

Variational Analysis and Applications to Group Dynamics

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Abstract

In this paper, we establish a new version of Ekeland's variational principle in a new setting of cone pseudo-quasimetric spaces. In contrast to metric spaces, we do not require that each forward Cauchy sequence is forward convergent and that each forward convergent sequence has the unique forward limit. The motivation of this paper comes from applications in Behavioral Sciences since a cone pseudo-quasimetric helps to modelize multidimensional aspects of resistance to change for a group in a simplified but benchmark behavioral context, where each agent have his own resistance and advantage to change, and where the group has a vector of advantages and resistances to change.

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1 Introduction

In mathematical analysis, Ekeland's variational principle (abbr. EVP), discovered by Ivar Ekeland in [1], is one of the most useful tools in nonlinear analysis and variational analysis. It allows us to study minimization problems in which the lower level set of the problem is not compact; in the other words, when the Bolzano-Weierstrass theorem can not be applied. More precisely, it asserts that there exist nearly optimal solutions to some optimization problems for the class of functionals, defined on a complete metric space being lower semi-continuous, bounded below, and not identically equal to plus infinity. This paper develops a generalized version of the original EVP in the context of cone pseudo-quasimetric spaces. A cone pseudo-quasimetric refers to a vectorial version of scalar pseudo-quasimetrics, where distance is non negative, distance from one point to itself is zero, and satisfies the triangular inequality, without the symmetric axiom of distances. Furthermore, pseudo-quasidistances of two distinct points may be zero. Few papers exist in this direction (as far as we know, let us cite [2, 3, 4]). They focus the attention, in the given order, on Caristi's fixed point theorem, Ekeland's variational principle, and other fixed points results. However, they presented no justification for these extensions. Even more, although all these papers and all the following papers merit attention,

- i) the vast majority of generalizations of EVP done in the less general context of cone metric spaces, where the symmetric axiom is preserved, present no more real justification (see, among others, [5, 6]);

- ii) some papers show that the case of cone metric spaces does not offer a true generalization of metric spaces when the cone have the usual nice properties [7, 8, 9].

On the contrary, our motivation to offer this generalization is very precise. In the context of Behavioral Sciences, it wants to start to modelize the famous but informal theory of Groups dynamics [13, 14], taking advantage of a recent Variational Rationality (abbrev. VR) approach of human behaviors [10, 11, 12]. The pionner of group dynamic, action research, and change processes is the famous psychologist Kurt Lewin [13, 14] who states that human behavior is a function of one person in his group environment and that “a group is composed of more than the sum of its individual members”. He focuses the attention on the way groups and individuals act and react to changing circumstances. His “Force field analysis” looks at two kinds of factors (forces) that influence a temporary situation: forces that drive movement toward a goal (motivational and helping forces) and forces which block movement toward a goal (resistance to change and hindering forces). This balancing principle between driving and restraining forces describes human behavior as a succession of temporary equilibria. It had a profound influence in the fields of social science, psychology, social psychology, organizational development, process management, and change management. In this context, Lewin [13, 14] emphasizes the major role of resistance to change, using, for a typical unit change, a three stage model of change. The first stage, named “unfreezing”, involves overcoming inertia, dismantling the existing “mind set” and bypassing defense mechanisms. In this inital stage agents have to unfreeze the driving and restraining forces that hold the initial situation in a state of quasi-equilibrium. The second stage is change. This is a transition stage of confusion where “the old ways are being challenged but we do not have a clear picture as to what we are replacing them with yet” [13]. To be able to change requires to create an imbalance between forces, i.e., to increase the drivers, and to reduce the restraints. The third and final stage is “freezing”. The new mindset is crystallizing and one’s comfort level is returning to previous levels.

In the same spirit, the VR approach [10, 11, 12] offers, in a dynamical context, a general model of human behaviors. It modelizes a lot of human behaviors as worthwhile stay and change approach or avoidance dynamics. These dynamics include a starting point, a transition and some end (if any). They start from some undesirable initial states. They follow acceptable transitions made of a succession of acceptable stays and changes, because agents do not always stay or always change. To be more concrete, stays refer to habits, routines, norms, rules, exploitation phases, while changes refer to exploration, search, learning and innovations. These transitions can end (if any) in some desired ends, or much before, in some traps, where agents, being stucked in the middle, can fail to reach their desires. Such transitions are worthwhile, hence acceptable, when each successive stay or change is itself worthwhile. This means that, each period, an agent prefers to change rather than to stay if his motivation to change is not lower than his resistance to change; otherwise, he will prefer to stay. An end is a (variational) trap which is worthwhile to approach and reach, but not worthwhile to leave. Quite surprisingly, the VR approach, which initial goal was to propose a general model of human behavior in a dynamical context, not only unified a lot of stay and change dynamics in Behavioral Sciences, but also gave a completely new interpretation of EVP and other famous variational principles in applied mathematics. In a simple context where motivation to change and resistance to change are identified to advantages and inconveniences to change, and when experience does not matter too much, etc., it showed that this famous theorem gives, in a (scalar) quasimetric space, sufficient conditions for the

existence of variational traps.

Our present paper shows how a cone pseudo-quasimetric helps to modelize multidimensional aspects of resistance to change for a group in a simplified but benchmark behavioral context, where each agent have his own resistance and advantage to change, and where the group has a vector of advantages and resistances to change. Such a mathematical model for group dynamics uses the whole machinery of VR concepts.

This paper is organized as follow. Section 2 presents basic definitions and preliminary results in pseudo-quasimetric spaces and cone pseudo-quasimetric spaces. In Section 3, we establish enhanced versions of EVP in cone pseudo-quasimetric spaces which may fail to be complete and whose limit might not be unique. The final Section 4 gives an application to group dynamics.

2 Basic Definitions and Preliminaries

2.1 Pseudo-quasimetric spaces

We use the definitions of pseudo-quasimetric spaces and the corresponding notions of closedness, compactness, and completeness in these spaces in the recent publications [15, 16], where the reader can find more details, discussions, and references; cf. [18, 19, 20, 17].

Definition 2.1. (pseudo-quasimetric spaces). (i) A PSEUDO-QUASIMETRIC on a nonempty set X is a bifunction $q : X \times X \rightarrow \mathbb{R}$, where for all $x, y, z \in X$ the following conditions are satisfied:

(Q1) $q(x, x) \geq 0$ (nonnegativity);

(Q2) $x = y \implies q(x, y) = 0$ (equality implies indistancy);

(Q3) $q(x, z) \leq q(x, y) + q(y, z)$ (triangle inequality).

The pair (X, q) is called a PSEUDO-QUASIMETRIC SPACE.

(ii) A pseudo-quasimetric q is called a QUASIMETRIC, if it satisfies $q(x, y) = 0 \implies x = y$ (indistancy-implies-equality). In the other words, q satisfies (Q1), (Q3), and

(Q2') $x = y \iff q(x, y) = 0$ (identity of indiscernibles).

(iii) A quasimetric q is called a METRIC, if it satisfies the symmetry condition

(Q4) $q(x, y) = q(y, x)$ (symmetry axiom).

It is noted that the notation d is used for metrics instead of q .

It is well known that every pseudo-quasimetric space (X, q) can be considered as a topological space in which a ball center at x with a radius r is defined by $\mathbb{B}_q(x, r) := \{u \in X \mid q(x, u) < r\}$, and that the conjugate \bar{q} of a pseudo-quasimetric q with $\bar{q}(x, y) = q(y, x)$ is also a pseudo-quasimetric. Kelly [18] called the triple (X, q, \bar{q}) a bitopological space. In this paper and others, we try to express all definitions and results in terms of q only because, in behavioral applications, using the whole machinery of the VR approach, $q(x, y)$ and $q(y, x)$ refer to inconveniences to change from x to y , and from y to x . Then, there is no need to consider the conjugate pseudo-quasimetric.

Due to the lack of the symmetry axiom in pseudo-quasimetric spaces, each concept in metric spaces could be splitted into several different notions, but only what will be used in the rest of the paper are presented.

