LOCAL MONOTONICITY AND FULL STABILITY
FOR PARAMETRIC VARIATIONAL SYSTEMS

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Abstract. The paper introduces and characterizes new notions of Lipschitzian and Hölderian full stability of solutions to general parametric variational systems described via partial subdifferential of prox-regular functions acting in finite-dimensional and Hilbert spaces. These notions, postulated certain quantitative properties of single-valued localizations of solution maps, are closely related to local strong maximal monotonicity of associated set-valued mappings. Based on advanced tools of variational analysis and generalized differentiation, we derive verifiable characterizations of the local strong maximal monotonicity and full stability notions under consideration via some positive-definiteness conditions involving second-order constructions of variational analysis.

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1. Introduction. The paper belongs to the area of modern variational analysis, which has been well recognized as a fruitful field of mathematics with numerous applications; see, e.g., the books [6, 15, 27] and the references therein. We pursue here a twofold goal: to study *local strong maximal monotonicity* of set-valued operators in Hilbert spaces and the usage of ideas and results on strong maximal monotonicity to introduce and characterize new notions of *quantitative stability*, in both Lipschitzian and Hölderian settings, for *parametric variational systems* (PVS) given in the form

$$v \in f(x, p) + \partial_x g(x, p), \tag{1.1}$$

where $x \in X$ is the *decision* variable from a Hilbert space X; $(v, p) \in X \times P$ is a pair of perturbation *parameters*, where $v \in X$ signifies *canonical* perturbations while $p \in P$ stands for *basic* perturbations taking values in a metric space P; the single-valued *base* mapping $f \colon X \times P \to X$ is smooth around the reference point $(\bar{x}, \bar{p})$; and the *potential* $g \colon X \times P \to \overline{I\!R} := (-\infty, \infty]$ is an extended-real-valued and lower semicontinuous function with the symbol ∂_x indicating the set of its *partial limiting subgradients* with respect to the decision variable; see Section 2 for more details.

It has been realized over the years that model (1.1) and its various specifications (known, in particular, as variational and quasi-variational inequalities, generalized equations, and variational conditions) provide convenient frameworks for the study and applications of many important issues of nonlinear analysis, partial differential equations, optimization, equilibria, control theory, numerical algorithms, etc.; see, e.g., [6, 7, 8, 9, 15, 19, 24, 26, 27] and the bibliographies therein.

The vast majority of research on parametric variational systems revolves around establishing certain stability properties of the *solution map*

$$S(v, p) := \left\{ x \in X \,\middle|\, v \in f(x, p) + \partial_x g(x, p) \right\}, \quad (v, p) \in X \times P, \tag{1.2}$$

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to PVS (1.1) with respect to perturbations of the reference parameter pair $(\bar{v}, \bar{p})$. Starting with the pioneering papers by Stampacchia [28] for variational inequalities (motivated by applications to partial differential equations) and by Robinson [24] for generalized equations (motivated by applications to optimization), many publications in this direction have been devoted to deriving efficient conditions ensuring the *single-valuedness* and *continuity* or *Lipschitz continuity* of solution maps (1.2) to important specifications of PVS with their further applications to various fields of mathematics. It is worth mentioning that the main tools of analysis in the aforementioned developments were related to the usage of fundamental results from implicit function and topological degree theories; see, e.g., the books [6, 7] and their references.

Another approach to Lipschitzian stability of solution maps to (1.1) and more general types of parameter-dependent generalized equations was initiated by the first author [14] who employed the machinery of nonsmooth variational analysis based on his *coderivative characterization* of the *Lipschitz-like* property of multifunctions together with well-developed coderivative calculus; see [15, 27]. However, the main results of [14] were concerned with Lipschitzian stability of *set-valued* solution maps while their *single-valuedness* was established therein only by imposing rather restrictive *monotonicity assumptions* on the initial data of generalized equations. On the other hand, it has been shown in [5] that, under some smoothness requirements on f, the Lipschitz-like property is *equivalent* to the simultaneous validity of the local single-valuedness and the classical Lipschitz continuity of solution maps to finite-dimensional variational inequalities over *polyhedral* convex sets, which correspond to (1.2) in the case when $g(x) = \delta_C(x)$ is the indicator function of a convex polyhedron $C \subset \mathbb{R}^n$. Furthermore, a certain "critical face" characterization of these equivalent properties of solution maps to such variational inequalities was established in [5] by using the aforementioned coderivative criterion via a suitable linearization procedure.

Our major goal here is to employ advanced tools of first-order and second-order variational analysis and generalized differentiation to deriving verifiable *characterizations* of new notions of quantitative stability for PVS (1.1). These stability notions *imply* (being properly stronger than) the local *single-valuedness* and *Lipschitz* or *Hölder continuity* of the solution maps (1.2) without a priori monotonicity assumptions imposed on the initial data of PVS. When $f(x, p) = 0$ in (1.1), the obtained characterizations allow us to conclude that the new stability notions for PVS are equivalent to Lipschitzian (resp. Hölderian) full stability for local minimizers of g introduced by Levy, Poliquin and Rockafellar [12] (resp. by Mordukhovich and Nghia [16]). Based on this, we also label the new stability notions for PVS as "full Lipschitzian and Hölderian stability" of the corresponding solution maps, observing that these concepts are different from the standard local single-valuedness and Lipschitz or Hölder continuity of (1.2); see Section 4 for details. Furthermore, it occurs that in the absence of the parameter p in (1.1) both notions of Lipschitzian and Hölderian full stability for (1.1) reduce to the local strong maximal monotonicity of the inverse solution mapping S^{-1} characterized in this paper. It indicates that this kind of *local monotonicity* is behind the full quantitative stability notions.

It is worth mentioning that the full stability notions and the developed approach for parametric *variational systems* introduced below are significantly different from the corresponding notions and results for various classes of *minimization* problems investigated in [12, 16, 18, 20, 21] by using different techniques, which essentially exploit the nature of scalar minimization and specific features of the optimization problems therein. Proceeding in this way, second-order characterizations of Lipschitzian and

Hölderian full stability were obtained in [12] and [16], respectively, in the extended-real-valued framework of unconstrained minimization and then applied to characterizing Lipschitzian full stability for the following classes of constrained optimization problems with $\mathcal{C}^2$-smooth data: for classical nonlinear programs and mathematical programs with polyhedral constraints in [21], for semilinear elliptic partial differential equations in [16], for second-order cone programs in [20], and for general conic and semidefinite cone programs in [18]. Our ongoing research [17] applies the results of this paper to characterizing full stability for parametric variational inequalities in Hilbert spaces [3, 9] and for variational conditions [13, 26] in finite dimensions.

The rest of our paper is organized as follows. In Section 2 we recall some notions of variational analysis and generalized differentiation used in the paper. Section 3 is devoted to a systematic study of *local strong maximal monotonicity* of set-valued mappings in Hilbert spaces. We establish there several neighborhood and pointwise *coderivative characterizations* of this property and discuss some related results. The section is self-contained, while its results are very instrumental to proceed further with the study of quantitative full stability of general parametric variational systems. Note that the local monotonicity properties are significantly different from global ones with many applications to optimization problems [11, 22, 23, 16, 32] and thus require for their study more involved tools of analysis.

Section 4 is central in the paper. We introduce and discuss there the new notions of *Hölderian* and *Lipschitzian full stability* for general PVS of (1.1) and then derive complete *second-order characterizations* of these stabilities in both the *neighborhood form* (using generalized differential constructions in a neighborhood of the reference point) in the case of infinite-dimensional spaces and the *pointwise form* (using only the point in question) when the decision and parameter spaces are finite-dimensional. The obtained characterizations are expressed in terms of certain *positive-definiteness* conditions involving appropriate second-order subdifferential constructions.

Notation and terminology of the paper are standard in variational analysis and generalized differentiation; cf. [15, 27]. Unless otherwise stated, throughout the paper we assume that the *decision* space X is *Hilbert* being identified with its dual space X^*. As usual, the symbol $\langle \cdot, \cdot \rangle$ indicates the canonical pairing in X with the norm $\|x\| := \sqrt{\langle x, x \rangle}$ while the symbol $\overset{w}{\to}$ signifies the weak convergence in X. We denote by $I\!B$ the closed unit ball, and thus $I\!B_\eta(x) := x + \eta I\!B$ stands for the closed ball centered at x with radius $\eta > 0$. Given a set-valued mapping $F\colon X \rightrightarrows X$ from X into itself, the symbol

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \Big\{ v \in X \ \Big| \ \exists \text{ sequences } x_k \to \bar{x}, \ v_k \overset{w}{\to} v \text{ such that} \\ v_k \in F(x_k) \text{ for all } k \in I\!N := \{1, 2, \ldots\} \Big\} \tag{1.3}$$

signifies the (sequential) Painlevé-Kuratowski outer limit of $F(x)$ as $x \to \bar{x}$. Recall that the *parameter* space (P, d) is *metric*, and we denote by $I\!B_\eta(p) := \{q \in P \mid d(q, p) \leq \eta\}$ the closed ball centered at p with radius $\eta > 0$. Finally, the symbol $x \overset{f}{\to} \bar{x}$ for a function $f\colon X \to \overline{I\!R}$ indicates that $x \to \bar{x}$ with $f(x) \to f(\bar{x})$.

2. Generalized Differentiation and Preliminary Material. First we present here the generalized differential constructions for functions, sets, and set-valued mappings widely implied in the paper; see [15, 27] for more details. Given an extended-real-valued function $f\colon X \to \overline{I\!R}$ on a Hilbert space X, suppose unless otherwise stated that it is lower semicontinuous (l.s.c.) around the reference points. The (Fréchet) *reg-*

ular subdifferential of f at $\bar{x} \in \text{dom}\, f := \{x \in X | \ f(x) < \infty\}$ is

$$\widehat{\partial} f(\bar{x}) := \left\{ v \in X \,\Big|\, \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}, \qquad (2.1)$$

while the (Mordukhovich) *limiting subdifferential* and *singular subdifferential* of f at $\bar{x} \in \text{dom}\, f$ are defined respectively via the sequential outer limit (1.3) by

$$\partial f(\bar{x}) := \underset{x \xrightarrow{f} \bar{x}}{\text{Lim}\sup}\ \widehat{\partial} f(x) \ \text{ and } \ \partial^{\infty} f(\bar{x}) := \underset{x \xrightarrow{f} \bar{x}, \lambda \downarrow 0}{\text{Lim}\sup}\ \lambda \widehat{\partial} f(x). \qquad (2.2)$$

If f is convex, both regular and limiting subdifferentials reduce to the subdifferential of convex analysis. Furthermore, $\partial^{\infty} f(\bar{x}) = \{0\}$ if f is locally Lipschitzian around $\bar{x}$.

Given a nonempty set $\Omega \subset X$ locally closed around $\bar{x} \in \Omega$, the regular and limiting *normal cones* to Ω at $\bar{x} \in \Omega$ are defined, respectively, by

$$\widehat{N}_{\Omega}(\bar{x}) := \widehat{\partial}\delta_{\Omega}(\bar{x}) \ \text{ and } \ N_{\Omega}(\bar{x}) := \partial\delta_{\Omega}(\bar{x}) \qquad (2.3)$$

via the corresponding subdifferential constructions (2.1) and (2.2) applied to the indicator function $\delta_{\Omega}(x)$ of Ω equal to 0 for $x \in \Omega$ and to ∞ otherwise.

Let $F : X \rightrightarrows Y$ be a set-valued mapping between Hilbert spaces with the domain $\text{dom}\, F := \{x \in X | \ F(x) \neq \emptyset\}$ and the graph $\text{gph}\, F := \{(x, y) \in X \times Y | \ y \in F(x)\}$. Assume that $\text{gph}\, F$ is locally closed around $(\bar{x}, \bar{y}) \in \text{gph}\, F$ and define the *regular coderivative* and the *limiting coderivative* of F at $(\bar{x}, \bar{y})$ by using the corresponding normal cone (2.3) to the graph of F by, respectively,

$$\widehat{D}^* F(\bar{x}, \bar{y})(w) := \left\{ z \in X \,\middle|\, (z, -w) \in \widehat{N}_{\text{gph}F}(\bar{x}, \bar{y}) \right\}, \quad w \in Y, \qquad (2.4)$$

$$D^* F(\bar{x}, \bar{y})(w) := \left\{ z \in X \,\middle|\, (z, -w) \in N_{\text{gph}F}(\bar{x}, \bar{y}) \right\}, \quad w \in Y. \qquad (2.5)$$

When F is single-valued around $\bar{x}$, we skip $\bar{y} = F(\bar{x})$ from the coderivative notation. It has been strongly recognized that the coderivatives (2.4) and (2.5) are appropriate tools for the study and characterizations of well-posedness and sensitivity in nonlinear and variational analysis; see [15, Chapter 4] for more details and references. Recall to this end the *Lipschitz-like* (pseudo-Lipschitz, Aubin) property of F around $(\bar{x}, \bar{y}) \in \text{gph}\, F$ defined in the case when X is a metric space with metric d while Y is a normed space as follows: there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ as well as a constant $\ell > 0$ such that

$$F(x) \cap V \subset F(u) + \ell d(x, u) I\!B \ \text{ for all } \ x, u \in U. \qquad (2.6)$$

When both X and Y are finite-dimensional, the Lipschitz-like property of F admits a *pointwise characterization* known as the *coderivative/Mordukhovich criterion*

$$D^* F(\bar{x}, \bar{y})(0) = \{0\} \qquad (2.7)$$

presented in [15, Theorem 4.10] and [27, Theorem 9.40]; see also the references therein.

The next definition of single-valued localization plays an important role in this paper. Note that our definition is slightly different from the one in [6, p. 4], where it is not required that the single-valued localization $\widehat{T}$ has the full domain in U. The reason why the condition $\text{dom}\,\widehat{T} = U$ is needed is that most of the time we consider single-valued and *continuous* localizations.

DEFINITION 2.1. **(single-valued localizations).** *Let* $T : X \rightrightarrows Y$ *be a set-valued mapping between metric spaces, and let* $(\bar{x}, \bar{y}) \in \text{gph}\, T$. *We say that* T *admits a*

SINGLE-VALUED LOCALIZATION *around* $(\bar{x}, \bar{y})$ *if there is a neighborhood* $U \times V \subset X \times Y$ *of* $(\bar{x}, \bar{y})$ *such that the mapping* $\widehat{T} : U \to V$ *defined by* $\mathrm{gph}\,\widehat{T} := \mathrm{gph}\,T \cap (U \times V)$ *is single-valued on* U *with* $\mathrm{dom}\,\widehat{T} = U$. *In this case we say that* $\widehat{T}$ *is a single-valued localization of* T *relative to* $U \times V$. *If in addition* $\widehat{T}$ *is (Lipschitz) continuous on* U, *then we say that* T *admits a* (LIPSCHITZ) CONTINUOUS SINGLE-VALUED LOCALIZATION *around* $(\bar{x}, \bar{y})$, *i.e.,* $\widehat{T}$ *is a (Lipschitz) continuous single-valued localization of* T *relative to* $U \times V$.

Note that in the above definitions we can equivalently replace the product neighborhood $U \times V$ by any open set W around $(\bar{x}, \bar{y})$. Indeed, define

$$\mathrm{Pr}_X(W) := \left\{ x \in X \,\middle|\, \exists\, y \in Y \ \text{with}\ (x, y) \in W \right\}$$

and observe that this set is a neighborhood of $\bar{x}$. Denoting now $\widehat{T} \colon \mathrm{Pr}_X(W) \rightrightarrows Y$ by

$$\widehat{T}(x) := \left\{ y \in Y \,\middle|\, y \in T(x),\ (x, y) \in W \right\},$$

we get that $\mathrm{gph}\,\widehat{T} = \mathrm{gph}\,T \cap W$, i.e., $\widehat{T}$ is a localization of T relative to W.

