A BASIS-FREE NULL SPACE METHOD FOR SOLVING GENERALIZED SADDLE POINT PROBLEMS*

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Abstract. Using an augmented Lagrangian matrix approach, we analytically solve in this paper a broad class of linear systems that includes symmetric and nonsymmetric problems in saddle point form. To this end, some mild assumptions are made and a preconditioning is specially designed to improve the sensitivity of the systems before the calculation of their solutions. In this way, they can be properly employed for computational purposes. As an example of such a procedure, we develop a direct method for solving the symmetric saddle point problem. Then we show that our method is intrinsically related with the null space method, but it doesn't depend on finding a basis of the null space of the $(2,1)$ block of the saddle point systems. For this reason, the proposed method can be seen as a basis-free null space method. This method is also extended to nonsymmetric problems. To the best of our knowledge, the analytical solution of the nonsymmetric linear systems considered here is not known. So it is of major interest, since it can lead to the development of efficient algorithms, besides our own.

Key words. direct methods, null space methods, saddle point problems, indefinite linear systems, analytical solutions, matrix inversion, conditioning, preconditioning

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1. Introduction.

Much attention has been given over the past decades to the problem of solving linear systems in saddle point form, since they naturally arise in many fields, such as computational fluid dynamics, constrained optimization and interpolation of scattered data. For this reason, nowadays it is possible to find in the literature many methods for solving them, each one with its own requirements and characteristics. In this paper we develop a method of our own that makes extensive use of the augmented Lagrangian technique to compute the solution of an important class of these systems.

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, with $m < n$, $M \in \mathbb{R}^{(n+m)\times(n+m)}$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The saddle point problem we are going to solve is the problem of finding the solution of a linear system of the following form

$$M \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{1.1}$$

The symbol := in this equation means that M is equal to the square matrix on its right. When $A^T \neq A$, this problem is also called generalized saddle point problem.

In the system above, we consider that B has full rank because this is a necessary condition for it to admit a unique solution (Theorem 3.3 in [1, p. 17]). Initially, we are also going to assume that the matrix A is symmetric and positive definite on $\ker(B)$, since M is nonsingular with its blocks having the mentioned properties (Theorem 3.2 in [1, p. 16]). But to this end, we could replace the positiveness of A on $\ker(B)$ by the positiveness of its symmetric part on this same vector space (Theorem 3.4 in

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[1, p. 17]). This other possibility will be treated at the end of this text, where we solve the generalized saddle point problem under also a particular condition on the skew-symmetric part of A.

The calculations presented here give us the analytical solution of (1.1). In the case of a symmetric matrix A, they can easily be translated into a direct method for solving such a system, which then helps to establish a direct method for solving the generalized saddle point problem. The former method involves the construction of a linear system equivalent to (1.1) that has improved sensitivity, in a sense that we'll make clear latter.

When A is symmetric, we are able to show that there is an intrinsic relation between the direct method developed for this situation and the null space method [1, p. 32]. From this, we note that both methods have several features in common, which is why we can see the first one as a variant of the second. In particular, we infer that this new method can serve as an alternative to the null space method that doesn't require the prior calculation of a basis of $\ker(B)$. Therefore, it is at least adequate to solve saddle point problems for which the performance of the null space method isn't satisfactory due to difficulties with the calculation of such a basis.

We observe that more elaborate constructions are required to solve the problem (1.1) in the absence of strong assumptions for it. Such constructions usually have higher computational cost. Even so, in the case that the difference $n - m$ is significantly large, this cost is comparable — with some advantage — to the null space method's cost. Besides, part of this cost can be alleviated by the exploitation of special structures possibly present in the matrices A and B, as sparsity — similar to what is done with the null space method itself. This requires, however, the specialization of some calculations to solve specific problems, which is not discussed in this paper, since such an adaptation diverges too much from its goals.

We introduce in Section 2 the version of the augmented Lagrangian technique that will be used in Section 3, along with other techniques, to nontrivially convert the linear system (1.1) into an equivalent system with better numerical properties. The intermediate result is a scheme that provides the solution of (1.1) iteratively. In Section 4 we list some technical results for future use. In Section 5 we explicitly find the limit of the iterative process from Section 3. In Section 6 we convert the solution of (1.1) into a direct method for solving this system and also give algorithms for this. We then show in Section 7 how to derive the null space method from the method developed, dwelling on the properties common to the two of them. In Section 8 we extend the main results of the previous sections to the nonsymmetric case of (1.1) and give algorithms for the solving of this problem. Finally, we summarize our findings in Section 9, pointing some directions for future work.

2. One way to see the augmented Lagrangian technique. The augmented Lagrangian technique applied to the saddle point problem considered in this paper consists of choosing a real and symmetric semi-definite positive matrix $W \in \mathbb{R}^{m\times m}$ and of replacing the linear system that makes up the problem at hand by an equivalent one, which is dependent of W and has the following form:

$$M_W \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} A + B^T W B & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a + B^T W b \\ b \end{bmatrix}.$$

Appropriate choices for the matrix W are those by which some algebraic or numeric property is added to the new system. Of course it isn't easy to spot them and so the choices are often trivial ones. Nevertheless, even simple choices can provide

interesting results. Hestenes, for example, made use of this technique in [8, pp. 76-77], although not originally in matrix language, to obtain a linear system as above, with coefficient matrix with positive definite $(1,1)$ block on all of $\mathbb{R}^n$. To this end, he chose

$$W = \gamma I,$$

for a sufficiently large $\gamma > 0$. In turn, Golub and Greif experimentally noticed in [4] that, for this same choice of W and primarily a positive definite matrix A, the value

$$\gamma = \|A\|_2 / \|B\|_2^2$$

approximately minimized, among the choices of $\gamma > 0$, the 2-norm condition numbers of at least two — else all three — matrices

$$M_W, \quad A + B^T W B \quad \text{and} \quad -B(A + B^T W B)^{-1} B^T.$$

Observe that, when the third matrix is defined, it is just the Schur complement of $A + B^T W B$ in M_W. This approach thus may alleviate the effect of a possibly ill-conditioning of A on many numerical methods.

Besides employing the augmented Lagrangian technique, we also do elementary row operations on the linear system (1.1), in a way that a real matrix $\tilde{W}$, this time of order $n \times m$, is chosen and the system is rewritten in the equivalent form

$$M_{\tilde{W}} \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} A + \tilde{W} B & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a + \tilde{W} b \\ b \end{bmatrix}.$$

The first technique is clearly a particular case of this one, in which emphasis is given to the preservation of the symmetry of M and the signal of its $(1,1)$ block.

3. A handy iterative scheme. Loosely speaking, our first step in the attempt to solve (1.1) is to somehow rectify the signal of the $(1,1)$ block of M so it becomes positive on all of $\mathbb{R}^n$. In order to achieve this, let's consider the matrices

$$B^\dagger = B^T (B B^T)^{-1}, \quad P = B^\dagger B, \quad \tilde{W} = (P - 2I) A B^\dagger \quad \text{and} \quad \tilde{A} = A + \tilde{W} B.$$

For any positive constant ν, let's also consider

$$W = \nu I, \quad A_\nu = \tilde{A} + B^T W B \quad \text{and} \quad a_\nu = a + (\tilde{W} + B^T W) b.$$

Evidently $I - P$ and P are, in this order, the matrix representations relative to the canonical basis of $\mathbb{R}^n$ of the orthogonal projections onto $\ker(B)$ and $\mathrm{ran}(B^T)$. But not only: the real quadratic functions associated with $\tilde{A}$ and A_ν, namely

$$\tilde{q}(x) = x^T \tilde{A} x \quad \text{and} \quad q_\nu(x) = x^T A_\nu x, \quad x \in \mathbb{R}^n,$$

are such that $\tilde{q}$ is identically null on $\mathrm{ran}(B^T)$ and positive on $\ker(B) \setminus \{0\}$, while q_ν is positive on $\mathbb{R}^n \setminus \{0\}$. To see this, decompose $\mathbb{R}^n$ into a direct sum of $\ker(B)$ and its orthogonal complement,

$$\mathbb{R}^n = \ker(B) \oplus \mathrm{ran}(B^T),$$

and take $x = \tilde{x} + B^T y \in \mathbb{R}^n$, with $\tilde{x} \in \ker(B)$ and $y \in \mathbb{R}^m$, so that $\tilde{x} = (I - P)x$. Then, employ the known fact that only the symmetric part of a matrix matters to the quadratic function it induces and also use that

$$\tilde{W} = -2(I - P) A B^\dagger - P A B^\dagger \tag{3.1}$$

to compute the matrix sum $\tilde{A} + \tilde{A}^T$, to find that

$$\tilde{q}(x) = x^T \left[\frac{1}{2}(\tilde{A} + \tilde{A}^T)\right] x = x^T \left[(I-P)A(I-P)\right] x = \tilde{x}^T A \tilde{x}.$$

This establishes the mentioned properties for $\tilde{q}$, because $\tilde{q}(x) = \tilde{q}(\tilde{x})$ then and A is positive definite on $\ker(B)$. In turn,

$$q_\nu(x) = x^T \tilde{A} x + \nu (Bx)^T (Bx) = \tilde{q}(\tilde{x}) + \nu \|BB^T y\|_2^2$$

and at least one of the addends of the last sum must be nonzero if x is a nonzero vector, since $\tilde{x} \neq 0$ or $y \neq 0$ in this case, B has full rank and we just observed that $\tilde{q}$ is positive on $\ker(B) \setminus \{0\}$. Therefore, the quadratic function q_ν is positive on $\mathbb{R}^n \setminus \{0\}$, as stated.