Definition 2.2. (convergence in pseudo-quasimetric spaces, [15, Definition 4]). Let $\{x_n\}$ be a sequence in a pseudo-quasimetric (X, q) . We say that:

- (i) The sequence $\{x_n\}$ is FORWARD CONVERGENT to x_* , if $\lim_{n \rightarrow \infty} q(x_n, x_*) = 0$.
- (ii) The sequence $\{x_n\}$ is BACKWARD CONVERGENT to x_* , if $\lim_{n \rightarrow \infty} q(x_*, x_n) = 0$.

Definition 2.3. (Cauchy sequences in pseudo-quasimetric spaces, [15, Definition 5]). Let $\{x_n\}$ be a sequence in a pseudo-quasimetric space (X, q) . We say that:

- (i) The sequence $\{x_n\}$ is FORWARD CAUCHY, if for each $\varepsilon > 0$ there is an integer $N_\varepsilon \in \mathbb{N}$ such that $q(x_n, x_{n+k}) < \varepsilon$ for all $n \geq N_\varepsilon$ and $k \in \mathbb{N}$.
- (ii) The sequence $\{x_n\}$ is BACKWARD CAUCHY, if for each $\varepsilon > 0$ there is an integer $N_\varepsilon \in \mathbb{N}$ such that $q(x_{n+k}, x_n) < \varepsilon$ for all $n \geq N_\varepsilon$ and $k \in \mathbb{N}$.

Definition 2.4. (completeness in pseudo-quasimetric spaces). Let (X, q) be a pseudo-quasimetric space. We say:

- (i) The space is FORWARD-FORWARD COMPLETE, if every forward Cauchy sequence is forward convergent to some forward limit.
- (ii) The space is BACKWARD-BACKWARD COMPLETE, if every backward Cauchy sequence is backward convergent to some backward limit.

Note that we could define forward-backward and backward-forward complete as well. Note also that we should not reduce ‘forward-forward’ and ‘backward-backward’ into ‘forward’ and ‘backward’ since it was defined in many publications that a (pseudo)-quasimetric space is forward complete (resp. backward complete) if every forward (resp. backward) Cauchy sequence is both forward and backward convergent. Note finally that in a (pseudo)-quasimetric, a sequence being both forward and backward convergent is called bi-convergent (i.e., convergent with respect to two topologies) or simply convergent. In a similar way, the concept of bi-Cauchy (or Cauchy) was defined. These concepts seem to be too restrictive in pseudo-quasimetric spaces. It is proved by Wilson in [17, Theorems I and II] that if $\{x_n\}$ is both forward and backward convergent to x_* in a quasimetric space (X, q) , then x_* is the only limit point of $\{x_n\}$ of any kind and $\{x_n\}$ is bi-Cauchy. Many examples in [19] showed that the forward and backward notions are distinct in (unsymmetric) pseudo-quasimetrics.

2.2 Cone pseudo-quasimetric spaces

Let Z be a topological vector space and Θ be an ordering cone of Z . Assume through out the paper that Θ is a proper ($\Theta \neq \mathbf{0}$ and $\Theta \neq Z$), solid ($\text{int } \Theta \neq \emptyset$), closed ($\Theta = \text{cl } \Theta$), and convex cone ($\Theta + \Theta = \Theta$). We do not required that the cone Θ is either pointed ($\Theta \cap (-\Theta) = \{\mathbf{0}\}$) or normal ($(\mathbb{B} + \Theta) \cap (\mathbb{B} - \Theta) \subset M\mathbb{B}$, where \mathbb{B} is a unit ball and M is a positive number). Recall the Pareto order \leq_Θ and the weak Pareto order $<_\Theta$ induced by the ordering cone Θ that for any $x, y \in Z$, x is said to be stricly less (resp. less) than y with respect to Θ , denoted by $x <_\Theta y$ (resp. $x \leq_\Theta y$), if $x \in y - \text{int } \Theta$ (resp. $x \in y - \Theta$). When Θ is not pointed ($\Theta \cap (-\Theta) \neq \{\mathbf{0}\}$), we could find $\bar{\theta}$ such that $\mathbf{0} \leq_\Theta \bar{\theta}$ and $\bar{\theta} \leq_\Theta \mathbf{0}$. In this case, \leq_Θ does not enjoy the antisymmetric axiom of a partial order.

Definition 2.5. (cone pseudo-quasimetrics). Let X be a nonempty set, Z be topological vector space, and Θ be an ordering cone of Z . Then:

(i) A vectorial function $q_\Theta : X \times X \rightarrow Z$ is called a CONE PSEUDO-QUASIMETRIC, if for any $x, y, z \in X$ the following conditions hold:

$$(Q1) \quad \mathbf{0} \leq_\Theta q_\Theta(x, y), \text{ i.e., } q_\Theta(x, y) \in \Theta;$$

$$(Q2) \quad q_\Theta(x, x) = \mathbf{0};$$

$$(Q3) \quad q_\Theta(x, y) \leq_\Theta q_\Theta(x, z) + q_\Theta(z, y).$$

The quadruple (X, Z, Θ, q_Θ) is called a CONE PSEUDO-QUASIMETRIC SPACE.

(ii) If a cone pseudo-quasimetric q_Θ satisfies

$$(Q2') \quad q_\Theta(x, y) = \mathbf{0} \text{ if and only if } x = y,$$

then it is called a CONE QUASIMETRIC.

(iii) If a cone quasimetric q_Θ enjoys the symmetry axiom

$$(Q4) \quad q_\Theta(x, y) = q_\Theta(y, x),$$

then it is called a CONE METRIC.

Note that a cone pseudo-quasimetric space $(X, \mathbb{R}, \mathbb{R}_+, q)$ is nothing but a (scalar) pseudo-quasimetric space (X, q) defined in Definition 2.1.

Example 2.6. (some cone pseudo-quasimetrics). — Let $X = \mathbb{R}^2, Z = \mathbb{R}^2, \Theta = \mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}, X = \mathbb{R}$, and let $q_\Theta : X \times X \rightarrow \Theta$ be defined by

$$q_\Theta(x, y) = (\max\{y_1 - x_1, 0\}, \max\{y_2 - x_2, 0\}),$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, (X, q_Θ) is a cone pseudo-quasimetric space with the normal cone Θ . It is not a cone quasimetric since $q_\Theta((1, -1), (0, -1)) = (\max\{0 - 1, 0\}, \max\{-1 + 1, 0\}) = (0, 0)$.

— Let $X = \mathbb{R}, Z = \mathbb{R}^2, \Theta = \mathbb{R}_+^2$, and let $q_\Theta : X \times X \rightarrow \Theta$ be defined by

$$q_\Theta(x, y) = (|x - y|, \max\{y - x, 0\}),$$

Then, (X, q_Θ) is a cone quasimetric space with the normal cone Θ .

— Let $X = \mathbb{R}, Z = \mathbb{R}^2, \Theta = \mathbb{R}_+^2$, and let $q_\Theta : X \times X \rightarrow \Theta$ be defined by

$$q_\Theta(x, y) = (|x - y|, \lambda|x - y|),$$

where $\lambda \geq 0$. Then, (X, q_Θ) is a cone-metric space with the normal cone Θ .

— See [33, pp.853-854] for many cone metric spaces and [31] for cone quasimetric spaces.

Recently, many authors generalized the notions in metric spaces in order to obtain principal results that are well known in metric spaces in a ‘more general’ cone metric setting. We simply adopt many of them in the cone pseudo-quasimetric setting.