Recall finally that $T \colon X \rightrightarrows Y$ is *strongly metrically regular* around $(\bar{x}, \bar{y})$ with modulus $\kappa > 0$ if its inverse $T^{-1} \colon Y \rightrightarrows X$ admits a Lipschitz continuous single-valued localization around $(\bar{y}, \bar{x})$ with constant $\kappa > 0$; see, e.g., [6, 7] for further details.

3. Characterizations of Local Strong Maximal Monotonicity. In this self-contained section we study some local monotonicity properties of set-valued operators in Hilbert spaces and obtain complete *coderivative characterizations* of *local strong maximal monotonicity*, which is actually behind the quantitative full stability notions for PVS (1.1) studied below.

Given a Hilbert space X, an operator $T : X \rightrightarrows X$ is (globally) *monotone* if

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \quad \text{whenever} \quad (u_1, v_1), (u_2, v_2) \in \mathrm{gph}\,T.$$

The monotone operator T is *maximally monotone* if $\mathrm{gph}\,T = \mathrm{gph}\,S$ for any monotone operator $S \colon X \rightrightarrows X$ satisfying the inclusion $\mathrm{gph}\,T \subset \mathrm{gph}\,S$. The next definition presents several types of *local* monotonicity (cf. [22, 23]) considered in this section.

DEFINITION 3.1. **(local monotonicity)** *Let* $T \colon X \rightrightarrows X$ *and let* $(\bar{x}, \bar{v}) \in \mathrm{gph}\,T$. *We say that:*

• T *is* LOCALLY MONOTONE *around* $(\bar{x}, \bar{v})$ *if there is a neighborhood* $U \times V$ *of* $(\bar{x}, \bar{v})$ *such that*

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \quad \textit{for all} \quad (u_1, v_1), (u_2, v_2) \in \mathrm{gph}\,T \cap (U \times V). \qquad (3.1)$$

T *is* LOCALLY MAXIMALLY MONOTONE *around* $(\bar{x}, \bar{v})$ *if there is a neighborhood* $U \times V$ *of* $(\bar{x}, \bar{v})$ *such that* (3.1) *holds and that* $\mathrm{gph}\,T \cap (U \times V) = \mathrm{gph}\,S \cap (U \times V)$ *for any monotone operator* $S : X \rightrightarrows X$ *satisfying the inclusion* $\mathrm{gph}\,T \cap (U \times V) \subset \mathrm{gph}\,S$.

• T *is* (locally) HYPOMONOTONE *around* $(\bar{x}, \bar{v})$ *if there exist a neighborhood* $U \times V$ *of this point and some positive number* r *such that*

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq -r \|u_1 - u_2\|^2 \quad \textit{for all} \quad (u_1, v_1), (u_2, v_2) \in \mathrm{gph}\,T \cap (U \times V). \ (3.2)$$

• T *is* LOCALLY STRONGLY MONOTONE *around* $(\bar{x}, \bar{v})$ *with modulus* $\kappa > 0$ *if there exists a neighborhood* $U \times V$ *of* $(\bar{x}, \bar{v})$ *such that*

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq \kappa \|u_1 - u_2\|^2 \quad \textit{for all} \quad (u_1, v_1), (u_2, v_2) \in \mathrm{gph}\,T \cap (U \times V). \ (3.3)$$

Finally, T is LOCALLY STRONGLY MAXIMALLY MONOTONE *around* $(\bar{x}, \bar{v})$ *with modulus* $\kappa > 0$ *if there exists a neighborhood* $U \times V$ *such that* (3.3) *holds and that* $\mathrm{gph}\, T \cap (U \times V) = \mathrm{gph}\, S \cap (U \times V)$ *for any monotone* $S : X \rightrightarrows X$ *satisfying the inclusion* $\mathrm{gph}\, T \cap (U \times V) \subset \mathrm{gph}\, S$.

First we briefly discuss local hypomonotonicity. It is shown in [11, 22, 23, 27] that this class of operators is rather broad including locally monotone operators and Lipschitzian mappings, limiting subgradient mappings for continuously prox-regular functions considered in Section 4, etc. The next proposition presents two useful relationships involving hypomonotonicity and localization. The first part of this proposition is taken from [11, Theorem 1.2], while the second part can be deduced directly from the definitions; so we omit details.

PROPOSITION 3.2. **(hypomonotonicity and single-valued localization).** *Let $T_1, T_2 : X \rightrightarrows X$ be set-valued mappings with $(\bar{x}, \bar{v}_1) \in \mathrm{gph}\, T_1$ and $(\bar{x}, \bar{v}_2) \in \mathrm{gph}\, T_2$. The following assertions hold:*
(i) *If T_1 admits a Lipschitz continuous single-valued localization around $(\bar{x}, \bar{v}_1)$, then T_1 is hypomonotone around this point.*
(ii) *If both T_1 and T_2 are hypomonotone around $(\bar{x}, \bar{v}_1) \in \mathrm{gph}\, T_1$ and $(\bar{x}, \bar{v}_2) \in \mathrm{gph}\, T_2$, respectively, then $T := T_1 + T_2$ is also hypomonotone around $(\bar{x}, \bar{v})$ with $\bar{v} := \bar{v}_1 + \bar{v}_2$ provided that T_1 has a continuous single-valued localization around $(\bar{x}, \bar{v}_1)$.*

The next result crucial in what follows describes local strong maximal monotonicity of operators in Hilbert spaces via single-valued localizations of their inverses. The global version of the equivalence between **(i)** and **(ii)** in this lemma can be tracked from [27, Proposition 12.54]. We are not familiar with any correspondence of this type for the local case in literature. The new characterization of local strong maximal monotone given in **(iii)** partially motivates our further developments in Section 4 for stabilities of variational systems.

LEMMA 3.3. **(local strong maximal monotonicity via single-valued localizations).** *Let $T : X \rightrightarrows X$ be a set-valued mapping with $(\bar{x}, \bar{v}) \in \mathrm{gph}\, T$. Then the following assertions are equivalent:*
(i) *T is locally strongly maximally monotone around $(\bar{x}, \bar{v})$ with modulus $\kappa > 0$.*
(ii) *T is locally strongly monotone around $(\bar{x}, \bar{v})$ with modulus κ and the inverse mapping T^{-1} admits a Lipschitz continuous single-valued localization around $(\bar{v}, \bar{x})$.*
(iii) *The mapping T^{-1} admits a single-valued localization ϑ relative to a neighborhood $V \times U$ of $(\bar{v}, \bar{x})$ such that for all $v_1, v_2 \in V$ we have the estimate*

$$\big\| (v_1 - v_2) - 2\kappa \big[\vartheta(v_1) - \vartheta(v_2) \big] \big\| \leq \| v_1 - v_2 \|. \qquad (3.4)$$

Consequently, if T is locally strongly maximally monotone around $(\bar{x}, \bar{v})$, then T is strongly metrically regular around $(\bar{x}, \bar{v})$ with modulus κ^{-1}.

Proof. To verify [**(i)**$\Longrightarrow$**(ii)**], suppose that T is locally strongly maximally monotone around $(\bar{x}, \bar{v})$ and then find a neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ such that (3.3) is satisfied and that the equality $\mathrm{gph}\, T \cap (U \times V) = \mathrm{gph}\, S \cap (U \times V)$ holds for any monotone operator $S : X \rightrightarrows X$ satisfying $\mathrm{gph}\, T \cap (U \times V) \subset \mathrm{gph}\, S$. Denote $W := J_\kappa(U \times V)$ with $J_\kappa(u, v) := (u, v - \kappa u)$ for $(u, v) \in X \times X$ and note from (3.3) that the set-valued mapping $F : X \rightrightarrows X$ defined by $\mathrm{gph}\, F := \mathrm{gph}\,(T - \kappa I) \cap W$ is monotone. Indeed, for any $(u_i, v_i) \in \mathrm{gph}\, F$, $i = 1, 2$ we have $(u_i, v_i + \kappa u_i) \in \mathrm{gph}\, T \cap J_\kappa^{-1}(W) = \mathrm{gph}\, T \cap (U \times V)$. It follows from (3.3) that

$$\langle v_1 + \kappa u_1 - v_2 - \kappa u_2, u_1 - u_2 \rangle \geq \kappa \| u_1 - u_2 \|^2,$$

which implies that $\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0$ and thus justifies the monotonicity of F. Accordingly, there exists a maximal monotone operator $R : X \rightrightarrows X$ extending F via

Zorn's lemma (see, e.g., [1, Theorem 20.21]), which means that $\operatorname{gph} F \subset \operatorname{gph} R$ and that R is maximal monotone. It yields

$$\operatorname{gph}(F + \kappa I) \cap (U \times V) = \operatorname{gph} T \cap (U \times V) \subset \operatorname{gph}(R + \kappa I).$$

The local maximality of T relative to $U \times V$ and the monotonicity of $R + \kappa I$ ensure the representation $\operatorname{gph} T \cap (U \times V) = \operatorname{gph}(R + \kappa I) \cap (U \times V)$, and thus we have

$$\operatorname{gph} T^{-1} \cap (V \times U) = \operatorname{gph}(R + \kappa I)^{-1} \cap (V \times U). \tag{3.5}$$

Applying the classical Minty theorem tells us that $\operatorname{dom}(R + \kappa I)^{-1} = X$ and that the mapping $(R + \kappa I)^{-1}$ is single-valued and Lipschitz continuous on X. This ensures

$$V_1 := (R + \kappa I)(U) \cap V = \left[(R + \kappa I)^{-1} \right]^{-1} (U) \cap V$$

is a neighborhood of $\bar{v}$ by noting from (3.5) that $(R + \kappa I)^{-1}(\bar{v}) = \bar{x}$. Moreover, it follows from (3.5) that $T^{-1}(v) = (R + \kappa I)^{-1}(v) \in U$ for all $v \in V_1$. Thus the localization $S : V_1 \to U$ with $\operatorname{gph} S = \operatorname{gph} T^{-1} \cap (V_1 \times U)$ and $\operatorname{dom} S = V_1$ is single-valued and Lipschitz continuous in V_1. This justifies the implication [(i)$\Longrightarrow$(ii)].

To prove [(ii)$\Longrightarrow$(iii)], suppose that T is locally strongly monotone around $(\bar{x}, \bar{v})$ and that T^{-1} admits a Lipschitz continuous single-valued localization ϑ relative to some neighborhood $V \times U$ of $(\bar{v}, \bar{x})$. By shrinking U, V if possible, we get that condition (3.3) is also valid on this neighborhood $U \times V$. For any $(v_1, u_1), (v_2, u_2) \in \operatorname{gph} \vartheta$, observe from (3.3) that

$$\begin{aligned} \|v_1 - v_2 - 2\kappa(u_1 - u_2)\|^2 &= \|v_1 - v_2\|^2 - 4\kappa \big[\langle v_1 - v_2, u_1 - u_2 \rangle - \kappa \|u_1 - u_2\|^2 \big] \\ &\leq \|v_1 - v_2\|^2, \end{aligned}$$

which therefore justifies assertion (iii).

It remains to verify [(iii)$\Longrightarrow$(i)]. Pick any $(u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap (U \times V)$, where $V \times U$ is the neighborhood of $(\bar{v}, \bar{x})$ on which T^{-1} admits a single-valued localization ϑ satisfying (3.4). Hence

$$0 \leq \|v_1 - v_2\|^2 - \|v_1 - v_2 - 2\kappa(u_1 - u_2)\|^2 = 4\kappa \big[\langle v_1 - v_2, u_1 - u_2 \rangle - \kappa \|u_1 - u_2\|^2 \big].$$

This clearly gives us the estimates

$$\|v_1 - v_2\| \cdot \|u_1 - u_2\| \geq \langle v_1 - v_2, u_1 - u_2 \rangle \geq \kappa \|u_1 - u_2\|^2, \tag{3.6}$$

which verify (3.3) and show that ϑ is Lipschitz continuous on V with constant κ^{-1}. Then we deduce from [16, Lemma 2.1] that ϑ is maximal monotone relative to $V \times U$. It justifies (i) and completes the proof of the equivalencies. The final consequence follows from (ii) by the definition of strong metric regularity. $\triangle$

Next we establish the main result in this section, which provides a characterization of local strong maximal monotonicity via the regular coderivative (2.4) for set-valued mappings in Hilbert spaces. The first result in this direction turns back to [23, Theorem 2.1], where Poliquin and Rockafellar obtained a necessary condition for the global maximal monotonicity in terms of the limiting coderivative (2.5) in finite dimensions. More recently [4], Chieu and Trang also established necessary and sufficient coderivative conditions for global monotonicity and strong monotonicity for single-valued and continuous mappings.

THEOREM 3.4. (**regular coderivative characterization of local strong maximal monotonicity of set-valued mappings**). *Let $T : X \rightrightarrows X$ be a set-valued mapping with closed graph around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. The following are equivalent:*

(i) T *is locally strongly maximally monotone around* $(\bar{x}, \bar{v})$ *with modulus* $\kappa > 0$.
(ii) T *is hypomonotone around* $(\bar{x}, \bar{v})$ *and there exists* $\eta > 0$ *such that*

$$\langle z, w \rangle \geq \kappa \|w\|^2 \quad \text{for all} \quad z \in \widehat{D}^* T(u, v)(w), \ (u, v) \in \operatorname{gph} T \cap I\!\!B_\eta(\bar{x}, \bar{v}). \qquad (3.7)$$

Consequently, the conditions in **(ii)** *ensure the strong metric regularity of* T *around* $(\bar{x}, \bar{v})$.

Proof. We start with justifying $[\textbf{(i)} \Longrightarrow \textbf{(ii)}]$. It is obvious that T is hypomonotone around $(\bar{x}, \bar{v})$ when **(i)** is satisfied. By Lemma 3.3 there is a single-valued localization ϑ of T^{-1} relative to some neighborhood $V \times U$ of $(\bar{v}, \bar{x})$ such that inequality (3.4) holds. Observe from (3.6), which is a consequence of (3.4) by the proof of Lemma 3.3, that ϑ is Lipschitz continuous on V with modulus κ^{-1}. Pick any $(u, v) \in \operatorname{gph} T \cap (U \times V)$ and $(w, z) \in X \times X$ with $z \in \widehat{D}^* T(u, v)(w)$. With any $\varepsilon > 0$ we find, by definition (2.4), some number $\delta > 0$ such that $I\!\!B_\delta(u, v) \subset U \times V$ and that

$$\varepsilon(\|x - u\| + \|y - v\|) \geq \langle z, x - u \rangle - \langle w, y - v \rangle \quad \text{for all} \quad (x, y) \in \operatorname{gph} T \cap I\!\!B_\delta(u, v). \quad (3.8)$$

When $t > 0$ is sufficiently small, define $u_t := \vartheta(v_t)$ with $v_t := v + t(z - 2\kappa w) \in V$ and get from the local Lipschitz continuity of ϑ that $(u_t, v_t) \to (u, v)$ as $t \downarrow 0$. Without loss of generality, suppose that $(u_t, v_t) \in I\!\!B_\delta(u, v)$ for all $t > 0$. Replacing (x, y) in (3.8) by (u_t, v_t) and using (3.4) yield

$$
\begin{aligned}
\varepsilon\big(\|u_t - u\| + \|v_t - v\|\big) & \geq \langle z, u_t - u \rangle - \langle w, v_t - v \rangle \\
& = \langle t^{-1}(v_t - v) + 2\kappa w, u_t - u \rangle - t \langle w, z - 2\kappa w \rangle \\
& = t^{-1} \langle v_t - v, u_t - u \rangle + 2\kappa \langle w, u_t - u \rangle - t \langle w, z \rangle + 2t\kappa \|w\|^2 \\
& \geq \kappa t^{-1} \|u_t - u\|^2 + 2\kappa \langle w, u_t - u \rangle - t \langle w, z \rangle + 2t\kappa \|w\|^2 \\
& \geq \kappa t^{-1} \|u_t - u\|^2 - 2\kappa \|w\| \|u_t - u\| + t\kappa \|w\|^2 - t \langle w, z \rangle + t\kappa \|w\|^2 \\
& \geq -t \langle z, w \rangle + t\kappa \|w\|^2.
\end{aligned}
$$

$$(3.9)$$

Since ϑ is Lipschitz continuous on V with modulus κ^{-1}, we have

$$
\begin{aligned}
\varepsilon\big(\|u_t - u\| + \|v_t - v\|\big) & = \varepsilon\big(\|\vartheta(v_t) - \vartheta(v)\| + \|v_t - v\|\big) \leq \varepsilon\big(\kappa^{-1} \|v_t - v\| + \|v_t - v\|\big) \\
& = \varepsilon(\kappa^{-1} + 1) \|v_t - v\| = \varepsilon(\kappa^{-1} + 1) t \|z - 2\kappa w\|,
\end{aligned}
$$

which together with (3.9) yields $\langle z, w \rangle + \varepsilon(\kappa^{-1} + 1) \|z - 2\kappa w\| \geq \kappa \|w\|^2$, and, by taking $\varepsilon \downarrow 0$, $\langle z, w \rangle \geq \kappa \|w\|^2$. This ensures (3.7) and completes the proof of $[\textbf{(i)} \Longrightarrow \textbf{(ii)}]$.