In view of the above observations and the techniques presented in Section 2 — which are applied here using W and $\tilde{W}$ —, we know that the linear system (1.1) is equivalent to this one other

$$M_\nu \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} A_\nu & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_\nu \\ b \end{bmatrix}, \tag{3.2}$$

whose coefficient matrix has "positive definite" $(1,1)$ block on all of $\mathbb{R}^n$. The quotation marks here are to say that this positiveness is considered in a more general sense, that doesn't ask for a symmetric A_ν. This generalization is needed because A_ν can be a nonsymmetric matrix. Indeed, for the skew-symmetric part of A_ν is just

$$A_{\nu,ss} = PA - AP$$

and we cannot guarantee that A and P commute under the assumptions made.

For obvious reasons, we wish to not lose the symmetry of the original saddle point problem. Thus, to ensure it, we begin a second step in the attempt to solve (1.1) that comes recovering the symmetry probably lost after the replacement of this system by the one above. To this extent, observe that the symmetric part of A_ν,

$$A_{\nu,s} = (I-P)A(I-P) + \nu B^T B,$$

is positive definite on all of $\mathbb{R}^n$ in the usual sense, as it is symmetric and the real quadratic function associated with it is exactly q_ν. With this property, we can relate the solving of (3.2) to the solving of the following iterative scheme,

$$M_{\nu,s} \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} := \begin{bmatrix} A_{\nu,s} & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} a_\nu - A_{\nu,ss} x^{(k)} \\ b \end{bmatrix}, \quad k \in \mathbb{N}_0, \tag{3.3}$$

defined by the parts of A_ν and any point $(x^{(0)}, y^{(0)}) \in \mathbb{R}^{n+m}$. Note that the coefficient matrix $M_{\nu,s}$ is symmetric and nonsingular, with positive definite $(1,1)$ block, all due to $A_{\nu,s}$. Such a transformation was discussed in details by Golub and Wathen in [6].

From the theory of splitting methods, we know that the sequence of solutions given by (3.3) converges to the solution of the system (1.1) if the spectral radius of

$$\begin{bmatrix} A_{\nu,s} & B^T \\ B & O \end{bmatrix}^{-1} \begin{bmatrix} -A_{\nu,ss} & O^T \\ O & O \end{bmatrix}.$$

is less than one (Theorem 7.19 in [2, pp. 442-443]). This criterion and its variants are, however, hardly of any use for the present moment, since we can't infer the sought

solution from them. For this reason, we seek to analyse in Section 5 the sequence $((x^{(k)}, y^{(k)}))$ without making use of any of them. This is a feasible possibility due to the preconditioning done, as we'll see.

Before that analysis, though, we would like to consider a last step in the attempt to solve (1.1), that aims to improve the sensitivity of the linear systems which compose the iterative process. To do so, notice that such systems were also purposely built to meet the assumptions made in [4]. In fact, that was one of the motives for step one in this section. Thus, these systems can be replaced by these others,

$$\begin{bmatrix} A_{\nu,s} + \gamma(\nu)B^TB & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} a_\nu + \gamma(\nu)B^Tb - A_{\nu,ss}x^{(k)} \\ b \end{bmatrix}, \quad k \in \mathbb{N}_0,$$

where $\gamma(\nu) = \|A_{\nu,s}\|_2/\|B\|_2^2$. As already mentioned, for this value of $\gamma(\nu)$, the 2-norm condition numbers of at least two of the matrices

$$(M_{\nu,s})_{\gamma(\nu)I}, \quad A_{\nu,s} + \gamma(\nu)B^TB \quad \text{and} \quad -B(A_{\nu,s} + \gamma(\nu)B^TB)^{-1}B^T$$

may have been approximately minimized among the choices of $\gamma(\nu) > 0$. In this situation, better numerical stability is expected during the solving of the latter systems.

The iterative process determined in this way depends on the positive parameter ν, but there isn't a preferential value for it. It is then possible to eliminate such a dependence by simply setting any positive value to this parameter. Even better: since ν can be made as small as desired and all the expressions that depend on it are also continuous on ν, we can define

$$\begin{aligned}
\gamma_* &= \lim_{\nu\to 0^+} \gamma(\nu) && = \|(I-P)A(I-P)\|_2/\|B\|_2^2, \\
A_* &= \lim_{\nu\to 0^+} \left(A_{\nu,s} + \gamma(\nu)B^TB\right) && = (I-P)A(I-P) + \gamma_* B^TB, \\
A_{**} &= \lim_{\nu\to 0^+} A_{\nu,ss} && = PA - AP, \\
a_*^{(k)} &= \lim_{\nu\to 0^+} \left(a_\nu + \gamma(\nu)B^Tb - A_{\nu,ss}x^{(k)}\right) && = a + (\tilde{W} + \gamma_* B^T)b - A_{**}x^{(k)}, \quad k \in \mathbb{N}_0,
\end{aligned}$$

and use these elements to eliminate such dependence. The advantage of doing so is to obtain a small simplification in some of the expressions of the above linear systems, which no longer counts on additional addends. The resultant scheme is

$$M_* \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} := \begin{bmatrix} A_* & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} a_*^{(k)} \\ b \end{bmatrix}, \quad k \in \mathbb{N}_0. \tag{3.4}$$

Note that γ_* is a positive constant or else the diagonalizable matrix $(I-P)A(I-P)$ would have only zero eigenvalues and, so, it would be the zero matrix. But such a thing cannot happen, because it contradicts the fact that this same matrix is positive definite on the nontrivial kernel of B. As a consequence of this remark, we have that $A_* = A_{\gamma_*,s}$ ($A_{\nu,s}$ with $\nu = \gamma_*$) and, thus, M_* remains being a matrix with symmetric positive definite $(1,1)$ block on all of $\mathbb{R}^n$. Since $A_* = A_{\gamma_*,s}$, we can also say that $A_{**} = A_{\gamma_*,ss}$ ($A_{\nu,ss}$ with $\nu = \gamma_*$).

It is the above scheme that we are in fact going to solve to find the solution of (1.1). As we will see, (3.4) terminates in its first iterations, which is why we are able to calculate the explicit solution of (1.1).

As a final note before ending this section, we would like to comment on the use of those limits. Someone can well argue against them by saying this whole step could be

skipped if we were to take $\nu = \gamma_*$ from the very beginning. Certainly this observation would lead this person to the same iterative process (3.4), but then a motivation for doing so would be missing. See that we rely on the work done in [4] for this and that the analysis carried out there is based on linear systems as (1.1), but mainly systems with positive definite $(1,1)$ blocks. Thus the constant γ_* makes little sense by itself, even though it is positive, because its numerator wasn't obtained from a positive definite matrix. Our procedure, on the other hand, reinforces such a choice, by determining it from more well reasoned ones. Also: it clears the way to us to explain why we expect that some of the 2-norm condition numbers of the matrices

$$M_*, \quad A_* \quad \text{and} \quad -BA_*^{-1}B^T$$

may have been approximately minimized with the use of γ_*. Well, that's because we can always see A_* as the matrix

$$\big[(I-P)A(I-P) + \epsilon B^T B\big] + (\gamma_* - \epsilon)B^T B \quad \text{for} \quad 0 < \epsilon < \gamma_* \quad \text{and} \quad \epsilon \ll 1,$$

where $\gamma_* - \epsilon$ approximates the value of

$$\gamma(\epsilon) = \|(I-P)A(I-P) + \epsilon B^T B\|_2 / \|B\|_2^2.$$

So, the observations presented in [4] back up our expectations. Besides, we can theoretically show that the condition number of the Schur complement of A_* in M_* is simply one. The proof of the last assertion is postponed for the next section, though.

4. Some technical results. Mostly those results that will prove of recurrent use are listed here. In what follows, $T_C : \mathbb{R}^q \to \mathbb{R}^p$ denotes the linear transformation naturally associated with a real matrix C of order $p \times q$.

PROPOSITION 4.1. *The subspaces of* $\mathbb{R}^n$, $\ker(B)$ *and* $\operatorname{ran}(B^T)$, *are invariants by the linear operators* T_{A_*} *and* $T_{A_*^{-1}}$.

Proof. Let $\tilde{x} \in \ker(B)$ and $\hat{x} \in \operatorname{ran}(B^T)$. Due to the expression of A_* and the orthogonality between the indicated spaces, we have that

$$T_{A_*}(\tilde{x}) = (I-P)(A\tilde{x}) \in \ker(B) \quad \text{and} \quad T_{A_*}(\hat{x}) = B^T(\gamma_* B\hat{x}) \in \operatorname{ran}(B^T).$$

So $\ker(B)$ and $\operatorname{ran}(B^T)$ are invariants by T_{A_*}. Now, let $K \in \mathbb{R}^{n\times(n-m)}$ be a matrix whose columns span $\ker(B)$. Then there are $\tilde{y}, \hat{y} \in \mathbb{R}^{n-m}$ and $\tilde{z}, \hat{z} \in \mathbb{R}^m$ such that

$$A_*^{-1}\tilde{x} = K\tilde{y} + B^T\tilde{z} \quad \text{and} \quad A_*^{-1}\hat{x} = K\hat{y} + B^T\hat{z},$$

or still

$$\tilde{x} = A_*(K\tilde{y}) + A_*(B^T\tilde{z}) \quad \text{and} \quad \hat{x} = A_*(K\hat{y}) + A_*(B^T\hat{z}).$$

Thus $B^T\tilde{z} = 0$ and $K\hat{y} = 0$, because the intersection of $\ker(B)$ and $\operatorname{ran}(B^T)$ is trivial and we just observed that these spaces are T_{A_*}-invariants. Hence $\tilde{z} = 0$ and $\hat{y} = 0$, since both matrices B^T and K have full column rank. Therefore,

$$T_{A_*^{-1}}(\tilde{x}) = K\tilde{y} \in \ker(B) \quad \text{and} \quad T_{A_*^{-1}}(\hat{x}) = B^T\hat{z} \in \operatorname{ran}(B^T),$$

from where we conclude that $\ker(B)$ and $\operatorname{ran}(B^T)$ are also $T_{A_*^{-1}}$-invariants. ∎

From now on we'll let K always denote a matrix whose columns span $\ker(B)$.

