consider the following subset of Δ_n:

$$S_n^r := \{p \in \Delta_n : rp \in \mathbb{N}^n\}, \quad r = 1, 2, \ldots. \tag{34}$$

Consider replacing the conic constraint $X \in \mathcal{CP}^n$ in (CP1) by $X \in \mathcal{I}(\mathcal{M}_{S_n^r})$, $r = 1, 2, \ldots$. Note that $F = \Delta_n$ in this case. Furthermore, for each $p \in S_n^r$, $(1/g^T p)p = p \in F$, which implies that $(S_n^r)' = S_n^r$, where $(S_n^r)'$ is defined similarly as in (27). By Proposition 4.2,

$$\nu(\mathcal{M}_{S_n^r}) = \min_{p \in (S_n^r)'} \left\{ \left(\frac{1}{g^T p} \right)^2 p^T Q p \right\} = \min_{p \in S_n^r} \left\{ p^T Q p \right\}.$$

We refer the reader to [4, 23] for an alternative derivation of the same result.

Furthermore, by [6],

$$\delta_3(F, S_n^r) = \sup_{x \in \Delta_n} \inf_{p \in S_n^r} \|x - p\|_\infty = \frac{1}{r} \left(1 - \frac{1}{n} \right), \quad r = 1, 2, \ldots.$$

Together with Proposition 4.3, we obtain

$$\begin{aligned} \nu(\mathcal{M}_{S_n^r}) - \nu^* &\leq 2 \left(\max_{z \in F} \|Qz\|_1 \right) \delta(F, S_n^r) \\ &= \left(\frac{2}{r} \right) \left(1 - \frac{1}{n} \right) \max_{i=1,\ldots,n} \|Q_{1:n,i}\|_1, \quad r = 1, 2, \ldots. \end{aligned}$$

We now compare this error bound by the previous error bounds in the literature. Bomze and de Klerk [4, Theorem 3.2] establish the following error bound on the same difference (see also [13, Corollary 1]):

$$\nu(\mathcal{M}_{S_n^r}) - \nu^* \le \frac{1}{r}\left(\max_{i=1,\dots,n} Q_{ii} - \nu^*\right), \le \frac{1}{r}\left(\max_{x\in\Delta_n} x^T Q x - \nu^*\right), r = 1, 2, \dots.$$

Since $|S_n^r| = O(n^r)$, the above error bound implies that we obtain a polynomial-time approximation scheme for (StQP) [4]. More recently, de Klerk et al. [11] sharpened this error bound in the following way:

$$\nu(\mathcal{M}_{S_n^r}) - \nu^* \le \frac{q}{r^2}\left(\max_{i=1,\dots,n} Q_{ii} - \nu^*\right), \quad r = 1, 2, \dots,$$

under the assumption that (StQP) has a global minimizer $x^* \in S_n^q$ (i.e., a global minimizer in which each component of x^* has denominator q). Using the inequality $\nu^* \ge \min\limits_{1\le i<j\le n} Q_{ij}$ (see [21, Lemma 1(i)]), one can verify that both of these error bounds are, in general, tighter than the bound obtained from Proposition 4.3. We remark, however, that the error bound of Proposition 4.3 applies to the completely positive reformulation of *any* optimization problem that can be formulated in the form of (BQP) with a bounded feasible region. On the other hand, the error bounds in [4, 11, 13] are obtained by exploiting the specific properties of the objective function and the feasible region of (StQP).

4.2.2 Box-Constrained Quadratic Programs

A box-constrained quadratic program is given by

$$\text{(BoxQP)} \quad \nu^* = \min\{x^T Q x + 2c^T x : 0 \le x \le e\},$$

where $Q \in \mathcal{S}^n$ and $c \in \mathbb{R}^n$. Box-constrained quadratic programs arise in several applications (see, e.g., [1]). Clearly, any quadratic program with box constraints can be equivalently reformulated in the form of (BoxQP) by a simple change of variables. Furthermore, any binary quadratic program can also be reformulated as an instance of (BoxQP), as we outline below. Consider the binary quadratic optimization problem given by

$$\text{(BQP)} \quad \min\{x^T Q x + 2c^T x : x_j \in \{0,1\}, \quad j = 1, \dots, n\}.$$