To verify the converse implication $[\textbf{(ii)} \Longrightarrow \textbf{(i)}]$, observe by Lemma 3.3 that we only need to show that T^{-1} admits a Lipschitz continuous single-valued localization ϑ around $(\bar{v}, \bar{x})$, which satisfies (3.4). It is done below in the following two claims.

Claim 1. T^{-1} *admits a Lipschitz continuous localization* ϑ *around* $(\bar{v}, \bar{x})$.

By choosing $\eta > 0$ to be sufficiently small, we may always assume that the set $\operatorname{gph} T \cap I\!\!B_\eta(\bar{x}, \bar{v})$ is closed and there is a number $r > 0$ such that

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -r \|x_1 - x_2\|^2 \quad \text{for all} \quad (x_1, v_1), (x_2, v_2) \in \operatorname{gph} T \cap I\!\!B_\eta(\bar{x}, \bar{v}). \quad (3.10)$$

Pick any $s > r$ and define $J_s(u, v) := (v + su, u)$ for $(u, v) \in X \times X$. Denote further $W_s := J_s(I\!\!B_\eta(\bar{x}, \bar{v}))$ and observe that $\operatorname{int} W_s = J_s(\operatorname{int} I\!\!B_\eta(\bar{x}, \bar{v}))$ is a neighborhood of $(\bar{v} + s\bar{x}, \bar{x})$. It follows from (3.10) that for all $(v_i, x_i) \in \operatorname{gph}(T + sI)^{-1} \cap W_s := \operatorname{gph} F$, $i = 1, 2$ we have $(x_i, v_i - sx_i) \in \operatorname{gph} T \cap J_s^{-1}(W_s) = \operatorname{gph} T \cap I\!\!B_\eta(\bar{x}, \bar{v})$. It allows us to deduce from (3.10) that

$$\|v_1 - v_2\| \cdot \|x_1 - x_2\| \geq \langle v_1 - v_2, x_1 - x_2 \rangle \geq (s - r) \|x_1 - x_2\|^2. \qquad (3.11)$$

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This tells us that the localization F of $(T + sI)^{-1}$ is single-valued and Lipschitz continuous on its domain with constant $(s-r)^{-1}$. Taking any $(v, u) \in \operatorname{gph} F \cap (\operatorname{int} W_s)$ and $(w, z) \in X \times X$ with $w \in \widehat{D}^* F(v, u)(z)$, we get that $w \in \widehat{D}^*(T + sI)^{-1}(v, u)(z)$ and $-z \in \widehat{D}^*(T + sI)(u, v)(-w)$. It follows from the coderivative sum rule [15, Theorem 1.62] that $-z + sw \in \widehat{D}^* T(u, v - su)(-w)$. Since $(u, v - su) = J_s^{-1}(u, v) \in J_s^{-1}(\operatorname{int} W_s) = \operatorname{int} I\!B_\eta(\bar{x}, \bar{v})$, we deduce from (3.7) that $\langle -z + sw, -w \rangle \geq \kappa \|w\|^2$ and

$$\|z\| \cdot \|w\| \geq \langle z, w \rangle \geq (\kappa + s)\|w\|^2. \tag{3.12}$$

To proceed further, for any $z \in I\!B$ define the extended-real-valued function

$$f_z(v) := \begin{cases} \langle z, F(v) \rangle & \text{if } v \in \operatorname{dom} F, \\ \infty & \text{otherwise.} \end{cases} \tag{3.13}$$

Since $\operatorname{gph} T \cap I\!B_\eta(\bar{x}, \bar{v})$ is closed in $X \times X$, $\operatorname{gph} F$ is also closed on $X \times X$. Furthermore, it is easy to check from the Lipschitz continuity of F on its domain that f_z is l.s.c. on X.

Now fix a positive number $\delta < \frac{\eta}{3}$ and pick any $(u_i, v_i) \in \operatorname{gph} T \cap I\!B_\delta(\bar{x}, \bar{v})$, $i = 1, 2$. Then we have $(y_i, u_i) \in \operatorname{gph} F$ with $y_i := v_i + su_i$. Choosing $\varepsilon \in (0, \delta)$ and applying the mean value inequality [15, Corollary 3.50] to the l.s.c. function f_z give us that

$$|f_z(y_1) - f_z(y_2)| \leq \|y_1 - y_2\| \sup\{\|w\| \mid w \in \widehat{\partial} f_z(y), \ y \in [y_1, y_2] + \varepsilon I\!B\}, \tag{3.14}$$

where $[y_1, y_2] := \{ty_1 + (1 - t)y_2 \mid t \in [0, 1]\}$. Note also that $\widehat{\partial} f_z(y) \subset \widehat{\partial} \langle z, F \rangle(y)$ if $y \in \operatorname{dom} f_z = \operatorname{dom} F$ and $\widehat{\partial} f_z(y) = \emptyset$ if $y \notin \operatorname{dom} F$. It follows from (3.14) that

$$|f_z(y_1) - f_z(y_2)| \leq \|y_1 - y_2\| \sup\{\|w\| \mid w \in \widehat{\partial} \langle z, F \rangle(y), \ y \in \operatorname{dom} F \cap ([y_1, y_2] + \varepsilon I\!B)\}. \tag{3.15}$$

For any $y \in \operatorname{dom} F \cap ([y_1, y_2] + \varepsilon I\!B)$ there are some $t \in [0, 1]$ and $y_0 \in \varepsilon I\!B$ such that $y = ty_1 + (1 - t)y_2 + y_0$. It follows from (3.11) that

$$\begin{aligned} \|y - \bar{v} - s\bar{x}\| &= \|ty_1 + (1 - t)y_2 + y_0 - \bar{v} - s\bar{x}\| \\ &= \|t(y_1 - \bar{v} - s\bar{x}) + (1 - t)(y_2 - \bar{v} - s\bar{x}) + y_0\| \\ &= \|t(v_1 + su_1 - \bar{v} - s\bar{x}) + (1 - t)(v_2 + su_2 - \bar{v} - s\bar{x}) + y_0\| \\ &\leq t(\|v_1 - \bar{v}\| + s\|u_1 - \bar{x}\|) + (1 - t)(\|v_2 - \bar{v}\| + s\|u_2 - \bar{x}\|) + \|y_0\| \\ &\leq t(\delta + s\delta) + (1 - t)(\delta + s\delta) + \varepsilon = (1 + s)\delta + \varepsilon < (2 + s)\delta. \end{aligned}$$

We get from the latter estimate and (3.11) that

$$\|F(y) - \bar{x}\| = \|F(y) - F(\bar{v} + s\bar{x})\| \leq (s-r)^{-1}\|y - \bar{v} - s\bar{x}\| \leq (s-r)^{-1}(2 + s)\delta. \tag{3.16}$$

Furthermore, it follows from the above that

$$\begin{aligned} \|y - sF(y) - \bar{v}\| &= \|y - \bar{v} - s\bar{x} - s(F(y) - \bar{x})\| \leq \|y - \bar{x} - s\bar{x}\| + s\|F(y) - \bar{v}\| \\ &\leq (2 + s)\delta + s(s-r)^{-1}(2 + s)\delta. \end{aligned}$$

By choosing δ sufficiently small, we get from (3.16) and the latter that $J_s^{-1}(y, F(y)) = (F(y), y - sF(y)) \in \operatorname{int} I\!B_\eta(\bar{x}, \bar{v})$, which yields $(y, F(y)) \in J_s(\operatorname{int} I\!B_\eta(\bar{x}, \bar{v})) = \operatorname{int} W_s$. Moreover, note that

$$\widehat{\partial} \langle z, F \rangle(y) \subset \widehat{D}^* F(y)(z) = \widehat{D}^* F(y, F(y))(z).$$

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Since $(y, F(y)) \in \mathrm{gph}\,(T + sI)^{-1} \cap \mathrm{int}\,W_s$, we have $\widehat{D}^* F(y, F(y))(z) = \widehat{D}^*(T + sI)^{-1}(y, F(y))(z)$. This together with (3.15), (3.13), and (3.12) shows that

$$|\langle z, F(y_1) - F(y_2)\rangle| = |f_z(y_1) - f_z(y_2)| \le \|y_1 - y_2\| \cdot (\kappa + s)^{-1}\|z\| \quad \text{for all} \quad z \in I\!\!B.$$

It allows us to conclude that

$$
\begin{aligned}
\|u_1 - u_2\| &= \|F(y_1) - F(y_2)\| \le (\kappa + s)^{-1}\|y_1 - y_2\| = (\kappa + s)^{-1}\|v_1 + su_1 - v_2 - su_2\| \\
&\le (\kappa + s)^{-1}(\|v_1 - v_2\| + s\|u_1 - u_2\|).
\end{aligned}
$$

Thus we arrive at the estimate

$$\kappa\|u_1 - u_2\| \le \|v_1 - v_2\| \quad \text{for all} \quad (u_1, v_1), (u_2, v_2) \in \mathrm{gph}\,T \cap I\!\!B_\delta(\bar{x}, \bar{v}). \quad (3.17)$$

It remains to check that the inverse mapping T^{-1} admits a Lipschitzian localization around $(\bar{v}, \bar{x})$. Observe to this end from (3.7) that

$$\|z\| \ge \kappa\|w\| \quad \text{for all} \quad z \in \widehat{D}^* T(u, v)(w), \ (u, v) \in \mathrm{gph}\,T \cap I\!\!B_\eta(\bar{x}, \bar{v}).$$

It follows from the regular coderivative criterion in [15, Theorem 4.7] that T^{-1} is Lipschitz-like around $(\bar{v}, \bar{x})$ with some modulus $\ell > 0$. By definition (2.6) we find $\nu > 0$ such that $I\!\!B_{\ell\nu}(\bar{x}) \times I\!\!B_\nu(\bar{v}) \subset I\!\!B_\delta(\bar{x}, \bar{v})$ and

$$\bar{x} \in T^{-1}(v) + \ell\|v - \bar{v}\| \quad \text{for all} \quad v \in I\!\!B_\nu(\bar{v}),$$

which yields $T^{-1}(v) \cap \mathrm{int}\,I\!\!B_{\ell\nu}(\bar{x}) \ne \emptyset$ as $v \in \mathrm{int}\,I\!\!B_\nu(\bar{v})$. Defining the mapping ϑ from $\mathrm{int}\,I\!\!B_\nu(\bar{v})$ to $\mathrm{int}\,I\!\!B_{\ell\nu}(\bar{x})$ by $\mathrm{gph}\,\vartheta := \mathrm{gph}\,T^{-1} \cap (\mathrm{int}\,I\!\!B_\nu(\bar{v}) \times \mathrm{int}\,I\!\!B_{\ell\nu}(\bar{x}))$ gives us $\mathrm{dom}\,\vartheta = \mathrm{int}\,I\!\!B_\nu(\bar{v})$. Moreover, it follows from (3.17) that ϑ is single-valued. Hence (3.17) shows that ϑ is locally Lipschitzian around $\bar{v}$ with constant κ^{-1}.

Claim 2. *The single-valued localization ϑ of T^{-1} defined in* Claim 1 *satisfies* (3.4).

For any $z \in I\!\!B$ we define $\xi_z(v) := \langle z, v - 2\kappa\vartheta(v)\rangle$, $v \in I\!\!B_\nu(\bar{v})$. Fix $\alpha, \beta > 0$ with $\alpha + \beta < \nu$ and $v_1, v_2 \in I\!\!B_\alpha(\bar{v})$. Similarly to (3.15) we get from the mean value inequality [15, Corollary 3.50] that

$$|\xi_z(v_1) - \xi_z(v_2)| \le \|v_1 - v_2\| \sup\{\|w\| \mid w \in \widehat{\partial}\xi_z(v), \ v \in [v_1, v_2] + \beta I\!\!B\}. \quad (3.18)$$

Since $v \in \mathrm{int}\,I\!\!B_\nu(\bar{v})$ for each $v \in [v_1, v_2] + \beta I\!\!B$ due to $\alpha + \beta < \nu$, it tells us that

$$w \in \widehat{\partial}\xi_z(v) \subset z - 2\kappa\widehat{D}^*\vartheta(v)(z) = z - 2\kappa\widehat{D}^* T^{-1}(v)(z),$$

which yields $(2\kappa)^{-1}(z - w) \in \widehat{D}^* T^{-1}(v)(z)$, i.e., $-z \in \widehat{D}^* T(\vartheta(v), v)((2\kappa)^{-1}(w - z))$. By (3.7) we get

$$\langle -z, (2\kappa)^{-1}(w - z)\rangle \ge \kappa\|(2\kappa)^{-1}(w - z)\|^2,$$

which implies in turn the relationships

$$2(\|z\|^2 - \langle z, w\rangle) \ge \|w - z\|^2 = \|w\|^2 - 2\langle w, z\rangle + \|z\|^2.$$

Hence we have $\|z\|^2 \ge \|w\|^2$, i.e., $\|z\| \ge \|w\|$. This together with (3.18) ensures that

$$|\xi_z(v_1) - \xi_z(v_2)| \le \|v_1 - v_2\| \cdot \|z\| \quad \text{for all} \quad z \in I\!\!B.$$

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Remembering the definition of ξ_z, we arrive at the estimate

$$\|v_1 - v_2 - 2\kappa[\vartheta(v_1) - \vartheta(v_2)]\| \leq \|v_1 - v_2\| \quad \text{whenever} \quad v_1, v_2 \in I\!\!B_\alpha(\bar{v}),$$

which verifies (3.4) and thus justifies Claim 2. This completes the proof of the theorem by combining the results given in Claim 1 and Claim 2. $\triangle$

Observe from the proof of [(i)$\Longrightarrow$(ii)] in Theorem 3.4 that if T is locally strongly maximally monotone around $(\bar{x}, \bar{v}) \in \mathrm{gph}\, T$, i.e., there is a single-valued localization ϑ of T^{-1} relative to some neighborhood $V \times U$ of $(\bar{v}, \bar{x})$ such that (3.4) holds, then we have the estimate

$$\langle z, w \rangle \geq \kappa \|w\|^2 \quad \text{for all} \quad z \in \widehat{D}^* T(u, v)(w),\; (u, v) \in \mathrm{gph}\, T \cap (U \times V). \quad (3.19)$$

Note also that the strong metric regularity of T in Theorem 3.4 can be characterized by using the *strict graphical derivative* in finite-dimensions; see [27, Theorem 9.54]. Our result above provides a verifiable sufficient condition for this property in terms of the regular coderivative under the hypomonotonicity assumption. However, the *main trust* of Theorem 3.4 is a characterization of the local strong maximal monotonicity property, which significantly supersedes strong metric regularity and is needed in what follows.

