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Remark on multi-target, robust linear-quadratic control problem on semi-infinite interval

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Abstract We consider multi-target, robust linear-quadratic control problem on semi-infinite interval. Using functional-analytic approach developed in [2], we reduce this problem to a convex optimization problem on the simplex. Explicit procedure for the evaluation of the reduced objective function is described.

1 Introduction

In [1] a comprehensive robust optimal control theory has been developed. Though the linear-quadratic control problem with finite possible evolutionary scenarios on semi-infinite interval is a very particular case of this theory, it is nevertheless important one from the point of view of possible applications. On the other hand, in [2] the multi-target linear-quadratic control problem has been considered. In both cases, the objective function is a maximum of finite number of quadratic functionals. To this end, it seems to be quite natural to consider a robust version of multi-target LQ problem, just combining both approaches together.

More precisely, we consider the following optimal control problem:

$$\max_{i\in[1,s]}\int_0^{+\infty}\left[\left[x^{(i)}(t)-\bar{x}^{(i)}(t)\right]^T\left[x^{(i)}(t)-\bar{x}^{(i)}(t)\right]+u(t)^TR_iu(t)\right]dt\to\min \tag{1}$$

$$\dot{x}^{(i)}=A_ix^{(i)}+B_iu, \tag{2}$$

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$$x^{(i)}(0) = x_0^{(i)},\ i = 1, 2, 3, \cdots, s. \tag{3}$$

$A_i \in \Re^{n_i \times n_i},\ B_i \in \Re^{n_i \times m},\ R_i = R_i^T \in \Re^{m \times m}, i = 1, 2, 3, \cdots, s$. Here we assume that symmetric matrices R_i are positive definite and $\bar{x}^i(t),\ t \in [0, \infty)$ are given targets. Note that the robust version (in the sense of [1]) of the target LQ corresponds to the case $n_1 = n_2 = \cdots, n_s$; $\bar{x}^{(1)}(t) = \bar{x}^{(2)}(t) = \cdots = \bar{x}^{(s)}(t), t \in [0, \infty)$; $x^{(1)}(0) = x^{(2)}(0) = \cdots = x^{(s)}(0)$; $R_1 = R_2 = \cdots = R_s$. The optimal choice of the optimal control $u(\ )$ in 1-3 provides the "robust" behavior of our system under s different scenarios (corresponding to different choices of A_i, B_i). To address(1)-(3) we will use a functional-analytic approach developed in [2], [3]. Note that the robust target LQ problem on semi-infinite interval does not formally fit in the approach in [1], since it contains time-varying linear terms in quadratic functionals. Therefore, the limit procedure in [1] (see e.g. p.225) will not work.

2 Functional-analytic model and duality

Let $H_1, H_2, \cdots, H_s$, and U be Hilbert spaces. To simplify notations we will use $<,>$ for the scalar product in each of these spaces. Let $H = H_1 \times H_2 \times \cdots, \times H_s \times U$, Z be a closed vector subspace in H and $c \in H$. Given $\bar{h}_i \in H_i$, $M_i : U \to U, i = 1, 2, \cdots, s$, where M_i are self-adjoint positive definite linear operators, consider the following optimization problem:

$$\max_{i \in [1,s]} \left(\|h_i - \bar{h}_i\|^2 + \langle u, M_i u \rangle \right) \to \min, \tag{4}$$

$$\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_s \\ u \end{bmatrix} \in c + Z. \tag{5}$$

Here $\|h\| = \sqrt{< h, h >}$.

The problem (4) - (5) can be rewritten in the following equivalent form:

$$t \to \min, \tag{6}$$

$$\|h_i - \bar{h}_i\|^2 + \langle u, M_i u \rangle - t \leq 0, i = 1, 2, \cdots, s, \tag{7}$$

$$\begin{bmatrix} h \\ u \end{bmatrix} \in c + Z,$$

where $h = \begin{bmatrix} h_1 \\ \vdots \\ h_s \end{bmatrix}$.

Notice that despite the fact that the problem is infinite dimensional, the usual KKT theorem holds true. (see e.g. [1], page 72). It is also clear that Slater conditions are satisfied. Consider the Lagrange function

$$L(\lambda_1, \cdots, \lambda_s, h, u, t) = t + \sum_{i=1}^{s} \lambda_i (f_i(h,u) - t)) = (1 - \sum_{i=1}^{s} \lambda_i)t + \sum_{i=1}^{s} \lambda_i f_i(h,u)$$

where

$$f_i(h,u) = \|h_i - \bar{h}_i\|^2 + \langle u, M_i u \rangle . \tag{8}$$

The optimality conditions for (6)-(7) take the form:

