Weak Infeasibility in Second Order Cone Programming

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Abstract

The objective of this work is to study weak infeasibility in second order cone programming. For this purpose, we consider a relaxation sequence of feasibility problems that mostly preserve the feasibility status of the original problem. This is used to show that for a given weakly infeasible problem at most m directions are needed to approach the cone, where m is the number of Lorentz cones. We also tackle a closely related question and show that given a bounded optimization problem satisfying Slater's condition, we may transform it into another problem that has the same optimal value but it is ensured to attain it. From solutions to the new problem, we discuss how to obtain solution to the original problem which are arbitrarily close to optimality. Finally, we discuss how to obtain finite certificate of weak infeasibility by combining our own techniques with facial reduction. The analysis is similar in spirit to previous work by the authors on SDPs, but a different approach is required to obtain tighter bounds.
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1 Introduction

Second order cone programming is an important class of conic linear programming with many applications [7]. The problem is solved efficiently with interior-point algorithms [13, 18, 12] as long as regularity conditions are satisfied. In this paper, we deal with the issue of weak infeasibility in second order cone programming (SOCP). Our starting point is the feasibility problem

$$\text{find } x, \text{ such that } x \in K \cap (L + c), \qquad (\mathcal{F})$$

where $c \in \mathbb{R}^n$ and L is a linear subspace of $\mathbb{R}^n$ and K is a closed convex cone. When K is a product of Lorentz cones, this is the *second order cone feasibility problem* (SOCFP). (K, L, c) will be used as shorthand for the feasibility problem $(\mathcal{F})$. We say that (K, L, c) is: (i) *strongly feasible* when $(L + c) \cap \operatorname{ri} K \neq \emptyset$, where ri denotes the relative interior; (ii) *weakly feasible*, when $K \cap (L + c) \neq \emptyset$ but $(L + c) \cap \operatorname{ri} K = \emptyset$; (iii) *weakly infeasible*, when $K \cap (L + c) = \emptyset$ but the distance between K and $L + c$ is 0; (iv) *strongly infeasible*, when $K \cap (L + c) = \emptyset$ and the distance between $L + c$ and K is greater than 0.

A major difficulty in identifying the feasibility status is the existence of weak infeasibility, since weak infeasibility does not admit an apparent finite certificate. A natural certificate of weak infeasibility is an infinite sequence $u^{(k)}$ such that $u^{(k)} \in L + c$ and $\lim_{k\to\infty} \operatorname{dist}(u^{(k)}, K) = 0$, together with some certificate of the infeasibility of (K, L, c). The sequence $\{u^{(k)}\}$ is referred to as weakly infeasible sequence in this paper.

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Weak infeasibility of conic programs is mainly discussed in the context of duality theory of conic programs and regularization of ill-conditioned conic programs, e.g. [11, 4, 15, 19]. In addition, it is closely related to the issue of closedness of image of closed convex cones, e.g., [14, 2]. See [8] for a more detailed review about this issue.

There is an alternative way to certificate weak infeasibility as follows. A weakly infeasible problem is characterized as one that is infeasible but not strongly infeasible. It is known that a system *is* strongly infeasible iff the system has a dual improving direction, see Table 1 in [11]. Therefore, checking infeasibility of the original system and checking nonexistence of a dual improving direction correspond to two conic feasibility problems, both of which can be solved with the facial reduction algorithm. In this way we can detect weak infeasibility without relying on an infinite sequence. However, this approach is not direct in the sense that even if we know that the system is weakly infeasible, it is not clear how to construct a weakly infeasible sequence.

In this paper, we develop a procedure of detecting weak infeasibility which enable us to construct a weakly infeasible sequence. Specifically, we generate a set of at most m directions and show that we are able to construct a weakly infeasible sequence with these directions. A possible application of this result is to the analysis of SOCP with unattained optimal value. Knowing the optimal value, we are able to generate an approximate optimal solution whose objective value is arbitrarily close to the (unattained) optimal value without solving SOCPs repeatedly.

We have three main contributions in this paper. First, we develop a way of constructing weakly infeasible sequence and show that for weakly infeasible problems over a product of m Lorentz cones, at most m directions are needed to approach the second order cone. We will describe in Section 4 the precise meaning of that, but we note already that this is tighter than a recent bound obtained by Liu and Pataki [6] for general linear conic programs. Another new contribution is to show how strongly feasible optimization problems can be further regularized in order to ensure that the optimal value is attained. There is also discussion on how to obtain points that are arbitrarily close to optimality for the original SOCP. Finally, we discuss how to distinguish between the four different feasibility statuses, strong feasibility, weak feasibility, weak infeasibility and strong infeasibility without requiring any regularity condition.

The main tool we use is the set of directions $\{d^1, \ldots, d^\gamma\}$ contained in L which are obtained through the application of facial reduction to $(K^*, L^\perp, 0)$ in order to obtain the relative interior of the feasible region of the dual system $K^* \cap L^\perp$. In facial reduction theory, these directions correspond to a family of hyperplanes $\{\{d^1\}^\perp, \ldots, \{d^\gamma\}^\perp\}$ which contain $K^* \cap L^\perp$. As such, this gives an interpretation of these directions from the point of view of the dual problem. What is novel about our analysis is proving that these directions have other useful "primal" properties besides what is currently known through facial reduction theory. For instance, the relaxed problems induced by them have almost the same feasibility status as the original problem. For weakly infeasible problems, these directions are useful to generate points that are arbitrarily close to the cone. This can be applied to strongly feasible problems with unattained optimal value, thus enabling the algorithmic generation of feasible points that are close to optimality. Moreover, our sequence of directions is likely to be shorter than what would be otherwise obtained through plain facial reduction, since we show that is enough to focus on the nonlinear part of the cone.