Next we derive from Theorem 3.4 a *pointwise* characterization of the local strong maximal monotonicity property for single-valued Lipschitzian mappings in finite-dimensions via the limiting coderivative (2.5). This gives us a natural extension of the classical result stated that a $\mathcal{C}^1$-smooth mapping $F : I\!\!R^n \to I\!\!R^n$ is locally strongly monotone around $\bar{x}$ provided that $\nabla F(\bar{x})$ is positive-definite. Note further that in the latter case the local maximality and hypomonotonicity of F are automatic due to the Lipschitz continuity of this mapping.

COROLLARY 3.5. **(limiting coderivative characterization of local strong monotonicity for Lipschitz continuous mappings).** *Let X be a finite-dimensional space and let $T : X \to X$ be a single-valued mapping Lipschitz continuous around $\bar{x} \in \mathrm{dom}\, T$. The following are equivalent:*
(i) T *is locally strongly monotone around* $(\bar{x}, T(\bar{x}))$.
(ii) $D^* T(\bar{x})$ *is positive-definite in the sense that*

$$\langle z, w \rangle > 0 \quad \text{whenever} \quad z \in D^* T(\bar{x})(w),\; w \neq 0. \quad (3.20)$$

Proof. It suffices to check that condition (3.20) is equivalent to (3.7) under the assumptions made. By passing to limit, it is easy to derive implication [(3.7)$\Longrightarrow$(3.20)]. To justify the converse implication, we argue by contradiction and suppose that (3.20) is satisfied while (3.7) is not. This gives us a sequence (u_k, z_k, w_k) such that $u_k \to \bar{x}$, $z_k \in \widehat{D}^* T(u_k)(w_k)$, and $\langle z_k, w_k \rangle < \frac{1}{k} \|w_k\|^2$, which implies that $w_k \neq 0$. Define further $\bar{w}_k := \frac{w_k}{\|w_k\|}$ and $\bar{z}_k := \frac{z_k}{\|w_k\|}$. Since T is Lipschitz continuous around $\bar{x}$ with some modulus ℓ, we have $\|\bar{z}_k\| \leq \ell \|\bar{w}_k\| = \ell$ for sufficiently large k. By passing to subsequences, assume that $\bar{w}_k \to \bar{w}$ with $\|\bar{w}\| = 1$ and $\bar{z}_k \to \bar{z}$ as $k \to \infty$. It follows from definition (2.5) of the limiting coderivative that $\bar{z} \in D^* T(\bar{x})(\bar{w})$. Furthermore, by $\langle \bar{z}_k, \bar{w}_k \rangle < \frac{1}{k}$ we get the inequality $\langle \bar{z}, \bar{w} \rangle \leq 0$, which contradicts the positive-definiteness condition (3.20) and thus completes the proof of the corollary. $\triangle$

As a direct consequence of Corollary 3.5, observe that condition (3.20) is sufficient for the strong metric regularity of T around $(\bar{x}, T(\bar{x}))$. It has been proved by Kummer [10] that the latter property can be characterized by using Thibault's strict derivative

[29]. To this end we emphasize again that our positive-definiteness coderivative criterion (3.20) characterizes essentially more specific property of local strong maximal monotonicity of our main interest here.

Finally in this section, we formulate a conjecture for which the affirmative answer is achieved presented in Corollary 3.5 as well as in Corollary 4.9 below in the important set-valued case including subgradient mappings generated by a major and fairly broad class of extended-real-valued functions.

CONJECTURE 3.6. (**limiting coderivative characterization of local strong maximal monotonicity for set-valued mappings**). *Let X be a finite-dimensional space and let $T : X \rightrightarrows X$ be a set-valued mapping with closed graph around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. The following are equivalent:*
(i) *T is locally strongly maximally monotone around $(\bar{x}, \bar{v})$.*
(ii) *T is hypomonotone around $(\bar{x}, \bar{v})$ and $D^*T(\bar{x}, \bar{v})$ is positive-definite in the sense:*

$$\langle z, w \rangle > 0 \quad whenever \quad z \in D^*T(\bar{x}, \bar{v})(w), \ w \neq 0. \tag{3.21}$$

4. Characterizations of Full Stability in Variational Systems. In this section we turn to the main subject of the paper concerning full quantitative stability of the parametric variational systems (PVS) given by

$$v \in f(x, p) + \partial_x g(x, p) \ \text{ for } \ x \in X, \ p \in P \tag{4.1}$$

with the Hilbert decision space X and the metric parameter space (P, d), where $f : X \times P \to X$, $g : X \times P \to \overline{I\!R}$, and $\partial_x g$ stands for the partial limiting subdifferential of the function g with respect to the variable x. Denote $g_p(\cdot) := g(\cdot, p)$ and observe that $\partial_x g(x, p) = \partial g_p(x)$ for all $(x, p) \in X \times P$.

Fix $\bar{v} \in f(\bar{x}, \bar{p}) + \partial_x g(\bar{x}, \bar{p})$ and consider the solution map $S : X \times P \rightrightarrows X$ to (4.1)

$$S(v, p) := \big\{ x \in X \,\big|\, v \in f(x, p) + \partial_x g(x, p) \big\} \ \text{ with } \ \bar{x} \in S(\bar{v}, \bar{p}). \tag{4.2}$$

The underlying goal of this section is to introduce and efficiently characterize the following new notions of Hölderian and Lipschitzian full stability for PVS (4.1).

DEFINITION 4.1. (**Hölderian and Lipschitzian full stability of parametric variational systems**). *Given $\bar{x} \in S(\bar{v}, \bar{p})$ from (4.2), we say that:*
(i) *$\bar{x}$ is a HÖLDERIAN FULLY STABLE solution to PVS (4.1) corresponding to the pair $(\bar{v}, \bar{p})$ if the solution map (4.2) admits a single-valued localization ϑ relative to some neighborhood $V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ such that for any $(v_1, p_1), (v_2, p_2) \in V \times Q$ we have*

$$\big\| (v_1 - v_2) - 2\kappa[\vartheta(v_1, p_1) - \vartheta(v_2, p_2)] \big\| \leq \|v_1 - v_2\| + \ell d(p_1, p_2)^{\frac{1}{2}} \tag{4.3}$$

with some positive constants κ and ℓ.
(ii) *$\bar{x}$ is a LIPSCHITZIAN FULLY STABLE solution to PVS (4.1) corresponding to the pair $(\bar{v}, \bar{p})$ if the solution map (4.2) admits a single-valued localization ϑ relative to some neighborhood $V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ such that for any $(v_1, p_1), (v_2, p_2) \in V \times Q$ we have*

$$\big\| (v_1 - v_2) - 2\kappa[\vartheta(v_1, p_1) - \vartheta(v_2, p_2)] \big\| \leq \|v_1 - v_2\| + \ell d(p_1, p_2) \tag{4.4}$$

with some positive constants κ and ℓ.

It is easy to see that the above stability notions imply the *local single-valuedness* and *Lipschitz* (resp. *Hölder*) *continuity* of the solution map S, which are conventional

definitions of quantitative stability of perturbed variational systems discussed, e.g., in [13, 25, 26, 30] for the Lipschitz case and in [31] for the Hölder one. However, full stability from Definition 4.1 is *much stronger* even in very simple settings. Consider, e.g., $g = 0$ and $f\colon I\!\!R^2 \to I\!\!R^2$ in (4.1) given by $f(x) := (x_1, -x_2)$. It is obvious that f^{-1} is single-valued and Lipschitz continuous around $(0,0)$. However, the Lipschitzian full stability property (4.4), which is equivalent to the local strong monotonicity of f around $((0,0),(0,0)) \in I\!\!R^2 \times I\!\!R^2$ by Lemma 3.3 fails.

More generally, we can observe to this end that when the basic parameter p is omitted in (4.1), both stability conditions (4.3) and (4.4) reduce to the one in (3.4), which is equivalent to the *local strong maximal monotonicity* of the mapping $T := f + \partial g$ by Lemma 3.3. We will see in what follows that the methods developed in Section 3 to characterize this monotonicity concept play a crucial role in deriving efficient second-order criteria for the full stability notions from Definition 4.1.

Furthermore, it will be shown below that the notions of full stability for PVS (4.1) are *equivalent* in the case of $f = 0$ to the following full stability definitions for *local minimizers* associated with g, which are initiated in [12] for the Lipschitz version and extended in [16] to the Hölder one.

DEFINITION 4.2. (**Lipschitzian and Hölderian fully stable local minimizers**).
(i) *We say that $\bar{x} \in X$ is a* LIPSCHITZIAN FULLY STABLE LOCAL MINIMIZER *associated with* $g\colon X \times P \to \overline{I\!\!R}$ *relative to* $\bar{p} \in P$ *with* $(\bar{x}, \bar{p}) \in \text{dom } g$ *and some "tilt" parameter $\bar{v} \in X$ if there exist positive numbers κ, ℓ, γ and a neighborhood $V \times Q$ of $(\bar{v}, \bar{p})$ such that the argminimum mapping*

$$(v, p) \mapsto M_\gamma(v, p) := \text{argmin}\left\{g(x, p) - \langle v, x \rangle \big| \ x \in I\!\!B_\gamma(\bar{x})\right\}$$

is single-valued on $V \times Q$ with $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$, satisfies the Lipschitz condition: for all $v_1, v_2 \in V,\ p_1, p_2 \in Q$,

$$\|M_\gamma(v_1, p_1) - M_\gamma(v_2, p_2)\| \leq \kappa\|v_1 - v_2\| + \ell d(p_1, p_2), \tag{4.5}$$

and in addition the value function

$$(v, p) \mapsto m_\gamma(v, p) := \inf\left\{g(x, p) - \langle v, x \rangle \big| \ x \in I\!\!B_\gamma(\bar{x})\right\} \tag{4.6}$$

is also locally Lipschitz continuous around $(\bar{v}, \bar{p})$.
(ii) *The point $\bar{x}$ is a* HÖLDERIAN FULLY STABLE LOCAL MINIMIZER *associated with g relative to $\bar{p}$ and $\bar{v}$ if there exist positive numbers κ, ℓ, γ such that the argminimum mapping M_γ is single-valued on some neighborhood $V \times Q$ of $(\bar{v}, \bar{p})$ with $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$, satisfies the condition: for all $v_1, v_2 \in V,\ p_1, p_2 \in Q$,*

$$\|M_\gamma(v_1, p_1) - M_\gamma(v_2, p_2)\| \leq \kappa\|v_1 - v_2\| + \ell d(p_1, p_2)^{\frac{1}{2}}, \tag{4.7}$$

and the function m_γ in (4.6) is locally Lipschitzian around $(\bar{v}, \bar{p})$.

We can see that definitions (4.5) and (4.7) of full stability for local minimizers are formulated differently in comparison with our new Definition 4.1 of full stability for parametric variational systems; the former ones essentially exploit specific features of scalar optimization. The equivalence between these types of full stability in the optimization framework is a nontrivial fact that follows from the criteria of full stability for PVS obtained below and those established earlier for local minimizers. Therefore the conditions formulated in Definition 4.1 can be treated as new full stability characterizations for local minimizers in problems of parametric optimization.

To proceed with the formulation and proof of our main result in this section, we first specify the class of functions g from (4.1) used in our analysis. In fact, it is

the major and fairly large collection of extended-real-valued functions employed in second-order variational analysis and parametric optimization; see, e.g., [2, 12, 23, 27] and the references therein. Given $g : X \times P \to \overline{\mathbb{R}}$ finite at $(\bar{x}, \bar{p})$ with $\widehat{v} := \bar{v} - f(\bar{x}, \bar{p}) \in \partial_x g(\bar{x}, \bar{p})$, we say by following [12] that g is *prox-regular* in x at $\bar{x}$ for $\widehat{v}$ with *compatible parameterization* by p at $\bar{p}$ if there are neighborhoods U of $\bar{x}$, V of $\widehat{v}$, and Q of $\bar{p}$ along with numbers $\varepsilon > 0$ and $r > 0$ such that

$$
\begin{aligned}
&g(x, p) \geq g(u, p) + \langle v, x - u \rangle - \tfrac{r}{2}\|x - u\|^2 \quad \text{for all} \quad x \in U, \\
&\text{when} \quad v \in \partial_x g(u, p) \cap V, \ u \in U, \ p \in Q, \ \text{and} \ g(u, p) \leq g(\bar{x}, \bar{p}) + \varepsilon.
\end{aligned}
\tag{4.8}
$$

Further, g is *subdifferentially continuous* in x at $\bar{x}$ for $\widehat{v}$ with compatible parameterization by p at $\bar{p}$ if the mapping $(x, p, v) \mapsto f(x, p)$ is continuous relative to the subdifferential graph gph $\partial_x g$ at $(\bar{x}, \bar{p}, \widehat{v})$. For simplicity we call g to be *parametrically continuously prox-regular* at $(\bar{x}, \bar{p})$ for $\widehat{v}$ when g is simultaneously prox-regular and subdifferentially continuous at $\bar{x}$ for $\widehat{v}$ with compatible parameterization by p at $\bar{p}$. In this case inequality "$g(u, p) \leq g(\bar{x}, \bar{p}) + \varepsilon$" can be omitted in (4.8).

Throughout this section we impose the following *standing assumptions*:

(A1) f is differentiable with respect to x around $(\bar{x}, \bar{p})$ uniformly in p and the partial Jacobian $\nabla_x f$ is continuous at $(\bar{x}, \bar{p})$. Furthermore, f is Lipschitz continuous with respect to p uniformly in x around $(\bar{x}, \bar{p})$, i.e., there exist a neighborhood $U \times Q$ of $(\bar{x}, \bar{p})$ and a constant $L > 0$ such that

$$
\|f(x, p_1) - f(x, p_2)\| \leq L d(p_1, p_2) \quad \text{for all} \quad x \in U, \ p_1, p_2 \in Q.
\tag{4.9}
$$

(A2) g is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\widehat{v}$.

(A3) The following *basic constraint qualification* (BCQ) holds at $(\bar{x}, \bar{p})$:

$$
\text{the mapping} \ p \mapsto \text{epi}\, g(\cdot, p) \ \text{is Lipschitz-like around} \ \left(\bar{p}, (\bar{x}, g(\bar{x}, \bar{p}))\right).
\tag{4.10}
$$

Note that assumption **(A1)** for f is classical in the study of generalized equations and turns back to the landmark paper by Robinson [25]. It follows from the mean value theorem that such a mapping is Lipschitz continuous around $(\bar{x}, \bar{p})$, i.e., there exist a neighborhood $U \times Q$ of $(\bar{x}, \bar{p})$ and a constant $L > 0$ for which

$$
\|f(x_1, p_1) - f(x_2, p_2)\| \leq L[\|x_1 - x_2\| + d(p_1, p_2)] \ \text{if} \ (x_1, p_1), (x_2, p_2) \in U \times Q.
\tag{4.11}
$$

Observe also that in the case when both spaces X and P are finite-dimensional, BCQ from **(A3)** can be equivalently described by the implication

$$
(0, q) \in \partial^\infty g(\bar{x}, \bar{p}) \Longrightarrow q = 0,
\tag{4.12}
$$

which follows from the coderivative criterion (2.7) for the Lipschitz-like property of the epigraphical mapping in (4.10); see [12]. It shown in [16, Proposition 4.2] that **(A3)** yields the local Lipschitz continuity of the value function m_γ in (4.6). Both assumptions **(A2)** and **(A3)** naturally hold for important special classes of functions in finite and infinite dimensions; see [12, 16]. It is worth mentioning furthermore that when the parameter p is not present, we have **(A3)** to fulfill automatically, **(A1)** means that f is smooth around $\bar{x}$, and **(A2)** reduces to the continuous prox-regularity of g at $\bar{x}$ for $\widehat{v}$.

