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Variational principles, completeness and the existence of traps in behavioral sciences

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Abstract In this paper, driven by Behavioral applications to human dynamics, we consider the characterization of completeness in pseudo-quasimetric spaces in term of a generalization of Ekeland's variational principle in such spaces, and provide examples illustrating significant improvements to some previously obtained results, even in complete metric spaces. At the behavioral level, we show that the completeness of a space is equivalent to the existence of traps, rather easy to reach (in a worthwhile way), but difficult (not worthwhile to) to leave. We first establish new forward and backward versions of Ekeland's variational principle for the class of strict-decreasingly forward (resp. backward)-lower-semicontinuous functions in pseudo-quasimetric spaces. We do not require that the space under consideration either be complete or to enjoy the limit uniqueness property since, in a pseudo-quasimetric space, the collections of forward-limits and backward ones of a sequence, in general, are not singletons.

Keywords Pseudo-quasimetric · Forward-completeness · Backward-completeness · Variational principle · Group dynamics · Existence of trap.

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1 Introduction

This paper, driven by Behavioral applications to human dynamics, considers the mathematical completeness problem. We provide a characterization of complete-

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ness in pseudo-quasimetric spaces in terms of a generalization of the Ekeland theorem in such spaces. Notice that a pseudo-quasimetric is a distance which satisfies the triangular inequality, the distance from point “y” to point “x” can be different from the distance from “x” to “y”, the distance from a point to itself is zero, and it is possible that the distance from a point to a different one be zero. Our paper also provides examples illustrating significant improvements to some previously obtained results, even in complete metric spaces. Our motivations are twofold.

On one side, in Mathematics, a big issue is the COMPLETENESS PROBLEM, that is, to know when a space is complete. In such a case, a Cauchy sequence converges. In a recent survey Cobzaş [10] presented various circumstances in which fixed point results and variational principles imply completeness. In this paper we will focus the attention on the Ekeland's variational principle [12], discovered by Ivar Ekeland in 1972. It is one of the most useful tools in nonlinear analysis and variational analysis. It allows us to study minimization problems in which the lower level set of the problem is not compact or, in other words, when the Bolzano-Weierstrass theorem can not be applied. More precisely, it asserts the existence of nearly optimal solutions to some optimization problems for the class of functions, defined on a complete metric space being lower semicontinuous, bounded below, and proper (i.e. not identically equal to infinity). Our paper develops generalized versions of Ekeland's variational principle in pseudo-quasimetric spaces which are not necessarily complete.

On the other side, in Behavioral Sciences, a big issue is THE END PROBLEM, that is, to know when and where a human dynamic, which starts from an initial position, and follows a transition, defined as a succession of positions (which can be doings, havings or beings) ends somewhere.

Strikingly, this End problem in Behavioral Sciences is exactly the big Completeness problem is Mathematics. This comes from two simple remarks. In Behavioral Sciences, a Cauchy sequence refers to a dynamic (sequence) which agglomerates, but without knowing where, while a convergent dynamic refers to a sequence which approaches some end. The end problem refers to three fundamental questions in Behavioral Sciences :
(1) to know when an individual behavior is predictable (stabilizes) in order to solve coordination problems between interrelated agents in game theory;
(2) to modelize habituation processes (habit formation and breaking) where agents and organizations, progressively, perform more and more similar individual and collective actions in similar contexts, ending in habits and routines;
(3) to know when and where human dynamics end in traps, rather easy to reach, but difficult to leave, before being able to reach their desires.

A recent variational rationality approach ([25, 26], and [27] for a revised version) allows us to make the connection between these two big problems – the mathematical Completeness problem and the behavioral End problem. This Variational Rationality (abbreviated VR) approach proposes a way to model human behaviors as worthwhile approach or avoidance dynamics (see also Sect. 5, *Conclusions*, in this paper). Such dynamics start from some undesirable initial states, follow acceptable transitions (which are defined as successions of worthwhile stays and changes), and make attempts to approach and reach desired ends (desires), or to avoid the undesirable initial states. To be more concrete, stays refer to habits, routines, norms, rules, exploitation phases, etc. and changes refer to exploration, search, learning, innovation, etc. In such dynamics, at each step, if motivation to change is proportionally higher than or as high as resistance to change, it is worthwhile to change. If not, the dynamic process ends up in some variational trap which is worthwhile to approach and reach, but not worthwhile to leave. This new model of human behavior allowed to give a striking and surprising new interpretation of Ekeland's variational principle and of other famous variational principles in the particular case where motivation to change and resistance to change are identified to advantage and inconvenience to change, experience does not matter too much, and a pseudo-quasimetric models inconvenience to change rather than to stay as the difference between costs to be able to change and costs to be able to stay. In this VR approach, Ekeland's variational principle gives sufficient conditions for the existence of variational traps. In his formal proofs, Soubeyran [25–27] considered only the case of quasimetrics (in the sense used in the present paper, see Definition 2). Here, we extend the mathematical aspect to pseudo-quasimetrics where inconveniences to change rather than to stay can be zero, when costs to be able to change are equal to costs to be able to stay. We do even more. Sullivan [28] showed that the original Ekeland's variational principle is equivalent to completeness of metric spaces. Karapinar and Romaguera [17] proved that their weak version of Ekeland's variational principle is equivalent to completeness of quasimetric spaces. In this paper, we extend their results to the case of pseudo-quasimetric spaces. At the end of the last section, we show, as a direct application, that completeness of a pseudo-quasimetric space X (the space of actions) is equivalent to the existence of variational traps (the VR meaning of Ekeland's variational principle), starting from any initial state. This is a very nice result in Behavioral Sciences: in a pseudo-quasimetric space, where the pseudo-quasimetric is an inconvenience to change, completeness of the space is equivalent to the fact that every worthwhile stay to change dynamic ends in some trap. Then, in such a complete space, no worthwhile dynamic can wander.

Ekeland's variational principle has been extended in many directions, without much justification, except for applications in Computer Science (see [1] and the references therein) and ones in Behavioral Sciences. For vector- and set-valued versions of Ekeland's variational principle, see [4–6]. Since the vector- and set-valued versions are out of the scope of this paper, we focus our attention on the following issues.

- **Topological spaces**. Ekeland's variational principles are still valid in quasimetric spaces [2–6, 8, 17, 29] and partial-metric spaces [1].
- **Lower semicontinuity of cost functions**. Ekeland's variational principle could be established for decreasingly lower-semicontinuous functions [2–5, 21] and nearly lower semicontinuous ones [17].
- **Limit uniqueness**. In the original Ekeland's variational principle this requirement is automatically satisfied since the limit (if exists) is unique in a metric space. This fact no longer holds true in (pseudo)-quasimetric spaces due to the lack of the symmetry axiom. A majority of publications in quasimetric spaces require that the space in question is complete and regular [29], complete and T_1 [8], complete and Hausdorff [3–6], or enjoys a less restrictive property: every generalized-Picard sequence has at most one forward-limit [7, Theorem 4.5].

In [17], Karapinar and Romaguera formulated for the first time a weak version of this principle in complete and possibly not T_1 quasimetric spaces.

The core of the present paper is to start with a hypothesis about series completeness (see Definition 8). Roughly speaking, in terms of the VR approach of human behaviors (see [27]), we suppose that if the resources spend to move, following a succession (path) of stays and changes is finite (a natural hypothesis), then, this human dynamic ends somewhere (converges). This gives a concrete meaning for the completeness assumption. More precisely, a sequence $\{x_n\}$ in a pseudo-quasimetric space (X, q) is said to be forward-distance-series-convergent if the forward-distance series $\Sigma_{n=0}^{+\infty} q(x_n, x_{n+1}) < +\infty$ is convergent. Then, a pseudo-quasimetric space is said to be forward-distance-series-complete if every forward-distance-series-convergent sequence is forward-convergent. This assumption must be related to the definition of Kasahara spaces as L-spaces endowed with our series completeness assumption (Filip [14], Definition 1.6.1).

The main purpose of this paper is to formulate new forward and backward versions of Ekeland's variational principle in possibly neither complete nor T_1 pseudo-quasimetric spaces. They reduce to the original principle when the space in question happens to be a complete metric space, while being able to be applied to more general settings when the existing results could not be used. Importantly, they give characterizations of the forward (resp. backward)-completeness of pseudo-quasimetric spaces.

The paper is organized as follows. Section 2 presents basic definitions and preliminary results for pseudo-quasimetrics. In Section 3 we establish enhanced forward and backward versions of Ekeland's variational principle in pseudo-quasimetric spaces which may fail to be complete. We also add some comments on applications to Behavioral Sciences.

2 Preliminaries

We present and discuss definitions of generalized distances (known also as metrics) and of notions of closedness and completeness in topological spaces whose topologies are induced by these distances; see, e.g., [7–9, 19, 24, 30].

There are numerous ways of relaxing the axioms of a metric space, leading to various notions of generalized metric spaces. Let us start with some types of metrics.

Definition 1 (types of metrics). (i) A METRIC on a nonempty set X is a bifunction (called the distance function or simply distance) $d : X \times X \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, and for all $x, y, z \in X$ the following conditions are satisfied:

(d1) $d(x, y) \geq 0$ (nonnegativity axiom);
(d2) $d(x, y) = 0$ if and only if $x = y$ (coincidence axiom);
(d3) $d(x, y) = d(y, x)$ (symmetry axiom);
(d4) $d(x, z) \leq d(x, y) + d(y, z)$ (subadditivity axiom).

The pair (X, d) is called a metric space.

(ii) A PSEUDO-METRIC on X is a bifunction $d : X \times X \to \mathbb{R}$ which satisfies the axioms for a metric, except that the coincidence axiom is replaced by the equality-implies-indistancy condition:

(d2′) $d(x,x)=0$ for all $x\in X$ (but possibly $d(x,y)=0$ for $x\neq y$).

(iii) A QUASIMETRIC is defined by a bifunction that satisfies all the axioms of a metric with the possible exception of the symmetry axiom.

(iv) A PRE-METRIC is defined by a bifunction that satisfies conditions (d1) and (d2′). A SEMI-METRIC is a bifunction that satisfies conditions (d1), (d2) and (d3) (i.e. the triangle inequality is not satisfied), see Wilson [31].

In this paper we work on the following spaces.

Definition 2 (pseudo-quasimetrics [19, Definition 2.1]). A PSEUDO-QUASIMETRIC[1] on a nonempty set X is a bifunction $q: X\times X\to\mathbb{R}_+ := [0,+\infty)$, where for all $x,y,z\in X$ the following conditions are satisfied:

(q1) $x=y \Longrightarrow q(x,y)=0$ (equality implies indistancy);

(q2) $q(x,z)\leq q(x,y)+q(y,z)$ (subadditivity, or triangle inequality).

Obviously, q satisfies the nonnegativity axiom of a metric. As always, (X,q) is called a PSEUDO-QUASIMETRIC SPACE.

If q also satisfies the condition

(q3) $q(x,y)=0 \Longrightarrow x=y$ (equality implies indistancy),

then q is called a QUASIMETRIC.

Remark 1 In [9] by a quasimetric one understands a bifunction $q: X\times X\to\mathbb{R}$ satisfying (q1),(q2) and

(q3′) $q(x,y)=q(y,x)=0 \implies x=y$.

This is partly justified by the fact that a pseudo-quasimetric q satisfies (q3′) if an only if the associated bifunction $q^s(x,y)=\max\{q(x,y),q(y,x)\}$, $x,y\in X$, is a metric on X. Notice that (q1) + (q3′) $\not\Rightarrow$ (d2).