COROLLARY 4.2. *Both the restrictions of* T_{A_*} *and* $T_{A_*^{-1}}$ *to* $\ker(B)$ *and* $\operatorname{ran}(B^T)$ *are self-adjoint operators.*

Proof. The linear transformations T_{A_*} e $T_{A_*^{-1}}$ are self-adjoint operators, because A_* and A_*^{-1} are symmetric matrices. And the restriction of any self-adjoint operator to a finite-dimensional invariant subspace of its domain is also self-adjoint. ∎

COROLLARY 4.3. *The restrictions of T_{BA_*} and $T_{BA_*^{-1}}$ to $\ker(B)$ are identically null, as are the restrictions of $T_{(I-P)A_*}$ and $T_{(I-P)A_*^{-1}}$ to $\mathrm{ran}(B^T)$.*

Proof. Just note that $T_{C_1C_2} = T_{C_1}T_{C_2}$ for any matrices C_1 and C_2 of appropriate dimensions and apply the latter proposition. ∎

Many matrix identities follow at once from the above corollary, all of which will be used plenty of times throughout this text:

$$BA_*(I-P) = O, \quad BA_*^{-1}(I-P) = O, \quad (I-P)A_*B^T = O, \quad (I-P)A_*^{-1}B^T = O,$$

and others similar to them, with $I-P$ replaced by K, or B replaced by P. We warn the reader to keep these identities in mind, because we usually don't refer them during our calculations, so to avoid unnecessary repetition of arguments.

LEMMA 4.4. *The matrices A_*, A_*^{-1}, P and $I-P$ commute pairwise.*

Proof. Only the nontrivial products need to be justified. We have that

$$A_*P = (I-P)A[(I-P)P] + \gamma_*B^T[BP] = \gamma_*B^TB$$

and, analogously, $PA_* = \gamma_*B^TB$. Thus, $A_*P = PA_*$. Knowing this, we see that

$$A_*(I-P) = A_* - A_*P = A_* - PA_* = (I-P)A_*.$$

Since $A_*A_*^{-1} = I$, it follows that

$$P = PA_*A_*^{-1} = A_*PA_*^{-1}.$$

Therefore, $A_*^{-1}P = PA_*^{-1}$. As in the case of A_* and $I-P$, this fact alone is sufficient to ensure that the matrices A_*^{-1} and $I-P$ also commute. ∎

The next result answers the assertion left open at the end of the previous section.

PROPOSITION 4.5. *The Schur complement of A_* in M_* is just a multiple of the identity matrix, namely: $-BA_*^{-1}B^T = -(1/\gamma_*)I$.*

Proof. The post-multiplication of the identity $I = A_*^{-1}A_*$ by $B^\dagger$ gives

$$B^\dagger = A_*^{-1}(A_*B^\dagger) = A_*^{-1}\{(I-P)A[(I-P)B^\dagger] + \gamma_*B^T[BB^\dagger]\} = \gamma_*A_*^{-1}B^T, \tag{4.1}$$

due to the expression of A_*. Therefore, $-BA_*^{-1}B^T = -(1/\gamma_*)I$. ∎

This proposition is quite interesting because it shows that the calculation of the Schur complement of A_* in M_* absolutely doesn't represent a difficulty to the method being developed. This contrasts with some other methods for which approximations to the Schur complement must be computed.

LEMMA 4.6. $BA_*^{-1}a_*^{(k)} = (1/\gamma_*)(B^\dagger)^T\{a - A[B^\dagger b + (I-P)x^{(k)}]\} + b$, $k \in \mathbb{N}_0$.

Proof. From (3.1) and the expression of A_{**} we have that

$$\begin{aligned} Pa_*^{(k)} &= Pa + (-PAB^\dagger + \gamma_*B^T)b - PA(I-P)x^{(k)} \\ &= P\{a - A[B^\dagger b + (I-P)x^{(k)}]\} + \gamma_*B^Tb. \end{aligned} \tag{4.2}$$

On the other hand,

$$BA_*^{-1}a_*^{(k)} = BA_*^{-1}[Pa_*^{(k)} + (I-P)a_*^{(k)}] = BA_*^{-1}(Pa_*^{(k)})$$

by one of the identities that follows Corollary 4.3. Therefore,

$$BA_*^{-1}a_*^{(k)} = (BA_*^{-1}P)\{a - A[B^\dagger b + (I-P)x^{(k)}]\} + \gamma_*(BA_*^{-1}B^T)b.$$

As $P = B^T(B^\dagger)^T$, we can then use Proposition 4.5 to conclude that

$$BA_*^{-1}a_*^{(k)} = (1/\gamma_*)(B^\dagger)^T\{a - A[B^\dagger b + (I-P)x^{(k)}]\} + b,$$

completing the proof. ∎

5. The analytical solution of the symmetric saddle point problem. We finally have the conditions to determine the solution of the linear system (1.1). As already said, we are going to solve the iterative process (3.4) to obtain it. Well, using Gaussian elimination, the equation that defines this scheme can be reformulated as:

$$\begin{cases} A_* x^{(k+1)} = a_*^{(k)} - B^T y^{(k+1)} \\ (BA_*^{-1}B^T) y^{(k+1)} = BA_*^{-1} a_*^{(k)} - b \end{cases}.$$

We see from the second equation above, Proposition 4.5 and Lemma 4.6 that

$$y^{(k+1)} = (B^\dagger)^T \{a - A[B^\dagger b + (I-P)x^{(k)}]\}.$$

So, we can replace $y^{(k+1)}$ in the first equation to find that

$$A_* x^{(k+1)} = a_*^{(k)} - P\{a - A[B^\dagger b + (I-P)x^{(k)}]\}$$

and even pre-multiply the latter expression by BA_*^{-1} to get that

$$Bx^{(k+1)} = BA_*^{-1} a_*^{(k)} - (BA_*^{-1}B^T)(B^\dagger)^T \{a - A[B^\dagger b + (I-P)x^{(k)}]\}.$$

Then it clearly follows from Proposition 4.5 and Lemma 4.6 that $Bx^{(k+1)} = b$. Thus,

$$Px^{(k+1)} = B^\dagger b, \quad k \in \mathbb{N}_0.$$

In particular, this implies that

$$(I-P)a_*^{(k)} = (I-P)a - 2(I-P)AB^\dagger b + (I-P)A(Px^{(k)}) = (I-P)(a - AB^\dagger b)$$

for all $k \in \mathbb{N}$. On the other hand, we see from (4.2) that

$$P\{a - A[B^\dagger b + (I-P)x^{(k)}]\} = Pa_*^{(k)} - \gamma_* B^T b.$$

The replacement of this expression in the equation we just found for $A_* x^{(k+1)}$ gives

$$A_* x^{(k+1)} = (I-P)a_*^{(k)} + \gamma_* B^T b.$$

Then we have from both observations that

$$A_* x^{(k+1)} = (I-P)(a - AB^\dagger b) + \gamma_* B^T b, \quad k \in \mathbb{N}.$$

So the sequence $(x^{(k)})$ is constant from its third term forward and convergent to

$$x^* = (I-P)A_*^{-1}(a - AB^\dagger b) + B^\dagger b.$$

Notice the use of Lemma 4.4 and of equation (4.1) in the last equality. We replace in the expression of $y^{(k+1)}$ the limit point just found to conclude that the sequence $(y^{(k)})$ is constant too — this time from its fourth term forward — and convergent to

$$\begin{aligned} y^* &= (B^\dagger)^T \{a - A[B^\dagger b + (I-P)A_*^{-1}(a - AB^\dagger b)]\} \\ &= (B^\dagger)^T [I - AA_*^{-1}(I-P)](a - AB^\dagger b). \end{aligned}$$

The last equality is again due to Lemma 4.4. If we now observe that

$$(I-P)A[(I-P)A_*^{-1}] = (I-P)A[(I-P)A_*^{-1}(I-P)]$$

due to, for instance, one of the many identities that follow Corollary 4.3, then

$$[(I-P)A(I-P)]A_*^{-1} = (A_* - \gamma_* B^T B)A_*^{-1}(I-P) = I - P. \tag{5.1}$$

In this way,

$$\begin{aligned} Ax^* + B^T y^* &= A(I-P)A_*^{-1}(a - AB^\dagger b) + AB^\dagger b + P[I - A(I-P)A_*^{-1}](a - AB^\dagger b) \\ &= [(I-P)A(I-P)A_*^{-1}](a - AB^\dagger b) + P(a - AB^\dagger b) + AB^\dagger b \\ &= a, \end{aligned}$$

Furthermore, $Bx^* = b$ trivially.