One can modify the objective function and the feasible region as follows:

$$\text{BoxQP}(\mu) \quad \min\{x^T Q x + 2c^T x - \mu\|x - (1/2)e\|^2 : 0 \le x \le e\}.$$

Note that the additional term in the objective function yields the same value of $-\mu(n/4)$ when evaluated at any vertex of the unit box, which consists of all binary vectors in $\mathbb{R}^n$. If $\mu \geq \lambda_{\max}(Q)$, where $\lambda_{\max}(Q)$ denotes the largest eigenvalue of Q, then the objective function of BoxQP(μ) is a concave function and the optimal solution is attained at a vertex of the unit box, which is necessarily a binary vector. This establishes the equivalence between binary quadratic programs and (BoxQP). It follows that problems like MAX-CUT can be formulated as an instance of (BoxQP). Therefore, (BoxQP) is, in general, an NP-hard problem.

By adding slack variables, (BoxQP) can be formulated in the form of (BQP), where $F = \{[x; s] \in \mathbb{R}^{2n} : x_i + s_i = 1, \ i = 1, \ldots, n\}$. Clearly, F is bounded and (26) is satisfied by defining $y_i = 1/n$ for $i = 1, \ldots, n$, in which case $g = (1/n)e \in \mathbb{R}^{2n}_+$. Therefore, (BoxQP) admits the following completely positive formulation in the form of (CP′):

$$
\begin{array}{llrcl}
\text{(CP2)} & \nu^* = \min \langle Q, X_{1:n,1:n} \rangle + (1/n) c^T X_{1:n,1:2n} e & & & \\
\text{s.t.} & & (Xe)_i + (Xe)_{n+i} & = & n, \quad i = 1, \ldots, n, \\
& & X_{ii} + 2X_{i,n+i} + X_{n+i,n+i} & = & 1, \quad i = 1, \ldots, n, \\
& & X & \in & \mathcal{CP}^{2n}.
\end{array}
$$

Let us consider polyhedral inner approximations corresponding to S_{2n}^r given by (34), i.e., we replace $X \in \mathcal{CP}^{2n}$ in (CP2) by $X \in \mathcal{I}(\mathcal{M}_{S_{2n}^r})$, $r = 1, 2, \ldots$. Let us fix r. By Proposition 4.2, the corresponding inner approximation has a nonempty feasible region if and only if there exists $p \in S_{2n}^r$ such that $(1/g^T p)p = np \in F$, or equivalently, $n(p_i + p_{n+i}) = 1$ for each $i = 1, \ldots, n$. By (34), $p \in S_{2n}^r$ if and only if $p_j = q_j / r$, where $q_j \in \{0, 1, \ldots, r\}$ for each $j = 1, \ldots, 2n$ and $\sum_{j=1}^{2n} q_j = r$. Therefore, the inner approximation is feasible if and only if $(n/r)(q_i + q_{n+i}) = 1$, or equivalently, $q_i + q_{n+i} = r/n$ for each $i = 1, \ldots, n$. Since $q_j \in \mathbb{N}$ for each $j = 1, \ldots, 2n$, this system has a solution if and only if $r/n \in \mathbb{N}$, or equivalently, $r = \kappa n$, where κ is a positive integer. This discussion shows that polyhedral inner approximations corresponding to S_{2n}^r are feasible if and only if $r = \kappa n$, where κ is a positive integer.

Let us next consider the upper bound arising from the polyhedral inner approximation corresponding to S_{2n}^r for $r = \kappa n$, where κ is a positive integer. Let us define

$$(S_{2n}^{\kappa n})' := \{p \in S_{2n}^{\kappa n} : np \in F\}.$$

Note that $p \in (S_{2n}^{\kappa n})'$ if and only if there exists $[x; s] \in F$ such that $[x; s] = np$, or equivalently, $(1/n)[x; s] \in S_{2n}^{\kappa n}$. By the preceding discussion, this holds if and only if there exists $q \in \mathbb{N}^{2n}$ such that $(1/n)x_i = q_i/(\kappa n)$ and $(1/n)s_i = q_{n+i}/(\kappa n)$, or equivalently, $x_i = q_i/\kappa$ and $s_i = q_{n+i}/\kappa$ for each $i = 1, \ldots, n$. Therefore, $p \in (S_{2n}^{\kappa n})'$ if and only if there exists $[x; s] \in F$ such that $\kappa[x; s] \in \mathbb{N}^{2n}$. Combining this observation with Proposition 4.2, we obtain