Example 1 **(some important pseudo-quasimetrics).** Let X be a subset of $\mathbb{R}$ and define $q_1,q_2,q_3: X\times X\to\mathbb{R}_+$ by

$$q_1(x,y)=\begin{cases} y-x & \text{if } y\geq x\\ 0 & \text{otherwise}\end{cases},\quad q_2(x,y)=\begin{cases} y-x & \text{if } y\geq x\\ 1 & \text{otherwise}\end{cases},\ \textit{and}$$

$$q_3(x,y)=\begin{cases} a(y-x) & \text{if } y\geq x\\ b(x-y) & \text{otherwise}\end{cases}\quad \text{where } a,b>0 \text{ are given.}$$

- All of these bifunctions are pseudo-quasimetrics.
- q_2 and q_3 are quasimetrics, but q_1 is not since $q_1(2,1)=0$.
- The distance q_1 does not satisfy the reverse implication of (q1), but it satisfies (q3′), i.e. it is a quasimetric space in the sense used in [9].

[1] In [2–7] and the references therein, by a pseudo-quasimetric space one understands a quasimetric space.

Every pseudo-quasimetric space (X, q) can be considered as a topological space. For $x \in X$ and $r > 0$, we could define an open ball in X by

$$\mathbb{B}_q(x, r) = \{y \in X |\ q(x, y) < r\}.$$

The topology $\tau(q)$ of a pseudo-quasimetric space (X, q) can be introduced by taking, for any $x \in X$, the collection $\{\mathbb{B}_q(x, r) \mid r > 0\}$ as a base of the neighborhood filter of the point x.

Following Kelly [19], consider a pseudo-quasimetric q over a nonempty set X. The conjugate of q, denoted by $\overline{q} : X \times X \to \mathbb{R}_+$, defined by $\overline{q}(x, y) = q(y, x), \forall\ x, y \in X$, is a pseudo-quasimetric as well, generating a topology $\tau(\overline{q})$. As a space equipped with two topologies, $\tau(q)$ and $\tau(\overline{q})$, the triple $(X, q, \overline{q})$ is a bitopological space[2]. The bifunction $q^s(x, y) = \max\{q(x, y), \overline{q}(x, y)\}$, $x, y \in X$, is a pseudo-metric which is a metric if and only if q satisfies condition (q3′) from Remark 1.

Let us describe the open balls in $\mathbb{R}$ with respect to the pseudo-quasimetrics in Example 1.

$$\mathbb{B}_{q_1}(x, r) = (-\infty, x + r), \quad \mathbb{B}_{\overline{q}_1}(x, r) = (x - r, +\infty),$$

$$\mathbb{B}_{q_2}(x, r) = [x, x + r) \text{ for all } r \in (0, 1) \text{ and } (-\infty, x + r) \text{ for all } r \in [1, +\infty),$$

$$\mathbb{B}_{\overline{q}_2}(x, r) = (x - r, x] \text{ for all } r \in (0, 1) \text{ and } (x + r, +\infty) \text{ for all } r \in [1, +\infty),$$

$$\mathbb{B}_{q_3}(x, r) = (x - br, x + ar),\ \mathbb{B}_{\overline{q}_3}(x, r) = (x - ar, x + br).$$

Notice that the quasimetric q_3 satisfies

$$\min\{a, b\} \cdot |y - x| \le q_3(x, y) \le \max\{a, b\} \cdot |y - x| \text{ for all } x, y \in \mathbb{R},$$

so that the topology defined by q_3 agrees with the usual topology of $\mathbb{R}$.

In contrast to metric spaces, in bitopological pseudo-quasimetric spaces there are two different notions of convergence (with respect to each topology)[3].

Definition 3 (convergence in pseudo-quasimetric spaces, [7, Definition 4]). Let $\{x_n\}$ be a sequence in a bitopological pseudo-quasimetric $(X, q, \overline{q})$. We say that:

(i) the sequence $\{x_n\}$ is FORWARD-CONVERGENT to x_*, if it converges to x_* with respect to $\tau(\overline{q})$, i.e., $\overline{q}(x_*, x_n) = q(x_n, x_*) \to 0$.
(ii) the sequence $\{x_n\}$ is BACKWARD CONVERGENT to x_*, if it converges to x_* with respect to the topology $\tau(q)$, i.e., $q(x_*, x_n) \to 0$;

We use the following notation:

(iii) $\overrightarrow{\{x_n\}}$ stands for the collection of all forward-limits of the sequence $\{x_n\}$ and thus $\overrightarrow{\{x\}} = \{y \in X |\ q(x, y) = 0\} = \overline{\{x\}}^{\overline{q}}$ is the set of all forward-limits of the constant sequence $x_n = x$ for all $n \in \mathbb{N}$;

[2] [19, page 71] "The notion of a bitopological space used in relation to semi-continuous functions restores sufficient symmetry to enable one to use some of the existing techniques of continuous functions."

[3] It has been also defined a sequence as being bi-convergent if it is both forward- and backward-convergent, (i.e. if it convergent with respect to the pseudo-metric q^s).

(iv) $\overleftarrow{\{x_n\}}$ stands for the collection of all backward limits of the sequence $\{x_n\}$ and thus $\overleftarrow{\{x\}} = \{y \in X |\ q(y, x) = 0\} = \overline{\{x\}}^q$.

Due to the lack of the symmetry axiom in pseudo-quasimetric spaces, the definition of Cauchy sequences could be generalized in many ways.

Definition 4 (Cauchy sequences in pseudo-quasimetrics [7, Definition 5]). Let $\{x_n\}$ be a sequence in a pseudo-quasimetric space (X, q). We say that:

(i) the sequence $\{x_n\}$ is FORWARD-CAUCHY [4], if for each $\varepsilon > 0$ there is an integer $N_\varepsilon \in \mathbb{N}$ such that $q(x_n, x_{n+k}) < \varepsilon$ for all $n \geq N_\varepsilon$ and $k \in \mathbb{N}$;
(ii) the sequence $\{x_n\}$ is BACKWARD CAUCHY [5], if for each $\varepsilon > 0$ there is an integer $N_\varepsilon \in \mathbb{N}$ such that $q(x_{n+k}, x_n) < \varepsilon$ for all for all $n \geq N_\varepsilon$ and $k \in \mathbb{N}$;
(iii) the sequence $\{x_n\}$ is CAUCHY [6], if it is both forward- and backward-Cauchy, i.e., for each $\varepsilon > 0$ there is an integer $N_\varepsilon \in \mathbb{N}$ such that $q(x_m, x_n) < \varepsilon$ for all $m, n \geq N_\varepsilon$. This is equivalent to the fact that it is Cauchy with respect to the associated pseudo-metric q^s.

Definition 5 (completeness in pseudo-quasimetric spaces). Let (X, q) be a pseudo-quasimetric space. We say:

(i) the space is FORWARD-FORWARD-COMPLETE (resp. backward-backward-complete), if every forward- (resp. backward-)Cauchy sequence is forward- (resp. backward-)convergent to some forward- (resp. backward-)limit;
(ii) the space is FORWARD-BACKWARD-COMPLETE (resp. backward-forward-complete), if every forward- (resp. backward-)Cauchy sequence is backward- (resp. forward-)convergent to some backward- (resp. forward-)limit;
(iii) the space is FORWARD-COMPLETE (resp. BACKWARD-COMPLETE), if every Cauchy sequence is forward- (resp. backward-)convergent to some forward- (resp. backward-)limit.

With two notions of convergence and three kinds of Cauchy sequences defined in a pseudo-quasimetric space, we could define six types of completeness. The reader is referred to [24, Examples 2–3] for differences between various Cauchy and completeness notions.

Remark 2 The notions with forward and backward used here differ from the notions of left and right used in [9]. For convenience we present the equivalences between the corresponding notions. For a sequence $\{x_n\}$ in a pseudo-quasimetric space (X, q):

- forward-Cauchy means left q-K-Cauchy, or equivalently, right $\overline{q}$-K-Cauchy.
- backward Cauchy means right q-K-Cauchy or, equivalently, left $\overline{q}$-K-Cauchy.

For the space X:

- forward-forward-complete means right $\overline{q}$-K-complete;
- backward-backward-complete means right q-K-complete;
- backward-forward-complete means left $\overline{q}$-K-complete;

[4] known also as left-Cauchy [4–6], q-Cauchy [19], and left K-Cauchy [24]
[5] known also as $\overline{q}$-Cauchy [19], and right K-Cauchy [24]
[6] known also as bi-Cauchy, or Cauchy in two topologies

– forward-backward-complete means left q-K-complete.

Remark 3 **(on 'left' terminologies).**
When calling the ball $\mathbb{B}(x,r)$ a left ball [8, 9], it would be logic to say that the topology $\tau(q)$ is a left topology, and a sequence $\{x_n\}$ converging to x_* with respect to the topology $\tau(q)$ is left-convergent. Observe that doing this would lead to some contradictions between known 'left' concepts in the literature.

Assume, in addition, that the forward distance series is convergent, i.e.,

$$\sum_{n=1}^{\infty} q(x_n, x_{n+1}) < +\infty.$$

It is easy to check (see, e.g., [7, Proposition 2.2]) that the sequence is forward-Cauchy, or left-Cauchy. It is natural to say that the space is left-complete if every left-Cauchy sequence is left-convergent. Doing so, the sequence under consideration is left-convergent to some x_*, i.e., $q(x_*, x_n) \to 0$ as $n \to \infty$. Unfortunately, it conflicts with the so-called left-convergence in [4–6] and earlier publications mentioned therein.

Another reason for not mentioning the topologies in the definitions of convergence and Cauchyness is that a backward-convergent sequence is related to the topology $\tau(q)$ ($\lim\limits_{n\to\infty} q(x_*, x_n) = 0 \iff \forall\varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} :\ \forall n \geq N_\varepsilon,\ x_n \in \mathbb{B}_q(x,\varepsilon)$), while a backward-Cauchy sequence is related to the topology $\tau(\overline{q})$ ($\{x_n\}$ is backward-Cauchy $\iff \forall\varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} :\ \forall k \in \mathbb{N},\ \forall n \geq N_\varepsilon,\ x_{n+k} \in \mathbb{B}_{\overline{q}}(x_n,\varepsilon)$).

Wilson [30] proved that if $\{x_n\}$ is both forward- and backward-convergent to x_* in a quasimetric space (X,q), then x_* is the only limit point of $\{x_n\}$ of any kind and $\{x_n\}$ is Cauchy. He also showed that if $\{x_n\}$ has more than one forward-(backward-)limit point then $\{x_n\}$ has no backward- (forward-)limit point. Note that Wilson used the terms of u-limit and l-limit (from 'upper' and 'lower') instead.

Theorem 1 ([30, Theorems I and II], see also [9, Proposition 1.2.4]). *Let $\{x_n\}$ be a sequence in a pseudo-quasimetric space (X,q).*

(1) *If $\{x_n\}$ is forward-convergent to x_* and backward-convergent to x^*, then $q(x^*,\ x_*) = 0$.*
(2) *If $\{x_n\}$ is forward-convergent to x_* and $q(x_*, y_*) = 0$, then $\{x_n\}$ is forward-convergent to y_*.*
(3) *If a forward-Cauchy sequence $\{x_n\}$ has a subsequence which is forward-(backward-)convergent to x_* (to x^*), then $\{x_n\}$ is forward- (backward-) convergent to x_* (to x^*).*

Proof For reader's convenience we include the simple proofs.
(1) If $q(x_n, x_*) \to 0$ and $q(x^*, x_n) \to 0$, then

$$q(x^*, x_*) \leq q(x^*, x_n) + q(x_n, x_*) \to 0 \quad \text{as} \quad n \to \infty\,.$$

(2) If $q(x_n, x_*) \to 0$ and $q(x_*, y_*) = 0$, then

$$q(x_n, y_*) \leq q(x_n, x_*) + q(x_*, y_*) = q(x_n, x_*) \to 0 \quad \text{as} \quad n \to \infty\,.$$

(3) Suppose that $\{x_n\}$ is a forward-Cauchy sequence having a subsequence $\{x_{n_k}\}$ forward-convergent to some $x_* \in X$, that is $\lim_k q(x_{n_k}, x_*) = 0$.