Now we are ready to formulate the main result of this section giving a second-order characterization of Hölder full stability for general parametric variational systems.

THEOREM 4.3. **(second-order characterization of Hölderian full stability for PVS).** *Consider the following two statements:*

(i) $\bar{x}$ *is a Hölderian fully stable solution of PVS* (4.1) *corresponding to the parameter pair* $(\bar{v}, \bar{p})$ *with the moduli* $\kappa, \ell > 0$ *taken from* (4.3).
(ii) *There are* $\eta, \kappa_0 > 0$ *such that whenever* $(u, p, v) \in \mathrm{gph}\, \partial_x g \cap \mathbb{B}_\eta(\bar{x}, \bar{p}, \widehat{v})$ *with* $\widehat{v} = \bar{v} - f(\bar{x}, \bar{p})$ *we have*

$$\langle \nabla_x f(\bar{x}, \bar{p})w, w \rangle + \langle z, w \rangle \geq \kappa_0 \|w\|^2 \quad \text{for all} \quad z \in (\widehat{D}^* \partial g_p)(u, v)(w),\ w \in X. \quad (4.13)$$

Then **(i)** *implies* **(ii)** *with constant* κ_0 *that can be chosen smaller than but arbitrarily close to* κ. *Conversely, the validity of* **(ii)** *ensures that* **(i)** *holds, where* κ *can be chosen smaller but arbitrarily close to* κ_0.

Consequently, (4.13) *implies that the solution map* (4.2) *admits a single-valued and Hölder continuous localization* ϑ *relative to a neighborhood* $V \times Q \times U$ *of* $(\bar{v}, \bar{p}, \bar{x})$, *i.e., for any* $(v_1, p_1), (v_2, p_2) \in V \times Q$ *we have*

$$\left\| \vartheta(v_1, p_1) - \vartheta(v_2, p_2) \right\| \leq \frac{1}{\kappa} \|v_1 - v_2\| + \frac{\ell}{2\kappa} d(p_1, p_2)^{\frac{1}{2}}. \quad (4.14)$$

Prior the proof of Theorem 4.3, we derive the following two lemmas that can be treated as significant steps in the proof of this theorem while being of their own interest. The first lemma establishes a certain "time propagation" of the aforementioned full stability properties in the case of linearized PVS of type (4.1).

LEMMA 4.4. **(propagation of full stability for linearized PVS).** *Denote* $A := \nabla_x f(\bar{x}, \bar{p})$ *and consider the one-parametric family of operators*

$$A_t := \frac{1}{2}\left(A + A^*\right) + tB \quad \text{with} \quad B := A - A^* \quad \text{and} \quad t \geq 0,$$

where A^* *is the adjoint operator of* A. *Define further the set-valued mapping* $G_t : X \times P \rightrightarrows X$ *by*

$$G_t(v, p) := \left\{ x \in X \mid v \in f(\bar{x}, \bar{p}) + A_t(x - \bar{x}) + \partial_x g(x, p) \right\} \quad \text{for} \quad (v, p) \in X \times P. \quad (4.15)$$

The following two assertions are satisfied:
(i) *Suppose that* G_τ *for some* $\tau \geq 0$ *has a single-valued localization* ϑ_τ *relative to a neighborhood* $V \times Q \times U$ *of* $(\bar{v}, \bar{p}, \bar{x})$ *such that for any* $(v_1, p_1), (v_2, p_2) \in V \times Q$ *it holds*

$$\left\| (v_1 - v_2) - 2\kappa \left[\vartheta_\tau(v_1, p_1) - \vartheta_\tau(v_2, p_2) \right] \right\| \leq \|v_1 - v_2\| + \ell d(p_1, p_2)^{\frac{1}{2}} \quad (4.16)$$

with some $\ell > 0$. *Then* G_t *also admits a single-valued localization* ϑ_t *relative to a neighborhood* $V_1 \times Q_1 \times U_1 \subset V \times Q \times U$ *of* $(\bar{v}, \bar{p}, \bar{x})$ *so that for any* $(v_1, p_1), (v_2, p_2) \in V_1 \times Q_1$ *we have*

$$\left\| (v_1 - v_2) - 2\kappa \left[\vartheta_t(v_1, p_1) - \vartheta_t(v_2, p_2) \right] \right\| \leq \|v_1 - v_2\| + 2\ell d(p_1, p_2)^{\frac{1}{2}} \quad (4.17)$$

whenever $t \in \left[\tau, \tau + \frac{\kappa}{2\|B\|} \right)$ *under the convention that* $1/0 := \infty$.
(ii) *Suppose that* G_τ *for some* $\tau \geq 0$ *has a single-valued localization* ϑ_τ *relative to a neighborhood* $V \times Q \times U$ *of* $(\bar{v}, \bar{p}, \bar{x})$ *such that for any* $(v_1, p_1), (v_2, p_2) \in V \times Q$ *we have*

$$\left\| (v_1 - v_2) - 2\kappa \left[\vartheta_\tau(v_1, p_1) - \vartheta_\tau(v_2, p_2) \right] \right\| \leq \|v_1 - v_2\| + \ell d(p_1, p_2) \quad (4.18)$$

with some $\ell > 0$. *Then* G_t *also admits a single-valued localization* ϑ_t *relative to a neighborhood* $V_1 \times Q_1 \times U_1 \subset V \times Q \times U$ *of* $(\bar{v}, \bar{p}, \bar{x})$ *satisfying the condition*

$$\left\| (v_1 - v_2) - 2\kappa \left[\vartheta_t(v_1, p_1) - \vartheta_t(v_2, p_2) \right] \right\| \leq \|v_1 - v_2\| + 2\ell d(p_1, p_2) \quad (4.19)$$

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for any $(v_1, p_1), (v_2, p_2) \in V_1 \times Q_1$ *provided that* $t \in \left[\tau, \tau + \frac{\kappa}{2\|B\|}\right)$.

Proof. To justify assertion **(i)**, observe first from (4.16) that

$$\|\vartheta_\tau(v_1, p_1) - \vartheta_\tau(v_2, p_2)\| \leq \ell_1 \|v_1 - v_2\| + \ell_2 d(p_1, p_2)^{\frac{1}{2}} \text{ with } \ell_1 := \frac{1}{\kappa}, \ \ell_2 := \frac{\ell}{2\kappa}. \quad (4.20)$$

Fixing $t \in \left[\tau, \tau + \frac{\kappa}{2\|B\|}\right)$, we get $\varepsilon := r(1 - \ell_1(t - \tau)\|B\|) \in (0, r]$ as $r > 0$. Choose now $r > 0$ so small that $I\!B_{r\ell_1}(\bar{x}) \subset U$, $I\!B_r(\bar{v}) \subset V$, and $I\!B_s(\bar{p}) \subset Q$ with $s := \frac{\varepsilon^2 \ell_1^2}{4\ell_2^2}$. Define the mapping ϑ_t by

$$\text{gph } \vartheta_t := \text{gph } G_t \cap \left(V_1 \times Q_1 \times U_1\right) \quad (4.21)$$

with $V_1 := \text{int } I\!B_{\frac{\varepsilon}{2}}(\bar{v}) \subset I\!B_r(\bar{v}) \subset V$, $Q_1 := \text{int } I\!B_s(\bar{p}) \subset Q$, $U_1 = \text{int } I\!B_{r\ell_1}(\bar{x}) \subset U$ and show that it is single-valued. To proceed, we pick $v \in V_1$ and $p \in Q_1$, form the single-valued mapping $T : I\!B_{r\ell_1}(\bar{x}) \to X$ by

$$T(x) := \vartheta_\tau\big(v - (t - \tau)B(x - \bar{x}), p\big) \quad \text{for all} \quad x \in I\!B_{r\ell_1}(\bar{x}), \quad (4.22)$$

and claim that T has the full domain. Indeed, it is easy to see that

$$\|v - (t - \tau)B(x - \bar{x}) - \bar{v}\| \leq \|v - \bar{v}\| + (t - \tau)\|B\| \cdot \|x - \bar{x}\| < \varepsilon + (t - \tau)r\ell_1\|B\| = r$$

for $x \in I\!B_{r\ell_1}(\bar{x})$, which amounts to $v - (t - \tau)B(x - \bar{x}) \in I\!B_r(\bar{v}) \subset V$. We get from the definition of ϑ_τ that the values $T(x) = \vartheta_\tau(v - (t - \tau)B(x - \bar{x}), p)$ are well defined, and thus $\text{dom } T = I\!B_{r\ell_1}(\bar{x})$.

Let us verify further that $u \in \vartheta_t(v, p)$ if and only if $u = T(u)$ and $u \in U_1$. Indeed, if $u \in \vartheta_t(v, p)$, we get from (4.21) and (4.15) that $u \in U_1$ and

$$\begin{aligned} v - (t - \tau)B(u - \bar{x}) \ &\in f(\bar{x}, \bar{p}) + A_t(u - \bar{x}) + \partial_x g(u, p) - (t - \tau)B(u - \bar{x}) \\ &= f(\bar{x}, \bar{p}) + A_\tau(u - \bar{x}) + \partial_x g(u, p), \end{aligned}$$

which ensures that $(v - (t - \tau)B(u - \bar{x}), p, u) \in \text{gph } G_\tau$. Knowing from the above proof that $v - (t - \tau)B(u - \bar{x}) \in I\!B_r(\bar{x}) \subset V$, we have $(v - (t - \tau)B(u - \bar{x}), p, u) \in \text{gph } G_\tau \cap (V \times Q \times U)$. This tells us that $u = \vartheta_\tau(v - (t - \tau)B(u - \bar{x}), p)$ giving us $u = T(u)$ by (4.22). Conversely, if $u = T(u)$ and $u \in U_1$, we have $u = \vartheta_\tau(v - (t - \tau)B(u - \bar{x}), p)$. It follows from (4.21) and (4.15) that

$$v \in f(\bar{x}, \bar{p}) + A_\tau(u - \bar{x}) + \partial_x g(u, p) + (t - \tau)B(u - \bar{x}) = f(\bar{x}, \bar{p}) + A_t(u - \bar{x}) + \partial_x g(u, p),$$

which clearly justifies $u \in \vartheta_t(v, p)$ due to the fact that $(v, p) \in V_1 \times Q_1$ and $u \in U_1$.

Next we show that $\vartheta_t(v, p)$ exists uniquely whenever $(v, p) \in V_1 \times Q_1$ by applying the classical *contraction principle* to T. To justify, observe first that the Hölder continuity of ϑ_τ in (4.20) yields the estimates

$$\begin{aligned} \|T(\bar{x}) - \bar{x}\| \ &= \|\vartheta_\tau(v, p) - \vartheta_\tau(\bar{v}, \bar{p})\| \leq \ell_1 \|v - \bar{v}\| + \ell_2 d(p, \bar{p})^{\frac{1}{2}} \\ &< \frac{\ell_1 \varepsilon}{2} + \ell_2 s^{\frac{1}{2}} = \frac{\ell_1 \varepsilon}{2} + \frac{\ell_1 \varepsilon}{2} = \ell_1 \varepsilon. \end{aligned} \quad (4.23)$$

It follows from (4.20) that for any $x_1, x_2 \in I\!B_{r\ell_1}(\bar{x})$ we have

$$\begin{aligned} \|T(x_1) - T(x_2)\| \ &= \|\vartheta_\tau(v - (t - \tau)B(x_1 - \bar{x}), p) - \vartheta_\tau(v - (t - \tau)B(x_2 - \bar{x}), p)\| \\ &\leq \ell_1(t - \tau)\|B(x_1 - x_2)\| \leq \ell_1(t - \tau)\|B\| \cdot \|x_1 - x_2\|. \end{aligned} \quad (4.24)$$

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Pick now any $x \in \mathbb{B}_{r\ell_1}(\bar{x})$ and deduce from (4.23) and (4.24) that

$$\begin{aligned} \|T(x) - \bar{x}\| &\leq \|T(x) - T(\bar{x})\| + \|T(\bar{x}) - \bar{x}\| < \ell_1(t - \tau)\|B\| \cdot \|x - \bar{x}\| + \ell_1 \varepsilon \\ &\leq \ell_1(t - \tau)\|B\| r\ell_1 + \ell_1 r(1 - \ell_1(t - \tau)\|B\|) = r\ell_1. \end{aligned}$$

Since $\ell_1(t-\tau)\|B\| < 1$ by the choice of $t \in [\tau, \tau + \frac{\kappa}{2\|B\|})$, we apply the contraction principle to the mapping T and find a unique fixed point u with $\|u - \bar{x}\| = \|T(u) - \bar{x}\| < r\ell_1$, i.e., $u = T(u)$ and $u \in U_1 = \text{int } \mathbb{B}_{r\ell_1}(\bar{x})$. By the claim proved above, it tells us that $u = \vartheta_t(v, p)$ is a *singleton* whenever $(v, p) \in V_1 \times Q_1$.

It remains to show that ϑ_t satisfies (4.17). Indeed, picking any $(v_i, p_i) \in V_1 \times Q_1$ and denoting $u_i := \vartheta_t(v_i, p_i)$, $i = 1, 2$, we have $u_i = \vartheta_\tau(v_i - (t - \tau)B(u_i - \bar{x}), p_i)$. Define $u_3 := \vartheta_t(v_1, p_2)$, i.e., $u_3 = \vartheta_\tau(v_1 - (t - \tau)B(u_3 - \bar{x}), p_2)$. Using (4.16) for $(v_1 - (t - \tau)B(u_3 - \bar{x}), p_2)$ and $(v_2 - (t - \tau)B(u_2 - \bar{x}), p_2)$ yields

$$\|v_1 - v_2 - (t - \tau)B(u_3 - u_2) - 2\kappa(u_3 - u_2)\| \leq \|v_1 - v_2 - (t - \tau)B(u_3 - u_2)\|,$$

which ensures in turn the conditions

$$\begin{aligned} 0 &\geq 4\kappa^2 \|u_3 - u_2\|^2 - 4\kappa \langle v_1 - v_2 - (t - \tau)B(u_3 - u_2), u_3 - u_2 \rangle \\ &= 4\kappa^2 \|u_3 - u_2\|^2 - 4\kappa \langle v_1 - v_2, u_3 - u_2 \rangle + 4\kappa(t - \tau)\langle B(u_3 - u_2), u_3 - u_2 \rangle \\ &= 4\kappa^2 \|u_3 - u_2\|^2 - 4\kappa \langle v_1 - v_2, u_3 - u_2 \rangle \\ &= \|v_1 - v_2 - 2\kappa(u_3 - u_2)\|^2 - \|v_1 - v_2\|^2, \end{aligned}$$

where the second equality is valid, since for any $x \in X$ we have

$$\langle Bx, x \rangle = \langle Ax, x \rangle - \langle A^*x, x \rangle = \langle x, A^*x \rangle - \langle A^*x, x \rangle = 0.$$

This gives us the norm relationship

$$\|v_1 - v_2\| \geq \|v_1 - v_2 - 2\kappa(u_3 - u_2)\|. \tag{4.25}$$

Then using (4.16) for $(v_1 - (t - \tau)B(u_1 - \bar{x}), p_1)$ and $(v_1 - (t - \tau)B(u_3 - \bar{x}), p_2)$ implies

$$\| - (t - \tau)B(u_1 - u_3) - 2\kappa(u_1 - u_3)\| \leq \|(t - \tau)B(u_1 - u_3)\| + \ell d(p_1, p_2)^{\frac{1}{2}}$$

with noting again that $\vartheta_\tau(v_1 - (t - \tau)B(u_1 - \bar{x}), p_1) = \vartheta_t(v_1, p_1) = u_1$ and $\vartheta_\tau(v_1 - (t - \tau)B(u_3 - \bar{x}), p_2) = \vartheta_t(v_1, p_2) = u_3$. Thus we arrive at the lower distance estimate

$$\begin{aligned} \ell d(p_1, p_2)^{\frac{1}{2}} &\geq 2\kappa \|u_1 - u_3\| - (t - \tau)\|B(u_1 - u_3)\| - \|(t - \tau)B(u_1 - u_3)\| \\ &\geq 2(\kappa - (t - \tau)\|B\|)\|u_1 - u_3\| \geq \kappa \|u_1 - u_3\|, \end{aligned}$$

where the last inequality is satisfied by $t \in [\tau, \tau + \frac{\kappa}{2\|B\|})$. It follows from (4.25) that

$$\begin{aligned} \|v_1 - v_2\| &\geq \|v_1 - v_2 - 2\kappa(u_1 - u_2)\| - 2\kappa \|u_1 - u_3\| \\ &\geq \|v_1 - v_2 - 2\kappa(u_1 - u_2)\| - 2\ell d(p_1, p_2)^{\frac{1}{2}}. \end{aligned}$$

It verifies (4.17) and completes the proof of **(i)**.