Definition 2.7. ([31, Definition 8], cf. [33, 32]). *Let (X, Z, Θ, d_Θ) be a cone pseudo-quasimetric space and $\{x_n\}$ be a sequence in X . Then:*

- (i) *the sequence $\{x_n\}$ is FORWARD CONVERGENT to x_* , if $\forall \mathbf{0} <_\Theta \mathbf{c}, \exists N_c \in \mathbb{N}: q_\Theta(x_n, x_*) <_\Theta \mathbf{c}, \forall n \geq N_c$;*
- (ii) *the sequence $\{x_n\}$ is BACKWARD CONVERGENT to x_* , if $\forall \mathbf{0} <_\Theta \mathbf{c}, \exists N_c \in \mathbb{N}: q_\Theta(x_*, x_n) <_\Theta \mathbf{c}, \forall n \geq N_c$;*
- (iii) *the sequence $\{x_n\}$ is FORWARD CAUCHY, if $\forall \mathbf{0} <_\Theta \mathbf{c}, \exists N_c \in \mathbb{N}: q_\Theta(x_n, x_{n+k}) <_\Theta \mathbf{c}, \forall n \geq N_c$ and $\forall k \in \mathbb{N}$;*
- (iv) *the sequence $\{x_n\}$ is BACKWARD CAUCHY, if $\forall \mathbf{0} <_\Theta \mathbf{c}, \exists N_c \in \mathbb{N}$ such that $q_\Theta(x_{n+k}, x_n) <_\Theta \mathbf{c}, \forall n \geq N_c$ and $\forall k \in \mathbb{N}$;*
- (v) *the space (X, Z, Θ, d_Θ) is FORWARD-FORWARD (RESP. BACKWARD-BACKWARD) COMPLETE, if every forward (resp. backward) Cauchy sequence is forward (resp. backward) convergent to some forward (resp. backward) limit in X ;*
- (vi) *a sequence is BI-CAUCHY (or simply CAUCHY), if it is not only forward but also backward Cauchy, i.e., $\forall \mathbf{0} <_\Theta \mathbf{c}, \exists N_c \in \mathbb{N}: q_\Theta(x_n, x_m) <_\Theta \mathbf{c}, \forall n, m \geq N_c$.*

In contrast to Shaddad and Noorani [31], we used the adjectives ‘forward’ and ‘backward’ instead of ‘left’ and ‘right’. The reader is referred to [15, Remark 3] for discussions on the choice of terminologies.

In [32] Jankovich et al. presented a brief reviews on cone-metric spaces in which and the references therein a cone metric space is said to be complete, if every bi-Cauchy sequence is forward convergent to some forward limit.

It is known from [32, 33] that the topological cone metric space (X, Z, Θ, d_Θ) is equivalent to some topological metric space (X, d) provided that Z is a Banach space and Θ is a proper, solid, pointed, closed and convex cone. Let us list several known forms for d :

$$d_1(x, y) = \inf\{\|u\| : u \in \Theta \text{ and } d_\Theta(x, y) <_\Theta u\},$$

$$d_2(x, y) = s_{\mathbf{e}, \Theta} \circ d_\Theta(x, y) \text{ with } s_{\mathbf{e}, \Theta}(z) := \inf\{t \in \mathbb{R} \mid z \in t\mathbf{e} - \Theta\},$$

$$d_3(x, y) = \|d_\Theta(x, y)\| \text{ provided additionally that } \Theta \text{ is normal}^1.$$

In this paper, we strive to establish a new and efficient vector version of EVP. It allows us to show how to construct a forward Cauchy sequence converging to some forward limit and satisfying conclusions in EVP. Among many aforementioned equivalent metric spaces, $d(x, y) = s_{\mathbf{e}, \Theta} \circ d_\Theta$ seems to be best fitted for us to accomplish the goal of this paper because of Propositions 1–4 given just later.

Let us recall several important properties of the nonlinear scalarization function $s_{\mathbf{e}, \Theta}$ introduced by Tammer and Weidner in [34].

Let A be a nonempty subset of Z and $\mathbf{e} \neq \mathbf{0}$ be an element of Z . The functional $s_{A, \mathbf{e}} : Z \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined by

$$s_{A, \mathbf{e}}(z) := \inf\{t \in \mathbb{R} \mid z \in t\mathbf{e} - A\} \tag{1}$$

is called a *nonlinear (separating) scalarization function* (with respect to the set A and the direction \mathbf{e}).

Lemma 2.8. ([35, Theorem 2.3.1]) *Let Z be a real topological linear space, Θ be a solid, closed, and convex cone in Z , and $\mathbf{e} \in \text{int } \Theta$. Then the following hold:*

- (a) $s_{\Theta, \mathbf{e}}(z + t\mathbf{e}) = s_{\Theta, \mathbf{e}}(z) + t, \forall z \in Z$ and $\forall t \in \mathbb{R}$.
- (b) $s_{\Theta, \mathbf{e}}(z) \leq r$ if and only if $z \in r\mathbf{e} - \Theta$.
- (c) $s_{\Theta, \mathbf{e}}(z) < r$ if and only if $z \in r\mathbf{e} - \text{int } \Theta$.
- (d) $s_{\Theta, \mathbf{e}}$ is positive homogeneous and continuous on Z .
- (e) $s_{\Theta, \mathbf{e}}$ is subadditive, i.e., $s_{\Theta, \mathbf{e}}(z_1 + z_2) \leq s_{\Theta, \mathbf{e}}(z_1) + s_{\Theta, \mathbf{e}}(z_2)$ for all $z_1, z_2 \in Z$.
- (f) $s_{\Theta, \mathbf{e}}$ is strictly Θ -monotone, i.e. $[a \leq_{\Theta} b \wedge a \neq b \implies s_{\Theta, \mathbf{e}}(a) < s_{\Theta, \mathbf{e}}(b)]$.

Below are some known results in cone metric spaces which are still valid in cone pseudo-quasimetric spaces.

Proposition 2.9. (equivalence between scalar and cone pseudo-quasimetrics).

(i) *Let $(X, Z, \Theta, q_{\Theta})$ be a cone pseudo-quasimetric space, and let $s := s_{\Theta, \mathbf{e}}$ be a nonlinear scalarization functional of the ordering cone Θ along the positive direction \mathbf{e} defined in (1). The scalarized functional $q = s \circ q_{\Theta} : X \times X \rightarrow \mathbb{R}_+$ is a pseudo-quasimetric.*

(ii) *Let (X, q) be a pseudo-quasimetric space, let Z be a vector space, let Θ be a solid and convex pointed cone of Z , and let $\mathbf{e} \in \text{int } \Theta$ be a positive direction of Z . Then, the bifunction $q_{\Theta} : X \times X \rightarrow Z$ defined by*

$$q_{\Theta}(x, y) \in \Theta \cap (q(x, y)\mathbf{e} - \text{bd } \Theta) \quad (2)$$

is a cone pseudo-quasimetric.

Proof. The proof of (i) is simple and omitted.

To prove (ii), we will check the validness of the three conditions (Q1)–(Q3) in Definition 2.1.

- (Q1) By the definition of q_{Θ} , we have $q_{\Theta}(x, y) \in \Theta \implies \mathbf{0} \leq_{\Theta} q_{\Theta}(x, y)$ for all $x, y \in X$.
- (Q2) By the definition of q_{Θ} and the pointedness of Θ , we have $q_{\Theta}(x, x) \in \Theta \cap (q(x, x)\mathbf{e} - \text{bd } \Theta) = \Theta \cap -\text{bd } \Theta = \{\mathbf{0}\}$ and thus $q_{\Theta}(x, x) = \mathbf{0}$ for all $x \in X$.
- (Q3) For any $x, y, z \in Z$, by the definition of q_{Θ} we can find $\theta_1, \theta_2, \theta_3 \in \Theta$ such that $q_{\Theta}(x, y) = q(x, y)\mathbf{e} - \theta_1$, $q_{\Theta}(y, z) = q(y, z)\mathbf{e} - \theta_2$, and $q_{\Theta}(x, z) = q(x, z)\mathbf{e} - \theta_3$. Taking into account the triangle inequality of the pseudo-quasimetric q ($q(x, z) \leq q(x, y) + q(y, z)$) and the convexity of the cone Θ ($\Theta + \Theta = \Theta$), we have

$$q(x, z)\mathbf{e} - \theta_3 \in q(x, y)\mathbf{e} - \theta_1 + q(y, z)\mathbf{e} - \theta_2 - \Theta$$

clearly verifying that $q_{\Theta}(x, z) \leq_{\Theta} q_{\Theta}(x, y) + q_{\Theta}(y, z)$ and the validity of the triangle inequality of the cone pseudo-quasimetric. □ \triangle

Proposition 2.10. (equivalence between forward convergence concepts). *Let $(X, Z, \Theta, q_{\Theta})$ be a cone pseudo-quasimetric space and $q = s \circ q_{\Theta}$ be the scalarized pseudo-quasimetric of q_{Θ} . Then, a sequence $\{x_n\}$ is forward convergent to x_* in $(X, Z, \Theta, q_{\Theta})$ if and only if it is forward convergent to x_* in (X, q) .*

Proof. Assume that $\{x_n\}$ is forward convergent to x_* in (X, Z, Θ, q_Θ) . By definition, for each $\mathbf{0} <_\Theta c$, there exists $N_c \in \mathbb{N}$ such that $q_\Theta(x_n, x_*) <_\Theta c$ for all $n \geq N_c$. Then, for every $\varepsilon > 0$, there exists $N_\varepsilon = N_{\varepsilon\mathbf{e}}$ such that for every $n \geq N_\varepsilon$ one has $q_\Theta(x_n, x_*) <_\Theta \varepsilon\mathbf{e} \xrightarrow{\text{Lem. 2.8}} q(x_n, x_*) = s(q_\Theta(x_n, x_*)) < s(\varepsilon\mathbf{e}) = \varepsilon$. This means that $\{x_n\}$ is forward convergent to x_* in (X, q) .