$$\lambda_i \geq 0, \lambda_i (f_i(h,u) - t) = 0, i = 1, 2, \cdots, s,$$

$\frac{\partial L}{\partial t} = 0$ i.e. $\sum_{i=1}^{s} \lambda_i = 1$,

$$\begin{bmatrix} h \\ u \end{bmatrix} \in c + Z,$$

$$\sum_{i=1}^{s} \lambda_i \nabla f_i(h,u) \in Z^{\perp} \tag{9}$$

where $Z^{\perp}$ stands for the orthogonal complement of Z in H. In particular (9) takes the form

$$\begin{bmatrix} \lambda_1 (h_1 - \bar{h}_1) \\ \vdots \\ \lambda_s (h_s - \bar{h}_s) \\ (\sum_{i=1}^{s} \lambda_i M_i)\, u \end{bmatrix} \in Z^{\perp}.$$

The Lagrange dual to (6) -(7):

$$\phi(\lambda_1, \cdots, \lambda_s) \to \max, \tag{10}$$

$$\lambda_i \geq 0, i = 1, 2 \cdots, s, \sum_{i=1}^{s} \lambda_i = 1.$$

where

$$\phi(\lambda_1, \cdots, \lambda_s) = \min\{L(\lambda_1, \cdots, \lambda_s, h, u, t) : \begin{bmatrix} h \\ u \end{bmatrix} \in c + Z, t \in \Re\}. \tag{11}$$

The problem (11) obviously takes the form

$$\sum_{i=1}^{s} \lambda_i f_i(h,u) \to \min \tag{12}$$

$$\begin{bmatrix} h \\ u \end{bmatrix} \in c + Z. \tag{13}$$

taking into account (8), we can rewrite (12) -(13) in the following form:

$$\sum_{i=1}^{s} \lambda_i \|h_i - \bar{h}_i\|^2 + \left\langle u, \sum_{i=1}^{s} \lambda_i M_i u \right\rangle \to \min$$

$$\begin{bmatrix} h \\ u \end{bmatrix} \in c + Z,$$

which is the same as:

$$\left\langle N(\boldsymbol{\lambda})(h - \bar{h}), (h - \bar{h}) \right\rangle + \langle u, M(\boldsymbol{\lambda})u \rangle \to \min \tag{14}$$

$$\begin{bmatrix} h \\ u \end{bmatrix} \in c + Z. \tag{15}$$

Here $N(\boldsymbol{\lambda})$ is a linear operator on $H_1 \times \cdots \times H_s$ defined as

$$N(\boldsymbol{\lambda}) \begin{bmatrix} h_1 \\ \vdots \\ h_s \end{bmatrix} = \begin{bmatrix} \lambda_1 h_1 \\ \vdots \\ \lambda_s h_s \end{bmatrix}, \ M(\boldsymbol{\lambda}) = \sum_{i=1}^{s} \lambda_i M_i.$$

3 Functional -analytic formulation of the original problem

The original problem (1) -(3) can be rewritten in the form:

$$\max_{i \in [1,s]} \int_0^\infty \left[(x^{(i)} - \bar{x}^{(i)})^T (x^{(i)} - \bar{x}^{(i)}) + u^T R_i u \right] dt \to \min$$

$$\dot{X} = AX + Bu, \ X(0) = X_0,$$

where

$$X = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(s)} \end{bmatrix}, \ A = \mathrm{diag}(A_1, \cdots, A_s),$$

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B^s \end{bmatrix}, \ X_0 = \begin{bmatrix} x_0^{(1)} \\ \vdots \\ x_0^{(s)} \end{bmatrix}, \ \bar{X} = \begin{bmatrix} \bar{x}^{(1)} \\ \vdots \\ \bar{x}^{(s)} \end{bmatrix}.$$

We will assume that $H = L_2^n[0,\infty) \times L_2^m[0,\infty)$, where $n = n_1 + n_s + \cdots + n_s$ and $L_2^n[0,\infty)$ stands for the Hilbert space of $\Re^n$ valued square integrable functions on $[0,\infty)$. Notice that if we take $H_i = L_2^{n_i}[0,\infty)$, then $H = H_1 \times H_2 \times \cdots \times H_s \times U$, with $U = L_2^m[0,\infty)$. Note that we assume that $\bar{X} \in L_2^n[0,\infty)$.
Let, further,

$$Z = \{(X,u) \in H; X \text{ is absolutely continuous}, \ \dot{X} \in L_2^n[0,\infty), \ \dot{X} = Ax + Bu\}.$$

In this way we cast the original problem (1) - (3) in the form (4), (5). Note that for $\alpha, \beta \in L_2^n[0, \infty)$,

$$< \alpha, \beta >= \int_0^\infty \alpha(t)^T \beta(t) dt$$

by definition.