This work is organized as follows. Section 2 describes the notation and the setting of this work. Section 3 discusses how to relax a SOCFP in a way that the feasibility properties are mostly preserved. Section 4 discussed the minimal number of directions needed to approach the second order cone. Section 5 contains a discussion on unattained strongly feasible problems and how to regularize for attainment. Section 6 describes a theoretical recipe to distinguish the four feasibility statuses. Section 7 contains a brief summary of this work.

2 Notation and preliminary considerations

For $d \in \mathbb{R}^n$, we define the closed half-space $H_d^n = \{x \in \mathbb{R}^n \mid d^T x \geq 0\}$ and the ray $\mathrm{h}_d^n = \{\alpha d \in \mathbb{R}^n \mid \alpha \geq 0\}$. We also write $x = (x_0, \ldots, x_{n-1})$ for the components of x. We use the notation $\overline{x}$ to denote the last $(n-1)$ components of x, i.e., $\overline{x} = (x_1, \ldots, x_{(n-1)})$. The Lorentz cone in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$, i.e., $\mathcal{K}^n = \{x \in$

$\mathbb{R}^n \mid x_0 \geq \|\overline{x}\|\}$, where $\|.\|$ is the usual Euclidean norm. We remark that $\mathcal{K}^1 = \{x \in \mathbb{R} \mid x \geq 0\}$, so the non-negative orthant in $\mathbb{R}^n$ can be written as a direct product of one-dimensional Lorentz cones. If $x \in \mathcal{K}^n$, we write x' for the reflection of x with respect to $\mathcal{K}^n$, i.e., $x' = (x_0, -\overline{x})$.

Our main object of study is the feasibility problem $(\mathcal{F})$, where K is a direct product of Lorentz cones. We will also consider problems where K also includes closed half-spaces, rays and subspaces. We have $K = K^{n_1} \times \ldots \times K^{n_m}$, where $K^{n_i} \subseteq \mathbb{R}^{n_i}$ for every i and $n_1 + \ldots + n_m = n$. The cone K induces a block division such that for $x \in \mathbb{R}^n$ we have $x \in K$ if and only if $x_{n_i} \in K^{n_i}$, for every i. Throughout the article we use the convention that a superscript over a set indicates the dimension of ambient space. For example, $\mathcal{K}^n, H_d^n, \mathrm{h}\,_d^n$ are all sets contained in $\mathbb{R}^n$. A single subscript under a point denotes a coordinate and double subscript denotes a block. For example x_i is the i-th coordinate of x, while x_{n_i} is the i-th block of x. Of course, it is implicitly understood that the division in blocks is induced by some cone $K \subseteq \mathbb{R}^n$.

Finally, we use $\operatorname{ri} C$, $\operatorname{rec} C$, $\operatorname{lin} C$ and $\operatorname{relbd} C$ to denote the relative interior, recession cone, lineality space and relative boundary of C, respectively. We denote the dual cone of K by $K^* = \{s \in \mathbb{R}^n \mid s^T x \geq 0, \forall x \in K\}$. See [17] for basic properties of those sets.

Two convex sets $C_1, C_2 \subseteq \mathbb{R}^n$ are said to be *properly separated* when there is a hyperplane such that C_1 and C_2 lie at opposite closed half-spaces but at least one of them is not entirely contained in the hyperplane. A necessary and sufficient condition for proper separation is that $\operatorname{ri} C_1 \cap \operatorname{ri} C_2 = \emptyset$, see Theorem 11.3 of [17]. The separation is said to be *strong* if there is a ball B centered in the origin such that $C_1 + B$ and $C_2 + B$ lie at opposite open half-spaces. A necessary and sufficient conditions for strong separation is $\operatorname{dist}(C_1, C_2) > 0$, see Theorem 11.6 of [17].

Note that strong feasibility admits an obvious certificate, since it is enough to obtain an element $x \in \operatorname{ri} K \cap (L + c)$. From these basic facts about separating hyperplanes we can characterize two of the remaining feasibility statuses. We have that (K, L, c) is: *i)* weakly feasible if and only if (iff) there is $x \in K \cap (L + c)$ and $w \in (K^* \setminus K^\perp) \cap L^\perp \cap \{c\}^\perp$, where $A^\perp$ indicates the set of elements orthogonal to a set A, *ii)* strongly infeasible iff there is $w \in K^* \cap L^\perp$ with $w^T c = -1$. No such simple characterization is known for weak infeasibility, and this can be traced down to its asymptotic nature; in spite of the absence of a feasible solution, the distance between K and $L + c$ is 0. Hence, there must be some sequence $\{x^k\}$ contained in $L + c$ satisfying $\lim_{K \to \infty} \operatorname{dist}(x^k, K) = 0$. However, such a sequence cannot have any converging subsequence, so $\|x^k\| \to +\infty$.