Assume now that $\{x_n\}$ is forward convergent to x_* in (X, q) . By definition, for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $q(x_n, x_*) < \varepsilon$ for all $n \geq N_\varepsilon$. Taking into account the properties of s we get from the last inequality that

$$q_\Theta(x_m, x_{m+n}) \in \varepsilon\mathbf{e} - \text{int } \Theta.$$

For each $\mathbf{0} <_\Theta \mathbf{c} \iff \mathbf{0} \in \mathbf{c} - \text{int } \Theta$, since the latter set is open we could find some $\varepsilon > 0$ such that $\varepsilon\mathbf{e} \in \mathbf{c} - \text{int } \Theta$. Thus, by choosing $N_c = N_\varepsilon$ one has

$$q_\Theta(x_m, x_{m+n}) \in \varepsilon\mathbf{e} - \text{int } \Theta \subset \mathbf{c} - \text{int } \Theta,$$

i.e., $q_\Theta(x_m, x_{m+n}) <_\Theta \mathbf{c}$ for all $n \geq N_c$ clearly justifying that $\{x_n\}$ is forward convergent to x_* in (X, Z, Θ, q_Θ) . The proof is complete. \square \triangle

Proposition 2.11. (equivalence between forward Cauchy sequences). *Let (X, Z, Θ, q_Θ) be a cone pseudo-quasimetric space and $q = s \circ q_\Theta$ be the scalarized pseudo-quasimetric of q_Θ . Then, a sequence $\{x_n\}$ is forward Cauchy in (X, Z, Θ, q_Θ) if and only if it is forward Cauchy in (X, q) .*

Proof. Assume that a sequence $\{x_n\}$ is forward Cauchy in (X, Z, Θ, q_Θ) . By definition, for each $\mathbf{0} <_\Theta \mathbf{c}$, there exists $N_c \in \mathbb{N}$ such that for all $n \geq N_c$ and for all $k \in \mathbb{N}$ one has $q_\Theta(x_n, x_{n+k}) <_\Theta \mathbf{c}$. Then, for every $\varepsilon > 0$, there exists $N_\varepsilon = N_{\varepsilon\mathbf{e}}$ such that for all $n \geq N_\varepsilon$ and for all $k \in \mathbb{N}$ one has $q_\Theta(x_n, x_{n+k}) <_\Theta \varepsilon\mathbf{e}$ and thus $q(x_n, x_{n+k}) = s(q_\Theta(x_n, x_{n+k})) < s(\varepsilon\mathbf{e}) = \varepsilon$ due to Lemma 2.8. This proves that $\{x_n\}$ is a forward Cauchy sequence in (X, q) .

Assume now that $\{x_n\}$ is forward Cauchy in (X, q) . By definition, for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$ and for all $k \in \mathbb{N}$ one has $q(x_n, x_{n+k}) < \varepsilon$. Taking into account the properties of s we get from the last inequality that

$$q_\Theta(x_n, x_{n+k}) \in \varepsilon\mathbf{e} - \text{int } \Theta.$$

Fix an arbitrary element $\mathbf{0} <_\Theta \mathbf{c} \iff \mathbf{0} \in \mathbf{c} - \text{int } \Theta$. Since $\mathbf{0}$ belongs to the open set $\mathbf{c} - \text{int } \Theta$, there exists $\varepsilon > 0$ such that $\varepsilon\mathbf{e} \in \mathbf{c} - \text{int } \Theta$. Taking $N_c = N_\varepsilon$ one has

$$q_\Theta(x_n, x_{n+k}) \in \varepsilon\mathbf{e} - \text{int } \Theta \subset \mathbf{c} - \text{int } \Theta,$$

i.e., $q_\Theta(x_m, x_{m+n}) \leq_\Theta \mathbf{c}$ for all $n \geq N_\varepsilon$ and for all $k \in \mathbb{N}$. Therefore, $\{x_n\}$ is a forward Cauchy sequence in (X, Z, Θ, q_Θ) . The proof is complete. \square \triangle

Proposition 2.12. (equivalence on completeness concepts). *A cone pseudo-quasimetric space (X, Z, Θ, q_Θ) is forward-forward complete if and only if the scalarized pseudo-quasimetric space (X, q) is forward-forward complete, where $q = s \circ q_\Theta$ is the scalarized pseudo-quasimetric of q_Θ .*

Proof. It is immediate from Propositions 2.10 and 2.11. \square \triangle

3 Cone Pseudo-Quasimetrics

In this section we establish a new version of EVP in cone pseudo-quasimetric spaces. In contrast to existing vectorial principles, which were derived by using a scalarization functional to convert a vector-valued function into an extended-real-valued functional, we combine the variational techniques used in [21, 22, 23] and the nonlinear scalarization functional discussed in Section 3. First, let us briefly explain why the scalarization approach is not applicable. Given a vector function $f : X \rightarrow Z$ and a cone pseudo-quasimetric $q_\Theta : X \times X \rightarrow Z$, we have a scalarized cost function $\varphi := s_{\Theta, \mathbf{e}} \circ f$ and a scalarized pseudo-quasimetric $q := s_{\Theta, \mathbf{e}} \circ q_\Theta$. Under appropriate assumptions, the original EVP ensures that for any $x_0 \in \text{dom } f$ there is x_* such that

- (i) $(s_{\Theta, \mathbf{e}} \circ f)(x_*) + \lambda(s_{\Theta, \mathbf{e}} \circ q_\Theta)(x_0, x_*) \leq (s_{\Theta, \mathbf{e}} \circ f)(x_0)$ and
- (ii) $(s_{\Theta, \mathbf{e}} \circ f)(x) + \lambda(s_{\Theta, \mathbf{e}} \circ q_\Theta)(x_*, x) > (s_{\Theta, \mathbf{e}} \circ f)(x_*)$, $\forall x \neq x_*$.

In contrast to one dimension vectorial metric $q(x, y)\mathbf{e}$ for some $q : X \times X \rightarrow \mathbb{R}$, in the cone pseudo-quasimetric setting it is impossible to represent assertions (i) and (ii) in terms of the given data f and q_Θ only; i.e., no composition is present.

To formulate and justify our version of EVP, we use the so-called dynamic system of f and q_Θ with respect to $\lambda > 0$ denoted by $S : X \rightrightarrows X$ and defined by

$$S_{f, \lambda}(x) := \{u \in X \mid f(u) + \lambda q_\Theta(x, u) \leq_\Theta f(x)\} \quad \forall x \in \text{dom } f. \quad (3)$$

Definition 3.1. (generalized-Picard sequences). *We say that a sequence $\{x_n\}$ in X is a GENERALIZED-PICARD SEQUENCE of $S_{f, \lambda}$, if $x_{n+1} \in S_{f, \lambda}(x_n)$ for all $n \in \mathbb{N}$. It is called Θ -DECREASING FORWARD-CAUCHY GENERALIZED-PICARD, if it is both forward-Cauchy and generalized-Picard and the sequence $\{f(x_n)\}$ is Θ -decreasing, i.e., $f(x_{n+1}) \leq_\Theta f(x_n)$ for all $n \in \mathbb{N}$.*

Theorem 3.2. (a cone pseudo-quasimetric version of EVP). *Let (X, Z, Θ, q_Θ) be a cone pseudo-quasimetric space, let $f : X \rightarrow Z$ be a vector-valued function with $\text{dom } f \neq \emptyset$, let $s := s_{\Theta, \mathbf{e}}$ be a nonlinear scalarization functional of the ordering cone Θ along the positive direction $\mathbf{e} \in \text{int } \Theta$ defined in (1), let $q := s \circ q_\Theta$ be the scalarized pseudo-quasimetric, and let $S_{f, \lambda} : X \rightrightarrows X$ be a set-valued mapping defined in (3). Given $x_0 \in \text{dom } f$ and $\lambda > 0$. Assume that Θ be a closed, solid and convex cone of Z . Assume also that*