Along with Theorem 3.2 from [1], we just proved that

THEOREM 5.1. *If A is a symmetric and positive definite matrix on* $\ker(B)$ *and B has full rank, then the system* (1.1) *admits a unique solution, which is*

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} (I-P)A_*^{-1}(a - AB^\dagger b) + B^\dagger b \\ (B^\dagger)^T[I - AA_*^{-1}(I-P)](a - AB^\dagger b) \end{bmatrix}.$$

COROLLARY 5.2. *Under the conditions of the Theorem 5.1, the inverse of M is*

$$\begin{bmatrix} F & (I-FA)B^\dagger \\ (B^\dagger)^T(I-AF) & -(B^\dagger)^T A(I-FA)B^\dagger \end{bmatrix},$$

where $F = (I-P)A_*^{-1}$ is a symmetric matrix.

Proof. To find the expression of M^{-1}, just replace in the equations of the latter theorem the letters a and b, in this order, by the matrices $I \in \mathbb{R}^{n\times n}$ and $O \in \mathbb{R}^{m\times n}$ or $O \in \mathbb{R}^{n\times m}$ and $I \in \mathbb{R}^{m\times m}$. To see that F is a symmetric matrix, remember that A_*^{-1} and $I - P$ commute and both are symmetric matrices. ∎

According to [1, p. 20], the expression shown above to the inverse of the matrix M is identical to that given by Gansterer *et al.* in [3], differentiating from it by the expression of the $(1,1)$ block of M^{-1}. For these authors, its expression is given by

$$K(K^T AK)^{-1}K^T,$$

where K is a real matrix of order $n\times(n-m)$ whose orthonormal columns span $\ker(B)$. Clearly, the two forms of F are equivalent, as the inverse of a nonsingular matrix is unique, which implies its blocks are too.

LEMMA 5.3. *The matrices $(I-P)A_*^{-1}$ and $K(K^TAK)^{-1}K^T$ are identical.*

Proof. Since K has orthonormal columns, the identity $A_*A_*^{-1} = I$ and the expression of A_* gives

$$I = K^T K = K^T[(I-P)A(I-P) + \gamma_* B^T B]A_*^{-1}K = K^T A(I-P)A_*^{-1}K.$$

As $I - P = K(K^TK)^{-1}K^T$ too, we then have that $K^TA_*^{-1}K = (K^TAK)^{-1}$. Thus,

$$(I-P)A_*^{-1} = (I-P)A_*^{-1}(I-P) = K(K^TA_*^{-1}K)K^T = K(K^TAK)^{-1}K^T,$$

completing the proof. ∎

As a mean of comparison with A_*, the next result is worth displaying on here.

COROLLARY 5.4. $A_*^{-1} = K(K^TAK)^{-1}K^T + (1/\gamma_*)B^\dagger(B^\dagger)^T$.

Proof. We have that

$$(I-P)A_*^{-1} = A_*^{-1} - B^\dagger(BA_*^{-1}) = A_*^{-1} - (1/\gamma_*)B^\dagger(B^\dagger)^T,$$

because A_*^{-1} is a symmetric matrix and $\gamma_* A_*^{-1}B^T = B^\dagger$ by (4.1). So,

$$A_*^{-1} = (I-P)A_*^{-1} + (1/\gamma_*)B^\dagger(B^\dagger)^T = K(K^TAK)^{-1}K^T + (1/\gamma_*)B^\dagger(B^\dagger)^T,$$

now due to the previous lemma. ∎

6. A direct method for solving the symmetric saddle point problem. In what follows, $\text{cond}_2(C)$, $\sigma_{max}(C)$ and $\sigma_{min}(C)$ denote, in this order, the 2-norm condition number of a matrix C and its largest and smallest singular values. Also: $\lambda_{max}(C)$ and $\lambda_{min}(C)$ represent, respectively, the largest and smallest eigenvalues of a square matrix C.

According to Theorem 5.1 and (4.1), we can think of a direct method for solving (1.1) that consists of the construction and resolution of the following linear system:

$$\begin{bmatrix} A_* & O^T \\ O & \gamma_* BB^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (I-P)(a-AB^\dagger b) + \gamma_* B^T b \\ \gamma_* B[I - AA_*^{-1}(I-P)](a - AB^\dagger b) \end{bmatrix}. \tag{6.1}$$

This system is interesting because $\text{cond}_2(A_*)$ is expected to be approximately minimum, among some choices previously considered in Section 3, and because the 2-norm condition number of its coefficient matrix is exactly $\text{cond}_2(A_*)$. Indeed, since all eigenvalues of $\gamma_* BB^T$ are also eigenvalues of A_* and the coefficient matrix is block diagonal — meaning its spectrum is determined by the isolated spectrums of the blocks in its diagonal. To understand the first assertion, just let $(\gamma_*\sigma^2, y)$ be an eigenpair of $\gamma_* BB^T$. Then note that $(\gamma_*\sigma^2, B^T y)$ is an eigenpair of A_*, since

$$A_*(B^T y) = (I-P)A[(I-P)B^T]y + B^T(\gamma_* BB^T y) = \gamma_*\sigma^2 B^T y.$$

In addition to the last observation, system (6.1) is also interesting because the value of $\text{cond}_2(A_*)$ is known in terms of the matrices A, B and K — the latter now with orthonormal columns. This allows us to further proceed with our analysis.

PROPOSITION 6.1. $\text{cond}_2(A_*) = \max\{\text{cond}_2(K^T AK), \text{cond}_2(BB^T)\}$.

Proof. Let $\tilde{T}_{A_*}$ and $\hat{T}_{A_*}$ be, in this order, the restrictions of the linear transformation T_{A_*} to $\ker(B)$ and $\text{ran}(B^T)$. Due to Corollary 4.2 and to the spectral theorem, there are orthonormal basis of $\ker(B)$ and $\text{ran}(B^T)$ consisting, respectively, of eigenvectors of $\tilde{T}_{A_*}$ and $\hat{T}_{A_*}$, all of which are associated with real eigenvalues. The union of these basis gives an orthonormal basis of $\mathbb{R}^n$, because $\ker(B)$ and $\text{ran}(B^T)$ are orthogonal spaces. Then it is possible to analyse the whole spectrum of T_{A_*} from the basis of $\mathbb{R}^n$ obtained in this way, since the eigenvectors of $\tilde{T}_{A_*}$ and $\hat{T}_{A_*}$ are obviously eigenvectors of T_{A_*} too.

Now, some of the eigenvalues of the operator T_{A_*} are already know, for we just observed in the paragraph previous to this proposition that the eigenvalues of T_{A_*} relative to such a basis of $\text{ran}(B^T)$ are of the form $\gamma_*\sigma^2$, where σ is a singular value of B. To find out the other eigenvalues of T_{A_*}, though, we need first to note that there also is an orthonormal basis of $\ker(B)$ consisting of eigenvectors of the restriction of $T_{(I-P)A(I-P)}$ to this space. And that all its eigenvalues are real. Indeed, since the transformation at hand is self-adjoint likewise in Corollary 4.2. So, consider an eigenpair $(\lambda, \tilde{x})$ of this restriction, with $\tilde{x} \in \ker(B)$. Clearly, $(\lambda, \tilde{x})$ is also an eigenpair of T_{A_*}, due to the definition of A_*, and $(\lambda, K^T\tilde{x})$, an eigenpair of $T_{K^T AK}$, this time due to $I - P = KK^T$. Thus, the other eigenvalues of T_{A_*} are eigenvalues of $T_{K^T AK}$ too. Besides,

$$\lambda_{max}(K^T AK) = \|(I-P)A(I-P)\|_2 = \gamma_*\|B\|_2^2 = \gamma_*\sigma_{max}^2(B),$$

due to the orthonormality of the columns of K and the definition of γ_*. Hence

$$\lambda_{max}(A_*) = \max\{\lambda_{max}(K^T AK), \gamma_*\sigma_{max}^2(B)\} = \lambda_{max}(K^T AK)$$

and, also,

$$\lambda_{min}(A_*) = \min\{\lambda_{min}(K^T AK), \gamma_* \sigma^2_{min}(B)\}.$$

Consequently,

$$\mathrm{cond}_2(A_*) = \max\{\mathrm{cond}_2(K^T AK), \mathrm{cond}_2(BB^T)\},$$

completing the proof. □

This proposition shows a first connection between the method being developed and the null space method, since the latter depends on the solving of linear systems with coefficient matrices $K^T AK$ and BB^T.

From (6.1), we see that $(I - P)(a - AB^\dagger b) = A_* x - \gamma_* B^T b$ and that

$$\begin{aligned}\gamma_* BB^T y &= \gamma_* B(a - AB^\dagger b) - \gamma_* BA[A_*^{-1}(I - P)(a - AB^\dagger b)] \\ &= \gamma_* B(a - AB^\dagger b) - \gamma_* BA[x - (\gamma_* A_*^{-1} B^T)b].\end{aligned}$$

Recalling (4.1), we then have that

$$\gamma_* BAx + \gamma_* BB^T y = \gamma_* Ba.$$

Thus the linear system (6.1) is equivalent to

$$\begin{bmatrix} A_* & O^T \\ \gamma_* BA & \gamma_* BB^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (I - P)(a - AB^\dagger b) + \gamma_* B^T b \\ \gamma_* Ba \end{bmatrix}. \tag{6.2}$$

The appearance of this system further reaffirms the relationship between the method being developed and the null space method, for it gives the last m components of the solution of the system (1.1) in the same way as the null space method does.