For $\varepsilon > 0$ let $k_0, n_0 \in \mathbb{N}$ be such that

$$q(x_{n_k}, x_*) < \varepsilon, \ \forall k \geq k_0, \quad \text{and} \quad q(x_n, x_m) < \varepsilon, \ \forall m, n \ \text{ with } \ n_0 \leq n < m.$$

For $n \geq n_0$ let $k \geq k_0$ be such that $n_k > n$. Then

$$q(x_n, x_*) \leq q(x_n, x_{n_k}) + q(x_{n_k}, x_*) < 2\varepsilon\,.$$

Suppose now the forward-Cauchy sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ backward-convergent to $x^* \in X$, that is $\lim_k q(x^*, x_{n_k}) = 0$. For $\varepsilon > 0$ let $n_0 \in \mathbb{N}$ be such that $q(x_n, x_m) < \varepsilon, \ \forall m, n$ with $n_0 \leq n < m$ and let $k_0 \in \mathbb{N}$ be such that $n_{k_0} \geq n_0$ and $q(x^*, x_{n_k}) < \varepsilon, \ \forall k \geq k_0$. Then for $n \geq n_{k_0}$, $q(x^*, x_n) \leq q(x^*, x_{n_{k_0}}) + q(x_{n_{k_0}}, x_n) < 2\varepsilon$. △

We present now the separation axioms in topological spaces.

Definition 6 A topological space (X, τ) is:

(i) T_0 if for every pair of distinct points $x, y \in X$, at least one of them has a neighborhood which does not contain the other;
(ii) T_1 if for every pair of distinct points $x, y \in X$, there exist the neighborhoods U of x and V of y such that $y \notin U$ and $x \notin V$;
(iii) T_2 (or HAUSDORFF) if for every pair of distinct points $x, y \in X$, there exist the neighborhoods U of x and V of y such that $U \cap V = \emptyset$;
(iv) REGULAR if for every point $x \in X$ and closed set A not containing x there exist the disjoint open sets U, V such that $x \in U$ and $A \subset V$; the space X is called T_3 if it is T_1 and regular;
(v) NORMAL if for every pair of disjoint closed subsets A, B of X there exist the disjoint open sets U, V such that $A \subset U$ and $B \subset V$; the space X is called T_4 if it is T_1 and normal.

Notice that some authors include in the definition of regular and normal spaces the separation condition T_1.

Remark 4

(a) A topological space (X, τ) is T_0 if and only if $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$ implies $x = y$. One defines an order on a T_0 topological space by $x \leq_\tau y$ if $x \in \overline{\{y\}}$. This combination of order and topology in T_0 spaces found applications in theoretical computer science, see [15] for a comprehensive presentation and [16] for a good introduction to the area.
(b) A topological space (X, τ) is T_1 if and only if $\overline{\{x\}} = \{x\}$, for all $x \in X$, i.e. the one-point sets are closed.

We present some topological properties of pseudo-quasimetric spaces.

Theorem 2 ([9]). *Let (X, q) be a pseudo-quasimetric space. Then, the following are true:*

(1) *The ball $\mathbb{B}_q(x, r)$ is open in the topology $\tau(q)$ and the ball $\mathbb{B}_q[x, r]$ is closed in the topology $\tau(\overline{q})$. The ball $\mathbb{B}_q[x, r]$ need not be closed in the topology $\tau(q)$.*

(2) *For every fixed element* $x \in X$, *the function* $q(x,\cdot) : (X,q) \to (\mathbb{R}, |\cdot|)$ *is usc with respect to the topology* $\tau(q)$ *and lsc with respect to the topology* $\tau(\overline{q})$.
(3) *For every fixed element* $y \in X$, *the mapping* $q(\cdot,y) : (X,q) \to (\mathbb{R}, |\cdot|)$ *is lsc with respect to the topology* $\tau(q)$ *and usc with respect to the topology* $\tau(\overline{q})$.
(4) *If* q *is a quasimetric, then the topology* $\tau(q)$ *is* T_0, *but not necessarily* T_1 [7]
(5) *If the function* $q(x,\cdot) : (X,q) \to (\mathbb{R}, |\cdot|)$ *is continuous with respect to the topology* $\tau(q)$ *for every* $x \in X$, *then the topology* $\tau(q)$ *is regular.*
(6) *If* $q(x,\cdot)$ *is continuous with respect to the topology* $\tau(\overline{q})$ *for every* $x \in X$, *then the topology* $\tau(\overline{q})$ *is semi-metrizable.*

We conclude this section with an important observation clearly implying that the results obtained in [2–7] are indeed established for quasimetric spaces in the sense of Definition 2.

Proposition 1 *Let* (X,q) *be a pseudo-quasimetric space.*

(1) *The space* $(X,\tau(q))$ *is* T_1 *if and only if* $q(x,y) > 0$ *for every pair* x,y *of distinct points in* X, *i.e. if* q *is a quasimetric. In this case* $(X,\tau(\overline{q}))$ *is* T_1 *as well.*
(2) *The space* $(X,\tau(\overline{q}))$ $((X,\tau(q)))$ *is* T_2 *if and only if every sequence in* X *has at most one forward- (backward-)limit.*

Proof (1) Suppose that q is a quasimetric. If $x,y \in X$ are distinct, then $r := q(x,y) > 0$ and $r' := q(y,x) > 0$. It follows that $U = \mathbb{B}_q(x,r)$ is a $\tau(q)$-neighborhood of x with $y \notin U$, and $V = \mathbb{B}_q(y,r')$ is a $\tau(q)$-neighborhood of y with $x \notin V$.

Conversely, if there exists a pair x,y of distinct points with $q(x,y) = 0$, then $y \in \mathbb{B}(x,\varepsilon)$ for every $\varepsilon > 0$, showing that the topology $\tau(q)$ is not T_1. Similarly, $\overline{q}(y,x) = q(x,y) = 0 < \varepsilon$ implies $x \in \mathbb{B}_{\overline{q}}(y,\varepsilon)$ for all $\varepsilon > 0$, showing that the topology $\tau(\overline{q})$ is not T_1.

(2) Suppose that $(X,\tau(\overline{q}))$ is T_2 and let $\{x_n\}$ be forward-convergent to x_*. If $y_* \neq x_*$, then there exists $r > 0$ such that $\mathbb{B}_{\overline{q}}(x,r) \cap \mathbb{B}_{\overline{q}}(y,r) = \emptyset$. By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $x_n \in \mathbb{B}_{\overline{q}}(x_*,r)$ for all $n \geq n_0$, implying $x_n \notin \mathbb{B}_{\overline{q}}(y_*,r)$, for all $n \geq n_0$, that is $\{x_n\}$ is not forward-convergent to y_*.

Suppose now that $(X,\tau(\overline{q}))$ is not T_2. Then there exist two points $x_* \neq y_*$ in X such that $\mathbb{B}_{\overline{q}}(x_*,1/n) \cap \mathbb{B}_{\overline{q}}(y_*,1/n) = \emptyset$ for all $n \in \mathbb{N}$. Choosing an x_n in this intersection, it follows that $q(x_n,x_*) < 1/n$ and $q(x_n,y_*) < 1/n$, for all n, which shows that the sequence $\{x_n\}$ is forward-convergent to both x_* and y_*. △

Example 2 In a case of a pseudo-metric space (X,d) the condition $d(x,y) > 0$ for all distinct x,y implies that the space X is normal, even T_4. The situation is different in the quasimetric case. We present an example of a quasimetric space which is not Hausdorff (i.e. T_2).

Let $X = \{x_n : n \in \mathbb{N}\} \cup \{x_*, y_*\}$ where all the points are pairwise distinct. Define

$$q(u,u) = 0,\ \forall u \in X,\ \ q(x_n,x_m) = 1,\ \forall m,n \in \mathbb{N},\ m \neq n,$$
$$q(x_n,x_*) = q(x_n,y_*) = 1/n \ \text{ and }\ q(x_*,x_n) = q(y_*,x_n) = 1,\ \forall n \in \mathbb{N},$$
$$q(x_*,y_*) = q(y_*,x_*) = 1\,.$$

[7] Recall that in [9] a quasimetric satisfies the condition: $q(x,y) = q(y,x) = 0$ implies $x = y$.

It is easy to check that q is a quasimetric. The sequence $\{x_n\}$ is forward-convergent to x_* and to y_*, so that the topology $\tau(q)$ is not Hausdorff, but it is obviously T_1, as $q(u,v)>0$ for all $u,v\in X$, $u\neq v$.

Incidentally, this gives also an example of a forward-convergent sequence which is not forward-Cauchy.

Remark 5 A deep result in functional analysis asserts that if a topological vector space X is T_0, then it is T_1 and regular (i.e. T_3), see [23, Theorem 2.2.14]. If X is a finite dimensional asymmetric locally convex (in particular, asymmetric normed) space satisfying T_1, then it is topologically and algebraically isomorphic to $\mathbb{R}^m$, where $m\in\mathbb{N}$ is the algebraic dimension of X (see [9, Proposition 1.1.68]

We don't know whether there exists or not an infinite dimensional asymmetric normed space which is T_1 but not T_2. For a characterization of Hausdorff property in asymmetric locally convex (in particular, in asymmetric normed) spaces, see [9, Proposition 1.1.63].

3 Variational principles in pseudo-quasimetric spaces

Ekeland's variational principle, first discovered by Ivar Ekeland in 1972, [12], is one of the most useful tools in nonlinear analysis and variational analysis. It can be used when the lower level set of a minimization problems is not compact. It leads to a quick proof of the Caristi fixed point theorem and it has been shown by F. Sullivan [28] to be equivalent to the completeness of the corresponding metric space.

Theorem 3 ([13]; cf. [12]). *Let $\varphi : X\to\mathbb{R}\cup\{+\infty\}$ be an extended real-valued function on a complete metric space (X,d) which is lower semicontinuous, bounded below, and not identically equal to $+\infty$. For any $\varepsilon>0$ consider $x_0\in\operatorname{dom}\varphi$ satisfying $\varphi(x_0)\leq\inf_{x\in X}\varphi(x)+\varepsilon$, i.e., x_0 is an ε-minimal solution of φ. Then for each $\lambda>0$, there exists a point $x_*\in\operatorname{dom}\varphi$ such that*

(i) $\varphi(x_*)\leq\varphi(x_0)$;
(ii) $d(x_0,x_*)\leq\lambda$;
(iii) $\varphi(x)+(\varepsilon/\lambda)d(x_*,x)>\varphi(x_*),\ \forall\, x\in X\setminus\{x_*\}$.

If for $\gamma>0$ the set-valued mapping $S_{\gamma,d}: X\rightrightarrows X$ is defined by

$$S_{\gamma,d}(x):=\{y\in X\mid \gamma\, d(x,y)\leq\varphi(x)-\varphi(y)\},\ \forall x\in X\ , \tag{3.1}$$

then the conclusions (i) and (ii) of Theorem 3 can be expressed in the form: there is $x^*\in S_{\varepsilon/\lambda,d}(x_0)$ such that $S_{\varepsilon/\lambda,d}(x_*)=\{x_*\}$.