- (H1) **(boundedness)** *f is bounded from below over the set $S_{f, \lambda}(x_0)$, i.e., there is some element $\underline{z} \in Z$ such that $\underline{z} \leq_\Theta f(x)$ for all $x \in S_{f, \lambda}(x_0)$;*
- (H2) **(nonempty intersection)** *for any strictly- Θ -decreasing forward-Cauchy generalized-Picard sequence $\{x_n\}_{n=0}^\infty$ of $S_{f, \lambda}$, there exists $y \in X$ such that $S_{f, \lambda}(y) \subset S_{f, \lambda}(x_n)$ for all $n \in \mathbb{N}$;*
- (H3) **(scalarized convergence)** *for any Θ -decreasing generalized-Picard sequence $\{x_n\}_{n=0}^\infty$ of $S_{f, \lambda}$, if the series $\sum_{n=1}^\infty q_\Theta(x_n, x_{n+1})$ is convergent in Z , i.e., there is $\bar{z} \in Z$ such that $\sum_{n=1}^\infty q_\Theta(x_n, x_{n+1}) \leq_\Theta \bar{z}$, then the scalarized distances $q(x_n, x_{n+1})$ tend to zero as $n \rightarrow \infty$.*

Then, there is a Θ -decreasing forward-Cauchy generalized-Picard sequence $\{x_n\}$ starting from x_0 and forward converging to a forward limit x_* in the scalarized pseudo-quasimetric (X, q) such that

(i) $f(x_*) + \lambda q_\Theta(x_0, x_*) \leq_\Theta f(x_0)$ and

(ii) $f(x) + \lambda q_\Theta(x_*, x) \not\leq_\Theta f(x_*)$, $\forall x \in X \setminus \overline{\{x_*\}}$,

where $\overline{\{x_*\}} = \{y_* \in X \mid q(x_*, y_*) = 0\} = \{y_* \in X \mid q_\Theta(x_*, y_*) = \mathbf{0}\}$.

Furthermore, (ii) reduces to

(ii') $f(x) + \lambda q_\Theta(x_*, x) \not\leq_\Theta f(x_*)$, $\forall x \neq x_*$

provided that the forward limit uniqueness condition

(H4) for any strictly- Θ -decreasing forward-Cauchy generalized-Picard sequence $\{x_n\}_{n=0}^\infty$ of $S_{f,\lambda}$, it has at most one forward limit, i.e.,

$$\lim_{n \rightarrow \infty} q(x_n, y_*) = \lim_{n \rightarrow \infty} q(x_n, x_*) = 0 \implies y_* = x_*,$$

holds.

Proof. Starting from the given element $x_0 \in \text{dom } f$, we construct a forward Cauchy generalized Picard sequence of $S_{f,\lambda}$ which converges to some forward limit x_* and satisfies both (i) and (ii).

Let x_n be given. The set $S_{f,\lambda}(x_n) \neq \emptyset$ since $x_n \in S_{f,\lambda}(x_n)$. If $\sup_{x \in S_{f,\lambda}(x_n)} q(x_n, x) = 0$, then $x_{n+1} = x_n$; otherwise the next iteration x_{n+1} is chosen from the nonempty set $S_{f,\lambda}(x_n)$, i.e., $x_{n+1} \in S_{f,\lambda}(x_n)$, such that

$$q(x_n, x_{n+1}) \geq \sup_{x \in S_{f,\lambda}(x_n)} q(x_n, x) - 2^{-n} \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (4)$$

Consider two cases:

Case 1. The sequence $\{x_n\}$ is eventually constant, i.e., there is $n_* \in \mathbb{N}$ such that $x_n = x_*$ for all $n \geq n_*$. In this case, $x_* \in S_{f,\lambda}(x_0)$ and $\sup_{x \in S_{f,\lambda}(x_*)} q(x_*, x) = 0$, i.e., $S_{f,\lambda}(x_*) \subset \overline{\{x_*\}}$. Obviously, they implies (i) and (ii) respectively.

Case 2. The sequence $\{x_n\}$ has all forward distances nonzero, i.e., $q(x_n, x_{n+1}) > 0$. By Lemma 2.8(c), $q_\Theta(x_n, x_{n+1}) \in \text{int } \Theta$. This together with $x_{n+1} \in S_{f,\lambda}(x_n)$ yields $f(x_{n+1}) <_\Theta f(x_n)$. Thus, $\{x_n\}$ is a strictly- Θ -decreasing generalized-Picard sequence of $S_{f,\lambda}$. In addition, it enjoys the following properties:

(a) $S_{f,\lambda}(x_{n+1}) \subset S_{f,\lambda}(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$. This is straightforward from the definition of $S_{f,\lambda}$ and the triangle inequality of q .

(b) $\lim_{n \rightarrow \infty} q(x_n, x_{n+1}) = 0$ and $\lim_{n \rightarrow \infty} \sup_{x \in S_{f,\lambda}(x_n)} q(x_n, x) = 0$.

For any $n \in \mathbb{N} \cup \{0\}$, $x_{n+1} \in S_{f,\lambda}(x_n)$ yields $f(x_{n+1}) + \lambda q_\Theta(x_n, x_{n+1}) \leq_\Theta f(x_n)$. Summing up these inequalities from $n = 0$ to $m - 1$ gives

$$\lambda \sum_{n=0}^{m-1} q_\Theta(x_n, x_{n+1}) \leq_\Theta f(x_0) - f(x_m) \leq_\Theta f(x_0) - \underline{z}, \quad (5)$$

where the last estimate holds due to the lower boundedness condition (H1). Passing to limit as $m \rightarrow \infty$ ensures the boundedness from above of the series $\sum_{n=0}^{\infty} q_{\Theta}(x_n, x_{n+1})$. By the convergence condition (H3), the series $\sum_{n=0}^{\infty} q(x_n, x_{n+1})$ is convergent and thus $\lim_{n \rightarrow \infty} q(x_n, x_{n+1}) = 0$ due to the divergence theorem. Then, we get from the inequality in (4) that

$$0 \leq \lim_{n \rightarrow \infty} \sup_{x \in S_{f,\lambda}(x_n)} q(x_n, x) \leq \lim_{n \rightarrow \infty} (q(x_n, x_{n+1}) + 2^{-n}) = 0.$$

(c) $\{x_n\}$ is a forward-Cauchy sequence in (X, q) . The convergence of the series $\sum_{n=0}^{\infty} q(x_n, x_{n+1})$ says that for every $\varepsilon > 0$, there is $N_{\varepsilon} \in \mathbb{N}$ such that for all $m \geq N_{\varepsilon}$ and for all $k \in \mathbb{N}$ we have

$$q(x_m, x_{m+k}) \leq \sum_{n=m}^{m+k-1} q(x_n, x_{n+1}) \leq \sum_{n=m}^{\infty} q(x_n, x_{n+1}) < \varepsilon,$$

clearly verifying that $\{x_n\}$ is a forward-Cauchy sequence in (X, q) .

(d) $\{x_n\}$ has a forward limit which satisfies (i) and (ii).

Employing the nonempty intersection condition (H2) to the chosen Θ -decreasing forward-Cauchy generalized-Picard sequence $\{x_n\}$, we obtain the existence of $x_* \in X$ such that

$$x_* \in S_{f,\lambda}(x_*) \subset S_{f,\lambda}(x_n) \forall n \in \mathbb{N} \cup \{0\}. \quad (6)$$

Obviously, (i) holds by taking $n = 0$ in (6). To justify the validity of (ii), it is sufficient to check $S_{f,\lambda}(x_*) \subset \overline{\{x_*\}}$. We get from (6), (a) and (b) that

$$\sup_{S_{f,\lambda}(x_*)} q(x_*, y_*) = 0$$

and thus $q(x_*, y_*) = 0$ for all $y_* \in S_{f,\lambda}(x_*)$, i.e., $S_{f,\lambda}(x_*) \subset \overline{\{x_*\}}$.