The proof of **(ii)** is similar. The only differences needed therein are the change of the neighborhood Q_1 above and the replacement of $d(p_1, p_2)^{\frac{1}{2}}$ by $d(p_1, p_2)$. Now we choose $Q_1 := \text{int } \mathbb{B}_s(\bar{p}) \subset Q$ with $s := \frac{\varepsilon \ell_1}{2\ell_2}$ for $r > 0$ sufficiently small. This allows us

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to show that the mapping T in (4.22) also satisfies (4.23), (4.24), and (4.25) for any $v \in V_1$ and $p \in Q_1$. The rest of the proof follows the lines in the proof of **(i)**. $\triangle$

The next lemma shows how to pass, after the parameter propagation of Lemma 4.4, from single-valued localizations of the linearized variational systems $G_{1/2}$ in (4.15) satisfying (4.16) and (4.18) to single-valued localizations of the solution map S to the original PVS (4.1). The linearization results of this type go back to Robinson [25, Theorem 2.1] in the case of local Lipschitz continuity for generalized equations with parameter-independent set-valued parts.

LEMMA 4.5. **(single-valued localizations of solutions maps to nonlinear PVS).** *In the setting of Lemma 4.4 the following assertions hold:*
(i) *Consider the mapping G_τ from (4.15) with $\tau = \frac{1}{2}$ and suppose that it admits a single-valued localization ϑ_τ relative to a neighborhood $V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ satisfying (4.16) with some moduli $\kappa, \ell > 0$. Then for any $\varepsilon \in (0, \kappa)$ the solution map S from (4.2) also admits a single-valued localization ϑ relative to some neighborhood $V_1 \times Q_1 \times U_1 \subset V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ such that whenever $(v_1, p_1), (v_2, p_2) \in V_1 \times Q_1$ we have*

$$\left\| v_1 - v_2 - 2(\kappa - \varepsilon)\big[\vartheta(v_1, p_1) - \vartheta(v_2, p_2)\big] \right\| \leq \|v_1 - v_2\| + (\ell + 2L\sqrt{2\varepsilon})d(p_1, p_2)^{\frac{1}{2}}, \quad (4.26)$$

where $L > 0$ is the constant taken from (4.9).
(ii) *Suppose that in the setting of **(i)** the mapping G_τ with $\tau = \frac{1}{2}$ has a single-valued localization ϑ_τ relative to a neighborhood $V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ satisfying (4.18) with moduli $\kappa, \ell > 0$. Then for any $\varepsilon \in (0, \kappa)$ the solution map S also admits a single-valued localization ϑ relative to some neighborhood $V_1 \times Q_1 \times U_1 \subset V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ so that*

$$\left\| v_1 - v_2 - 2(\kappa - \varepsilon)\big[\vartheta(v_1, p_1) - \vartheta(v_2, p_2)\big] \right\| \leq \|v_1 - v_2\| + (\ell + 2L)d(p_1, p_2) \quad (4.27)$$

for all pairs (v_1, p_1) and (v_2, p_2) from $V_1 \times Q_1$.

Proof. To justify **(i)**, assume that G_τ with $\tau = \frac{1}{2}$ admits a single-valued localization ϑ_τ relative to a neighborhood $V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ satisfying (4.16) with some moduli $\kappa, \ell > 0$. Thus we also have the Hölder continuity of ϑ_τ in (4.20) with $\ell_1 := \frac{1}{\kappa}$ and $\ell_2 := \frac{\ell}{2\kappa}$ as follows

$$\|\vartheta_\tau(v_1, p_1) - \vartheta_\tau(v_2, p_2)\| \leq \ell_1 \|v_1 - v_2\| + \ell_2 d(p_1, p_2)^{\frac{1}{2}} \quad \text{for all} \ \ (v_1, p_1), (v_2, p_2) \in V \times Q. \quad (4.28)$$

Furthermore, observe from the construction of G_τ and $A_\tau = A$ with $\tau = \frac{1}{2}$ that

$$G_\tau(v, p) = \big\{ x \in X \, \big| \, v \in f(\bar{x}, \bar{p}) + A(x - \bar{x}) + \partial_x g(x, p) \big\} \quad \text{when} \ (v, p) \in X \times P. \quad (4.29)$$

Define $r(x, p) := f(\bar{x}, \bar{p}) + A(x - \bar{x}) - f(x, p)$ for all $(x, p) \in X \times P$. It follows from the continuity of f around $(\bar{x}, \bar{p})$ that for any $\varepsilon \in (0, \kappa)$ there are $\rho, \eta \in (0, \varepsilon)$ with $\mathbb{B}_\rho(\bar{x}) \times \mathbb{B}_\eta(\bar{v}) \times \mathbb{B}_\eta(\bar{p}) \subset U \times V \times Q$ such that $v + r(x, p) \in V$ for $(x, v, p) \in \mathbb{B}_\rho(\bar{x}) \times \mathbb{B}_\eta(\bar{v}) \times \mathbb{B}_\eta(\bar{p})$. By the continuity of $\nabla_x f$ around $(\bar{x}, \bar{p})$ we may choose $\rho, \eta > 0$ so small that $\|\nabla_x f(x, p) - A\| \leq \varepsilon$ for all $(x, p) \in \mathbb{B}_\rho(\bar{x}) \times \mathbb{B}_\eta(\bar{p})$ and that

$$\ell_1 (\|f(\bar{x}, \bar{p}) - f(\bar{x}, p)\| + \|v - \bar{v}\|) + \ell_2 d(p, \bar{p})^{\frac{1}{2}} < (1 - \ell_1 \varepsilon)\rho. \quad (4.30)$$

Pick $(x_1, v, p), (x_2, v, p) \in \mathbb{B}_\rho(\bar{x}) \times \mathbb{B}_\eta(\bar{v}) \times \mathbb{B}_\eta(\bar{p})$ and deduce from the mean value theorem that

$$
\begin{aligned}
\|r(x_1, p) - r(x_2, p)\| &\leq \sup_{t \in [0,1]} \|r'(x_1 + t(x_2 - x_1), p)\| \cdot \|x_1 - x_2\| \\
&= \sup_{t \in [0,1]} \|A - \nabla_x f(x_1 + t(x_2 - x_1), p)\| \cdot \|x_1 - x_2\| \quad (4.31) \\
&\leq \varepsilon \|x_1 - x_2\|.
\end{aligned}
$$

Next we claim that the mapping ϑ with

$$\mathrm{gph}\,\vartheta := \mathrm{gph}\,S \cap \big(\mathrm{int}\,I\!\!B_\eta(\bar{v}) \times \mathrm{int}\,I\!\!B_\eta(\bar{p}) \times \mathrm{int}\,I\!\!B_\rho(\bar{x})\big) \tag{4.32}$$

is single-valued and satisfies (4.26). Proceeding in this way, let us first verify the single-valuedness of ϑ. Observe from (4.32) that for any $(v,p) \in \mathrm{int}\,I\!\!B_\eta(\bar{v}) \times \mathrm{int}\,I\!\!B_\eta(\bar{p})$ we have $u \in \vartheta(v,p)$ if and only if $u = \vartheta_\tau(v + r(u,p),p)$ and $u \in \mathrm{int}\,I\!\!B_\rho(\bar{x})$.

Consider the mapping $\Phi : I\!\!B_\rho(\bar{x}) \to X$ with $\Phi(x) := \vartheta_\tau(v + r(x,p),p)$ for all $x \in I\!\!B_\rho(\bar{x})$, which is well defined due to $v + r(x,p) \in V$. By the above equivalence, $\vartheta(v,p)$ is singleton whenever there is a unique u satisfying $u = \Phi(u)$ and $u \in \mathrm{int}\,I\!\!B_\rho(\bar{x})$. We verify such an existence of u by using the contraction principle on Φ. To proceed, pick any points $x_1, x_2 \in I\!\!B_\rho(\bar{x})$ and deduce from (4.28) and (4.31) that

$$\begin{aligned}\|\Phi(x_1) - \Phi(x_2)\| &= \|\vartheta_\tau(v + r(x_1,p),p) - \vartheta_\tau(v + r(x_2,p),p)\| \\ &\leq \ell_1\|r(x_1,p) - r(x_2,p)\| \leq \ell_1\varepsilon\|x_1 - x_2\|\end{aligned} \tag{4.33}$$

ensuring by $\ell_1\varepsilon < 1$ the contraction condition for Φ. It also follows from (4.28) and (4.30) that

$$\begin{aligned}\|\Phi(\bar{x}) - \bar{x}\| &= \|\vartheta_\tau(v + r(\bar{x},p),p) - \vartheta_\tau(\bar{v},\bar{p})\| \leq \ell_1\big(\|r(\bar{x},p)\| + \|v - \bar{v}\|\big) + \ell_2 d(p,\bar{p})^{\frac{1}{2}} \\ &= \ell_1\big(\|f(\bar{x},\bar{p}) - f(\bar{x},p)\| + \|v - \bar{v}\|\big) + \ell_2 d(p,\bar{p})^{\frac{1}{2}} < (1 - \ell_1\varepsilon)\rho,\end{aligned} \tag{4.34}$$

which together with the last estimate in (4.33) implies the relationships

$$\|\Phi(x) - \bar{x}\| \leq \|\Phi(x) - \Phi(\bar{x})\| + \|\Phi(\bar{x}) - \bar{x}\| < \ell_1\varepsilon\|x - \bar{x}\| + (1 - \ell_1\varepsilon)\rho \leq \rho \text{ for } x \in I\!\!B_\rho(\bar{x}). \tag{4.35}$$

Combining (4.34) and (4.35) with (4.33) tells us that there exists a unique fixed point u of Φ due to the classical contraction principle so that $\|u - \bar{x}\| = \|\Phi(u) - \bar{x}\| < \rho$. By using the equivalence proved above, we have $u = \vartheta(v,p)$ and hence verify the single-valuedness of the mapping ϑ.

The obtained single-valuedness of ϑ allows us to observe that

$$\vartheta(v,p) = \Phi(\vartheta(v,p)) = \vartheta_\tau(v + r(\vartheta(v,p),p),p) \text{ for all } (v,p) \in \mathrm{int}\,I\!\!B_\eta(\bar{v}) \times \mathrm{int}\,I\!\!B_\eta(\bar{p}). \tag{4.36}$$

Thus it remains to check condition (4.26). Picking any $(v_i,p_i) \in \mathrm{int}\,I\!\!B_\eta(\bar{v}) \times \mathrm{int}\,I\!\!B_\eta(\bar{p})$, $i = 1, 2$, and denoting $u_i := \vartheta(v_i,p_i)$ and $u_3 := \vartheta(v_1,p_2)$, we deduce from (4.36) that

$$u_i = \vartheta_\tau\big(v_i + r(u_i,p_i),p_i\big), \; i = 1,2, \text{ and } u_3 = \vartheta_\tau\big(v_1 + r(u_3,p_2),p_2\big).$$

Applying (4.16) to the pairs $(v_1 + r(u_3,p_2),p_2)$ and $(v_2 + r(u_2,p_2),p_2)$ gives us

$$\|v_1 - v_2 + [r(u_3,p_2) - r(u_2,p_2)] - 2\kappa(u_3 - u_2)\| \leq \|v_1 - v_2 + [r(u_3,p_2) - r(u_2,p_2)]\|,$$

which implies in turn the following relationships:

$$\begin{aligned}0 \; &\leq \|v_1 - v_2 + [r(u_3,p_2) - r(u_2,p_2)]\|^2 \\ &\quad - \|v_1 - v_2 + [r(u_3,p_2) - r(u_2,p_2)] - 2\kappa(u_3 - u_2)\|^2 \\ &= 4\kappa\langle v_1 - v_2 + [r(u_3,p_2) - r(u_2,p_2)], u_3 - u_2\rangle - 4\kappa^2\|u_3 - u_2\|^2 \\ &= 4\kappa\langle v_1 - v_2, u_3 - u_2\rangle - 4\kappa^2\|u_3 - u_2\|^2 + 4\kappa\langle r(u_3,p_2) - r(u_2,p_2), u_3 - u_2\rangle \\ &\leq 4\kappa\langle v_1 - v_2, u_3 - u_2\rangle - 4\kappa^2\|u_3 - u_2\|^2 + 4\kappa\|r(u_3,p_2) - r(u_2,p_2)\| \cdot \|u_3 - u_2\| \\ &\leq 4\kappa\langle v_1 - v_2, u_3 - u_2\rangle - 4\kappa^2\|u_3 - u_2\|^2 + 4\kappa\varepsilon\|u_3 - u_2\|^2 \\ &= 4\kappa\langle v_1 - v_2, u_3 - u_2\rangle - 4\kappa(\kappa - \varepsilon)\|u_3 - u_2\|^2,\end{aligned}$$

where the last inequality follows from (4.31). This tells us that

$$0 \leq \langle v_1 - v_2, u_3 - u_2 \rangle - (\kappa - \varepsilon)\|u_3 - u_2\|^2,$$

and therefore we arrive at the estimate

$$\|v_1 - v_2 - 2(\kappa - \varepsilon)(u_3 - u_2)\|^2 \leq \|v_1 - v_2\|^2. \qquad (4.37)$$

Applying further (4.16) to the pairs $(v_1 + r(u_3, p_2), p_2)$ and $(v_1 + r(u_1, p_1), p_1)$ yields

$$\|r(u_3, p_2) - r(u_1, p_1)\| + \ell d(p_1, p_2)^{\frac{1}{2}} \geq \|[r(u_3, p_2) - r(u_1, p_1)] - 2\kappa(u_3 - u_1)\|$$
$$\geq 2\kappa\|u_3 - u_1\| - \|r(u_3, p_2) - r(u_1, p_1)\|,$$

which readily imply the inequalities

$$\ell d(p_1, p_2)^{\frac{1}{2}} \geq 2\kappa\|u_3 - u_1\| - 2\|r(u_3, p_2) - r(u_1, p_1)\|$$
$$\geq 2\kappa\|u_3 - u_1\| - 2\|r(u_3, p_2) - r(u_1, p_2)\| - 2\|r(u_1, p_2) - r(u_1, p_1)\|$$
$$= 2\kappa\|u_3 - u_1\| - 2\|r(u_3, p_2) - r(u_1, p_2)\| - 2\|f(u_1, p_1) - f(u_1, p_2)\|$$
$$\geq 2\kappa\|u_3 - u_1\| - 2\varepsilon\|u_3 - u_1\| - 2Ld(p_1, p_2),$$

where the last one is a consequence of (4.31) and (4.9). Thus we get

$$2(\kappa - \varepsilon)\|u_3 - u_1\| \leq \ell d(p_1, p_2)^{\frac{1}{2}} + 2Ld(p_1, p_2) \leq \ell d(p_1, p_2)^{\frac{1}{2}} + 2L\sqrt{2\eta}d(p_1, p_2)^{\frac{1}{2}}.$$

This together with (4.37) ensures that

$$\|v_1 - v_2 - 2(\kappa - \varepsilon)(u_1 - u_2)\| \leq \|v_1 - v_2 - 2(\kappa - \varepsilon)(u_3 - u_2)\| + 2(\kappa - \varepsilon)\|u_3 - u_1\|$$
$$\leq \|v_1 - v_2\| + (\ell + 2L\sqrt{2\eta})d(p_1, p_2)^{\frac{1}{2}}$$
$$\leq \|v_1 - v_2\| + (\ell + 2L\sqrt{2\varepsilon})d(p_1, p_2)^{\frac{1}{2}},$$

which verifies (4.26) and completes the proof of (i). The proof of (ii) is quite similar with replacing all the terms $d(p_1, p_2)^{\frac{1}{2}}$ by $d(p_1, p_2)$ through the proof of (i). $\triangle$

Now we are ready to prove the main Theorem 4.3 by using the lemmas derived above, the coderivative criteria for local strong maximal monotonicity from Section 3, and characterizations of Hölderian full stability of local minimizers established in [16].