Many of the known characterizations of weak infeasibility involve, in a way or another, infinite sequences (see Table 1 of Luo, Sturm and Zhang [10]). As a computer cannot verify infinite sequences, it is very hard to distinguish numerically between weak infeasibility and weak feasibility, see, for instance, Pólik and Terlaky [16]. This motivates the search for ways of checking infeasibility without using sequences as in the recent work for semidefinite programming by Liu and Pataki [5], see Theorem 1 therein. See also Section 4.3 of [4] by Klep and Schweighofer. These characterizations are *finite* and no infinite sequences are needed. In Section 6, we will show that distinguishing weak feasibility/infeasibility for SOCFPs can also be performed without using sequences.

For convenience, we group together weak feasibility and weak infeasibility in a single status: *weak status*. We say the feasibility status of feasibility problems $\mathcal{A}$ and $\mathcal{B}$ are "mostly the same" if $\mathcal{A}$ and $\mathcal{B}$ are both strongly feasible, strongly infeasible or in weak status. Note that it is possible that $\mathcal{A}$ is weakly infeasible and $\mathcal{B}$ is weakly feasible (or vice-versa).

3 Relaxation of SOCFPs

In this section, we show how SOCFPs can be relaxed in a way that the feasibility properties are mostly preserved. Consider a feasibility problem of the form (K, L, c), where K is a direct product $K^{n_1} \times \ldots \times K^{n_m}$, where each K^{n_i} is: the trivial cone $\{0\}$; $\mathbb{R}^{n_i}$; a second order cone $\mathcal{K}^{n_i}$; a closed half-space defined by a supporting hyperplane to $\mathcal{K}^{n_i}$, i.e., $H_d^{n_i}$ for $d \in \mathcal{K}^{n_i} \setminus \{0\}$; or a half-line contained in $\mathcal{K}^{n_i}$, i.e., $\mathrm{h}\,_d^{n_i}$ for $d \in \mathcal{K}^{n_i}$.

Note that the family of cones having the format above is no more expressive than the family of products of second order cones. Still, for our purposes we need to consider this slightly more general situation because these cones will appear as byproducts of Theorem 2. We will call them *extended second order cones*. We remark that the dual cone K^* is the direct product of the duals of the cones K^{n_i} and it is also an extended second order cone. It is also clear that we have $(H_d^{n_i})^* = \mathrm{h}_d^{n_i}$.

Suppose that we have a non-zero element $a \in K$, then we define: *i)* $\mathcal{H}_1(a,K) = \{i \mid K^{n_i} = \mathcal{K}^{n_i}, a_{n_i} \in \mathrm{ri}\,\mathcal{K}^{n_i}\}$ and *ii)* $\mathcal{H}_2(a,K) = \{i \mid K^{n_i} = \mathcal{K}^{n_i}, a_{n_i} \in (\mathrm{relbd}\,\mathcal{K}^{n_i}) \setminus \{0\}\}$. We will omit K when it is clear from the context.

Lemma 1. *Let $x \in \mathbb{R}^n$ and $a \in \mathcal{K}^n$ be such that $x^T a' > 0$. Then $x + ta \in \mathrm{ri}\,\mathcal{K}^n$ for $t > 0$ sufficiently large.*

Proof. The point a must be non-zero and if it is an interior point, then the statement clearly holds. If a lies in the boundary, then

$$(x + ta)_0^2 - \|\overline{x + ta}\|^2 = 2t(a_0 x_0 - \overline{a}^T \overline{x}) + x_0^2 - \|\overline{x}\|^2.$$

However, $a_0 x_0 - \overline{a}^T \overline{x}$ is equal to $x^T a'$. So if t is large enough we have that $(x + ta)_0^2 - \|\overline{x + ta}\|^2$ will be greater than 0. □

Theorem 2. *Let (K, L, c) be a feasibility problem such that $K = K^{n_1} \times \ldots \times K^{n_m}$. Suppose that there is $a \in K \cap L$ such that $\mathcal{H}_1(a) \cup \mathcal{H}_2(a)$ is non-empty. Define the cone $\widetilde{K} = \widetilde{K}^{n_1} \times \ldots \times \widetilde{K}^{n_m}$ such that for every i:*

- $\widetilde{K}^{n_i} = \mathbb{R}^{n_i}$ *if* $i \in \mathcal{H}_1(a)$,
- $\widetilde{K}^{n_i} = H_d^{n_i}$ *where* $d = a'_{n_i}$, *if* $i \in \mathcal{H}_2(a)$,
- $\widetilde{K}^{n_i} = K^{n_i}$, *otherwise.*

Then

i. (K, L, c) *is strongly feasible if and only if* $(\widetilde{K}, L, c)$ *is strongly feasible;*

ii. (K, L, c) *is in weak status if and only if* $(\widetilde{K}, L, c)$ *is in weak status;*

iii. (K, L, c) *is strongly infeasible if and only if* $(\widetilde{K}, L, c)$ *is strongly infeasible.*

Proof. (i) If (K, L, c) is strongly feasible, then for a relative interior point $y \in L + c$, we have $y_{n_i}^T a'_{n_i} > 0$, for all $i \in \mathcal{H}_2(a)$. All the other coordinate blocks of y_{n_i} stay in the relative interior of the respective cones. So, $(\widetilde{K}, L, c)$ is strongly feasible.