Note that (ii) \iff (ii') since the fulfilment of (H4) implies that $\overline{\{x_n\}} = \{x_*\}$. The proof is complete. \square \triangle

Proposition 3.3. (several sufficient conditions for (H3)). *The fulfilment of one of the following conditions:*

(H3') $\mathbf{0} \notin \text{cl conv}(\Theta \setminus [\mathbf{0}, \mathbf{e}]);$

(H3'') $z^*(z) \geq \gamma \forall z \in \Theta \setminus [\mathbf{0}, \mathbf{e}]$ for some $z^* \in Z^*$ and $\gamma > 0$,

where $[\mathbf{0}, \mathbf{e}] := (\mathbf{0} + \Theta) \cap (\mathbf{e} - \text{int } \Theta)$, implies the validity of the convergence condition (H3) in Theorem 3.2.

Proof. It is obvious that (H3') \implies (H3'') thanks to the classical separation theorem. Therefore, it remains to show (H3'') \implies (H3).

Arguing by contradiction, assume that (H3'') holds, but (H3) does not. The latter ensures the existence of some $\bar{z} \in Z$ such that $\sum_{n=1}^{\infty} q_{\Theta}(x_n, x_{n+1}) \leq_{\Theta} \bar{z}$, and the existence of a positive number ε and an integer number N_{ε} such that

$$q(x_n, x_{n+1}) = s(q_{\Theta}(x_n, x_{n+1})) > \varepsilon \forall n \geq N_{\varepsilon}.$$

By Lemma 2.8 for any $n \geq N_\varepsilon$ we have

$$q_\Theta(x_n, x_{n+1}) \notin \varepsilon \mathbf{e} - \text{int } \Theta \implies \frac{1}{\varepsilon} q_\Theta(x_n, x_{n+1}) \in \Theta \cap [\mathbf{0}, \mathbf{e}].$$

This together with the imposed condition (H3') implies

$$\frac{1}{\varepsilon} z^* \left(\sum_{n=N_\varepsilon}^m q_\Theta(x_n, x_{n+1}) \right) = \sum_{n=N_\varepsilon}^m z^* \left(\frac{1}{\varepsilon} q_\Theta(x_n, x_{n+1}) \right) \geq (m - N_\varepsilon) \gamma$$

for all $m \geq N_\varepsilon$. Passing to limit as $m \rightarrow \infty$ we arrive at

$$z^* \left(\sum_{n=0}^{\infty} q_\Theta(x_n, x_{n+1}) \right) \geq z^* \left(\sum_{n=N_\varepsilon}^{\infty} q_\Theta(x_n, x_{n+1}) \right) = \infty$$

which contradicts the boundedness of the series $z^* \left(\sum_{n=1}^{\infty} q_\Theta(x_n, x_{n+1}) \right) \leq z^*(\bar{z}) < \infty$. This contradiction proves the validity of (H3). The proof is complete. \square \triangle

Next, we provide an example illustrating that the condition (H3) is satisfied by closed, solid, and convex cones being not pointed (and thus not normal).

Example 3.4. Let $Z = \mathbb{R}^2$ and $\Theta = \mathbb{R}_+ \times \mathbb{R}$. Obviously, Θ is not a pointed cone since $\Theta \cap (-\Theta) = \{0\} \times \mathbb{R}$. By choosing $\mathbf{e} = (1, 0) \in \text{int } \Theta$, we have $(0, 0) \notin \text{cl conv}(\Theta \setminus [\mathbf{0}, \mathbf{e}]) = [1, +\infty) \times (-\infty, +\infty)$ and thus the validity of (H3').

Now by taking $z^* = (1, 0)$, for any $z = (z_1, z_2) \in \text{cl conv}(\Theta \setminus [\mathbf{0}, \mathbf{e}]) = [1, +\infty) \times (-\infty, +\infty)$ implying that $z_1 \geq 1$ and $z_2 \in \mathbb{R}$, we have $z^*(z) = z_1 \geq 1$ clearly justifying the validity of (H3').

The next two propositions provide two special cone pseudo-quasimetrics which unconditionally satisfy the convergence condition (H3).

Proposition 3.5. Let q be a pseudo-quasimetric and Θ be an ordering cone (which is not necessarily solid). Then, the cone pseudo-quasimetric defined by $q_\Theta(x, y) := q(x, y)\mathbf{e}$ for some element $\mathbf{e} \in \Theta$ satisfies the scalarized convergence condition (H3).

Proof. It is immediate from the structure of the metric. \square \triangle

Proposition 3.6. Let q be a pseudo-quasimetric and Θ be an ordering cone. Then, the cone pseudo-quasimetric defined by $q_\Theta(x, y) := q(x, y)\mathbf{H}$, where $H \subset \Theta$ is a convex subset of Θ with $\mathbf{0} \notin \text{cl } H$, satisfies the scalarized convergence condition (H3).

Proof. Arguing by contradiction, assume that (H3') does not hold. Then, we could find some $\bar{\theta} \in \Theta$ such that $\sum_{n=1}^{\infty} q_\Theta(x_n, x_{n+1}) \leq_\Theta \bar{\theta}$, but $s_k := \sum_{n=1}^k q(x_n, x_{n+1}) \rightarrow \infty$ as $n \rightarrow \infty$. Fix a sequence $\{h_n\} \subset H$ such that $\sum_{n=1}^{\infty} q_\Theta(x_n, x_{n+1})h_n \in \bar{\theta} - \Theta$. Since H is a convex set, we have

$$\frac{a_k}{s_k} \in H \quad \text{with} \quad a_k := \sum_{n=1}^k q_\Theta(x_n, x_{n+1})h_n \leq \bar{\theta}, \quad \forall k \in \mathbb{N}.$$

Passing to limit as $k \rightarrow \infty$, we have $\mathbf{0} \in \text{cl } H$. Contradiction. Thus, (H3) holds true. \square \triangle

The next result is a sufficient condition for (H2) in terms of some type of lower-semicontinuity of f .

Definition 3.7. Let (X, Z, Θ, q_Θ) be a cone pseudo-quasimetric space. The function $f : X \rightarrow Z$ is said to be Θ -DECREASINGLY LOWER-SEMICONINUOUS on X if, for any Θ -decreasingly sequence $\{x_n\}$ in X being forward convergent to x_* , the validity of $f(x_n) \leq_\Theta c$ for some $c \in Z$ and for all $n \in \mathbb{N}$ implies $f(x_*) \leq_\Theta c$.

Proposition 3.8. (a sufficient condition for (H2)). Let (X, Z, Θ, q_Θ) be a cone pseudo-quasimetric space and $q = s \circ q_\Theta$ be the scalarized pseudo-quasimetric of q_Θ . Assume that (X, q) is forward-forward complete and that the vectorial function $f : X \rightarrow Z$ is Θ -decreasingly forward-lower-semicontinuous on $S_{f,\lambda}(x_0)$, where $S_{f,\lambda} : X \rightrightarrows X$ is the set-valued mapping defined by (3). Then, the condition (H2) in Theorem 3.2 holds.

Proof. Fix an arbitrary forward-Cauchy generalized Picard sequence $\{x_n\}_{n=0}^\infty$ of $S_{f,\lambda}$. By Proposition 2.12, since (X, Z, Θ, q_Θ) is forward-forward complete, so is the space (X, q) . Therefore, we could find some $x_* \in X$ such that $q(x_n, x_*) \rightarrow 0$ as $n \rightarrow \infty$. Since $\{x_n\}$ is a generalized Picard sequence of $S_{f,\lambda}$, we have $x_{n+1} \in S(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$. By the definition of $S_{f,\lambda}$ in (3), we have

$$f(x_{n+1}) + \lambda q_\Theta(x_n, x_{n+1}) \leq_\Theta f(x_n) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Summing up these inequalities while taking into account the triangle inequality of q_Θ , we have

$$f(x_{m+k+l}) + \lambda q_\Theta(x_m, x_{m+k}) \leq_\Theta f(x_{m+k+l}) + \lambda \sum_{n=m}^{m+k+l-1} q_\Theta(x_n, x_{n+1}) \leq_\Theta f(x_m)$$

for all $m, k, l \in \mathbb{N}$, which leads to

$$f(x_{m+k+l}) \leq_\Theta f(x_m) - \lambda q_\Theta(x_m, x_{m+k}) \quad \text{for all } l \in \mathbb{N}.$$