Recently, this principle has been extended to the class of pseudo-quasimetric spaces in which the symmetry axiom of the metric d is dropped and the coincidence one is weaken to the equality-implies-indistancy condition, i.e., $d(x,x)=0$ for all $x\in X$. It is important to emphasize that under the hypothesis made in a majority of extensions, the pseudo-quasimetric space is indeed a quasimetric one; see Proposition 1. We pay our attention to the following developments:

- In [29, Theorem 2.9], it is assumed that the space (X, q) is a forward-forward-complete[8] quasimetric space and that the function $q(x, \cdot) : X \to \mathbb{R}$ is lower semicontinuous. Note that it is not clear in the paper whether the function is lower-semicontinuous with respect to the topology $\tau(q)$ or the topology $\tau(\overline{q})$). As it is remarked in [8, Remark 1.4], if $q(x, \cdot)$ is lower semicontinuous with respect to the topology $\tau(q)$, then the topology $\tau(q)$ is regular.
- In [8, Theorem 2.4], the space (X, q) in question is assumed to be a T_1 quasimetric[9] space being either forward-forward-complete or backward-backward-complete[10]. Karapinar and Romaguera [17] further extended Cobzaş' result to a weak form of Ekeland's variational principle in (not necessarily T_1) pseudo-quasimetric spaces, yielding a characterization of backward-backward completeness of these spaces.
- In [2,4–6], Bao et al. worked with forward-forward-complete quasimetric spaces satisfying the forward Hausdorff property [11].

Remark 6 Unfortunately, these results can not be applied to the pseudo-quasimetric space $(\mathbb{R}, q_1)$, where q_1 is defined in Example 1, since the topology induced by q_1 is not T_1 and since it is not forward-forward-complete. The incompleteness of this space is obvious since the forward-Cauchy sequence $\{x_n\}$ with $x_n = -n$ has no forward-limit. It is a forward-Cauchy sequence since $q(x_n, x_{n+k}) = q(-n, -n - k) = 0$ for all $n, k \in \mathbb{N}$. Fix an arbitrary number $x_* \in \mathbb{R}$, then $x_* > x_n$ for all sufficiently large $n \in \mathbb{N}$, and thus $q(x_n, x_*) = x_* + n \to +\infty$ as $n \to \infty$. Since x_* was arbitrary, the sequence $\{x_n\}$ has no forward-limit.

In fact the following result holds in $(\mathbb{R}, q_1)$. For a sequence $\{x_n\}$ in $\mathbb{R}$

$$\overrightarrow{\{x_n\}} = \begin{cases} (-\infty, a] & \text{if } \ a \in \mathbb{R}, \\ \mathbb{R} & \text{if } \ a = +\infty, \\ \emptyset & \text{if } \ a = -\infty, \end{cases}$$

where $\overrightarrow{\{x_n\}}$ denotes the set of the forward-limits and $a = \liminf_{n \to \infty} x_n$.

The following examples partly motivate us to accomplish a study of Ekeland's variational principles in (not necessarily T_1) pseudo-quasimetric spaces.

Example 3 Consider a function $\varphi : (\mathbb{R}, q_1) \to (\mathbb{R}, |\cdot|)$ defined by $\varphi(x) = e^x$. Obviously, it is bounded from below by 0 and $S_{1,q_1}(x) = (-\infty, x]$ for all $x \in X$, where S_{1,q_1} is defined in (3.1). Therefore, there is no element $x_* \in \mathbb{R}$ such that $S_{1,q_1}(x_*) = \{x_*\}$ and the conclusion of Ekeland's variational principle does not hold true for this function φ in the pseudo-quasimetric space $(\mathbb{R}, q_1)$.

Example 4 Consider a function $\varphi : ([-1, 1], q_1) \to (\mathbb{R}, |\cdot|)$ defined by $\varphi(x) = e^x$. Obviously, it is bounded from below by 0 and $S_{1,q_1}(x) = [-1, x]$ for all $x \in [-1, 1]$. Therefore, $x_* = -1$ satisfies the conclusion of Ekeland's variational principle for any starting point $x_0 \in [-1, 1]$.

[8] left-complete

[9] A quasimetric in the sense of [8, Cobzaş] is a pseuso-quasimetric satisfying $q(x, y) = q(y, x) = 0 \Longrightarrow x = y$.

[10] there are two versions of Ekeland's variational principle with respect to each kind of completeness.

[11] A quasimetric space enjoys the forward Hausdorff property, i.e. a sequence being forward-convergent has a unique forward-limit.

Example 5 Consider a function $\varphi : (\mathbb{R}, \overline{q}_1) \to (\mathbb{R}, |\cdot|)$ defined by $\varphi(x) = e^x$. It is easy to check that $S_{\varphi,1}(1) = [0,1]$ and $S_{\varphi,1}(0) = \{0\}$ showing that Ekeland's variational principle holds with the desired point $x_* = 0$.

A question arises: under what requirements does Ekeland's variational principle hold in pseudo-quasimetric spaces? This question is answered in this section. We also prove that our new versions are characterizations of completeness of pseudo-quasimetric spaces.

Proposition 2 *Let (X, q) be a pseudo-quasimetric space and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function (i.e., with $\operatorname{dom}\varphi \neq \emptyset$). Given $\lambda > 0$, associate with φ and q the set-valued mapping $S_{\lambda,q} : X \rightrightarrows X$ defined by*

$$S_{\lambda,q}(x) = \{y \in X \mid \varphi(y) + \lambda q(x,y) \leq \varphi(x)\}, \quad \forall\, x \in X. \tag{3.2}$$

If $x \in X \setminus \operatorname{dom}\varphi$ (i.e. if $\varphi(x) = +\infty$), then $S_{\lambda,q}(x) = X$. For $x \in \operatorname{dom}\varphi$, $S_{\lambda,q}$ can be written as

$$S_{\lambda,q}(x) = \{y \in X \mid \lambda q(x,y) \leq \varphi(x) - \varphi(y)\}, \quad \forall\, x \in X. \tag{3.3}$$

Then $S_{\lambda,q}$ enjoys the following properties:

(1) *$x \in S_{\lambda,q}(x)$ for all $x \in \operatorname{dom}\varphi$;*
(2) *if $x \in \operatorname{dom}\varphi$ and $y \in S_{\lambda,q}(x)$, then $\varphi(y) \leq \varphi(x)$ and $S_{\lambda,q}(y) \subset S_{\lambda,q}(x)$.*

Proof The relation from (1) follows by the definition of $S_{\lambda,q}$.
(2) Since $y \in S_{\lambda,q}(x) \iff \lambda q(x,y) \leq \varphi(x) - \varphi(y)$, it follows $\varphi(y) \leq \varphi(x)$ for $y \in S_{\lambda,q}(x)$. Also, $z \in S_{\lambda,q}(y) \iff \lambda q(y,z) \leq \varphi(y) - \varphi(z)$ and

$$\lambda q(x,z) \leq \lambda q(x,y) + \lambda q(y,z) \leq \varphi(x) - \varphi(y) + \varphi(y) - \varphi(z) = \varphi(x) - \varphi(z)\,,$$

implies $z \in S_{\lambda,q}(x)$, that is $S_{\lambda,q}(y) \subset S_{\lambda,q}(x)$. △

Remark 7 Replacing the pseudo-quasimetric q by the equivalent one $\lambda \cdot q$, we can suppose in all proofs that $\lambda = 1$, that is we can work with the set-valued mapping S given by

$$S(x) := S_{1,q}(x) = \{y \in X \,|\, q(x,y) \leq \varphi(x) - \varphi(y)\}\,.$$

Definition 7 (generalized-Picard sequences, [11], see also [7]). Let X be a nonempty set, $S : X \rightrightarrows X$ a set-valued mapping, and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ a function. We say that a sequence $\{x_n\}$ in X is a GENERALIZED PICARD SEQUENCE[12] of S, if $x_{n+1} \in S(x_n)$ for all $n \in \mathbb{N}$. It is said to be STRICTLY φ-DECREASING if the sequence $\{\varphi(x_n)\}$ is strictly decreasing, i.e., $\varphi(x_{n+1}) < \varphi(x_n)$ for all $n \in \mathbb{N}$.

Definition 8 (series completeness). A sequence $\{x_n\}$ in a pseudo-quasimetric space (X, q) is said to be FORWARD-DISTANCE-SERIES-CONVERGENT if the forward-distance series $\sum_{n=1}^{\infty} q(x_n, x_{n+1})$ is convergent. A pseudo-quasimetric space is said to be FORWARD-DISTANCE-SERIES-COMPLETE if every forward-distance-series-convergent sequence is forward-convergent.

[12] The Picard-Lindelöf theorem, which shows that ordinary differential equations have solutions, is essentially an application of the Banach fixed point theorem to a special sequence of functions which forms a fixed point iteration, constructing the solution to the equation. Solving an ODE in this way is called Picard iteration, Picard's method, or the Picard iterative process.

Proposition 3 *Let (X, q) be a pseudo-quasimetric space. The following hold:*

(1) *Every forward-distance-series-convergent sequence $\{x_n\}$ in X is forward-Cauchy.*
(2) *The pseudo-quasimetric space (X, q) is forward-forward-complete if and only if it is forward-distance-series-complete.*

Proof (1) Consider an arbitrary forward-distance-series-convergent sequence $\{x_n\}$ in X, i.e., $\sum_{n=1}^{\infty} q(x_n, x_{n+1}) < \infty$. Then, for every $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that for all $m \geq N_\varepsilon$ and for all $k \in \mathbb{N}$ we have

$$q(x_m, x_{m+k}) \leq \sum_{n=m}^{m+k-1} q(x_n, x_{n+1}) \leq \sum_{n=m}^{\infty} q(x_n, x_{n+1}) < \varepsilon,$$

i.e., $\{x_n\}$ is a forward-Cauchy sequence in (X, q).

(2) Suppose that the space (X, q) is forward-forward-complete. If the sequence $\{x_n\}$ satisfies the condition $\sum_{n=1}^{\infty} q(x_n, x_{n+1}) < \infty$, then, by (1), it is forward-Cauchy, so it is forward-convergent to some $x_* \in X$.

Suppose now that the space (X, q) is forward-distance-series-complete and let $\{x_n\}$ be forward-Cauchy. Let $n_1 \in \mathbb{N}$ be such that

$$q(x_n, x_m) < \frac{1}{2}, \ \forall m, n \in \mathbb{N}: \ n_1 \leq n < m.$$

Suppose that $n_1 < n_2 < \cdots < n_k$ are such that

$$q(x_n, x_m) < \frac{1}{2^i}, \ \forall m, n \in \mathbb{N}: \ n_i \leq n < m\,, 1 \leq i \leq k.$$

It follows $q(x_{n_i}, x_{n_{i+1}}) < \frac{1}{2^i}$ for $1 \leq i \leq k-1$. Let n'_{k+1} be such that

$$q(x_n, x_m) < \frac{1}{2^{k+1}}, \ \forall m, n \in \mathbb{N}: \ n'_{k+1} \leq n < m\,. \tag{3.4}$$

Taking $n_{k+1} := 1 + \max\{n_k, n'_{k+1}\} > n_k$, it follows $q(x_{n_k}, x_{n_{k+1}}) < \frac{1}{2^k}$. Also (3.4) holds with n_{k+1} instead of n'_{k+1}.

Continuing in this manner, we find the numbers $n_1 < n_2 < \ldots$ such that $q(x_{n_k}, x_{n_{k+1}}) < 1/2^k$ for all $k \in \mathbb{N}$, implying $\sum_{k=1}^{\infty} q(x_{n_k}, x_{n_{k+1}}) < 1 < \infty$. By hypothesis, the subsequence $\{x_{n_k}\}$ is forward- convergent to some $x_* \in X$. But then, by Proposition 1, the sequence $\{x_n\}$ is forward-convergent to x_*. △

Remark 8 The construction from the last part of the proof is standard in metric spaces. In the case of quasimetric spaces, we have to take into account the asymmetry of the metric.