Now, if $(\widetilde{K}, L, c)$ is strongly feasible we pick $y \in L + c$ such that y lies in the relative interior of $\widetilde{K}$. For $i \in \mathcal{H}_1(a)$ we have $a_{n_i} \in \mathrm{ri}\,\mathcal{K}^{n_i}$ and for $i \in \mathcal{H}_2(a)$ we have $y_{n_i}^T a'_{n_i} > 0$. Hence if t is sufficiently large we have $(y + ta)_{n_i}^T a'_{n_i} \in \mathrm{int}(\mathcal{K}^{n_i})$, for all $i \in \mathcal{H}_1(a) \cup \mathcal{H}_2(a)$, by Lemma 1. It is also clear that adding ta does not affect the fact that $y_{n_i} \in \mathrm{ri}(K^{n_i})$ for $i \notin \mathcal{H}_1(a) \cup \mathcal{H}_2(a)$.

(iii) If $(\widetilde{K}, L, c)$ is strongly infeasible then (K, L, c) also is because $K \subseteq \widetilde{K}$. Let us prove the converse now. We have that (K, L, c) is strongly infeasible if and only if there exists s such that $s \in L^\perp \cap K^*$ and $s^T c < 0$ (see Lemma 5 of [11]). In particular, $s^T a = 0$. This means that $s_{n_i}^T a_{n_i} = 0$ for every i, because $s \in K^*$ and $a \in K \cap L$. It follows that for $i \in \mathcal{H}_1(a)$ we have $s_{n_i} = 0$. Also, for $i \in \mathcal{H}_2(a)$ we have that s_{n_i} is a non-negative multiple of a'_{n_i} (including, of course, the possibility that s_{n_i} is 0)[1]. We conclude that s also produces strong separation for $(\widetilde{K}, L, c)$ because $s \in \widetilde{K}^*$. So $(\widetilde{K}, L, c)$ is strongly infeasible.

Finally, (ii) follows by elimination.

□

[1] Recall that if $x, y \in \mathcal{K}^n$ satisfy $x^T y = 0$, then $x_0 \overline{y} + y_0 \overline{x} = 0$.

3.1 Relaxation sequence

After applying Theorem 2 to (K, L, c), it might still be possible to relax it further. This motivates the next definition.

Definition 3 (Relaxation sequence)**.** *A relaxation sequence for (K, L, c) is a finite sequence of conic feasibility problems $\{(K_1, L, c), \ldots, (K_\gamma, L, c)\}$ such that $K_1 = K$ and:*

1. *Every K_i is an extended second order cone, see the beginning of Section 3.*
2. *For $i > 1$, there is $d^{i-1} \in K_{i-1} \cap L$ such that (K_i, L, c) is obtained as a result of applying Theorem 2 to $K_{i-1} \cap L$ and d^{i-1}. In addition, we must have $K_{i-1} \subsetneq K_i$ (i.e., we do not admit trivial relaxations).*

The vectors in $\{d^1, \ldots, d^{\gamma-1}\}$ are called reducing directions*, due to the fact that they came from the application of facial reduction to the dual system $(K^*, L^\perp, 0)$. A relaxation sequence is* maximal *if it does not admit non-trivial relaxations. The problem (K_γ, L, c) is called* the last problem *of the sequence. The length of the sequence is defined to be γ.*

We now attempt to clarify the connection between relaxation sequences and facial reduction. Given a conic linear program (K, L, c), facial reduction algorithms (FRAs) [1, 3, 15, 20] aim at identifying the minimal face $\mathcal{F}_{\min}$ of K which contains the feasible region $K \cap (L + c)$. This is done by generating a sequence of faces ending at $\mathcal{F}_{\min}$. In this respect, FRA and relaxation sequences accomplish different goals. However, it can be shown that the cones appearing in a relaxation sequence correspond to the *dual* of the *faces* obtained by applying FRA to $(K^*, L^\perp, 0)$. As we are considering dual of faces instead of the faces themselves, this is more akin to the conic expansion algorithm as described in Section 4 of [20]. While this allows us to cast our techniques in the conic expansion framework, as it is a more indirect route, it seems that not much geometric insight is gained by doing that. Moreover, it is not obvious that results such as Proposition 7 and Theorem 9 hold. In the next two sections, we will use relaxation sequences to prove basic properties of weakly infeasible problems and unattained problems.

Since every reducing direction is responsible for relaxing at least one Lorentz cone, the maximum length of a relaxation sequence is $m+1$, where m is the number of second order cones appearing in K. Each relaxed problem almost preserves the feasibility status of the original, in the sense of Theorem 2. We will prove that when the relaxation sequence is maximal, the last problem cannot be weakly infeasible. Before we go further, we need a detour which we believe might be of independent interest.

Theorem 4. *Let C_1 and C_2 be non-empty convex sets in $\mathbb{R}^n$ such that C_1 is polyhedral, C_2 is closed. Suppose that*

$$\operatorname{rec} C_1 \cap -\operatorname{rec} C_2 \subseteq \operatorname{lin} C_2,$$

where $\operatorname{rec} C = \{x \in \mathbb{R}^n \mid x + C \subseteq C\}$ is the recession cone of a closed convex set C. Then $C_1 + C_2$ is closed.