By the assumed Θ -decreasingly forward-lower-semicontinuity of f on $S_{f,\lambda}(x_0)$, we have

$$f(x_*) \leq_\Theta f(x_m) - \lambda q_\Theta(x_m, x_{m+k}).$$

Using the triangle inequality of q_Θ , we get from the last inequality that

$$f(x_*) + \lambda(q_\Theta(x_m, x_*) - q_\Theta(x_{m+k}, x_*)) \leq_\Theta f(x_*) + \lambda q_\Theta(x_m, x_{m+k}) \leq_\Theta f(x_m) \quad \forall k \in \mathbb{N}.$$

Passing to limit as $k \rightarrow \infty$ while taking into account the closeness of Θ , we get

$$f(x_*) + \lambda q_\Theta(x_m, x_*) \leq_\Theta f(x_*),$$

i.e., $x_* \in S_{f,\lambda}(x_m)$. Since m was arbitrary, the nonempty intersection condition (H2) holds. The proof is complete. □ △

It is ready for us to formulate a simple vectorial version of EVP in cone pseudo-quasimetric spaces (X, Z, Θ, q_Θ) which reduces to the original one when $(X, Z, \Theta, q_\Theta) = (X, \mathbb{R}, \mathbb{R}_+, |\cdot|)$.

Corollary 3.9. (an enhanced version of EVP for cone pseudo-quasimetrics). Let (X, Z, Θ, q_Θ) , $f : X \rightarrow Z$, Θ , \mathbf{e} , $s = s_{\Theta, \mathbf{e}}$, $q = s \circ q_\Theta$ and $S_{f,\lambda} : X \rightrightarrows X$ as in Theorem 3.2. Given $x_0 \in \text{dom } f$ and $\lambda > 0$. Assume that

(H1) f is Θ -bounded from below over $S_{f,\lambda}(x_0)$.

(H2') (X, Z, Θ, q_Θ) is forward-forward complete and f is Θ -decreasingly forward-lower-semicontinuous on $S_{f,\lambda}(x_0)$.

(H3'') $\mathbf{0} \notin \text{cl conv}(\Theta \setminus [\mathbf{0}, \mathbf{e}])$ with $[\mathbf{0}, \mathbf{e}] = (\mathbf{0} + \Theta) \cap (\mathbf{e} - \text{int}\Theta)$.

hold. Then, there is $x_* \in X$ such that

(i) $f(x_*) + \lambda q_\Theta(x_0, x_*) \leq_\Theta f(x_0)$ and

(ii) $f(x) + \lambda q_\Theta(x_*, x) \not\leq_\Theta f(x_*)$, $\forall x \notin \overline{\{x_*\}}$,

where $\overline{\{x_*\}} = \{u \in X \mid q(x_*, u) = 0\}$.

Proof. It is immediate from Theorem 3.2, and Propositions 3.8 and 3.3. \square \triangle

Corollary 3.10. (a forward version of EVP). *Let (X, q) be a pseudo-quasimetric space and let $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended-real-valued functional. Assume that the space (X, q) is strictly- φ -decreasingly forward-complete, the functional φ is proper, bounded from below, and strictly-decreasing forward-lower-semicontinuous. Given $x_0 \in \text{dom } \varphi$ and $\lambda > 0$, consider a set-valued mapping $S_{\varphi,\lambda} : X \rightrightarrows X$ defined by (3) with $f = \varphi$. Then, there is $x_* \in X$ such that*

(i) $\varphi(x_*) + \lambda q(x_0, x_*) \leq \varphi(x_0)$;

(ii) $\varphi(x) + \lambda q(x_*, x) > \varphi(x_*)$, $\forall x \in X \setminus \overline{\{x_*\}}$,

where $\overline{\{y_*\}} = \{u \in X \mid q(y_*, u) = 0\}$.

Proof. It is straightforward from Corollary 3.9 in the space $(X, \mathbb{R}, \mathbb{R}_+, q)$, $\Theta = \mathbb{R}_+$, $\mathbf{e} = 1$, and $s_{e,\Theta}(t) = t$ for all $t \in \mathbb{R}$. \square \triangle

Let us conclude this section and the paper with some comparisons and remarks.

Remark 3.11. (a) *To the best of our knowledge, Theorem 3.2 and Corollary 3.9 are new in two regards: (1) the results are established for cone pseudo-quasi-metric spaces in which a forward-convergent sequence may have two distinct forward limits; (2) the cost vector-valued function enjoys the strictly- Θ -decreasing forward-lower-semicontinuity which is strictly broader than the existing kinds of lower semicontinuity. Corollary 3.10 is not as strong as [16, Corollary 3.14] and [28, Theorem 2(3)]. They were characterizations for the completeness of pseudo-quasimetric spaces and quasimetric spaces, respectively.*

(b) *Although cost functions and cone pseudo-quasimetrics are single-valued, the obtained results can be extended to set-valued mappings and set-valued cone pseudo-quasimetrics, respectively. Recall from [23, Definition 5.1] that a mapping $D : X \times X \rightrightarrows Z$ is said to be a set-valued Θ -quasimetric if it satisfies*

(D1) $D(x_1, x_2) \subset \Theta$ for all $x_1, x_2 \in X$.

(D2) $\mathbf{0} \in D(x, x)$ for all $x \in X$.

(D3) $D(x_1, x_2) + D(x_2, x_3) \subset D(x_1, x_3) + \Theta$ for all $x_1, x_2, x_3 \in X$.

(c) The convergence condition (H3) has its root in [36], where one of the first vectorial versions of EVP was established in terms of vector-valued distances. Precisely, consider an ordered space (Z, \preceq) and a distance $d : X \times X \rightarrow Z$. The space X is called to be d -complete if

$$\sum_{n=1}^m q(x_n, q_{n+1}) \preceq \bar{z} \text{ for some } \bar{z} \in Z, \text{ for all } m \in \mathbb{N}$$

implies that the sequence $\{x_n\}$ converges to some $x_* \in X$.

In this direction, Nemeth further developed them in [37] for distances $d_\Theta : G \times G \rightarrow \Theta$ from an ordered topological Abelian group G with closed positive quasiwedge Θ in G . A similar condition of (H3)-type is that there exists a neighborhood U of the zero element such that any sequence $\{x_n\}$ with the property $d_\Theta(x_n, x_{n+1}) \in \Theta \setminus U$ implies that the set

$$\left\{ \sum_{n=1}^m d_\Theta(x_n, x_{n+1}), n \in \mathbb{N} \right\}$$

cannot be Θ -bounded from above.

(d) In contrast to our approach, the authors in [23, 38, 39, 40] worked with two given spaces: a pseudo-quasimetric (X, q) with $q : X \times X \rightarrow \mathbb{R}_+$ and a cone quasimetric $D : X \times X \rightarrow \Theta$ defined in (a), where Θ is an ordering cone of some normed space Z , and imposed the so-called convergence comparison condition; see [23, Theorem 5.2 (F3)]: for any Θ -decreasing sequence $\{x_n\}$, the upper bound of the series

$$\sum_{n=1}^{\infty} D(x_n, x_{n+1}) \subset M - \Theta,$$

with some bounded set M in Z ensures that $q(x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. Obviously, if q happens to be the scalarized pseudo-quasimetric of D , then this condition is identical to the condition (H3). It is proved in [23, Theorem 5.3] that the convergence comparison condition is satisfied provided that there exists $z^* \in \Theta^+$ and $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\inf \left\{ z^*(z) \mid z \in \cup_{q(x_1, x_2) \geq \delta} D(x_1, x_2) \right\} \geq \eta(\delta) > 0 \text{ for all } \delta > 0.$$

It seems to be more complicated than the sufficient conditions presented in Proposition 3.3.