The following proposition contains some properties of generalized Picard sequences. We use the notation $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Proposition 4 *Let (X, q) be a pseudo-quasimetric space, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ and $\{x_n\}_{n=0}^{\infty}$ a generalized Picard sequence of $S_{\lambda,q}$, with $x_0 \in \operatorname{dom} \varphi$. Then, the following hold:*

(1) *The sequence $\{\varphi(x_n)\}_{n=0}^{\infty}$ is decreasing, i.e., $\varphi(x_{n+1}) \leq \varphi(x_n)$, $\forall n \in \mathbb{N}_0$.*
(2) *If φ is bounded from below on $S_{\lambda,q}(x_0)$, then the sequence $\{x_n\}$ is also forward-distance-series-convergent.*

Proof Put $S(x) = S_{1,q}(x)$. (1) The inequalities

$$q(x_k, x_{k+1}) \le \varphi(x_k) - \varphi(x_{k+1}) \quad (\iff x_{k+1} \in S(x_k))$$

imply $\varphi(x_{k+1}) \le \varphi(x_k)$, $\forall k \in \mathbb{N}_0$.

(2) By Proposition 2, $S(x_{n+1}) \subset S(x_n) \subset S(x_0)$, so that $\inf \varphi(S(x_0)) \le \inf \varphi(S(x_n))$ for all $n \in \mathbb{N}$, and the above inequalities yield by summation

$$\sum_{k=0}^{n} q(x_k, x_{k+1}) \le \varphi(x_0) - \varphi(x_{n+1}) \le \varphi(x_0) - \inf \varphi(S(x_{n+1})) \le \varphi(x_0) - \inf \varphi(S(x_0)), \tag{3.5}$$

showing that the sequence $\{x_n\}_{n\in\mathbb{N}_0}$ is forward-distance-series-convergent, provided φ is bounded below on $S(x_0)$. △

We can state now the first version of Ekeland's variational principle.

Theorem 4 (a forward version of Ekeland's variational principle) *Let (X, q) be a pseudo-quasimetric space and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Suppose that there exists $x_0 \in \operatorname{dom}(\varphi)$ such that*

(C1) $\inf \varphi(S_{\lambda,q}(x_0)) > -\infty$ *(i.e. φ is bounded from below on $S_{\lambda,q}(x_0)$);*

(C2) *for every sequence $\{x_n\}_{n=0}^{\infty}$ which is forward-distance-series-convergent, generalized Picard with respect to $S_{\lambda,q}$ (starting with x_0), and strictly φ-decreasing, there exists $y \in X$ such that $S_{\lambda,q}(y) \subset \bigcap_{n=0}^{\infty} S_{\lambda,q}(x_n)$.*

Then there exists a forward-distance-series-convergent sequence $\{x_n\}_{n\in\mathbb{N}_0}$ which is generalized Picard with respect to $S_{\lambda,q}$ and forward-convergent to some $x_* \in X$ such that for every $y_* \in S_{\lambda,q}(x_*)$ the following conditions hold:

$$\begin{aligned}
&\text{(i)} && \varphi(y_*) + \lambda q(x_0, y_*) \le \varphi(x_0)\,;\\
&\text{(ii)} && \varphi(x) + \lambda q(y_*, x) > \varphi(y_*),\ \forall\, x \in X \setminus S_{\lambda,q}(y_*)\,;\\
&\text{(iii)} && q(x_*, y_*) = 0,\ S_{\lambda,q}(y_*) \subset \overline{\{y_*\}}^{q},\\
& && \varphi(x) = \varphi(y_*) = \varphi(x_*),\ \forall x \in S_{\lambda,q}(y_*), \quad \text{and}\\
& && \varphi(x) > \varphi(y_*),\ \forall x \in \overline{\{y_*\}}^{q} \setminus S_{\lambda,q}(y_*).
\end{aligned} \tag{3.6}$$

Proof Replacing, if necessary, q by λq, we can suppose $\lambda = 1$. Put also $S(x) = S_{1,q}(x)$, $x \in X$.

Case I. We shall define inductively a forward-distance-series-convergent generalized Picard sequence which will satisfy all the requirements of the theorem.

Start with x_0 and suppose that $\alpha_0 := \inf \varphi(S(x_0)) < \varphi(x_0)$. Choose $x_1 \in S(x_0)$ such that

$$\alpha_0 \le \varphi(x_1) < \alpha_0 + \frac{1}{2}(\varphi(x_0) - \alpha_0) = \frac{1}{2}(\alpha_0 + \varphi(x_0)) < \varphi(x_0)\,.$$

Suppose that we have found $x_0, x_1, \ldots, x_n$ satisfying

$$\begin{aligned}
&x_{k+1} \in S(x_k),\\
&\alpha_k \le \varphi(x_{k+1}) < \alpha_k + \frac{1}{2}(\varphi(x_k) - \alpha_k) = \frac{1}{2}(\alpha_k + \varphi(x_k)) < \varphi(x_k)\,,
\end{aligned}$$

for $k = 0, 1, \ldots, n-1$, and $\varphi(x_k) > \alpha_k := \inf \varphi(S(x_k))$, $k = 0, 1, \ldots, n$.

Pick then $x_{n+1} \in S(x_n)$ such that

$$\alpha_n \leq \varphi(x_{n+1}) < \alpha_n + \frac{1}{2}(\varphi(x_n) - \alpha_n) = \frac{1}{2}(\alpha_n + \varphi(x_n)) < \varphi(x_n)\,. \tag{3.7}$$

Supposing that we can do indefinitely this procedure, we find a generalized Picard sequence $x_{n+1} \in S(x_n)$ satisfying (3.7) for all $n \in \mathbb{N}_0$. Let us show that the conditions from (3.6) are satisfied by this sequence.

Since the sequence $\{\varphi(x_n)\}$ is strictly decreasing and bounded from below (by condition (C1)), there exists $\alpha_* := \lim_{n\to\infty} \varphi(x_n) = \inf_{n\to\infty} \varphi(x_n)$.

By Proposition 2, $x_{n+1} \in S(x_n)$ implies $S(x_{n+1}) \subset S(x_n) \subset S(x_0)$, so that $\alpha_{n+1} \geq \alpha_n$, implying the existence of $\alpha = \lim_{n\to\infty} \alpha_n$. The inequalities

$$\varphi(x_{n+1}) < \frac{1}{2}(\alpha_n + \varphi(x_n)) < \varphi(x_n)$$

yield for $n \to \infty$, $\alpha_* \leq \frac{1}{2}(\alpha + \alpha_*) \leq \alpha_*$, implying $\alpha = \alpha_*$. Consequently

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \varphi(x_n) = \alpha_*\,.$$

Since φ is bounded below on $S(x_0)$, Proposition 4 implies that the generalized Picard sequence $\{x_n\}$ is forward-distance-series-convergent. Since the sequence $\{\varphi(x_n)\}$ is also strictly decreasing, condition (C2) implies the existence of a point $x_* \in X$ such that

$$S(x_*) \subset \bigcap_{n=0}^{\infty} S(x_n)\,.$$

Since $x_* \in S(x_*) \subset S(x_n)$, it follows that, for all $n \in \mathbb{N}_0$,

$$0 \leq q(x_n, x_*) \leq \varphi(x_n) - \varphi(x_*)\,. \tag{3.8}$$

Consequently,

$$\alpha_n \leq \varphi(x_*) \leq \varphi(x_n),\ \forall n \in \mathbb{N}_0,$$

yielding for $n \to \infty$, $\varphi(x_*) = \alpha_*$. But then, the inequalities (3.8) imply $\lim_{n\to\infty} q(x_n, x_*) = 0$, that is the sequence $\{x_n\}$ is forward-convergent to x_*.

Let $y_* \in S(x_*)$. Condition (i) is equivalent to $y_* \in S(x_0)$, which is true because $y_* \in S(x_*) \subset S(x_0)$. Condition (ii) follows from the definition of the set $S(y_*)$. Let us show that the conditions from (iii) are also satisfied.

The relations $y_* \in S(x_*) \subset S(x_n)$ imply $\alpha_n \leq \varphi(y_*) \leq \varphi(x_*) = \alpha_*$, $\forall n \in \mathbb{N}_0$. Letting $n \to \infty$, one obtains $\varphi(y_*) = \alpha_* = \varphi(x_*)$. Also

$$y_* \in S(x_*) \iff 0 \leq q(x_*, y_*) \leq \varphi(x_*) - \varphi(y_*) = 0\,,$$

so that $q(x_*, y_*) = 0$.

If $x \in S(y_*) \subset S(x_*)$, then, as above, $\alpha_n \leq \varphi(x) \leq \varphi(x_*) = \alpha_*$, $\forall n \in \mathbb{N}_0$, yields for $n \to \infty$, $\varphi(x) = \varphi(x_*) = \varphi(y_*)$. The inequality $0 \leq q(y_*, x) \leq \varphi(y_*) - \varphi(x) = 0$ implies $q(y_*, x) = 0$, that is $x \in \overline{\{y_*\}}^q$.

If $x \in \overline{\{y_*\}}^q \setminus S_{\lambda,q}(y_*)$, then $q(y_*, x) = 0$ and, by (ii), $\varphi(y_*) < \varphi(x) + \lambda q(y_*, x) = \varphi(x)$.

Case II. Suppose that, for some $n_0 \in \mathbb{N}_0$, $\varphi(x_{n_0}) = \alpha_{n_0} = \inf \varphi(S(x_{n_0}))$. Then take $x_{n_0+1} = x_{n_0}$ and, by induction, $x_{n_0+k} = x_{n_0}$ for all $k \in \mathbb{N}$.

Then the sequence $\{x_n\}$ is forward-distance-series-convergent with

$$\sum_{k=0}^{\infty} q(x_k, x_{k+1}) = \sum_{k=0}^{n_0-1} q(x_k, x_{k+1}),$$

and forward-convergent to x_{n_0}. Also, for $x \in S(x_{n_0})$, $\varphi(x) \geq \alpha_{n_0} = \varphi(x_{n_0})$, so that the inequalities

$$0 \leq q(x_{n_0}, x) \leq \varphi(x_{n_0}) - \varphi(x) \leq 0,$$

imply $q(x_{n_0}, x) = 0$ and $\varphi(x_{n_0}) = \varphi(x)$. It follows also that $x \in \overline{\{x_{n_0}\}}^q$, that is $S(x_{n_0}) \subset \overline{\{x_{n_0}\}}^q$.

These show that condition (iii) is satisfied. The validity of conditions (i) and (ii) follows as in Case I. △

Remark 9 Conditions (ii) and (iii) from Theorem 4 imply

(ii′) $\varphi(y_*) < \varphi(x) + \lambda\rho(y_*, x),\ \forall x \in X \setminus \overline{\{y_*\}}^q$.

(iii′) $\varphi(y_*) \leq \varphi(x),\ \forall x \in \overline{\{y_*\}}^q$.

Similar conditions appear in [17, Theorem 2].

Corollary 1 (a simple version in pseudo-quasimetric spaces). *Assume that all the hypotheses in Theorem 4 hold. Assume in addition that*

(C3) **(forward-limit uniqueness)** *any generalized-Picard sequence $\{x_n\}$ of $S_{\lambda,q}$, being strictly φ-decreasing and forward-distance-series-convergent, has at most one forward-limit.*

Then, there is $x_* \in S_{\lambda,q}(x_0)$ such that $S_{\lambda,q}(x_*) = \{x_*\}$, conditions which are respectively equivalent to

(i) $\varphi(x_*) + \lambda q(x_0, x_*) \leq \varphi(x_0)$;

(ii) $\varphi(x) + \lambda q(x_*, x) > \varphi(x_*), \quad \forall\, x \neq x_*$.

Proof It is straightforward from Theorem 4. △

Employing Theorem 4 in the context of the pseudo-quasimetric space $(X, \overline{q})$, where $\overline{q}(x, y) = q(y, x)$, $x, y \in X$, we obtain also a backward version of Ekeland's variational principle.