Proof. See Theorem 20.3 in [17]. □

We will show that if C_2 is the direct product of a closed convex set and a polyhedral set, we may weaken the assumptions of the Theorem 4.

Proposition 5. *Let C_1 and $C_2 \times P$ be non-empty convex sets in $\mathbb{R}^n$ such that C_1 and P are polyhedral, and C_2 is closed. Suppose that*

$$\operatorname{rec} C_1 \cap -(\operatorname{rec} C_2 \times \operatorname{rec} P) \subseteq \operatorname{lin} C_2 \times -\operatorname{rec} P. \tag{1}$$

Then $C_1 + (C_2 \times P)$ is closed.

Proof. We have that $C_1 + C_2 \times P = (C_1 + \{0\} \times P) + C_2 \times \{0\}$. Since C_1 and P are polyhedral sets, $(C_1 + (\{0\} \times P))$ is also polyhedral. We would like to use Theorem 4 with $(C_1 + \{0\} \times P)$ and $C_2 \times \{0\}$. For that purpose, we are required to check that

$$(\operatorname{rec} C_1 + (\{0\} \times \operatorname{rec} P)) \cap -(\operatorname{rec} C_2 \times \{0\}) \subseteq \operatorname{lin} C_2 \times \{0\}, \tag{2}$$

because, due to polyhedrality, $\operatorname{rec}(C_1+(\{0\}\times P)) = \operatorname{rec} C_1 + (\{0\}\times \operatorname{rec} P)$. Let (x,y) be a point that belongs to the set at the left-hand side of Equation (2), then $x \in -\operatorname{rec} C_2$ and $y = a + p = 0$, where $p \in \operatorname{rec} P$ and $(x,a) \in \operatorname{rec} C_1$. It follows that $(x,a) \in -(\operatorname{rec} C_2 \times \operatorname{rec} P)$. Since we are under the assumption that Equation (1) holds, $x \in \operatorname{lin} C_2$. Hence, $(x,y) \in \operatorname{lin} C_2 \times \{0\}$ and we are done. □

The following proposition is a small modification of Corollary 20.3.1 of [17].

Proposition 6. *Let C_1 and $C_2 \times P$ be non-empty convex sets in $\mathbb{R}^n$ such that C_1 and P are polyhedral, and C_2 is closed. Suppose that*

$$\operatorname{rec} C_1 \cap (\operatorname{rec} C_2 \times \operatorname{rec} P) \subseteq \operatorname{lin} C_2 \times \operatorname{rec} P. \tag{3}$$

and that $C_1 \cap (C_2 \times P) = \emptyset$. Then C_1 and $C_2 \times P$ can be strongly separated.

Proof. Since $C_1 \cap (C_2 \times P) = \emptyset$, we have that $0 \notin C_1 - (C_2 \times P)$. Applying Proposition 5 to C_1 and $-(C_2 \times P)$ we find that $C_1 - (C_2 \times P)$ is closed. Therefore, both sets can be strongly separated. □

Proposition 7. *If $\{(K_1, L, c), \dots, (K_\gamma, L, c)\}$ is a maximal relaxation sequence for (K, L, c) then we have:*

i. (K, L, c) *is strongly feasible if and only if* (K_γ, L, c) *is strongly feasible;*

ii. (K, L, c) *is in weak status if and only if* (K_γ, L, c) *is weakly feasible;*

iii. (K, L, c) *is strongly infeasible if and only if* (K_γ, L, c) *is strongly infeasible.*

Proof. By induction and using Theorem 2, items (i) and (iii) follow. We can also conclude that (K, L, c) is in weak status if and only if (K_γ, L, c) is in weak status. Now, suppose that (K_γ, L, c) is infeasible and that (K, L, c) is in weak status. To finish the proof, we have to show that (K_γ, L, c) cannot be weakly infeasible.

Reordering if necessary, we may assume that $K_\gamma = \widetilde{K} \times \widetilde{P}$, where $\widetilde{K}$ is the direct product of Lorentz cones and P is a polyhedral cone. In this case, $\widetilde{P}$ is a direct product of half-spaces and vector spaces. Now, we would like to use Proposition 6 by setting $C_1 = L + c$, $C_2 = \widetilde{K}$ and $P = \widetilde{P}$. Let us check that Equation (3) is satisfied. We have $\operatorname{rec} C_1 \cap (\operatorname{rec} C_2 \times \operatorname{rec} P) = L \cap (\widetilde{K} \times \widetilde{P})$ and $\operatorname{lin} C_2 \times \operatorname{rec} P = \{0\} \times \widetilde{P}$.

Pick an element $x \in L \cap (\widetilde{K} \times \widetilde{P})$. We must have $x \in \{0\} \times P$, otherwise we would be able to apply Proposition 2 one more time, which would contradict the assumption of maximality. Since Equation (3) is satisfied, it follows that if (K_γ, L, c) is infeasible, it must be strongly infeasible.