Algorithm 1 shown below provides the solution of (1.1) by solving (6.1). It is possible and advisable in this algorithm — and in all other algorithms of this paper — to force matrix operations which result in a symmetric matrix to take into account the symmetry of the result to reduce the number of floating point operations performed.

Observe that, in Algorithm 1, the calculation of A_c can be more efficiently performed directly in Step 3, but we prefer to keep the steps in the way presented, in order to return A_c, if desired, and to reuse this algorithm in the next section. Note also that, although not mandatory, the use of a small positive-valued parameter δ in Step 3 of Algorithm 1 turns A_* into a slightly more diagonally dominant matrix than predicted, which then assists the computation of its Cholesky factor in that same step.

As it should be clear from Proposition 6.1, a matrix B very close to a numerically rank deficient matrix can significantly impair the performance of the method. If this is the case, we eliminate the problem by doing a traditional change of variables that has not a prohibitive cost in the usual situation where $m \ll n$. As we will conclude in the next section, this is the ideal situation for the application of the method.

Let $Q \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{m\times m}$ be the factors from the thin QR factorization of B^T, where Q has orthonormal columns and R is upper triangular. Then the linear system (1.1) can be easily placed in the following form

$$\begin{bmatrix} A & Q \\ Q^T & O \end{bmatrix} \begin{bmatrix} x \\ y_Q \end{bmatrix} = \begin{bmatrix} a \\ b_Q \end{bmatrix}, \quad Ry = y_Q \in \mathbb{R}^m, \quad b_Q = R^{-T} b \in \mathbb{R}^m. \tag{6.3}$$

just by noticing that

$$\begin{bmatrix} A & Q \\ Q^T & O \end{bmatrix} = \begin{bmatrix} I & O^T \\ O & R^{-T} \end{bmatrix} \begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} I & O^T \\ O & R^{-1} \end{bmatrix}.$$

Algorithm 1 IT CALCULATES THE SOLUTION OF (1.1) BY SOLVING (6.1).

Entry data:
m, n positive integers such that $m < n$,
$B \in \mathbb{R}^{m\times n}$ of full rank, $A \in \mathbb{R}^{n\times n}$ symmetric positive definite on $\ker(B)$,
$a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $\delta \in \mathbb{R}$ such that $0 \leq \delta \ll 1$.

Workspace: $\gamma_*, \sigma_* \in \mathbb{R}$; $x^*, x_c^*, x_{2c}^*, x_{3c}^* \in \mathbb{R}^n$; $y^* \in \mathbb{R}^m$; $R \in \mathbb{R}^{m\times m}$; $S \in \mathbb{R}^{m\times n}$; $P, A_*, A_c, R_* \in \mathbb{R}^{n\times n}$.

Step 1: Calculate the Cholesky factor R of the product BB^T by means of the QR factorization of B^T. Using R, calculate the solution of the matrix system $(BB^T)S = B$ and then estimate the value of $\sigma_* = \|B\|_2^2 = \|R^T R\|_2$ (for example, with the power method).

Step 2: Calculate the product $P = B^T S$ (symmetric product) and then replace it by $P - I$ (just update the main diagonal of P). Also calculate the matrix $A_c = B^T B$ (symmetric product).

Step 3: Calculate $A_* = PAP$ (symmetric product) and estimate the largest eigenvalue γ_* of A_* (for example, with the power method). Replace γ_* by γ_*/σ_* and A_*, by $A_* + \gamma_*(A_c + \delta I)$ (symmetric sum). Calculate the Cholesky factor R_* of A_*.

Step 4: Calculate $x_c^* = S^T b$ and $x_{2c}^* = Ax_c^* - a$. Then take x_{3c}^* as the solution of the system $A_* x = Px_{2c}^*$, solved by using R_*, and take x^* as $x_{3c}^* + x_c^*$.

Step 5: Replace x_{3c}^* by $-(x_{2c}^* + Ax_{3c}^*)$ and take y^* as the solution of the system $(\gamma_* BB^T)y = \gamma_* Bx_{3c}^*$, computed using the Cholesky factor $\sqrt{\gamma_*}R$ of $\gamma_* BB^T$, or, alternatively, by Sx_{3c}^*.

Return data:
The solution of (1.1): (x^*, y^*). Optionally: A_*, A_c, R_*, γ_*, R, S, $P + I$, $\sqrt{\sigma_*}$.

Thus, a system similar to (6.1) can be trivially built from (6.3):

$$\begin{bmatrix} A_* & O^T \\ O & \gamma_* I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (I-P)(a - AQR^{-T}b) + \gamma_* QR^{-T}b \\ \gamma_* R^{-1}Q^T[I - AA_*^{-1}(I-P)](a - AQR^{-T}b) \end{bmatrix}, \tag{6.4}$$

for which

$$P = QQ^T, \quad \gamma_* = \|(I-P)A(I-P)\|_2 \quad \text{and} \quad A_* = (I-P)A(I-P) + \gamma_* P.$$

Note that the right hand side of (6.4) explicitly implies the storing of the inverse of the Cholesky factor of B^T, instead of R itself. This is acceptable when $m \ll n$, because R is a triangular matrix and the condition number of the coefficient matrix of the above system can be controlled this way. Indeed, since $A_* Q = \gamma_* Q$, all the eigenvalues of $\gamma_* I$ are eigenvalues of A_* as well. This fact and Proposition 6.1 then guarantee that such number is $\text{cond}_2(A_*) = \text{cond}_2(K^T AK)$, for any K as in the mentioned proposition.

From the last paragraph we immediately see a number of simplifications in the elements of the method developed, which compensate the additional cost required to compute and store the factor Q of B^T. Algorithm 2 considers these changes.

Algorithm 2 IT CALCULATES THE SOLUTION OF (1.1) BY SOLVING (6.4).

Entry data:
m, n positive integers such that $m \ll n$,
$B \in \mathbb{R}^{m\times n}$ of full rank, $A \in \mathbb{R}^{n\times n}$ symmetric and positive definite on $\ker(B)$,
$a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $\delta \in \mathbb{R}$ such that $0 \le \delta \ll 1$.

Workspace: $\gamma_* \in \mathbb{R}$; $x^*, x_c^*, x_{2c}^*, x_{3c}^* \in \mathbb{R}^n$; $y^* \in \mathbb{R}^m$; $Q \in \mathbb{R}^{n\times m}$; $R \in \mathbb{R}^{m\times m}$; $P, A_*, R_* \in \mathbb{R}^{n\times n}$.

Step 1: Calculate the factors Q and R of the thin QR factorization of B^T by the Householder method. Then replace R by its inverse.

Step 2: Calculate the product $P = QQ^T$ (symmetric product) and then replace it by $P - I$ (just update the main diagonal of P).

Step 3: Calculate $A_* = PAP$ (symmetric product) and then estimate the largest eigenvalue γ_* of A_* (for example, with the power method). Now replace A_* by $A_* + \gamma_* P$ (symmetric sum) and then, by $A_* + (1+\delta)\gamma_* I$ (just update the main diagonal of A_*). Calculate the Cholesky factor R_* of A_*.

Step 4: Take y^* as $R^T b$ and compute $x_c^* = Qy^*$. Also calculate $x_{2c}^* = Ax_c^* - a$ and the solution x_{3c}^* of the system $A_* x = P x_{2c}^*$, using R_*. Finally, let $x^* = x_{3c}^* + x_c^*$.

Step 5: Replace x_{3c}^* by $-(x_{2c}^* + Ax_{3c}^*)$ and y^*, by $Q^T x_{3c}^*$. Then replace y^* by $(1/\gamma_*)(\gamma_* R y^*)$ or, alternatively, replace y^* by Ry^*.

Return data:
The solution of (1.1): (x^*, y^*). Optionally: A_*, R_*, γ_*, Q, R, $P + I$.

Notice that, in the first step of Algorithm 2, we explicitly indicate the orthogonalization of the columns of the matrix B^T by the Householder method. This is because the orthogonal factor $\check{Q}$ of this factorization, computed in finite-precision arithmetic, really is, for practical purposes, a matrix with orthonormal columns, since

$$\check{Q}^T\check{Q} = I + E_H \quad \text{and} \quad \|E_H\|_2 \approx u,$$

where u is the rounding machine unit [5, p. 255]. If we want to solve the problem (1.1) with Algorithm 2 and the condition number of the matrix B is not a concern, then a QR factorization method more suitable is the modified Gram-Schmid method. For it we have that

$$\check{Q}^T\check{Q} = I + E_{MGS} \quad \text{and} \quad \|E_{MGS}\|_2 \approx u\,\mathrm{cond}_2(B),$$

but on the other hand $\check{Q}$ can be obtained with approximately half of the floating point operations required to the Householder's orthogonalization [5, p. 255].

7. An alternative derivation of the null space method. In this section, the matrix representation of a linear transformation $T : \mathbb{R}^p \to \mathbb{R}^q$ relative to the basis *out* of $\mathbb{R}^p$ and *in* of $\mathbb{R}^q$ is denoted by $[T]_{in}^{out}$, while the coordinates of vectors $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$ relative to these same basis are denoted, respectively, by $[u]_{out}$ and $[v]_{in}$.