Corollary 2 (a backward version of Ekeland's variational principle). *Let (X, q) be a pseudo-quasimetric space, and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ a proper extended-real-valued function. Given $x_0 \in \operatorname{dom}\varphi$ and $\lambda > 0$, consider the set-valued mapping $S_{\lambda,\overline{q}} : X \rightrightarrows X$ defined by*

$$S_{\lambda,\overline{q}}(x) := \{y \in X \mid \varphi(y) + \lambda q(y, x) \leq \varphi(x)\},\ \forall x \in X.$$

Impose the boundedness from below condition (C1) *from Theorem 4 and the backward nonempty intersection condition*

(C2B) *For any generalized-Picard sequence $\{x_n\}_{n=0}^{\infty}$ of $S_{\lambda,\overline{q}}$, being strictly φ-decreasing and backward-distance-series-convergent (i.e. $\sum_{n=0}^{\infty} q(x_{n+1}, x_n) < \infty$), there exists $y \in X$ such that $S_{\lambda,\overline{q}}(y) \subset S_{\lambda,\overline{q}}(x_n)$ for all $n \in \mathbb{N}_0$.*

Then, there is $x_* \in S_{\lambda,\overline{q}}(x_0)$ *such that for every* $y_* \in S_{\lambda,\overline{q}}(x_*)$ *one has*

(i) $\varphi(y_*) + \lambda q(y_*, x_0) \leq \varphi(x_0)$;

(ii) $\varphi(x) + \lambda q(x, y_*) > \varphi(y_*), \quad \forall\, x \in X \setminus \overline{\{y_*\}}^q$:

(iii) $q(x_*, y_*) = 0,\ S_{\lambda,\overline{q}}(y_*) \subset \overline{\{y_*\}}^q$,
$\varphi(x) = \varphi(y_*) = \varphi(x_*),\ \forall x \in S_{\lambda,\overline{q}}(y_*)$, *and*
$\varphi(x) > \varphi(y_*),\ \forall x \in \overline{\{y_*\}}^q \setminus S_{\lambda,\overline{q}}(y_*)$.

where $\overline{\{y_*\}}^q = \{u \in X \mid q(u, y_*) = 0\}$.

Proof It follows from Theorem 4 by using the pseudo-quasimetric space $(X, \overline{q})$ instead of (X, q). △

Let us revisit Examples 3–5.

Example 6 **(Examples 3–5 (revisited)).**

– Since Condition (C2) in Example 3 does not hold, both Theorem 4 and its Corollary 1 are not applicable. Precisely, we have $\varphi(x) = e^x$, $S_{1,q_1}(x) = (-\infty, x]$, $\{x_n\}$ with $x_n = -n$ is a strict-φ-decreasingly forward-distance-series-convergent generalized-Picard sequence of S_{1,q_1}, but for any number $y \in \mathbb{R}$ one has

$$S_{1,q_1}(y) = (-\infty, y] \not\subset S_{1,q_1}(x_n) = (-\infty, -n] \text{ whenever } \quad -n < y$$

clearly verifying that the nonempty intersection condition (C2) is not fulfilled.

– In Example 4, it is easy to check the validity of the bounded condition (C1) and the nonempty intersection condition (C2) with $y = -1$ for any strict-φ-decreasingly forward-distance-series-convergent generalized-Picard sequence of S_{λ,q_1}.

Take $\lambda = 2$ and $x_0 = 1$. Then, the sequence $x_n \equiv x_*$ for all $n \in \mathbb{N}$, where x_* and 1 are two distinct solutions of the equation $e^x = 2(x-1) + e$, satisfies both the assertions (i) and (ii) in Theorem 4 which, respectively, reduce to

(i) $e^{x_*} + 2q_1(1, x_*) = 2^1$;

(ii) $\varphi(x) + 2q_1(x_*, x) > \varphi(x_*), \quad \forall\, x \in [-1, 1] \setminus \overrightarrow{\{x_n\}} = (x_*, 1]$.

Note that excluding all the forward-limits in (ii) is essential since for any $x < x_*$, one has $\varphi(x) + 2q_1(x_*, x) = \varphi(x) < \varphi(x_*)$, i.e., (ii) is not valid.

Take $\lambda = 1$ and $x_0 = 0$. Then, the desired point of Theorem 4 is then $x_* = 0$.

– In Example 5, it is easy to check the validity of condition (C2). For any strict-φ-decreasingly forward-distance-series-convergent generalized-Picard sequence $\{x_n\}$ in $\mathbb{R}$, the sequence $\{x_n\}$ is a Cauchy sequence in the complete metric space $(\mathbb{R}, |\cdot|)$. Therefore, the classical Ekeland's variational principle applied to the closed set $\{x_n\} \cup \{x_*\}$ ensures the existence of some element x_* satisfying condition (C2).

Next, we will provide an efficient sufficient condition for the nonempty intersection condition (C2) which is less restrictive than lower semicontinuity.

Definition 9 (strict-decreasingly forward-lower-semicontinuous functions). Let (X, q) be a pseudo-quasimetric space and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function on X.

(i) The function φ is said to be STRICT-DECREASINGLY FORWARD-LOWER-SEMICONTINUOUS, if for every forward-convergent sequence $\{x_n\}$ such that $\{\varphi(x_n)\}$ is strictly decreasing, one has

$$\varphi(y_*) \leq \lim_{n\to\infty} \varphi(x_n), \ \forall\, y_* \in \overrightarrow{\{x_n\}},$$

where $\overrightarrow{\{x_n\}} = \{y_* \in X \mid \lim_{n\to\infty} q(x_n, y_*) = 0\}$ is the collection of forward-limits of the sequence $\{x_n\}$.

(ii) A related notion is that of DECREASINGLY FORWARD-LSC FUNCTION considered by Kirk and Saliga [21] (called by them *lower-semicontinuity from above*) meaning that $\varphi(x_*) \leq \lim_{n\to\infty} \varphi(x_n)$ for every sequence $\{x_n\}$ forward-convergent to x_* and such that $\varphi(x_{n+1}) \leq \varphi(x_n), \forall n \in \mathbb{N}$.

(iii) Following [17] we call the function φ FORWARD-NEARLY-LOWER-SEMICONTINUOUS if, whenever a sequence $\{x_n\}$ of distinct points in X is forward-convergent to a forward-limit x_*, then $\varphi(x_*) \leq \liminf_{n\to\infty} \varphi(x_n)$.

Similarly, we could define the concepts of strict-decreasing backward-lower-semicontinuity, of decreasing backward-lower-semicontinuity, and of backward-near-lower-semicontinuity.

It is worth mentioning that the class of strict-decreasingly forward-lower-semicontinuous functions is broader than the union of that of the decreasingly forward-lower-semicontinuous functions and that of the nearly lower-semicontinuous functions. Since the strict-φ-decreasing requirement of the sequence $\{x_n\}$ implies that $x_n \neq x_m$ for all $n \neq m$, every nearly lower-semicontinuous function is strict-φ-decreasingly forward-lower-semicontinuous.

Let us provide some examples with functions satisfying these conditions.

Example 7 **(decreasingly forward-lsc, strict-decreasingly forward-lsc and lsc functions).**

– Consider the functions φ, $\varphi_1 : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ given by

$$\varphi(x) := \begin{cases} x & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \quad \text{and} \quad \varphi_1(x) := \begin{cases} -x & \text{if } x > 0 \\ 1 & \text{if } x \leq 0\,. \end{cases}$$

Then φ is strict-decreasingly forward[13]-lsc at 0, but not decreasingly lsc (and so not lsc) at 0, since the sequence $x_n = -1/n$, $n \in \mathbb{N}$, is convergent to 0 and $\varphi(x_n) = -1$ for all $n \in \mathbb{N}$, so that $\lim_{n\to\infty} \varphi(x_n) = -1 < 0 = \varphi(0)$. The function φ_1 is decreasingly forward-lsc at 0, but not lsc. Indeed, there are no sequences $x_n \to 0$ with $\{\varphi_1(x_n)\}$ strictly decreasing. If $x_n \to 0$ and $\varphi_1(x_{n+1}) \leq \varphi_1(x_n)$ for all n, then $\varphi_1(x_n) = 1$ for sufficiently large n, so that $\lim_{n\to\infty} \varphi_1(x_n) = 1 = \varphi(0)$. The function φ_1 is not lsc at 0 because $\lim_{x\searrow 0} \varphi_1(x) = 0 < 1 = \varphi_1(0)$.

– The function $\varphi(x) = x$ is not strict-decreasingly forward-lower-semicontinuous in the quasimetric space $([0,1], q_4)$[14], where q_4 is defined by

$$q_4(x,y) = \begin{cases} x - y & \text{if } x \geq y; \\ 1 + x - y & \text{if } x < y \text{ but } (x,y) \neq (0,1); \\ 1 & \text{if } (x,y) = (0,1). \end{cases}$$

[13] The space in question is a metric space and thus there is no difference between two topologies.

[14] the conjugate quasimetric of the one studied in [22, Example 3.16]

In this space, the strict-φ-decreasingly forward-Cauchy sequence $\{x_n\}$ with $x_n = \frac{1}{n}$ has two forward-limits $x_* = 0$ and $y_* = 1$. Obviously,

$$\liminf_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \frac{1}{n} = 0 = \varphi(0) = \varphi(x_*) \neq \varphi(1) = \varphi(y_*).$$

This proves that φ is not strict-decreasingly lower-semicontinuous in (X, q_4). Incidentally, this furnishes another example of a quasimetric space (hence T_1) where the uniqueness condition for forward-limits does not hold (see Example 2). It is easy to check that the sequence $x_n = 1/n$ is also forward-Cauchy.

– The everywhere discontinuous function $\varphi(x) = 0$ for $x \in \mathbb{Q}$ and $\varphi(x) = 1$ for $x \in \mathbb{R} \setminus \mathbb{Q}$, defined on $(\mathbb{R}, |\cdot|)$, is strict-decreasingly lower-semicontinuous because there are no strictly φ-decreasing sequences. The function φ is not lower-semicontinuous at every point $x \in \mathbb{R}\setminus\mathbb{Q}$, because $\varphi(x) = 1 > 0 = \liminf_{u\to x} \varphi(u)$. Also, it is not upper-semicontinuous at every $x \in \mathbb{Q}$.

We shall present now some completeness conditions guaranteeing the validity of condition (C2) from Theorem 4. Consider also the condition

(C2′) for every sequence $\{x_n\}_{n=0}^{\infty}$, generalized Picard with respect to $S_{\lambda,q}$, there exists $y \in X$ such that $S_{\lambda,q}(y) \subset \bigcap_{n=0}^{\infty} S_{\lambda,q}(x_n)$,

where $S_{\lambda,q}(x)$ is defined by (3.2).

By Proposition 3, (C2′) $\Rightarrow$ (C2).

Lemma 1 *Let (X, q) be a pseudo-quasimetric space, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ a proper function, $x_0 \in \operatorname{dom}\varphi$, and $\{x_n\}_{n=0}^{\infty}$ a generalized Picard sequence of $S_{\lambda,q}$ forward-convergent to some x_*.*

(1) *If the function φ is decreasingly forward-lower-semicontinuous on* $\operatorname{dom}\varphi$ *then*

$$x_* \in \bigcap_{n=0}^{\infty} S_{\lambda,q}(x_n)\,. \tag{3.9}$$

By Proposition 2 this implies $S_{\lambda,q}(x_*) \subset \bigcap_{n=0}^{\infty} S_{\lambda,q}(x_n)$.