□

Example 8. *Let (K, L, c) be such that $K = \mathcal{K}^3 \times \mathcal{K}^3$ and $L + c = \{(t,t,s) \times (s,s,1) \mid (t,t,s) \in \mathcal{K}^3, (s,s,1) \in \mathcal{K}^3\}$. Then, $a = (1,1,0) \times (0,0,0) \in K \cap L$. Thus, we can relax the cone constraint from $\mathcal{K}^3 \times \mathcal{K}^3$ to $H^3_{a'_{n_1}} \times \mathcal{K}^3$. Now, $b = (0,0,1) \times (1,1,0) \in H^3_{a'_{n_1}} \times \mathcal{K}^3$. Thus, we can relax the problem from $H^3_{a'_{n_1}} \times \mathcal{K}^3$ to $H^3_{a'_{n_1}} \times H^3_{b'_{n_2}}$. The problem $(H^3_{a'_{n_1}} \times H^3_{b'_{n_2}}, L, c)$ is weakly feasible, because no point in $L + c$ strictly satisfies the inequalities which define $H^3_{a'_{n_1}} \times H^3_{b'_{n_2}}$. This implies that (K, L, c) is in weak status.*

4 The minimal number of directions needed to approach K

Let K be an extended second order cone, therefore it is a direct product of Lorentz cones and polyhedral cones. Suppose that there are m Lorentz cones among them. In this section, we will show that given a weakly infeasible feasibility problem (K, L, c) there is $c' \in \mathbb{R}^n$, a subspace L' contained in L of dimension at most m such that (K, L', c') is weakly infeasible. This means that starting at c', at most m directions are needed to approach the cone. One application of this result is on the study of problems with unattained optimal value, as in Section 5. Note that, *a priori*, the number of direction needed to approach the cone could be up to the dimension of the affine space $L + c$. Theorem 9 states, however, it is bounded by m, regardless of the dimension of $L + c$.

Theorem 9. *Let (K, L, c) be a weakly infeasible problem. Then there are a subspace $L' \subseteq L$ and $c' \in L + c$ such that (K, L', c') is weakly infeasible and dimension of $L' + c'$ is at most m, where m is the number of Lorentz cones.*

Proof. Let $\{(K_1, L, c), \ldots, (K_\gamma, L, c)\}$ be a maximal relaxation sequence and $\{d^1, \ldots, d^{\gamma-1}\}$ the associated set of reducing directions. Each d^i is responsible for relaxing at least one Lorentz cone. Since there are most m of them, there are at most m directions. Due to Proposition 7, the last problem is weakly feasible, so it admits a feasible solution c'.

If L' is the space spanned by $\{d^1, \ldots, d^{\gamma-1}\}$ then (K, L', c') is weakly infeasible. After all, (K, L', c') shares the same maximal relaxation sequence and Proposition 7 implies that (K, L', c') has weak status. Also, $L' + c'$ is an affine subspace of $L + c$, so (K, L', c') is an infeasible problem. □

Let $\mathcal{S}^n_+$ denote the cone of $n \times n$ positive semidefinite matrices. In [8], it was shown that given a weakly infeasible semidefinite feasibility problem (SDFP) $(\mathcal{S}^n_+, L, c)$, there is an affine space $L' + c'$ contained in $L + c$ of dimension at most $n - 1$ such that $(\mathcal{S}^n_+, L', c')$ is weakly infeasible. Transforming a weakly infeasible SOCFP into a $n \times n$ dimensional SDFP, immediately yields the bound $n - 1$. Note that this a much worse bound since $n - 1$ is typically larger than the number of Lorentz cones m.

Recently, Liu and Pataki generalized the results in [8] and showed that the dimension of $L' + c'$ can be taken to be less or equal than $\ell_{K^*} - 1$, see items *ii.* and *iii.* of Theorem 9 in [6]. The quantity ℓ_{K^*} corresponds to the length of the longest chain of faces of K^*. A chain of faces of K is a finite sequence of faces satisfying $F_1 \subsetneq \ldots \subsetneq F_\ell$ and the length is defined to be ℓ. For the SDP case, they showed that the bound can be refined to $\ell_{\mathcal{S}^n_+} - 2$, which matches the bound discussed in [8], since $\ell_{\mathcal{S}^n_+} = n + 1$.

If $K = \mathcal{K}^{n_1} \times \ldots \times \mathcal{K}^{n_m}$, then the largest chain of faces of K^* has length $2m + 1$. Liu and Pataki's result implies the bound $2m$ on the dimension of $L' + c'$, which is too pessimistic in view of Theorem 2.

5 Finding the optimal value in a strongly feasible unattained problem

Consider a pair of primal and dual SOCPs problems:

$$\inf_x \quad c^T x \qquad \text{subject to} \quad \mathcal{A}x = b, \quad x \in K^* \tag{P}$$

$$\sup_y \quad b^T y \qquad \text{subject to} \quad c - \mathcal{A}^T y \in K, \tag{D}$$

where $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ is linear map, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $\mathcal{A}^T$ is the adjoint of $\mathcal{A}$. We will denote by θ_P and θ_D, the primal and dual optimal values, respectively. Now, suppose that one of them (but not both) satisfy Slater's condition. This assumption can be satisfied by applying facial reduction to (D), for instance. Under these conditions, if the objective function of (D) is bounded above, then $\theta_P = \theta_D$ and the primal optimal value is attained. However, the dual optimal value may not be attained. It is natural then to consider whether (P) and (D) can be regularized so that the common optimal value is attained at both sides. Moreover, one may be interested in finding dual feasible points for (D) which are arbitrarily close to optimality. In this section, we will show how to use the techniques developed so far to accomplish both tasks.