Let $K \in \mathbb{R}^{n\times(n-m)}$ be, as usual, a matrix whose columns $K_1, \ldots, K_{n-m}$ span $\ker(B)$, and $B_1, \ldots, B_m \in \mathbb{R}^n$ be the rows of the matrix B. Then, in view of the direct sum decomposition exhibit to $\mathbb{R}^n$ in Section 3, the set $\{K_1, \ldots, K_{n-m}, B_1^T, \ldots, B_m^T\}$ is an ordered basis for this space. We denote this basis by *alt*, from alternative, and the canonical basis of $\mathbb{R}^n$, by *can*.

Now let $K^\dagger = (K^TK)^{-1}K^T$. Since $\ker(B)$ and $\mathrm{ran}(B^T)$ are T_{A_*}-invariant spaces, the linear operator T_{A_*} has a block-diagonal matrix representation with respect to the basis *alt*, namely

$$[T_{A_*}]_{alt}^{alt} = \begin{bmatrix} K^\dagger \\ (B^\dagger)^T \end{bmatrix} A_* \begin{bmatrix} K & B^T \end{bmatrix} = \begin{bmatrix} K^\dagger AK & O^T \\ O & \gamma_* BB^T \end{bmatrix}.$$

The coordinates of the vector $\bar{a} = (I-P)(a - AB^\dagger b) + \gamma_* B^T b \in \mathbb{R}^N$, with respect to this same basis, also are

$$[\bar{a}]_{alt} = \begin{bmatrix} K^\dagger \\ (B^\dagger)^T \end{bmatrix} \bar{a} = \begin{bmatrix} K^\dagger(a - AB^\dagger b) \\ \gamma_* b \end{bmatrix}.$$

From (6.2) we have that $A_* x = \bar{a}$, or what is equivalent, $[T_{A_*}]_{alt}^{alt}\,[x]_{alt} = [\bar{a}]_{alt}$. Thus $[x^*]_{alt}$ is the solution of the linear system

$$\begin{bmatrix} K^TAK & O^T \\ O & BB^T \end{bmatrix} [x]_{alt} = \begin{bmatrix} K^T(a - AB^\dagger b) \\ b \end{bmatrix}. \tag{7.1}$$

If $[x^*]_{alt}^T = \begin{bmatrix} \tilde{x}^T & \hat{x}^T \end{bmatrix}$, with $\tilde{x} \in \mathbb{R}^{n-m}$ and $\hat{x} \in \mathbb{R}^m$, then

$$x^* = [x^*]_{can} = \begin{bmatrix} K & B^T \end{bmatrix} [x^*]_{alt} = K\tilde{x} + B^T\hat{x} = K\tilde{x} + B^\dagger b. \tag{7.2}$$

This remark and the fact known from the linear system (6.2) that y^* is solution of

$$BB^T y = B(a - Ax^*), \tag{7.3}$$

establish a strategy that describes the null space methods in general, up to the choice of a particular solution of the system $Bx = b$, which, in our case, turned out to be set at the most natural choice: $B^T\hat{x} = B^\dagger b$.

Based on comparisons of (7.1)-(7.3) with the value of $\mathrm{cond}_2(A_*)$, the format of the linear system (6.2) and the expression of the solution of (1.1) itself, we believe it is correct to see the method developed in the last section as a null space method that does not depend on any basis of $\ker(B)$. This is why we say that that method is a basis-free method. So, due to its nature, the method developed has essentially the same applications as the null space method. In particular, it is possible to employ it in applications that require the solution of a sequence of saddle point problems,

$$\begin{bmatrix} A^{(k)} & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a^{(k)} \\ b^{(k)} \end{bmatrix}, \quad k \in \mathbb{N}_0,$$

where the submatrix $A^{(k)} \in \mathbb{R}^{n\times n}$ and the vector $(a^{(k)}, b^{(k)}) \in \mathbb{R}^{n+m}$ can change with k, but the matrix B remains fixed. For such applications, it is of course important to exploit the possible special structures present in the matrices $A^{(k)}$ and B, so the products $(I-P)A^{(k)}(I-P)$ and constants $\gamma_*^{(k)} \in \mathbb{R}$ are calculated with a smaller number of floating point operations. Unfortunately, we cannot guarantee that the

matrix A_* has, in the general case, a special structure that benefits the computation of its Cholesky factor, which is, therefore, the main difficulty for applying the developed method — or even the null space method.

An algorithm for solving the problem above is likewise Algorithm 1 or 2, iteratively repeating Steps 3 to 5, to new input data $A^{(k)}$, $a^{(k)}$ and $b^{(k)}$ at each iteration.

We close this section by noting that, the smaller the difference $n - m$, the more appropriate is the application of the null space method for solving saddle point problems, due to the lower cost involved in the calculation of the products $K^T A^{(k)} K$, $k \in \mathbb{N}_0$, and in the resolution of the linear systems which involve these matrices. But as the indicated difference increases, the null space method loses performance both in the initial determination of a basis for $\ker(B)$, as in the resolution of the systems itself. Our approach, on the other hand, only loses performance in the latter, having a negligible cost to the implementation of their initial steps in the usual situation in which $m \ll n$, case in which it end up having a performance similar to the null space method, regarding the calculation of

$$A_*^{(k)} = (I - P)A^{(k)}(I - P) + \gamma_*^{(k)} B^T B, \quad k \in \mathbb{N}_0,$$

and the resolution of the systems involving these matrices. True, since $n - m \approx n$ in the situation described. So we can have the method developed in this paper in mind if the number of iterations to be performed is also to be kept small compared to n.

8. The analytical solution of the nonsymmetric saddle point problem. The already quite mild assumptions made to the linear system (1.1) in the previous sections are relaxed in this one so we can analyze a problem similar to the resolution of this system, but with a lack of symmetry. We consider here, thus, the generalized saddle point problem, that is, the problem of determining the solution of (1.1) with a nonsymmetric matrix A. We repeat it below for convenience:

$$M \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{8.1}$$

Besides being possible and interesting, an extension of everything that was done before to this new problem is desired because there are important problems with its format. An example is the problem of solving the linearized Navier-Stokes equations, which linearizations were obtained by Picard iterations or by some variation of the Newton's method from stable discretizations of the Navier-Stokes equations (in the sense explained in [7, p. 12]), performed with the finite element method [7, pp. 81-96].

As noted in the introduction of this paper, for the problem of this section to admit a unique solution, it is sufficient that the symmetric part A_s of A is positive definite on $\ker(B)$ and that B has full rank. These are, therefore, two of the three hypotheses that we assume from now on. Observe that the assumption made on A_s is weaker than that which is usually found in similar works, for example [6], where it is assumed that A_s is a positive definite matrix on all of $\mathbb{R}^n$. The third hypothesis to which we refer is a condition related to the nonsymmetric part A_{ss} of A, namely: that the matrix

$$(I - P)A_{ss}(A_s)_*^{-1}(I - P) \tag{8.2}$$

has no eigenvalue -1. It will be needed to ensure the nonsingularity of a given matrix.

In what follows, we will employ the conditions above and the analytical solution of the symmetric case of (1.1) to also solve (8.1) analytically. And then we will take

the direct method developed in Section 6, along with this other solution, to establish a direct method for solving (8.1). Well, we start by defining

$$F_s = (I-P)(A_s)_*^{-1} \quad \text{and} \quad G_s = (B^\dagger)^T(I - A_sF_s),$$

which are matrices related with the inverse of the matrix M from the symmetric case of (1.1), taken with $A = A_s$. For F_s, we have in particular that

$$(I-P)(A_s)_*^{-1} = (I-P)(A_s)_*^{-1}(I-P) = (A_s)_*^{-1}(I-P),$$

due to the identities that follow Corollary 4.3 (look at them with A_s taking the role of A). Now let's consider the following linear system,

$$\bar{M}\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} \bar{A} & B^T \\ B & O \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{a} \\ b \end{bmatrix}, \tag{8.3}$$

where $\bar{A} = A_s + PA_{ss}$ e $\bar{a} \in \mathbb{R}^n$. Then, the symmetric and skew-symmetric parts of $\bar{A}$ are, respectively,

$$\bar{A}_s = A_s + \frac{1}{2}(PA_{ss} - A_{ss}P) \quad \text{and} \quad \bar{A}_{ss} = \frac{1}{2}(PA_{ss} + A_{ss}P),$$

and $\bar{A}_s$ is a positive definite matrix on $\ker(B)$, because

$$\tilde{x}^T\bar{A}_s\tilde{x} = \tilde{x}^TA_s\tilde{x} + \frac{1}{2}[(\tilde{x}^TP)A_{ss}\tilde{x} - \tilde{x}^TA_{ss}(P\tilde{x})] = \tilde{x}^TA_s\tilde{x} > 0$$

for all nonzero vector $\tilde{x}$ on $\ker(B)$. Under these conditions, (8.3) admits a unique solution and we can — following [6] — associate with this system the iterative scheme

$$\bar{M}_s\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} := \begin{bmatrix} \bar{A}_s & B^T \\ B & O \end{bmatrix}\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} \bar{a} - \bar{A}_{ss}x^{(k)} \\ b \end{bmatrix}, \quad k \in \mathbb{N}_0,$$

defined by $\bar{A}_s$, $\bar{A}_{ss}$ and any point $(x^{(0)}, y^{(0)})$ of $\mathbb{R}^n$. The solutions of each one of its iterations are given by Theorem 5.1. Before stating them, however, we can see that

$$(I-P)\bar{A}_s(I-P) = (I-P)A_s(I-P)$$

and so $(\bar{A}_s)_* = (A_s)_*$. Thus,

$$\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} F_s(\bar{a} - \bar{A}_sB^\dagger b - \bar{A}_{ss}x^{(k)}) + B^\dagger b \\ (B^\dagger)^T(I - \bar{A}_sF_s)(\bar{a} - \bar{A}_sB^\dagger b - \bar{A}_{ss}x^{(k)}) \end{bmatrix}, \quad k \in \mathbb{N}_0.$$