Correspondingly, the problem (14)-(15) takes the form

$$\int_0^{+\infty} \left[(x - \bar{x})^T Q(\boldsymbol{\lambda}) (x - \bar{x}) + u^T \left(\sum_{i=1}^{s} \lambda_i R_i \right) u \right] dt \to \min \tag{16}$$

$$\dot{X} = AX + Bu, \ X(0) = X_0, \tag{17}$$

where $Q(\boldsymbol{\lambda}) = \mathrm{diag}(\lambda_i I_i, \lambda_2 I_s, \cdots, \lambda_s I_s)$ with I_i to be the identity matrix in $\Re^{n_i \times n_i}$, $i = 1, 2, \cdots, s$.

We will assume that the algebraic Riccati equation

$$KA + A^T K + KL(\boldsymbol{\lambda})K - Q(\boldsymbol{\lambda}) = 0, \tag{18}$$

associated with (16) - (17) (here $L(\boldsymbol{\lambda}) = BR(\lambda)^{-1}B^T$, $R(\boldsymbol{\lambda}) = \sum_{i=1}^{s} \lambda_i R_i$) has a symmetric solution $K_{st}(\boldsymbol{\lambda})$ such that the matrix $A + L(\boldsymbol{\lambda})K_{st}(\lambda)$ is stable for all $\boldsymbol{\lambda}$ from the simplex

$$\sum_{i=1}^{s} \lambda_i = 1, \lambda_i \geq 0, i = 1, \dots s.$$

In particular, such a stabilizing solution exists provided the pair (A, B) is stabilizable and $\lambda_1, \lambda_2, \cdots \lambda_s > 0$.

See e.g. discussion in [3] and references therein. The solution to (16) -(17) is described in the following theorem.

Theorem 1 *Under assumptions made above there exists a unique solution* $\rho_0 \in L_2^n[0, \infty)$ *satisfying the differential equation*

$$\dot{\rho} = -(A + L(\boldsymbol{\lambda})K_{st})^T \rho - Q(\boldsymbol{\lambda})\bar{X}. \tag{19}$$

The optimal solution (X, u) *to (16), (17) has the form*

$$\dot{X} = (A + L(\boldsymbol{\lambda})K_{st}(\boldsymbol{\lambda})) + L(\boldsymbol{\lambda})\rho_0, \ X(0) = X_0, \tag{20}$$

$$u = R(\boldsymbol{\lambda})^{-1}B^T(K_{st}(\boldsymbol{\lambda})X + \rho_0). \tag{21}$$

The optimal value of (16), (17) has the form:

$$\phi(\boldsymbol{\lambda}) = X_0^T K_{st}(\boldsymbol{\lambda}) X_0 - 2\rho_0^T X_0 + \int_0^\infty \left[X^T Q(\boldsymbol{\lambda}) X - \rho_0^T L(\boldsymbol{\lambda}) \rho_0 \right] dt. \tag{22}$$

Proof For the proof see Theorem 2.1 in [3].

4 Discrete multi-target, robust linear-quadratic control problem on semi-infinite interval

It is natural to consider the discrete version for the problem (1)-(3). In this case, the problem can be reformulated as follows:

$$\max_{i\in[1,s]} \frac{1}{2}\sum_{k=0}^{\infty}[(x_k^{(i)}-\bar{x}_k^{(i)})^T(x_k^{(i)}-\bar{x}_k^{(i)})+u_k^T R_i u_k] \to \min, \tag{23}$$

$$X_{k+1}=AX_k+Bu_k,\ X_0=C, \tag{24}$$

where C is a constant vector and

$$X_k=\begin{bmatrix} x_k^{(1)} \\ \vdots \\ x_k^{(s)} \end{bmatrix},\ A=\mathrm{diag}(A_1,\cdots,A_s),$$

$$B=\begin{bmatrix} B_1 \\ \vdots \\ B_s \end{bmatrix},\ X_0=\begin{bmatrix} x_0^{(1)} \\ \vdots \\ x_0^{(s)} \end{bmatrix},\ \bar{X}_k=\begin{bmatrix} \bar{x}_k^{(1)} \\ \vdots \\ \bar{x}_k^{(s)} \end{bmatrix}.$$

Here we let $x^{(i)}$ denoted a sequence $\{x_k^{(i)}\}\subset\Re^n$ for $i=1,2,\cdots,s$ and for $k=0,\cdots,\infty$. We say that $x\in l_2^n(\mathbb{N})$ if $\sum_{i=0}^{\infty}\|x_i\|^2<\infty$ where $\|\cdot\|$ is a norm induced by an inner product $<,>$ in $\Re^n$. If we take $H_i=l_2^{n_i}(\mathbb{N})$, then we let $H=H_1\times\cdots\times H_s\times U$ with $U=l_2^m(\mathbb{N})$. Hence $H=l_2^n(\mathbb{N})\times l_2^m(\mathbb{N})$ where $n=n_1+\cdots+n_s$. Note that $X=\{X_k\}$ and $\bar{X}=\{\bar{X}_k\}\in l_2^n(\mathbb{N})$.