Proof of Theorem 4.3. First suppose that (i) is satisfied, i.e., the solution map S admits a single-valued localization ϑ relative to a neighborhood $V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ such that we have

$$\big\|(v_1 - v_2) - 2\kappa[\vartheta(v_1, p_1) - \vartheta(v_2, p_2)]\big\| \leq \|v_1 - v_2\| + \ell d(p_1, p_2)^{\frac{1}{2}}$$

for $(v_1, p_1), (v_2, p_2) \in V \times Q$. We want to prove that (ii) holds. The Hölder continuity of ϑ as in (4.28) allows us to suppose without loss of generality that $V = \text{int } \mathbb{B}_\nu(\bar{v})$, $Q = \text{int } \mathbb{B}_\nu(\bar{p})$, and $U = \text{int } \mathbb{B}_\rho(\bar{x})$ provided that the numbers $\nu, \rho > 0$ are sufficiently small. Fix $p \in Q$ and get from (4.3) with $\vartheta_p(v) := \vartheta(v, p)$ that

$$\big\|(v_1 - v_2) - 2\kappa\big[\vartheta_p(v_1) - \vartheta_p(v_2)\big]\big\| \leq \|v_1 - v_2\| \quad \text{for all} \quad v_1, v_2 \in V. \quad (4.38)$$

Define $T_p(x) := f_p(x) + \partial g_p(x)$ for $x \in X$ and note that $\text{gph } \vartheta_p = \text{gph } T_p^{-1} \cap (V \times U)$ satisfying (4.38). Applying the remark prior (3.19) to the mapping T_p tells us that

$$\langle z, w \rangle \geq \kappa\|w\|^2 \quad \text{for} \quad z \in \widehat{D}^* T_p(u, v)(w), \ (u, v) \in \text{gph } T_p \cap (U \times V), \ w \in X. \quad (4.39)$$

Now we justify the coderivative condition (4.13). It follows from **(A1)** that there exists $\eta \in (0, \min\{\nu, \rho\})$ such that for any $(u, p, v_1) \in \mathrm{gph}\, \partial_x g \cap I\!\!B_\eta(\bar{x}, \bar{p}, \widehat{v})$ we have

$$v_1 + f(u, p) = v_1 - \widehat{v} + f(u, p) - f(\bar{x}, \bar{p}) + \bar{v} \in \mathrm{int}\, I\!\!B_\nu(\bar{v}) = V. \qquad (4.40)$$

Pick any $(w, z_1) \in X \times X$ with $z_1 \in (\widehat{D}^* \partial g_p)(u, v_1)(w)$ and get from the coderivative sum rule given in [15, Theorem 1.62] that

$$\nabla_x f(u, p)^* w + z_1 \in \nabla_x f(u, p)^* w + (\widehat{D}^* \partial g_p)(u, v_1)(w) = \widehat{D}^* T_p(u, v_1 + f(u, p))(w),$$

which implies together (4.40) and (4.39) that

$$\kappa \|w\|^2 \leq \langle \nabla_x f(u, p)^* w, w \rangle + \langle z_1, w \rangle = \gamma(\nu, \rho) \|w\|^2 + \langle \nabla_x f(\bar{x}, \bar{p})^* w, w \rangle + \langle z_1, w \rangle,$$

where the inequality holds with some $\gamma(\nu, \rho) \downarrow 0$ as $\nu, \rho \downarrow 0$ due to **(A1)**. This justifies condition (4.13) when ν, ρ are small enough to ensure that $\kappa_0 := \kappa - \gamma(\nu, \rho) > 0$ is smaller than but arbitrarily close to κ. We complete the proof of [**(i)** $\Longrightarrow$ **(ii)**].

Let us show next that condition (4.13) is sufficient for the Hölderian full stability of $\bar{x} \in S(\bar{v}, \bar{p})$ while supposing that **(ii)** is satisfied with $\kappa_0, \eta > 0$. Define the l.s.c. function $h \colon X \times P \to \overline{I\!\!R}$ by

$$h(x, p) := \langle f(\bar{x}, \bar{p}), x - \bar{x} \rangle + \frac{1}{2} \langle A(x - \bar{x}), x - \bar{x} \rangle + g(x, p) \quad \text{for all} \quad (x, p) \in X \times P \quad (4.41)$$

with $A = \nabla_x f(\bar{x}, \bar{p})$. Note that $\partial_x h(x, p) = f(\bar{x}, \bar{p}) + \frac{1}{2}(A + A^*)(x - \bar{x}) + \partial_x g(x, p)$ and that $\bar{v} \in \partial_x h(\bar{x}, \bar{p}) = f(\bar{x}, \bar{p}) + \partial_x g(\bar{x}, \bar{p})$. For any $(u, p, v) \in \mathrm{gph}\, \partial_x h$ we get from the coderivative sum rule [15, Theorem 1.62] that

$$(\widehat{D}^* \partial h_p)(u, v)(w) = \frac{1}{2}(A + A^*)w + (\widehat{D}^* \partial g_p)\left(u, v - f(\bar{x}, \bar{p}) - \frac{1}{2}(A + A^*)(u - \bar{x})\right)(w). \qquad (4.42)$$

Select $\delta > 0$ to be sufficiently small to ensure that

$$\left(u, p, v - f(\bar{x}, \bar{p}) - \frac{1}{2}(A + A^*)(u - \bar{x})\right) \in I\!\!B_\eta(\bar{x}, \bar{p}, \widehat{v}) \quad \text{for} \quad (u, p, v) \in I\!\!B_\delta(\bar{x}, \bar{p}, \bar{v}). \qquad (4.43)$$

For $(u, p, v) \in \mathrm{gph}\, \partial_x h \cap I\!\!B_\delta(\bar{x}, \bar{p}, \bar{v})$ and $z \in (\widehat{D}^* \partial h_p)(u, v)(w)$ we deduce from (4.42):

$$z - \frac{1}{2}(A + A^*)w \in (\widehat{D}^* \partial g_p)\left(u, v - f(\bar{x}, \bar{p}) - \frac{1}{2}(A + A^*)(u - \bar{x})\right)(w).$$

This together with (4.43) and (4.13) gives us that

$$\langle z, w \rangle \geq \frac{1}{2} \langle (A + A^*)w, w \rangle + \kappa_0 \|w\|^2 - \langle Aw, w \rangle = \kappa_0 \|w\|^2. \qquad (4.44)$$

We may check from the definition of Lipschitz-like property and assumption **(A3)**:

the mapping $p \mapsto \mathrm{epi}\, h(\cdot, p)$ is Lipschitz-like around $\left(\bar{p}, (\bar{x}, h(\bar{x}, \bar{p}))\right)$,

which means that the basic qualification condition in **(A3)** holds also for the function h at $(\bar{x}, \bar{p})$. By [16, Theorem 4.7] it ensures together with (4.44) that the point $\bar{x}$ is a Hölderian fully stable local minimizer associated with the function h from (4.41) relative to the parameter pair $(\bar{v}, \bar{p})$ in the sense of Definition 4.2**(ii)**. Furthermore,

[16, Theorem 3.3] applied to the function h tells us that the Hölderian full stability at $\bar{x}$ yields the *uniform second-order growth condition*. This means that the mapping G_0 in (4.15) with $G_0(v,p) = \{x \in X | \ v \in \partial_x h(x,p)\}$ for $(v,p) \in X \times P$ admits a single-valued localization ϑ_0 relative to some neighborhood $V_0 \times Q_0 \times U_0$ of $(\bar{v}, \bar{p}, \bar{x})$ such that for any triple $(v,p,u) \in \operatorname{gph} \vartheta_0$ we have

$$h(x,p) \geq h(u,p) + \langle v, x - u \rangle + \frac{\kappa_0}{2} \|x - u\|^2 \quad \text{whenever} \quad x \in U_0 \qquad (4.45)$$

and there exists a positive number ℓ_0 for which if $(v_1,p_1), (v_2,p_2) \in V_0 \times Q_0$ we have

$$\|\vartheta_0(v_1,p_1) - \vartheta_0(v_2,p_2)\| \leq \kappa_0^{-1}\|v_1 - v_2\| + \ell_0 d(p_1,p_2)^{\frac{1}{2}}. \qquad (4.46)$$

To apply subsequently the propagation from Lemma 4.4**(i)**, we first check that the starting function ϑ_0 satisfies the Hölder condition (4.16). Indeed, for any (u_1,p,v_1), $(u_2,p,v_2) \in \operatorname{gph} \partial_x h \cap (U_0 \times Q_0 \times V_0)$ the second-order growth (4.45) yields

$$h(u_2,p) \geq h(u_1,p) + \langle v_1, u_2 - u_1 \rangle + \frac{\kappa_0}{2}\|u_2 - u_1\|^2,$$
$$h(u_1,p) \geq h(u_2,p) + \langle v_2, u_1 - u_2 \rangle + \frac{\kappa_0}{2}\|u_1 - u_2\|^2.$$

Summing them up gives us the inequality

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq \kappa_0 \|u_1 - u_2\|^2, \qquad (4.47)$$

which in turn ensures the estimate

$$\|v_1 - v_2 - 2\kappa_0(u_1 - u_2)\|^2 \leq \|v_1 - v_2\|^2.$$

It readily follows from the above that

$$\|v_1 - v_2 - 2\kappa_0(\vartheta_0(v_1,p) - \vartheta_0(v_2,p))\| \leq \|v_1 - v_2\| \quad \text{for all} \quad v_1, v_2 \in V_0, \ p \in Q_0. \quad (4.48)$$

To proceed further, pick any pairs $(v_i,p_i) \in (V_0 \times Q_0)$, $i = 1,2$, and define $u_i := \vartheta_0(v_i,p_i)$, $u_3 := \vartheta_0(v_1,p_2)$. Then we deduce from (4.48) and (4.46) the relationships

$$\begin{aligned}\|v_1 - v_2 - 2\kappa_0(u_1 - u_2)\| &\leq \|v_1 - v_2 - 2\kappa_0(u_3 - u_2)\| + 2\kappa_0\|u_3 - u_1\| \\ &\leq \|v_1 - v_2\| + 2\kappa_0\ell_0 d(p_1,p_2)^{\frac{1}{2}},\end{aligned}$$

which show that the starting single-valued localization ϑ_0 satisfies (4.16).

Now we are in a position to apply Lemma 4.4**(i)** and to do propagation from G_0 to G_τ with $\tau = \frac{1}{2}$. Taking into account that the length of the propagation interval by Lemma 4.4 is $\theta = \kappa_0/(2\|B\|)$ and that the modulus ℓ in (4.16) is doubled at each step, we need to make n steps for reaching $G_{1/2}$ from G_0, where $n \in \mathbb{N}$ is chosen from the interval $\frac{1}{2\theta} \leq n < \frac{1}{2\theta} + 1$. In this way we get from Lemma 4.4**(i)** that G_τ with $\tau = \frac{1}{2}$ admits a single-valued localization ϑ_τ relative to a neighborhood $V_\tau \times Q_\tau \times U_\tau$ of $(\bar{v}, \bar{p}, \bar{x})$ satisfying by (4.17) the following inequality:

$$\left\|v_1 - v_2 - 2\kappa_0\big[\vartheta_\tau(v_1,p_1) - \vartheta_\tau(v_2,p_2)\big]\right\| \leq \|v_1 - v_2\| + 2^n(2\kappa_0\ell_0)d(p_1,p_2)^{\frac{1}{2}} \quad (4.49)$$

for all $(v_1,p_1), (v_2,p_2) \in V_\tau \times Q_\tau$. Finally, we employ Lemma 4.5**(i)** to pass from the linearization $G_{1/2}$ to the solution map S in (4.2). Since $\kappa_0 > 0$ in (4.49), we choose any $0 < \varepsilon < \kappa_0$ in (4.26) and arrive at the Hölder property (4.3) with the assigned modulus $\kappa = \kappa_0 - \varepsilon$ in Definition 4.1**(i)**, which is smaller than but arbitrarily close to κ_0. This verifies the relationships between **(i)** and **(ii)** claimed in Theorem 4.3.

To finish the proof of the theorem, it remains to justify the last statement of the theorem. Suppose that (4.13) is satisfied. Then the map S in (4.2) admits a single valued localization ϑ relative to some neighborhood $V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ such that

$$\|(v_1 - v_2) - 2\kappa[\vartheta(v_1, p_1) - \vartheta(v_2, p_2)]\| \leq \|v_1 - v_2\| + \ell d(p_1, p_2)^{\frac{1}{2}} \text{ for } (v_1, p_1), (v_2, p_2) \in V \times Q$$

with some positive constants κ and ℓ. It follows that

$$-\|v_1 - v_2\| + 2\kappa\|\vartheta(v_1, p_1) - \vartheta(v_2, p_2)\| \leq \|v_1 - v_2\| + \ell d(p_1, p_2)^{\frac{1}{2}},$$

which is equivalent to (4.14) and thus completes the proof of the theorem. $\triangle$

As a consequence of Theorem 4.3, we get the equivalence between Hölderian full stability of PVS (4.1) when $f = 0$ and the notion with the same name for local minimizers of g defined in Definition 4.2(ii) following [16]. In this way, the full stability conditions from Definition 4.1(i) can be seen as a new characterization of Hölderian full stability of local minimizers in scalar optimization.

COROLLARY 4.6. **(Hölderian full stability of local minimizers).** *The point $\bar{x}$ is a Hölderian fully stable local minimizer associated with $g \colon X \times P \to \overline{\mathbb{R}}$ relative to the parameter pair $(\bar{v}, \bar{p})$ if and only if it is a Hölderian fully stable solution to PVS (4.1) with $f = 0$ corresponding to $(\bar{v}, \bar{p})$.*

Proof. It follows from the direct comparison of the characterization of Hölderian full stability obtained in Theorem 4.3 when $f = 0$ and that for Hölderian full stability of local minimizers associated with g established in [16, Theorem 4.7]. $\triangle$

Next we establish second-order characterizations of Lipschitzian full stability of solutions to PVS (4.1) in the sense of Definition 4.1(ii).