(2) *If the function φ is strict-decreasingly forward-lower-semicontinuous on* $\operatorname{dom}\varphi$*, then* (3.9) *holds if, in addition, the sequence $\{x_n\}$ is strictly φ-decreasing.*

Proof Work again with the sets $S(x) = S_{1,q}(x)$ (as in the proof of Theorem 4).

Since

$$x_{k+1} \in S(x_k) \iff q(x_k, x_{k+1}) \le \varphi(x_k) - \varphi(x_{k+1}),$$

it follows that the sequence $\{\varphi(x_n)\}_{n=0}^{\infty}$ is decreasing and

$$q(x_n, x_{n+k}) \le \varphi(x_n) - \varphi(x_{n+k})$$

so that

$$q(x_n, x_*) \le q(x_n, x_{n+k}) + q(x_{n+k}, x_*) \le \varphi(x_n) - \varphi(x_{n+k}) + q(x_{n+k}, x_*),\ \forall n, k \in \mathbb{N}_0.$$

Letting $k \to \infty$ and taking into account that $\varphi(x_*) \le \lim_{k\to\infty} \varphi(x_{n+k})$ and $\lim_{k\to\infty} q(x_{n+k}, x_*) = 0$, one obtains

$$q(x_n, x_*) \le \varphi(x_n) - \varphi(x_*) \iff x_* \in S(x_n),\ \forall n \in \mathbb{N}_0.$$

The same proof works in the case that the function φ is strict-decreasingly forward-lower-semicontinuous. △

Proposition 5 (sufficient conditions for (C2) **and** (C2′)**).** *Let* (X, q) *be a pseudo-quasimetric space and* $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *a proper function.*

(1) *If every forward-distance-series-convergent and strictly* φ*-decreasing sequence in the space* (X, q) *is forward-convergent and the function* φ *is strict-decreasingly forward-lower-semicontinuous on* $\operatorname{dom}\varphi$*, then condition (C2) is satisfied.*
(2) *If the space* (X, q) *is forward-distance-series-complete and the function* φ *is decreasingly forward-lower-semicontinuous on* $\operatorname{dom}\varphi$ *and bounded from below, then condition (C2′) is satisfied.*

Proof Let $x_0 \in \operatorname{dom}\varphi$ and $\{x_n\}_{n=0}^{\infty}$ a generalized Picard sequence with respect to $S_{\lambda,q}$.

(1) If the sequence $\{x_n\}$ is forward-distance-series-convergent and strictly φ-decreasing, then, by the completeness hypothesis, it has a forward-limit x_*. By Lemma 1 it follows $S_{\lambda,q}(x_*) \subset \bigcap_{n=0}^{\infty} S_{\lambda,q}(x_n)$, i.e. condition (C2) is satisfied.

(2) The generalized Picard sequence $\{x_n\}$ is φ-decreasing, so that, by Proposition 4, it is forward-distance-series-convergent. By hypothesis it has a forward-limit x_*. We can apply again Lemma 1 to conclude that $S_{\lambda,q}(x_*) \subset \bigcap_{n=0}^{\infty} S_{\lambda,q}(x_n)$, i.e. condition (C2′) is satisfied. △

Corollary 3 *Let* (X, q) *be a pseudo-quasimetric space and* $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *a proper extended-real-valued function. Assume that every forward-distance-series-convergent and strictly* φ*-decreasing sequence in the space* (X, q) *is forward-convergent, the function* φ *is proper, bounded from below, and strict-decreasingly forward-lower-semicontinuous. Then, for any* $x_0 \in \operatorname{dom}\varphi$ *and* $\lambda > 0$*, there is* $x_* \in S_{\lambda,q}(x_0)$ *such that for every* $y_* \in S_{\lambda,q}(x_*)$*, one has*

$$
\begin{aligned}
&\text{(i)} \quad \varphi(y_*) + \lambda q(x_0, y_*) \le \varphi(x_0);\\
&\text{(ii)} \quad \varphi(x) + \lambda q(y_*, x) > \varphi(y_*),\ \forall x \in X \setminus \overline{\{y_*\}}^{q};\\
&\text{(iii)} \quad q(x_*, y_*) = 0,\ S_{\lambda,q}(y_*) \subset \overline{\{y_*\}}^{q},\\
&\qquad \varphi(x) = \varphi(y_*) = \varphi(x_*),\ \forall x \in S_{\lambda,q}(y_*), \quad \textit{and}\\
&\qquad \varphi(x) > \varphi(y_*),\ \forall x \in \overline{\{y_*\}}^{q} \setminus S_{\lambda,q}(y_*).
\end{aligned}
$$

Assume, in addition, that the uniqueness condition (C3) from Corollary 1 holds. Then, for any $x_0 \in \operatorname{dom}\varphi$ *and* $\lambda > 0$*, there is* $x_* \in X$ *that satisfies conditions (i) and (ii) of the same corollary.*

Proof By Proposition 5, condition (C2) is satisfied. As the function φ is bounded from below, we can apply Theorem 4 to conclude. △

Remark 10 The forward-limit uniqueness of strictly φ-decreasing sequences allows us to use the pseudo-quasimetric versions of Ekeland's variational principle when the space is not forward-complete; for example, the space $(\mathbb{R}, q_1)$, where $q_1(x, y) = y - x$ if $x \le y$ and $q_1(x, y) = 0$ otherwise (see Example 1). Consider the sequence $x_n = -n$ for all $n \in \mathbb{N}$. Then $q_1(x_n, x_{n+k}) = 0$ for all $n, k \in \mathbb{N}$, so that $\{x_n\}$ is forward-Cauchy. However, this sequence has no forward-limit. Given $x^* \in \mathbb{R}$, $q_1(x_n, x_*) = x_* + n$ for all $n > x_*$ and thus $\lim_{n\to\infty} q_1(x_n, x_*) = +\infty$. This proves that x^* is not a forward-limit of $\{x_n\}$. Since x^* was arbitrary, the sequence has no forward-limit.

Remark 11 (a) Corollary 3 is more general than both [17, Theorem 2(3)] and [5, Corollary 3.3].

(b) Proceeding in a similar way, the backward nonempty intersection condition (C2B) is fulfilled provided that the cost function is strict-decreasingly backward-lower-semicontinuous. Thus, we could formulate a backward version of Corollary 3.

(c) The nonempty intersection condition (C2) is similar to the one used in [7, Theorem 4.1]. Assume that (X, q) is a complete metric space, condition (C2) reduces to

(C2$''$) for any strict-φ-decreasingly forward-Cauchy generalized-Picard sequence $\{x_n\}$, one has $S_{\lambda,q}(y_*) \subset S_{\lambda,q}(x_n)$ for all $n \in \mathbb{N}$, where y_* is the unique limit of the sequence $\{x_n\}$

since there is no difference between forward-Cauchy and backward-Cauchy sequences and the limit (if it exists) is unique. Condition (C2$''$) was first used in [20].

(d) Theorem 4 and its Corollaries 1 and 3 can be seen as far-going extended versions of Ekeland's variational principle in pseudo-quasi-metric spaces which might not be forward Hausdorff. In several less general settings, versions of Ekeland's variational principle were established by Bao et al. [2–6] (forward Hausdorff[15] quasimetric spaces), by Cobzaş [8] (T_1 quasimetric space[16] is T_1 - for any distinct points x and y in X each of them has a neighborhood not containing the other), by Ume [29] (regular[17] quasimetric spaces).

(e) Based on the fact that if (X, p) is a partial pseudo-metric space, then the space (X, q) with $q(x, y) = p(x, y) - p(y, y)$ is a pseudo-quasimetric space, it is possible to formulate new versions of Ekeland's variational principle as well as their equivalent fixed point theorems in partial pseudo-metric spaces. They are more general than the existing results in partial metric spaces; see [1].

It is well known that the original Ekeland's variational principle has been shown by F. Sullivan to be equivalent to the completeness of metric spaces. For an analysis of connections between various types of completeness, variational principles and fixed point theorems see the survey paper [10].

In the rest of this section, we will show that Corollary 3 yields a characterization of forward completeness of pseudo-quasimetric spaces.

To the best of our knowledge, Karapinar and Romaguera were the firsts who established a version of Ekeland's variational principle in [17] for non-necessarily T_1 quasimetric[18] spaces, and proved that their weak version of Ekeland's variational principle is equivalent to the completeness of the corresponding quasimetric space.

Theorem 5 ([17, Theorem 2]) *For a pseudo-quasimetric space (X, q) the following conditions are equivalent.*

(1) (X, q) *is forward-forward complete.*

[15] known as 'left Hausdorff' in the cited papers

[16] the topology $\tau(\overline{q})$

[17] Cobzaş pointed in [8, Remark 1.4] that the topology $\tau(q)$ is regular under the assumptions made in the cited paper.

[18] A quasimetric in the sense of Karapinar et al. is a pseudo-quasimetric defined in Definition 2 satisfying that $x = y$ if and only if $q(x, y) = q(y, x) = 0$.

(2) *For every proper bounded below and forward-nearly-lsc function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *and for every* $\varepsilon > 0$ *there exists* $y_\varepsilon \in X$ *such that*

$$\begin{aligned}
&\text{(i)}\ f(y_\varepsilon) \le \inf f(X) + \varepsilon;\\
&\text{(ii)}\ f(y_\varepsilon) < f(x) + \varepsilon q(y_\varepsilon, x),\ \forall x \in X \setminus \overline{\{y_\varepsilon\}}^{q}, \quad \textit{and}\\
&\text{(iii)}\ f(y_\varepsilon) \le f(x),\ \forall x \in \overline{\{y_\varepsilon\}}^{q}.
\end{aligned}$$

In fact, Karapinar and Romaguera [17] formulated a backward version (i.e. for right q-K-completeness) of Theorem 5. We presented here its forward analog.

The following theorem shows that Corollary 3 (see also Remark 9) gives a characterization of completeness of pseudo-quasimetric spaces.

Theorem 6 (a characterization of forward complete pseudo-quasimetric spaces). *For a pseudo-quasimetric space* (X, q) *the following statements are equivalent.*

(1) (X, q) *is forward-forward complete.*
(2) *For every proper, bounded from below and strictly-decreasing forward-lower-semicontinuous functional* $\varphi : X \to \mathbb{R}\cup\{+\infty\}$*, and for any* $x_0 \in \operatorname{dom}\varphi$*, there is* $x_* \in X$ *such that for every* $y_* \in S_{1,q}(x_*) = \{u \in X \mid \varphi(u) + q(x_*, u) \le \varphi(x_*)\}$ *one has*

$$\begin{aligned}
&\text{(i)}\ \ \varphi(y_*) + q(x_0, y_*) \le \varphi(x_0);\\
&\text{(ii)}\ \ \varphi(x) + q(y_*, x) > \varphi(y_*), \quad \forall\, x \in X \setminus \overline{\{y_*\}}^{q}, \quad \textit{and}\\
&\text{(iii)}\ \ \varphi(y_*) \le \varphi(x), \quad \forall\, x \in \overline{\{y_*\}}^{q}.
\end{aligned}$$

Proof (1) $\Rightarrow$ (2) By Proposition 3, the space (X, q) is forward-distance-series-complete. Consequently, the hypotheses of Corollary 3 are fulfilled. Taking into account Remark 9, it follows that condition (iii) from (2) holds too.

(2) $\Rightarrow$ (1) Assume that (X, q) is not forward-forward complete and show that there exists a function φ satisfying the hypotheses from (2), but for which the conclusions (i)-(iii) fail.

By Proposition 3, (X, q) is not forward-distance-series-complete so that there exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X with $\sum_{n=0}^{\infty} q(x_n, x_{n+1}) < \infty$, which has no forward limit. By Proposition 3 the sequence $\{x_n\}_{n\in\mathbb{N}_0}$ is forward-Cauchy, so that, by Theorem 1), it has no forward convergent subsequences.