The vector subspace Z now takes the form:

$$Z=\{(X,u)\in H:\ X_{k+1}=AX_k+Bu_k, k=0,1,\cdots,\ X_0=0\}.$$

As in the continuous case, we can rewrite the problem (23) - (24) in the form (4) - (5). Note that for α and $\beta\in l_2^n(\mathbb{N})$

$$<\alpha,\beta>=\sum_{k=0}^{\infty}<\alpha_k,\beta_k>.$$

Hence, the problem (14)-(15) takes the form

$$\frac{1}{2}\sum_{k=0}^{\infty}\left[\left(X_k-\bar{X}_k\right)^T Q(\boldsymbol{\lambda})\left(X_k-\bar{X}_k\right)+u_k^T\left(\sum_{i=1}^{s}\lambda_i R_i\right)u_k\right]\to\min \tag{25}$$

$$X_{k+1}=AX_k+Bu_k,\ X_0=C, \tag{26}$$

where $Q(\boldsymbol{\lambda})=\mathrm{diag}(\lambda_i I_i,\lambda_2 I_s,\cdots,\lambda_s I_s)$ with I_i to be the identity matrix in $\Re^{n_i\times n_i}$, $i=1,2,\cdots,s$ and C is a constant vector.

We assume that the following discrete algebraic Riccati equation (DARE) associated with (23) - (24) has a stabilizing solution $K_{st}(\boldsymbol{\lambda})$.

$$K = A^T K A - (A^T K B)(R(\boldsymbol{\lambda}) + B^T K B)^{-1}(A^T K B)^T + Q(\boldsymbol{\lambda}),$$

where $R(\boldsymbol{\lambda}) = \sum_{i=1}^{s} \lambda_i R_i$. The solution to (25) - (26) is described in the following theorem.

Theorem 2 *Under assumptions made above there exists a unique solution* $\rho = \{\rho_k\} \in l_2^n(\mathbb{N})$ *satisfying following recurrence relations*

$$\rho_k = [A^T - (A^T K_{st}(\lambda) B)\bar{R}^{-1}(\boldsymbol{\lambda}) B^T]\rho_{k+1} + Q(\boldsymbol{\lambda})\bar{X}_k.$$

The optimal solution $(X, u) = (\{X_k\}, \{u_k\})$ *has the following form:*

$$X_{k+1} = (A^T - A^T K_{st}(\boldsymbol{\lambda}) L)^T X_k + L\rho_{k+1}, \tag{27}$$

$$u_k = -\bar{R}^{-1}(\boldsymbol{\lambda}) B^T K_{st}(\boldsymbol{\lambda}) A x_k + \bar{R}^{-1}(\boldsymbol{\lambda}) B^T \rho_{k+1}, \tag{28}$$

where $\bar{R}(\boldsymbol{\lambda}) = (R(\boldsymbol{\lambda}) + B^T K_{st} B)$ *and* $L(\boldsymbol{\lambda}) = B\bar{R}^{-1}\boldsymbol{\lambda} B^T$.

The optimal value of (25) - (26) has the form:

$$\phi(\boldsymbol{\lambda}) = \frac{1}{2} X_0^T K_{st}(\boldsymbol{\lambda}) X_0 - \rho_0^T X_0 + \frac{1}{2}\sum_{k=0}^{\infty}[2\bar{X}_k^T Q(\boldsymbol{\lambda})\bar{X}_k - \rho_{k+1}^T L(\boldsymbol{\lambda})\rho_{k+1}]. \tag{29}$$

Proof For the proof see Theorem 4.1 in [3].

5 Discussion

Formula (22) provides a value $\phi(\boldsymbol{\lambda})$ of the objective function ϕ of the dual problem (10). Problem (10) is a convex optimization problem on a simplex. Given the optimal solution $\boldsymbol{\lambda}_*$ of the dual problem (10) , one can find the optimal solution (given by (20) - (19)) of the original problem. Comparing with [2], it is quite obvious that the evaluation of the objective function of the dual problem is much more difficult in robust optimization case. In this respect, it is worthwhile to mention that the original problem (1) - (3) is a (highly nontrivial) example of second-order cone programming problem in (infinite dimensional) Hilbert space. Primal-dual algorithms for such problem has been developed in [5], [4] and could be considered as an alternative to the approach described here. The comments above are equally applicable to the discrete case.

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