THEOREM 4.7. **(neighborhood second-order characterization of Lipschitz full stability of PVS).** *Let $\bar{x} \in S(\bar{v}, \bar{p})$ be the solution to (4.1) corresponding to $(\bar{v}, \bar{p})$. Consider the following statements:*
(i) *The second-order condition (4.13) holds for some $\eta, \kappa_0 > 0$.*
(ii) *The graphical partial subdifferential mapping*

$$K \colon p \mapsto \operatorname{gph} \partial_x g(\cdot, p) \text{ is Lipschitz-like around } (\bar{p}, \bar{x}, \hat{v}). \tag{4.50}$$

*Then $\bar{x}$ is Lipschitzian fully stable for PVS (4.1) if and only if both **(i)** and **(ii)** hold.*

Proof. To justify the "only if" part of the theorem, it remains to show in view of Theorem 4.3 that the Lipschitzian full stability of $\bar{x}$ for (4.1) implies **(ii)**. To proceed, fix the neighborhoods U, Q, V from Definition 4.1(ii) and recall that ϑ the single-valued localization of solution map S (4.2) relative to $V \times Q \times U$ satisfying (4.4). Find a neighborhood $U_1 \times Q_1 \times V_1 \subset U \times Q \times X$ of $(\bar{x}, \bar{p}, \hat{v})$ so that

$$v + f(u, p) = v - \hat{v} + f(u, p) - f(\bar{x}, \bar{p}) + \bar{v} \in V \quad \text{for all} \quad (u, p, v) \in U_1 \times Q_1 \times V_1.$$

Picking any $p_1, p_2 \in Q_1$ and $(u_1, v_1) \in K(p_1) \cap (U_1 \times V_1)$ gives us

$$V \ni v_1' := f(u_1, p_1) + v_1 \in f(u_1, p_1) + \partial_x g(u_1, p_1)$$

and yields $u_1 = \vartheta(v_1', p_1)$. Define $u_2 := \vartheta(v_1', p_2) \in S(v_1', p_2)$, which implies that $v_1' \in f(u_2, p_2) + \partial_x g(u_2, p_2)$ and thus $v_2 := v_1' - f(u_2, p_2) \in \partial_x g(u_2, p_2)$ ensuring therefore that $(u_2, v_2) \in K(p_2)$. Since $u_1 = \vartheta(v_1', p_1)$ and $u_2 = \vartheta(v_1', p_2)$, we deduce from (4.4) that

$$2\kappa\|u_1 - u_2\| = \|(v_1' - v_1') - 2\kappa(u_1 - u_2)\| \leq \|v_1' - v_1'\| + \ell d(p_1, p_2) = \ell d(p_1, p_2). \tag{4.51}$$

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Moreover, it follows from the Lipschitzian property of f in (4.11) that

$$\|v_1 - v_2\| = \|v_1 - v_1' + f(u_2, p_2)\| = \|-f(u_1, p_1) + f(u_2, p_2)\| \le L\big(\|u_1 - u_2\| + d(p_1, p_2)\big)$$

if U_1, Q_1 are chosen to be sufficiently small. This together with (4.51) tells us that

$$
\begin{aligned}
\|u_1 - u_2\| + \|v_1 - v_2\| &\le \|u_1 - u_2\| + L\big(\|u_1 - u_2\| + d(p_1, p_2)\big) \\
&= (1 + L)\|u_1 - u_2\| + L d(p_1, p_2) \\
&\le \tfrac{(1+L)\ell}{2\kappa} d(p_1, p_2) + L d(p_1, p_2).
\end{aligned}
$$

Thus we arrive at the inclusion

$$K(p_1) \cap (U_1 \times V_1) \subset K(p_2) + \left[\frac{(1 + L)\ell}{2\kappa} + L\right] d(p_1, p_2)\,I\!\!B \quad \text{for all} \quad p_1, p_2 \in Q_1,$$

which verifies the claimed Lipschitz-like property of the mapping K in (4.50).

Conversely, let us prove the sufficiency of conditions **(i)** and **(ii)** for the Lipschitzian full stability of $\bar{x} \in S(\bar{v}, \bar{p})$ for (4.1). It follows from Theorem 4.3 and **(i)** that there exist $\kappa, \ell > 0$ and a neighborhood $U \times Q \times V$ of $(\bar{x}, \bar{p}, \bar{v})$ such that the Hölderian condition (4.3) is satisfied. Since K is Lipschitz-like around $(\bar{p}, \bar{x}, \widehat{v})$ by **(ii)**, we find $L_1 > 0$ and a neighborhood $U_1 \times Q_1 \times V_1 \subset U \times Q \times X$ of $(\bar{p}, \bar{x}, \widehat{v})$ for which $v + f(u, p) \in V$ whenever $(u, p, v) \in U_1 \times Q_1 \times V_1$ and

$$K(p_1) \cap (U_1 \times V_1) \subset K(p_2) + L_1 d(p_1, p_2)\,I\!\!B \quad \text{for all} \quad p_1, p_2 \in Q_1. \qquad (4.52)$$

Take the localization ϑ from Definition 4.1**(i)**, which is Hölderian continuous around $(\bar{v}, \bar{p})$ as in (4.28), and get a neighborhood $U_2 \times Q_2 \times V_2 \subset U_1 \times Q_1 \times V$ of $(\bar{x}, \bar{p}, \bar{v})$ such that $\vartheta(V_2 \times Q_2) \subset U_2$ and that

$$v - f(u, p) = v - \bar{v} - (f(u, p) - f(\bar{x}, \bar{p})) + \widehat{v} \subset V_1 \quad \text{for all} \quad (u, p, v) \in U_2 \times Q_2 \times V_2.$$

Pick now any $(v_1, p_1), (v_2, p_2) \in V_2 \times Q_2$ and define $u_1 := \vartheta(v_1, p_1) \in U_2$ and $u_2 := \vartheta(v_2, p_2) \in U_2$. Therefore we have $v_1' := v_1 - f(u_1, p_1) \in \partial_x g(u_1, p_1) \cap V_1$, i.e., $(u_1, v_1') \in K(p_1) \cap (U_1 \times V_1)$. It follows from (4.52) that there is $(u, v) \in K(p_2)$ satisfying the condition

$$\|u - u_1\| + \|v - v_1'\| \le L_1 d(p_1, p_2). \qquad (4.53)$$

Define $v' := f(u, p_2) + v \in f(u, p_2) + \partial_x g(u, p_2)$ and observe from (4.11) and (4.53):

$$
\begin{aligned}
\|v' - v_1\| &= \|f(u, p_2) + v - f(u_1, p_1) - v_1'\| \le \|v - v_1'\| + \|f(u, p_2) - f(u_1, p_1)\| \\
&\le L_1 d(p_1, p_2) + L\big(\|u - u_1\| + d(p_1, p_2)\big) \\
&\le L_1 d(p_1, p_2) + L\big(L_1 d(p_1, p_2) + d(p_1, p_2)\big) \\
&= (L_1 + L L_1 + L) d(p_1, p_2).
\end{aligned}
\qquad (4.54)
$$

Hence we have $v' \in V$ by choosing Q_2 to be sufficiently small, which implies that $u = \vartheta(v', p_2)$. Using now (4.3) for the pairs (v', p_2) and (v_2, p_2) gives us the inequality

$$\|(v' - v_2) - 2\kappa(u - u_2)\| \le \|v' - v_2\|.$$

Combining this with (4.54) and (4.53) ensures the estimates

$$
\begin{aligned}
\|(v_1 - v_2) - 2\kappa(u_1 - u_2)\| &\le \|(v' - v_2) - 2\kappa(u - u_2)\| + \|v' - v_1\| + 2\kappa\|u_1 - u\| \\
&\le \|v' - v_2\| + \|v' - v_1\| + 2\kappa\|u_1 - u\| \\
&\le \|v_1 - v_2\| + \|v' - v_1\| + \|v' - v_1\| + 2\kappa\|u_1 - u\| \\
&\le \|v_1 - v_2\| + 2(L_1 + L L_1 + L) d(p_1, p_2) + 2\kappa L_1 d(p_1, p_2),
\end{aligned}
$$

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which verify (4.4) and thus complete the proof of the theorem. △

Similarly to Corollary 4.6, we can establish the equivalence between the Lipschitzian full stability of solutions to PVS (4.1) with $f = 0$ and Lipschitzian full stability of local minimizers (4.5) for the corresponding optimization problem associated with g. This follows from the comparison of the second-order characterizations obtained in Theorem 4.7 and in [16, Corollary 4.8], respectively.

Observe by the coderivative criterion (2.7) that in the case of $\dim X, \dim P < \infty$ (4.50) is equivalent to the following pointwise condition, see, e.g., [12, Proposition 4.3],

$$(0, q) \in (D^* \partial_x g)(\bar{x}, \bar{p}, \widehat{v})(0) \Longrightarrow q = 0 \qquad (4.55)$$

Our next result provides a complete pointwise characterization of Lipschitzian full stability of PVS via the limiting second-order subdifferential constructions for g.

THEOREM 4.8. **(pointwise characterization of Lipschitzian full stability of PVS in finite dimensions).** *Let X, P be two finite-dimensional spaces and let $\bar{x} \in S(\bar{v}, \bar{p})$. Then the Lipschitzian full stability of $\bar{x}$ for (4.1) is equivalent to the simultaneous validity of (4.55) and the condition*

$$\langle \nabla_x f(\bar{x}, \bar{p})w, w \rangle + \langle z, w \rangle > 0 \;\; \text{for all} \;\; (z, q) \in (D^* \partial_x g)(\bar{x}, \bar{p}, \widehat{v})(w), \; w \neq 0. \quad (4.56)$$

Consequently, conditions (4.55) and (4.56) imply that the solution map S from (4.2) admits a Lipschitz continuous and single-valued localization around $(\bar{v}, \bar{p}, \bar{x})$.

Proof. Similarly to the proof of Theorem 4.3, consider the l.s.c. function $h \colon X \times P \to \overline{\mathbb{R}}$ defined in (4.41) and easily get from the elementary sum rule that

$$\partial_x h(x, p) = f(\bar{x}, \bar{p}) + \frac{1}{2}\big(A + A^*\big)(x - \bar{x}) + \partial_x g(x, p) \;\; \text{and} \;\; \bar{v} \in \partial_x h(\bar{x}, \bar{p}).$$

It follows from [15, Theorem 1.62] the representation

$$\big(D^* \partial_x h\big)(\bar{x}, \bar{p}, \bar{v})(w) = \Big(\frac{1}{2}(A + A^*)w, 0\Big) + \big(D^* \partial_x g\big)(\bar{x}, \bar{p}, \widehat{v})(w). \qquad (4.57)$$

To justify the "only if" part of the theorem, suppose that $\bar{x}$ is a Lipschitzian full stable solution to (4.1) corresponding to $(\bar{v}, \bar{p})$ and find a single-valued localization ϑ of S satisfying (4.4). It follows from Theorem 4.7 that condition (4.50) holds and so does (4.55). By (4.57) we easily have

$$(0, q) \in \big(D^* \partial_x h\big)(\bar{x}, \bar{p}, \bar{v})(0) \Longrightarrow q = 0, \qquad (4.58)$$

which implies the Lipschitz-like property (4.50) as discussed before (4.55). Note also that the Lipschitzian full stability of $\bar{x}$ ensures the validity of condition (4.13) in Theorem 4.3. Furthermore, it is shown in the proof of Theorem 4.3 that the uniform second-order growth condition (4.45) holds. When both properties (4.50) and (4.45) are satisfied, we get from [16, Theorem 4.6] that $\bar{x}$ is a Lipschitzian fully stable local minimizer of h relative to $\bar{p}$ and $\bar{v}$. Then the coderivative criterion of Lipschitzian full stability for the function h from [16, Corollary 4.10] tells us that

$$\langle z, w \rangle > 0 \quad \text{for all} \quad (z, q) \in \big(D^* \partial_x h\big)(\bar{x}, \bar{p}, \bar{v})(w), \; w \neq 0. \qquad (4.59)$$

This together with equality (4.57) implies that

$$0 < \frac{1}{2}\langle (A + A^*)w, w \rangle + \langle z, w \rangle = \langle Aw, w \rangle + \langle z, w \rangle \text{ for } z \in \big(D^* \partial_x g\big)(\bar{x}, \bar{p}, \widehat{v})(w), \; w \neq 0,$$

which ensures (4.56) and thus completes the proof of the necessity part of the theorem.

To verify the sufficiency of conditions (4.55) and (4.56) for the Lipschitz full stability, suppose that these conditions are satisfied and mention again that (4.55) implies (4.58) due to equality (4.57). Similarly to the above but in the opposite direction, we can verify the validity of (4.59) from (4.56) and (4.57). When (4.59) and (4.58) are satisfied, [16, Corollary 4.10] exactly states that $\bar{x}$ is a Lipschitzian fully stable local minimizer associated with h relative to $\bar{p}$ and $\bar{v}$. Then [16, Theorem 3.4] tells us that the uniform second-order growth condition (4.45) holds, i.e., the mapping G_0 in (4.15) admits a single-valued and Lipschitz continuous localization ϑ_0 relative to a neighborhood $V \times Q \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ with some modulus $\kappa_0 > 0$ such that for any triple $(u, p, v) \in \text{gph}\,\vartheta_0$ we have

$$h(x, p) \geq h(u, p) + \langle v, x - u \rangle + \frac{\kappa_0}{2}\|x - u\|^2 \quad \text{whenever} \quad x \in U$$

and there exists a positive number ℓ_0 for which

$$\|\vartheta_0(v_1, p_1) - \vartheta_0(v_2, p_2)\| \leq \kappa_0^{-1}\|v_1 - v_2\| + \ell_0 d(p_1, p_2) \ \text{ if } \ (v_1, p_1), (v_2, p_2) \in V \times Q.$$

By following the lines of the proof after (4.45) and (4.46), we also obtain from the above two facts that the mapping G_0 admits a single-valued localization ϑ_0 satisfying condition (4.18). Employing in this vein the propagation from Lemma 4.4(ii)s finitely many times shows that the mapping G_τ with $\tau = \frac{1}{2}$ also has a single-valued localization ϑ_τ satisfying (4.18). Finally, Lemma 4.5(ii) allows us to pass to the solution map S in (4.2) and completes the proof of the theorem. $\triangle$

When the parameter p is omitted in (4.1), we derive from Theorem 4.8 the following new characterization of local strong maximal monotonicity for the important class of set-valued mappings, which is another evidence of the validity of Conjecture 3.6.

COROLLARY 4.9. *Let X be a finite-dimensional space, let $f \colon X \to X$ be continuously differentiable around $\bar{x}$, and let $g \colon X \to \overline{I\!R}$ be prox-regular and subdifferentially continuous at $\bar{x}$ for some $\widehat{v} \in \partial g(\bar{x})$. The following assertions are equivalent:*
(i) *The set-valued map $T = f + \partial g : X \rightrightarrows X$ is locally strongly maximally monotone around $(\bar{x}, \bar{v})$ with $\bar{v} := f(\bar{x}) + \widehat{v}$.*
(ii) *The limiting coderivative $D^*T(\bar{x}, \bar{v})$ is positive-definite in the sense of (3.21):*

$$\langle z, w \rangle > 0 \quad \text{whenever} \quad z \in D^*T(\bar{x}, \bar{v})(w), \ w \neq 0.$$

Proof. It follows from Lemma 3.3 that **(i)** of this corollary is equivalent to condition (3.4), which exactly is the Lipschitzian full stability condition from Definition 4.1(ii) in the nonparametric case. Theorem 4.8 tells us that the latter is equivalent to

$$\langle \nabla f(\bar{x})w, w \rangle + \langle z, w \rangle > 0 \quad \text{for all} \quad z \in (D^*\partial g)(\bar{x}, \widehat{v})(w), \ w \neq 0,$$

which reduces to (3.21) for $T = f + \partial g$ by the sum rule in [15, Theorem 1.62]. $\triangle$

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