We shall distinct two cases.

Case I. Suppose that

$$\exists m,\ \forall k \ge m,\ \exists n_k > k,\ \ \rho(x_k, x_{n_k}) > 0\,. \tag{3.10}$$

Then, for $n_0 = m$ there exists $n_1 > n_0$ such that $\rho(x_{n_0}, x_{n_1}) > 0$. Taking $k = n_1$ it follows the existence of $n_2 > n_1$ such that $\rho(x_{n_1}, x_{n_2}) > 0$. Continuing in this manner we obtain a sequence $n_0 < n_1 < \dots$ such that $\rho(x_{n_k}, x_{n_{k+1}}) > 0$ for all $k \in \mathbb{N}_0$.

Taking into account this subsequence and relabeling, we can suppose that

$$\rho(x_n, x_{n+1}) > 0,\ \forall n \in \mathbb{N}_0. \tag{3.11}$$

For

$$s := \sum_{k=1}^{\infty} q(x_n, x_{n+1}) < \infty \, .$$

consider the function $\varphi : X \to \mathbb{R}$ defined by $\varphi(x) = 2s$ for all $x \in X \setminus \{x_0, x_1, x_2, \ldots\}$, $\varphi(x_0) = s$ and $\varphi(x_{n+1}) := \varphi(x_n) - q(x_n, x_{n+1})$ for all $n \geq 0$. Obviously, the sequence $\{x_n\}$ is strictly φ-decreasing, since $q(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. One obtains successively $\varphi(x_1) = \varphi(x_0) - q(x_0, x_1)$, $\varphi(x_2) = \varphi(x_1) - q(x_1, x_2) = \varphi(x_0) - q(x_0, x_1) - q(x_1, x_2)$, and, by induction,

$$\varphi(x_n) = \varphi(x_0) - \sum_{i=0}^{n-1} q(x_i, x_{i+1}) \, .$$

Taking into account that $\varphi(x_0) = s$, it follows

$$\varphi(x_n) = \sum_{i=n}^{\infty} q(x_i, x_{i+1}) > 0 \, , \tag{3.12}$$

which shows that the functional φ is bounded from below by 0. Since any strictly φ-decreasing sequence must be a subsequence of $\{x_n\}$, it follows that there are no strictly φ-decreasing forward convergent sequences, implying that the functional φ is strictly φ-decreasing forward-lower-semicontinuous. By hypothesis, there exists $x_* \in X$ satisfying the conditions (i)-(iii). Condition (i) implies $q(x_0, x_*) \leq \varphi(x_0) - \varphi(x_*)$, hence $\varphi(x_*) \leq \varphi(x_0) = s$, so that $x_* = x_m$ for some $m \in \mathbb{N}_0$. Condition (ii) for $x = x_{m+1}$ yields

$$\varphi(x_{m+1}) > \varphi(x_m) - q(x_m, x_{m+1}) \, ,$$

in contradiction to the definition of φ.

Case II. Suppose that (3.10) does not hold, that is

$$\forall m, \; \exists k \geq m, \;\; \text{s.t.} \;\; \forall n > k, \;\; \rho(x_k, x_n) = 0 \, . \tag{3.13}$$

For $m = 0$ let $k = n_0 \geq 0$ be such that $\rho(x_{n_0}, x_n) = 0$ for all $n > n_0$. Now, for $m = 1 + n_0$ let $n_1 > n_0$ be such that $\rho(x_{n_1}, x_n) = 0$ for all $n > n_1$. It follows $\rho(x_{n_0}, x_{n_1}) = 0$.

Continuing in this manner we obtain a sequence $n_0 < n_1 < \ldots$ such that $\rho(x_{n_k}, x_{n_{k+1}}) = 0$ for all $k \in \mathbb{N}_0$. Relabeling, we can suppose that the sequence (x_n) satisfies

$$\rho(x_n, x_{n+1}) = 0, \; \forall n \in \mathbb{N}_0. \tag{3.14}$$

Let $B := \{x_n : n \in \mathbb{N}_0\}$, and define the function $\varphi : X \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} \frac{1}{2^n} & \text{if} \quad x = x_n \;\text{ for some } \quad n \in \mathbb{N}_0, \\ 2 & \text{for } \; x \in X \setminus B. \end{cases}$$

Again, any strictly φ-decreasing sequence must be a subsequence of $\{x_n\}$, implying that there are no strictly φ-decreasing forward convergent sequences, so that the functional φ is strictly φ-decreasing forward-lower-semicontinuous.

Suppose that there exists $x_* \in X$ satisfying the conditions (i)-(iii). By (i), $\varphi(x_*) \leq \varphi(x_0) = 1$, so that $y = x_m \in B$ for some $m \in \mathbb{N}_0$.

If (3.14) holds, then, by the triangle inequality,

$$\rho(x_m, x_{m+k}) \leq \sum_{i=1}^{k} \rho(x_{m+i-1}, x_{m+i}) = 0\,,$$

i.e. $x_n \in \overline{\{x_m\}}^q$, for all $n \geq m$. By (iii),

$$\varphi(x_m) \leq \varphi(x_n) = \frac{1}{2^n},\ \forall n \geq m,$$

implying $\varphi(x_m) = 0$, a value not taken by φ. △

Remark 12 The function considered in Case II in the above proof is inspired by one considered in [17, Theorem 2]. Note that the authors of [17] did not consider this case (i.e. the possibility that $q(x_n, x_{n+1}) = 0$ for all n) in their proof.

The following example shows that condition (iii) in Theorem 6 is essential for the completeness of the pseudo-quasimetric space (X, q).

Example 8 Let $x_n = -n$, $n \in \mathbb{N}_0$, and $X = \{x_n : n \in \mathbb{N}_0\}$ with the metric $q(x_n, x_m) = (x_m - x_n)^+ = (-m + n)^+ = n - m$ if $n > m$ and $= 0$ if $n \leq m$. Then $q(x_n, x_{n+1}) = 0$ for all $n \in \mathbb{N}_0$ and the space (X, q) is not forward-complete (see Remark 6). Let $\varphi : X \to [0, \infty)$ be an arbitrary function. For $x_* = x_0$,

$$\varphi(x_*) = \varphi(x_0) \leq \varphi(x_0) = \varphi(x_0) + q(x_*, x_0)\,.$$

Since $q(x_0, x_n) = 0$ for all $n \in \mathbb{N}_0$ the condition

$$\varphi(x_0) < \varphi(x_n) + q(x_0, x_n),\ \forall n \in \mathbb{N}_0 : q(x_0, x_n) > 0$$

is trivially satisfied.

4 Behavioral applications. When completeness is equivalent to the existence of traps

Soubeyran [25–27] showed that, quite surprisingly, Ekeland's variational principle offers a prototype of the variational rationality model. In this simple but very important benchmark case, we have:

- The space (X, q) is a quasimetric or a pseudo-quasimetric space. It refers to a space of actions (doings), havings or beings, where $q : X \times X \to \mathbb{R}$ is a quasimetric or a pseudo-quasimetric.
- The scalar function $g : X \to \mathbb{R}$ is a "to be increased payoff", some profit or satisfaction level, $\overline{g} = \sup_{x \in X} g(x) < +\infty$ is the highest payoff the agent can expect to realize (a maximum seen as an aspiration level), while $\varphi(x) = \overline{g} - g(x) \geq 0$ refers to his "to be decreased" residual dissatisfaction to fail to reach $\overline{g}$.
- Advantages to change are $A(x, y) = g(y) - g(x) = \varphi(x) - \varphi(y)$
- Inconveniences to change are $I(x, y) = C(x, y) - C(x, x) \geq 0$, where $C(x, y) \geq 0$ refers to costs to be able to change and $C(x, x) \geq 0$ represents costs to be able to stay.

– The worthwhile-to-change ratio λ which defines how, each period, it is worthwhile to change, is constant all along the transition.

Soubeyran [25–27] has shown, in great details, in which circumstances inconveniences to change can be modeled as pseudo-quasi-metrics $q(x, y) = I(x, y) = C(x, y) - C(x, x) \geq 0$. In this case, inconveniences to change can be zero even when $y \neq x$. This means that costs to be able to change $C(x, y)$ can be equal to costs to be able to stay $C(x, x)$ for some change $x \curvearrowright y \neq x$. Be cautious, these costs to be able to do are very different from traditional costs to do (execution costs).

Then, in the current period $n+1$, a change which moves from repeating the last action $x = x_n \in X$ to perform the current action $y = x_{n+1} \in X$ is worthwhile if $A(x, y) \geq \lambda I(x, y)$, where the highest is $\lambda > 0$, the more it is worthwhile to change. A worthwhile change $x \curvearrowright y$ satisfies $\varphi(x) - \varphi(y) \geq \lambda q(x, y)$. Given $\varphi(.), x$ and λ, the (VR) approach defines, each current period, the worthwhile to change set, $S_{\lambda,q}(x) = \{y \in X \mid \varphi(x) - \varphi(y) \geq \lambda q(x, y)\}$ and the related worthwhile stay and change dynamic $x_{n+1} \in S_{\lambda,q}(x_n)$. A current stay is such that $x_{n+1} = x_n$, while a current change is such that $x_{n+1} \neq x_n$. A variational trap $x_* \in X$ is worthwhile to approach, i.e., $x_{n+1} \in S_{\lambda,q}(x_n)$ and worthwhile to reach, i.e., $x_* \in S_{\lambda,q}(x_n)$ for all $n \in \mathbb{N}$ but not worthwhile to leave, that is $S_{\lambda,q}(x_*) = \{x_*\}$. Then, it is easy to see that Ekeland's variational principle gives sufficient conditions for the existence of variational traps, starting from any initial action.

Our paper gives sufficient conditions for the existence of variational traps in the case where inconveniences to change are pseudo-quasimetrics. Furthermore, our paper shows that completeness of the pseudo-quasimetric space X is equivalent to the existence of variational traps. It means that in such pseudo-quasimetric spaces, the fact that every worthwhile stay and change dynamic ends in some trap is equivalent to completeness. Then, in such complete spaces, no worthwhile stay and change dynamic can wander. Then, each period, if an agent prefers to change from his previous position to a new one, because such change is worthwhile, sooner or later, he will end in a trap where it is not worthwhile to change. This is a very nice result in Behavioral Sciences.

5 Conclusions

In this paper, we establish new versions of Ekeland's variational principle for strict-decreasingly forward- (resp. backward-)lower-semicontinuous functions in possible incomplete pseudo-quasi- metric spaces; in particular, in $\mathbb{R}$ with $q(x, y) = y - x$ if $y \geq x$ and 0 otherwise. It is important to emphasize that the motivation of this research comes from applications in behavioral sciences.

We further extend the characterization of the completeness of T_1-quasimetric (resp. quasimetric) spaces in [8] (resp. [17]).

A natural question arise: why matters the use of pseudo-quasimetrics instead of quasimetrics in the statements of variational principles? That is, why the existence of y different from x such that $q(x, y) = 0$ does matter so much?

The motivation comes from its interest in applications to Behavioral Sciences, based on the VR approach, where passing from quasidistances to pseudo-quasidistances as a model for inconveniences to change is essential, as explained below.

In this case a change from x to y, y different from x, can generate no inconveniences, that is $I(x,y) = C(x,y) - C(x,x) = q(x,y) = 0$, because costs of being able to change $C(x,y)$ equal costs of being able to stay, $C(x,x)$. In this case a worthwhile change from x to y is such that $g(y) - g(x) \geq hq(x,y) = 0$, that is such that $g(y) \geq g(x)$. Then, in this case, an improving change, and even more a strictly increasing change, becomes worthwhile.

This is also why our assumptions on strictly φ-decreasing sequences and strict-decreasingly forward-lower-semicontinuous functions are natural and matter much.

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