$$\begin{aligned} \nu\left(\mathcal{M}_{S_{2n}^{\kappa n}}\right) &= \min_{p \in \left(S_{2n}^{\kappa n}\right)'} \left\{ n^2 p_{1:n}^T Q p_{1:n} + 2nc^T p_{1:n} \right\} \\ &= \min_{[x;s] \in F : \kappa[x;s] \in \mathbb{N}^{2n}} \left\{ x^T Q x + 2c^T x \right\}. \end{aligned}$$

Finally, we consider the approximation error for $r = \kappa n$, where κ is a positive integer. We need to characterize $\delta_3(F, S_{2n}^{\kappa n})$. By the preceding discussion, $p \in (S_{2n}^{\kappa n})'$ if and only if there exists $[x; s] \in F$ such that $\kappa[x; s] \in \mathbb{N}^{2n}$, or equivalently, $x_i = q_i/\kappa$ and $s_i = (\kappa - q_i)/\kappa$, where $q_i \in \{0, 1, \ldots, \kappa\}$ for $i = 1, \ldots, n$. Therefore, for any feasible solution $[\hat{x}; \hat{s}] \in \mathbb{R}^{2n}$ of (BoxQP), there exists $p \in (S_{2n}^{\kappa n})'$ such that $\|[\hat{x}; \hat{s}] - np\|_\infty \leq 1/(2\kappa)$. In fact, it is straightforward to verify that we can have equality by using a feasible solution with $\hat{x}_1 = 1/(2\kappa)$. Therefore, it follows that $\delta_3(F, S_{2n}^{\kappa n}) = 1/(2\kappa)$. By Proposition 4.3, we obtain

$$\begin{aligned} \nu\left(\mathcal{M}_{S_{2n}^{\kappa n}}\right) - \nu^* &\leq 2\left(\max_{0 \leq x \leq e} \|Qx + c\|_1\right) \delta_3(F, S_{2n}^{\kappa n}) \\ &\leq \left(\frac{1}{\kappa}\right)\left(\max_{0 \leq x \leq e} \|Qx\|_1 + \|c\|_1\right) \\ &= \left(\frac{1}{\kappa}\right)\left(\max_{0 \leq x \leq e} \left\|\sum_{i=1}^n Q_{1:n,i} x_i\right\|_1 + \|c\|_1\right) \\ &\leq \left(\frac{1}{\kappa}\right)\left(\max_{0 \leq x \leq e} \sum_{i=1}^n \|Q_{1:n,i}\|_1 |x_i| + \|c\|_1\right) \\ &\leq \left(\frac{1}{\kappa}\right)\left(\sum_{i=1}^n \sum_{j=1}^n |Q_{ij}| + \|c\|_1\right), \quad \kappa = 1, 2, \ldots. \end{aligned}$$

In contrast with standard quadratic programs, this error bound does not translate into a polynomial-time approximation scheme since $|\,(S_{2n}^{\kappa n})'\,| = (\kappa + 1)^n$, which is not polynomial for fixed κ. In fact, this is an expected result since hardness of approximation results are known for the max-cut problem [17]. We refer the reader to [12] for other approximation and inapproximability results for (BoxQP).

5 Concluding Remarks

In this paper, we focused on inner approximations of the cone of completely positive matrices on instances of completely positive optimization problems that arise from the reformulation of mixed binary quadratic programs. We provided characterizations of the feasibility of such inner approximations and the optimal value of the corresponding approximation. We presented an upper bound on the gap between the optimal value of an inner approximation and that of the original instance. We considered general inner approximations as well as those generated by a set of nonnegative rank one matrices. We also discussed the special case of polyhedral approximations.

For inner approximations generated by nonnegative rank one matrices, our characterization reveals that the corresponding inner approximation reduces to sampling a subset of points from the feasible region of the original mixed binary quadratic program. An interesting research direction is the investigation of problem-specific inner approximation schemes that would provide provably good approximation guarantees for a given class of mixed binary quadratic programs.

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