Hence $Px^{(k+1)} = B^\dagger b$ for all $k \in \mathbb{N}_0$. But then,

$$\begin{aligned} x^{(k+1)} &= (A_s)_*^{-1}\{(I-P)\bar{a} - [(I-P)\bar{A}_s]B^\dagger b - [(I-P)\bar{A}_{ss}]x^{(k)}\} + B^\dagger b \\ &= F_s[\bar{a} - (A_s - {}^1\!/_2A_{ss}P)\,B^\dagger b - {}^1\!/_2A_{ss}Px^{(k)}] + B^\dagger b \\ &= F_s(\bar{a} - A_sB^\dagger b) + B^\dagger b, \quad k \in \mathbb{N}. \end{aligned}$$

So the sequence $(x^{(k)})$ is constant from its third term forward and convergent to

$$x^{**} = F_s(\bar{a} - A_sB^\dagger b) + B^\dagger b \in \mathbb{R}^n.$$

We replace this limit point on the expression of $y^{(k+1)}$ to conclude that the sequence $(y^{(k)})$ is also constant — this time from its fourth term forward — and convergent to

$$y^{**} = (B^\dagger)^T(I - \bar{A}_s F_s)(\bar{a} - \bar{A}_s B^\dagger b - \bar{A}_{ss} x^{**}) \in \mathbb{R}^m.$$

The last expression can be rewritten regardless of the terms $\bar{A}_s$ and $\bar{A}_{ss}$. Indeed, for

$$\begin{aligned}(B^\dagger)^T \bar{A}_s F_s &= (B^\dagger)^T[(A_s + {}^1\!/_2 P A_{ss})\,(I-P)(A_s)_*^{-1}] \\ &= (B^\dagger)^T\left[(I-P)A_s + PA_s + {}^1\!/_2 P A_{ss}\right] F_s = (B^\dagger)^T\,(A_s + {}^1\!/_2 A_{ss})\,F_s\end{aligned}$$

and $Px^{**} = B^\dagger b$, from where we also note that

$$\begin{aligned}\bar{a} - \bar{A}_s B^\dagger b - \bar{A}_{ss} x^{**} &= \bar{a} - A_s B^\dagger b - {}^1\!/_2(PA_{ss} - A_{ss})B^\dagger b - {}^1\!/_2(PA_{ss}x^{**} + A_{ss}B^\dagger b) \\ &= \bar{a} - A_s B^\dagger b - {}^1\!/_2 P A_{ss}(x^{**} + B^\dagger b).\end{aligned}$$

Not only: $(B^\dagger)^T A_{ss}(x^{**} + B^\dagger b) = (B^\dagger)^T A_{ss} F_s(\bar{a} - A_s B^\dagger b) + 2(B^\dagger)^T A_{ss} B^\dagger b$. So,

$$\begin{aligned}y^{**} &= \{(B^\dagger)^T - [(B^\dagger)^T \bar{A}_s F_s]\}(\bar{a} - \bar{A}_s B^\dagger b - \bar{A}_{ss} x^{**}) \\ &= (B^\dagger)^T[I - (A_s + {}^1\!/_2 A_{ss})F_s][\bar{a} - A_s B^\dagger b - {}^1\!/_2 P A_{ss}(x^{**} + B^\dagger b)] \\ &= (B^\dagger)^T[I - (A_s + {}^1\!/_2 A_{ss})F_s](\bar{a} - A_s B^\dagger b) - {}^1\!/_2 (B^\dagger)^T A_{ss}(x^{**} + B^\dagger b) \\ &= (B^\dagger)^T\left[I - (A_s + {}^1\!/_2 A_{ss})\,F_s - {}^1\!/_2 A_{ss} F_s\right](\bar{a} - A_s B^\dagger b) - (B^\dagger)^T A_{ss} B^\dagger b \\ &= (B^\dagger)^T(I - A_s F_s)(\bar{a} - A_s B^\dagger b) - (B^\dagger)^T A_{ss}[F_s(\bar{a} - A_s B^\dagger b) + B^\dagger b] \\ &= (B^\dagger)^T[(I - AF_s)(\bar{a} - A_s B^\dagger b) - A_{ss} B^\dagger b]\end{aligned}$$

is independent on $\bar{A}_s$ and $\bar{A}_{ss}$, as we would like.

It follows that $(x^{**}, y^{**}) \in \mathbb{R}^{n+m}$ is the solution of (8.3), because

$$\begin{aligned}\bar{A}x^{**} + B^T y^{**} &= A_s x^{**} + PA_{ss}x^{**} + B^T y^{**} \\ &= A_s F_s(\bar{a} - A_s B^\dagger b) + A_s B^\dagger b \\ &\quad + PA_{ss}F_s(\bar{a} - A_s B^\dagger b) + PA_{ss}B^\dagger b \\ &\quad + P(\bar{a} - A_s B^\dagger b) - PA_s F_s(\bar{a} - A_s B^\dagger b) \\ &\quad - PA_{ss}F_s(\bar{a} - A_s B^\dagger b) - PA_{ss}B^\dagger b \\ &= [(I-P)A_s F_s](\bar{a} - A_s B^\dagger b) + P\bar{a} + (I-P)A_s B^\dagger b \\ &= (I-P)(\bar{a} - A_s B^\dagger b) + P\bar{a} + (I-P)A_s B^\dagger b \\ &= \bar{a}\end{aligned}$$

and $Bx^{**} = b$ trivially. Note the use of (5.1) in the fourth equality of the first identity.

Returning now to the solving of (8.1), we see that

$$Ax^{**} + B^T y^{**} = \bar{A}x^{**} + (I-P)A_{ss}x^{**} + B^T y^{**} = \bar{a} + (I-P)A_{ss}x^{**}.$$

However, we would like the last equation to result in a. This will be the case if $\bar{a}$ is the solution of the linear system

$$[I + (I-P)A_{ss}F_s]\bar{a} = a - (I-P)A_{ss}G_s^T b, \tag{8.4}$$

since

$$\begin{aligned}\bar{a} + (I-P)A_{ss}x^{**} &= \bar{a} + (I-P)A_{ss}F_s(\bar{a} - A_s B^\dagger b) + (I-P)A_{ss}B^\dagger b \\ &= [I + (I-P)A_{ss}F_s]\bar{a} + (I-P)A_{ss}(I - F_s A_s)B^\dagger b = a.\end{aligned}$$

A unique solution exists for (8.4) because the eigenvalues of its coefficient matrix are of the form $1+\lambda$, where λ is an eigenvalue of (8.2), and we assumed that $\lambda \neq -1$.

Along with Theorem 3.4 from [1], we then just proved that

THEOREM 8.1. *If the matrix A_s is positive definite on* $\ker(B)$*, B has full rank and the matrix* (8.2) *has no eigenvalue -1, then the system* (8.1) *admits a unique solution, which is*

$$\begin{bmatrix} x^{**} \\ y^{**} \end{bmatrix} = \begin{bmatrix} F_s(\bar{a} - A_s B^\dagger b) + B^\dagger b \\ (B^\dagger)^T[(I - AF_s)(\bar{a} - A_s B^\dagger b) - A_{ss} B^\dagger b] \end{bmatrix} = \begin{bmatrix} x^* \\ y^* - (B^\dagger)^T A_{ss} x^* \end{bmatrix},$$

where $\bar{a} \in \mathbb{R}^n$ and $(x^*, y^*) \in \mathbb{R}^{n+m}$ are, respectively, the solutions of (8.4) and (1.1) *— the latter taken with $A = A_s$ and $a = \bar{a}$.*

Thus, our major difficulty to develop a method that solves (8.1) is, according to the last theorem, to solve the auxiliary linear system (8.4). We can explicitly compute the inverse of its coefficient matrix for a general A and still apply Theorem 8.1 by assuming the spectral radius of $(I-P)A_{ss}F_s$ to be less than one. Indeed, because in this case such matrix has no eigenvalue -1 and we can use the Neumann series of its additive inverse (Lemma 7.18 in [2, p. 442]) and the fact that $I - P$ is an idempotent matrix to see that

$$[I + (I-P)A_{ss}F_s]^{-1} = I + (I-P)\left[\sum_{i=1}^{\infty}(-1)^i (A_{ss}F_s)^i\right]. \tag{8.5}$$

Therefore, by truncating the series above, we can obtain a more or less accurate value to $\bar{a}$ using just multiplication of matrix by vector. This situation — that leads to the solution of (8.1) — is, by itself, much better than the iterated solving of an indefinite number of linear systems of order $n+m$, as originally suggested in [6].