4 An Application to Group Dynamics

4.1 A worthwhile stay or change group dynamic.

Let us define the simplest (not hierarchical) worthwhile stay or change group dynamic, using the VR concepts and modelization, see [12] for a general formulation. Consider a group of agents $j \in J \subset \mathbb{N}$ and its dynamic. In the last and current periods $n, n+1$, each agent j carries out a last action x_n^j and a current action x_{n+1}^j . Then, in the current period $n+1$, given their past experiences $e = e_n = (e_n^j)_{j \in J}$, agents moves from the profile of last actions $x = x_n = (x_n^j)_{j \in J}$ to carry out the current profile of actions $y = x_{n+1} = (x_{n+1}^j)_{j \in J}$. The current move $x \rightsquigarrow y$ is a change (resp. stay), if $y \neq x$ (resp. $y = x$). The current motivation to change of agent j is $M_e^j(x, y) = U_e^j(A_e^j(x, y))$, where $U_e^j(A_e^j)$ is the utility

of his advantages to change $A_e^j = A_e^j(x, y)$. His current resistance to change is $R_e^j(x, y) = D_e^j(I_e^j(x, y))$, where $D_e^j(I_e^j)$ is the disutility of his inconveniences to change $I^j = I^j(x, y)$. In the current period, each agent j will find that such an organizational change is worthwhile if his current motivation to change is not lower than his current resistance to change, i.e., $M_e^j(x, y) \geq \lambda^j R_e^j(x, y)$ for all $j \in J$, where the higher is the current individual worthwhile to change ratio $\lambda^j = \lambda_{n+1}^j > 0$, the more it is worthwhile to change for agent j . In this case, no agent will resist to change or block change. Let

$$W_{e,\lambda}(x) = \{y \in X \mid M_e^j(x, y) \geq \lambda^j R_e^j(x, y), j \in J\}$$

be the worthwhile to change set of the group at x . Our present paper considers a specific instance of this general model. To see how this is the case, please, see subsection 4.2 just below. In this case, among other aspects, experience does not matter much and all worthwhile to change ratios are equal to a constant. Notice that experience have nothing to do with the vector \mathbf{e} given in Section 3. Then, a worthwhile stay and change approach or avoidance group dynamic is defined by a succession of worthwhile stays and changes for the group, $x_{n+1} \in W_{e_n, \lambda_{n+1}}(x_n)$, for all $n \in \mathbb{N}$, where $\lambda_{n+1} = (\lambda_{n+1}^j)_{j \in J}$.

In this general (VR) context, a worthwhile stay and change group dynamic can end in a variational trap, before being able to reach a desired end for the group. The (VR) approach defines variational traps $x_* \in X$ for a group as, both,

- i) aspiration points, worthwhile to approach, $x_{n+1} \in W_{e_n, \lambda_{n+1}}(x_n)$, $n \in \mathbb{N}$, $x_n \rightarrow x_*$, $n \rightarrow \infty$ and reach, $x_* \in W_{e_n, \lambda_{n+1}}(x_n)$ for all $n \in \mathbb{N}$, from any or some successive position of the group and,
- ii) stationary traps, not worthwhile to leave to any member of the group, i.e., $W_{e_*, \lambda_*}(x_*) = \{x_*\}$, where e_* and λ_* are the current experiences and worthwhile to change ratio at x_* .

4.2 A benchmark behavioral context

Let us show how our extension of EVP gives sufficient conditions for the existence of variational traps for a group which can meet desires or not, depending of the context. The present mathematical extension of EVP considers a specific but very important benchmark version of the previous general (VR) model of human behavior. It supposes that,

- i) experience does not matter much, i.e., $M_e^j(x, y) = U^j [A^j(x, y)]$, and $R^j(x, y) = D^j [I^j(x, y)]$, for all possible experience e and each period $n + 1$,
- ii) motivation and resistance to change are identified to advantages and inconveniences to change, i.e., $U^j(A^j) = A^j$ and $D^j(I^j) = I^j$,
- iii) advantages to change are separable, i.e., $A^j(x, y) = g^j(y) - g^j(x) = f^j(x) - f^j(y)$ for all $j \in J$,
- iv) inconveniences to change are separable and non negative, i.e., $I^j(x, y) = C^j(x, y) - C^j(x, x) \geq 0$, for all $j \in J$,
- v) worthwhile to change ratios are the same for each agent, and do not change from period to period, i.e., $\lambda_{n+1}^j = \lambda$ for all $j \in J$ and all $n \in \mathbb{N}$.

Let us explain a little more point iii) and iv). For much more explanations and examples, see [12]. Individual advantages to change $A^j(x, y)$ refer, each period, for each agent j , to the difference between his “to be increased” payoff $g^j(y)$ when all agents move and his to be increased “payoff $g^j(x)$ when all agents stay (repeat their last action). For example, $g^j(x)$ can be a profit. Let $\bar{g}^j = \sup \{g^j(z), z \in X\} < +\infty$ be the highest payoff agent j can get within the group. Then, $f^j(y) = \bar{g}^j - g^j(y)$ defines his degree of dissatisfaction (a “to be decreased” payoff) and $A^j(x, y) = g^j(y) - g^j(x) = f^j(x) - f^j(y)$. When they are non negative, inconveniences to change $I^j(x, y) \geq 0$ refer, each period, for each agent j , to the difference between his costs to be able to change $C^j(x, y) \geq 0$ and his costs to be able to stay $C^j(x, x) \geq 0$. The (VR) approach has shown, very carefully, how each individual inconveniences to change function $I^j(x, y) = C^j(x, y) - C^j(x, x) \geq 0$ can be modelized as a scalar pseudo-quasidistance, which is zero for a stay and is non negative for a change, with the possibility that $I^j(x, y) = 0$ when $y \neq x$. Hence, if we want to modelize group dynamics, the space of actions or situations X must be endowed with a cone pseudo-quasimetric $q_\Theta(x, y) = I(x, y) = (I^j(x, y))_{j \in J} = C(x, y) - C(x, x)$, which modelizes vectors of inconveniences to change, where $C(x, y) = (C^j(x, y))_{j \in J}$ and $C(x, x) = (C^j(x, x))_{j \in J}$ refer to vectorial costs to be able to change and to stay. The cone puts weights on the vector of motivations and resistances to change and helps to say when the vector of motivations to change is not lower than the vector of resistance to change with respect to some order relation.

Let $g(x) = (g^j(x))_{j \in J}$, $f(x) = (f^j(x))_{j \in J}$, be “to be increased” and “to be decreased” vectorial payoffs, and $A(x, y) = g(y) - g(x) = f(x) - f(y)$ be vectorial advantages to change. Then, it is easy to see that our extension of EVP to cone pseudo-quasimetric spaces gives sufficient conditions for the existence of variational traps. More precisely, Theorem 3.2 shows that,

- i) $f(x_0) - f(x_*) \geq_\Theta \lambda q_\Theta(x_0, x_*)$ i.e., $A(x_0, x_*) \geq_\Theta \lambda I(x_0, x_*) \iff x_* \in S_{f, \lambda}(x_0)$. This means that it is worthwhile to move from x_0 to x_* . The proof of Theorem 3.2 shows more (see Proposition 3.8, a sufficient condition for (H2)): $x_* \in S_{f, \lambda}(x_n)$ for all $n \in \mathbb{N}$, i.e., x_* is an aspiration point,
- ii) $f(x_*) - f(x) \not\geq_\Theta \lambda q_\Theta(x_*, x)$ i.e., $A(x_*, x) \not\geq_\Theta \lambda I(x_*, x) \iff x \notin S_{f, \lambda}(x_*)$ for all $x \in X \setminus \{x_n\}$. Then, it is not worthwhile to leave x_* , except to move to $\{x_n\}$.

Then, x_* is a variational trap.

5 Conclusions

The cone pseudo-quasimetric versions of EVP obtained in this paper are motivated by a wide range of applications in Group Dynamics. They reduce to the corresponding results in [16] when the cone pseudo-quasimetric in question is $(X, \mathbb{R}, \mathbb{R}_+, q)$. Therefore, they can be viewed as characterizations of the completeness of cone pseudo-quasimetric spaces since the (scalar) pseudo-quasimetric versions of EVP are characterizations of the completeness of (scalar) pseudo-quasimetric spaces. These extensions can give immediate extensions to a lot of results such as Caristi, Danacs et al, Takahashi, Oettli-Thera, Sullivan completeness, etc. equivalence and equivalent variational principles in the setting in cone pseudo-quasimetric spaces.

Although we do not require that the ordering cone is either normal or pointed, it is assumed to be solid. In future research, we would study the possibility of dropping this requirement since many versions of EVP in [21, 22, 23] were formulated for nonsolid ordering cones by using a variational approach.

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