The first step is to consider the subspace $L = (\operatorname{range} \mathcal{A}^T) \cap \{\mathcal{A}^T y \mid \langle b, y \rangle = 0\}$. Let $a = -\mathcal{A}^T y^1 \in L \cap K$ be a nonzero point and let $\widetilde{K}$ be cone obtained as a result of applying Theorem 2 to (K, L, c). We have the following lemma.

Lemma 10. *Suppose that* (D) *is strongly feasible, i.e., there is y such that $c - \mathcal{A}^T y \in \operatorname{ri} K$. Let $\widehat{\theta}_D = \sup\{b^T y \mid c - \mathcal{A}^T y \in \widetilde{K}\}$. Then $\theta_D = \widehat{\theta}_D$.*

Proof. As $\widetilde{K} \supseteq K$, we have $\theta_D \leq \widehat{\theta}_D$. We will now show that $\theta_D \geq \widehat{\theta}_D$ holds as well. The first observation is that Theorem 2 ensures that $(\widetilde{K}, \operatorname{range} \mathcal{A}^T, c)$ is strongly feasible as well. This, together with the definition of $\widehat{\theta}_D$, implies that for every $\mu < \widehat{\theta}_D$, there is y_μ such that $s_\mu = c - \mathcal{A}^T y_\mu \in \operatorname{ri} \widetilde{K}$ and $\langle b, y_\mu \rangle \geq \mu$.

Because s_μ is a relative interior point of $\widetilde{K}$, if α_1 is positive and sufficiently large, it will be the case that $s_\mu + \alpha_1 a^1 \in \operatorname{ri} K$, by Lemma 1. As $s_\mu + \alpha_1 a^1 = c - \mathcal{A}^T(y_\mu + \alpha_1 y^1)$ and $\langle b, y^1 \rangle = 0$, we conclude that $y_\mu + \alpha_1 y^1$ is a feasible solution for (D) whose value is at least μ. This readily shows that $\widehat{\theta}_D = \theta_D$.

□

Under the conditions of Lemma 10, it is possible that the optimal value of the relaxed problem is attained even if θ_D is not attained for (D). We will now show that if former is attained, then it is possible to construct solutions close to optimality for the latter in a natural way. The recipe is as follows. Suppose that y^* is an optimal solution for $\sup\{\langle b, y \rangle \mid c - \mathcal{A}^T y \in \widetilde{K}\}$ and let $\hat{y}$ be *any* point such that $c - \mathcal{A}^T \hat{y} \in \operatorname{ri} \widetilde{K}$. For every $\beta \in [0, 1)$, the point $s_\beta = c - \mathcal{A}^T((1-\beta)\hat{y} + \beta y^*)$ lies in the relative interior of $\widetilde{K}$. Following the proof of Lemma 10, for fixed β, there will be some α^1 such that $s_\beta + \alpha_1 a^1 \in K$. Note that $s_\beta + \alpha_1 a^1$ corresponds to a feasible solution having value $\langle b, (1-\beta)\hat{y} + \beta y^* \rangle$. As β goes to 1, $s_\beta + \alpha_1 a^1$ approaches optimality, at the cost of, perhaps, making α^1 large. The next step is to show that if we keep relaxing the problem, we will must eventually reach some problem whose optimal value is attained if θ_D is finite.

Theorem 11. *Suppose that* (D) *is strongly feasible. Let* $L = (\operatorname{range} \mathcal{A}^T) \cap \{\mathcal{A}^T y \mid b^T y = 0\}$ *and consider a maximal relaxation sequence for* (K, L, c). *Let* K_γ *be the cone corresponding to the last subproblem. Consider the following SOCP in dual format.*

$$\begin{aligned} \sup_{y} \quad & b^T y \\ \textit{subject to} \quad & c - \mathcal{A}^T y \in K_\gamma, \end{aligned} \tag{D'}$$

The following properties hold.

i. (D') *has a relative interior feasible solution, so the corresponding primal problem (P') is attained.*

ii. The optimal value $\theta_{D'}$ *of* (D') *satisfies* $\theta_{D'} = \theta_D$. *If* $\theta_{D'}$ *is finite, then it is attained.*

Proof. Successive applications of Theorem 2 yield the first item. Using Lemma 10 and induction, we can also conclude that $\theta_{D'} = \theta_D$. We now suppose that $\theta_{D'}$ is finite and assume, for the sake of contradiction, that the optimal value is not attained for (D').

We first observe that the affine space $\mathcal{F} = (c + \operatorname{range} \mathcal{A}^T) \cap (\{\mathcal{A}^T y \mid b^T y = \theta_{D'}\}$ is non-empty. Also, the Euclidean distance between $\mathcal{F}$ and K_γ must be zero. Both observations follow from the definition of $\theta_{D'}$, which ensures the existence of a sequence $\{y^k\}$ contained in $(c + \operatorname{range} \mathcal{A}^T) \cap K_\gamma$ which satisfies $b^T y^k \to \theta_{D'}$.