From Theorem 8.1, we can naturally think of a direct method for solving (8.1) that consists of the calculation of $\bar{a}$ — as just indicated —, the solving of the linear system (6.1) — taken with $A = A_s$ and $a = \bar{a}$ —, and the application of

$$\begin{bmatrix} I & O^T \\ -(B^\dagger)^T A_{ss} & I \end{bmatrix}$$

to the solution thus computed. As it is clear from the expressions of (8.4), (8.5) and the solution of (8.1), a large part of the calculations done to the solving of (6.1) — as described in this paragraph — can actually be reused on the calculation of $\bar{a}$, so the two steps can be simultaneously done without incurring further significant computational costs. Moreover, the execution of such a method involves solving a linear system with coefficient matrix

$$\begin{bmatrix} (A_s)_* & O^T \\ \gamma_* B A_{ss} & \gamma_* B B^T \end{bmatrix} = \begin{bmatrix} (A_s)_* & O^T \\ O & \gamma_* B B^T \end{bmatrix} \begin{bmatrix} I & O^T \\ (B^\dagger)^T A_{ss} & I \end{bmatrix}$$

and right hand side given by the right hand side of (6.1) — but with $A = A_s$ and $a = \bar{a}$. Let's say it is $(\dot{a}, \dot{b}) \in \mathbb{R}^{n+m}$. Then this is a system equivalent to

$$\begin{bmatrix} (A_s)_* & O^T \\ O & \gamma_* B B^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dot{a} \\ \dot{b} - \gamma_* B A_{ss} (A_s)_*^{-1} \dot{a} \end{bmatrix}, \tag{8.6}$$

where

$$\begin{aligned} \dot{b} - \gamma_* B A_{ss} (A_s)_*^{-1} \dot{a} &= \gamma_* B (I - A_s F_s - A_{ss} F_s)(\bar{a} - A s B^\dagger b) - \gamma_* B A_{ss} B^\dagger b \\ &= \gamma_* B[(I - AF_s)(\bar{a} - A_s B^\dagger b) - A_{ss} B^\dagger b]. \end{aligned} \tag{8.7}$$

Notice that its coefficient matrix has 2-norm condition number equal to

$$\mathrm{cond}_2((A_s)_*) = \max\{\mathrm{cond}_2(K^T A_s K), \mathrm{cond}_2(BB^T)\},$$

due to the analysis presented in Section 6. So we expect this system to have improved sensitivity by the results shown in [4]. Algorithm 3 takes into account these remarks. Its nonnegative parameter i_{max} is to truncate the series exhibit in (8.5). Despite this truncation, the solution of the linear system (8.1) remains being computed by means of its analytical expression.

Again, if the numerical rank of the matrix B is a concern, we can compute its thin QR factorization and solve a linear system based on (6.3), (8.6) and (8.7),

$$\begin{bmatrix} (A_s)_* & O^T \\ O & \gamma_* I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} (I-P)(\bar{a} - A_s Q b_Q) + \gamma_* Q b_Q \\ \gamma_* R^{-1} Q^T [(I - AF_s)(\bar{a} - A_s Q b_Q) - A_{ss} Q b_Q] \end{bmatrix}, \tag{8.8}$$

where $b_Q = R^{-T} b$ and the coefficient matrix of the system has 2-norm condition number equal to $\mathrm{cond}_2((A_s)_*) = \mathrm{cond}_2(K^T A_s K)$. Algorithm 4 considers this case.

Algorithm 3 IT CALCULATES THE SOLUTION OF (8.1) BASED ON (8.6) AND (8.7).

Entry data:
m, n positive integers such that $m < n$, a nonnegative integer i_{max},
$B \in \mathbb{R}^{m\times n}$ of full rank, $A \in \mathbb{R}^{n\times n}$ such that A_s is positive definite on $\ker(B)$ and the spectral radius of $(I-P)A_{ss}F_s$ is less than 1,
$a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $\delta \in \mathbb{R}$ such that $0 \le \delta \ll 1$.

Workspace: $\gamma_*, \sigma_* \in \mathbb{R}$; $x^*, x_c^*, x_{2c}^*, x_{3c}^*, a_c \in \mathbb{R}^n$; $y^* \in \mathbb{R}^m$; $R \in \mathbb{R}^{m\times m}$; $S \in \mathbb{R}^{m\times n}$; $P, A_*, A_c, R_* \in \mathbb{R}^{n\times n}$.

Step 1 to Step 3: Execute Steps 1 to 3 as in Algorithm 1, taking $A = A_s$ there.

Step 4a: Calculate $x_c^* = S^T b$ and $x_{2c}^* = A_s x_c^*$.
Step 4b: Using R_*, take a_c as the solution of $(A_s)_* x = P x_{2c}^*$ and then replace a_c by $a + PA_{ss}(a_c + x_c^*)$.
Step 4c: Initialize x_{3c}^* and x^* with a_c and 0, respectively. Then, for i running from 0 to i_{max}, repeat: replace x_{3c}^* by the solution of $(A_s)_* x = P x_{3c}^*$, computed using R_*, and again, by $A_{ss}x_{3c}^*$; then add $(-1)^{i+1}x_{3c}^*$ to x^*. Once the $(i_{max}+1)$-th iteration is complete, replace a_c by $a_c + Px^*$.

Step 5: Replace x_{2c}^* by $a_c - x_{2c}^*$ and then x_{3c}^*, by the solution of $(A_s)^* x = P x_{2c}^*$. Finally replace x^* by $x_c^* - x_{3c}^*$.

Step 6: Replace x_{3c}^* by $x_{2c}^* + Ax_{3c}^* - A_{ss}x_c^*$ and take y^* as the solution of the system $(\gamma_* BB^T)y = \gamma_* B x_{3c}^*$, computed using the Cholesky factor $\sqrt{\gamma_*}R$ of $\gamma_* BB^T$, or, alternatively, by Sx_{3c}^*.

Return data:
The solution of (8.1): (x^*, y^*). Optionally: A_*, A_c, R_*, γ_*, R, S, $P+I$, $\sqrt{\sigma_*}$, a_c.

Algorithm 4 IT CALCULATES THE SOLUTION OF (8.1) BY SOLVING (8.8).

Entry data:
m, n positive integers such that $m \ll n$, a nonnegative integer i_{max},
$B \in \mathbb{R}^{m\times n}$ of full rank, $A \in \mathbb{R}^{n\times n}$ such that A_s is positive definite on $\ker(B)$ and the spectral radius of $(I-P)A_{ss}F_s$ is less than 1,
$a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $\delta \in \mathbb{R}$ such that $0 \leq \delta \ll 1$.

Workspace: $\gamma_* \in \mathbb{R}$; $x^*, x_c^*, x_{2c}^*, x_{3c}^*, a_c \in \mathbb{R}^n$; $y^* \in \mathbb{R}^m$; $Q \in \mathbb{R}^{n\times m}$; $R \in \mathbb{R}^{m\times m}$; $P, A_*, R_* \in \mathbb{R}^{n\times n}$.

Step 1 to Step 3: Execute Steps 1 to 3 as in Algorithm 2, taking $A = A_s$ there.

Step 4: Execute Steps 4a to 4c as in Algorithm 3 but compute $x_c^* = QR^Tb$ there.

Step 5 to Step 6: Execute Steps 5 to 6 as in Algorithm 2, taking $a = a_c$ there.

Return data:
The solution of (8.1): (x^*, y^*). Optionally: A_*, R_*, γ_*, Q, R, $P+I$, a_c.

9. Conclusions and future work. Many real applications lead to saddle point problems. The knowledge of their solutions — expressed with numerical considerations in mind — is thus very important, since it allows the development of efficient methods for solving such problems. Under only very mild assumptions, we analytically solved in this paper a broad class of them, that includes symmetric and nonsymmetric saddle point problems. Care was taken during the computation of their solutions to improve the sensitivity of the linear systems which compose the problems. This guided us, in particular, to a robust extension of the null space method for solving common generalized saddle point problems. We give some algorithms here to calculate the mentioned solutions, all of which we intend to specialize to relevant problems in the future, so to make them computationally cheaper and faster.

REFERENCES

[1] M. BENZI, G. H. GOLUB AND J. LIESEN, *Numerical solution of saddle point problems*, Acta Numerica, 14 (2005), pp. 1-137.
[2] R. L. BURDEN AND J. D. FAIRES, *Numerical Analysis*, 8th ed., Thomson Brooks/Cole, Belmont, CA, 2005.
[3] W. N. GANSTERER, AND J. SCHNEID AND C. W. UEBERHUBER, *Mathematical properties of equilibrium systems*, Technical Report AURORA TR2003-13, University of Vienna and Vienna University of Technology, 2003.
[4] G. H. GOLUB AND C. GREIF, *On solving block-structured indefinite linear systems*, SIAM J. Sci. Comput., 24(6) (2003), pp. 2076-2092.
[5] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, 4th ed., The Johns Hopkins University Press, Baltimore, MD, 2013.
[6] G. H. GOLUB AND A. J. WATHEN, *An iteration for indefinite systems and its application to the Navier-Stokes equations*, SIAM J. Sci. Comput., 19(2) (1998), pp. 530-539.
[7] M. D. GUNZBURGER, *Finite Element Methods for Viscous Incompressible Flows: A Guide to Theory, Practice, and Algorithms*, Academic Press, San Diego, CA, 1989.
[8] M. R. HESTENES, *Conjugate Direction Methods in Optimization*, Springer-Verlag, New York, NY, 1980.