Having observed that, we will proceed as in the proof of item *ii* of Proposition 7. Since there is no optimal solution for (D'), we have $\mathcal{F} \cap K_\gamma = \emptyset$. Reordering the coordinates if necessary, we can write K_γ as the direct product of a closed convex cone $\widetilde{K}$ and a polyhedral cone $\widetilde{P}$. Note that the recession cone of $\mathcal{F}$ is L and the recession cone of K_γ is K_γ. By the maximality of the relaxation sequence, the nonzero part of any point in the intersection $L \cap K_\gamma$ must be contained in the polyhedral portion $\widetilde{P}$. Therefore, Equation (3) is satisfied and Proposition 6 ensures that $\mathcal{F}$ and K_γ can be strongly separated. Thus the Euclidean distance between these two sets must be positive, which is a contradiction.

□

We can now extend the recipe discussed after Lemma 10. Let $\{a^1, \ldots, a^\gamma\}$ the corresponding reducing directions which produces the cone K_γ. Pick an optimal solution y^* for (D') and let $\hat{y}$ be any solution such that $c - \mathcal{A}^T \hat{y} \in \operatorname{ri} K_\gamma$. By induction, for a fixed $\beta \in [0, 1)$ there are positive constants $\alpha_1, \ldots, \alpha_\gamma$ such that $z_\beta = c - \mathcal{A}^T((1-\beta)\hat{y} + \beta y^*) + \sum_{i=1}^{\gamma} \alpha_1 a^i$ corresponds to a feasible solution to (D) having value equal to $b^T((1-\beta)\hat{y} + \beta y^*)$. As before, when β approaches 1, the value of z_β approaches θ_D. This shows very clearly how the reducing directions can be used to construct a path towards optimality. Moreover, the discussion on Section 4 shows that at most m directions are needed to build points close to optimality, where m is the number of Lorentz cones.

To wrap up this section, we remark that we proved similar results for SDP in Section 5 of [9], where it is shown how to obtain a pair of primal and problems such that *both* are strongly feasible. Therefore, a key difference here is that Theorem 11 does not ensure that (P') has a relative interior point. However, this is not necessary to ensure attainment, due to Proposition 6 and the fact that we have a mixture of nonlinear and polyhedral cones.

6 Determining the Feasibility Status

Let (K, L, c) be an arbitrary SOCFP. According to Theorem 2 the feasibility status of (K, L, c) and the last problem (K_γ, L, c) is exactly the same, except, perhaps, if (K, L, c) is weakly infeasible. As mentioned in Section 2, there are simple (finite) certificates for three of the feasibility statuses and these are exactly the three statuses that are possible for (K_γ, L, c).

Therefore, to determine the feasibility status of (K, L, c), one can first compute a relaxation sequence of (K, L, c) and seek for appropriate certificates for (K_γ, L, c). If (K_γ, L, c) is weakly feasible then we have ahead of ourselves the task of distinguishing between weak feasibility and weak infeasibility of (K, L, c). To do that, it is enough to produce a finite certificate of infeasibility for (K, L, c), which can be done through facial reduction [1, 3, 15, 20] as follows.

The variant in [20] is capable of detecting infeasibility. Starting with $F_0 = K$, facial reduction algorithms (FRAs) successively identify elements $d^i \in (F^*_{i-1} \setminus F^\perp_{i-i}) \cap L^\perp$ satisfying $\langle d^i, c\rangle \leq 0$. After d^i is found, we define $F_i = F_{i-1} \cap \{d^i\}^\perp$ and repeat. At any step, if no d^i exists, then the minimal face of K which contains the feasible region is precisely F_{i-1}. As the F_i form a strictly descending chain of faces of K, dimensional considerations readily imply that FRA must end in a finite number of steps. Moreover, it can be shown that (K, L, c) is infeasible if and only if $\langle d^i, c\rangle < 0$ at some iteration. For more details, see Section 3 in [20]. If K is an extended second order cone, it is possible to show that the search for d^i can be cast as a SOCP as well. The upshot is that we may do facial reduction without ever leaving the SOCP world and the vectors d^i serve as witnesses of the infeasibility of (K, L, c).

The certificate coming from facial reduction, the set of reducing directions associated to a relaxation sequence and a feasible solution to the last problem neatly summarize all aspects of weak infeasibility. Note that Theorem 9 and its proof show that the directions and a feasible solution to the last problem can be used to construct points in $L + c$ which are arbitrarily close to K. This fulfills the goal of producing a finite certificate that $\mathrm{dist}(K, L + c) = 0$ in spite of the fact that $K \cap (L + c) = \emptyset$.

Example 12. *Let (K, L, c) be as in Example 8. The last problem obtained was $(H^3_{a'_{n_1}} \times H^3_{b'_{n_2}}, L, c)$, which was a weakly feasible problem. We now have to find out whether (K, L, c) is weakly infeasible or feasible. In this case, it is easy because the point $(s, s, 1)$ can never belong to $\mathcal{K}^3$. If the problem were more complicated, one could formally do facial reduction and check its infeasibility. Note that, following Theorem 9 the vectors a, b and $(0, 0, 1) \times (0, 0, 1)$ attest that dist$(\mathcal{K}^3 \times \mathcal{K}^3, L + c) = 0$.*

7 Conclusion

In this paper, we presented an analysis of weakly infeasible problems via relaxation sequences. We used that to prove a basic result on the existence of affine subspaces that preserve the weak infeasibility of the problem (Theorem 9) and we showed how to regularize a strongly feasible problem in order to guarantee that it is attained.

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