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A polynomial primal-dual affine scaling algorithm for symmetric conic optimization

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Abstract The primal-dual Dikin-type affine scaling method was originally proposed for linear optimization and then extended to semidefinite optimization. Here, the method is generalized to symmetric conic optimization using the notion of Euclidean Jordan algebras. The method starts with an interior feasible but not necessarily centered primal-dual solution, and it features both centering and reducing the duality gap simultaneously. The method's iteration complexity bound is analogous to the semidefinite optimization case. Numerical experiments demonstrate that the method is viable and robust when compared to SeDuMi.

Keywords interior-point method · Dikin-type affine scaling method · symmetric conic optimization · Euclidean Jordan algebra

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1 Introduction

Following the generalization of interior point methods (IPMs) to convex optimization, conic optimization, as a special case, has received special attention. A conic optimization problem minimizes a linear objective function over a

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pointed closed convex cone intersected with affine constraints. From an algebraic point of view, linear optimization (LO), second-order conic optimization (SOCO), and semidefinite optimization (SDO) are considered as special cases of conic optimization.

Let $\mathcal{K} \subset \mathcal{J}$ be a closed pointed convex cone and $\mathcal{K}_+$ be the interior of $\mathcal{K}$, where $\mathcal{J}$ is a vector space over the field of real numbers, see Definition A.1. Then, the *dual* cone of $\mathcal{K}$ is defined as

$$\mathcal{K}^* = \{s : \langle x, s\rangle \geq 0, \quad \text{for all } x \in \mathcal{K}\},$$

where $\langle x, s\rangle$ stands for an inner product of x and s defined over $\mathcal{J}$. Cone $\mathcal{K}$ is called *self-dual* if $\mathcal{K} = \mathcal{K}^*$. Further, $\mathcal{K}$ is referred to as a *homogeneous* cone if for every $x, s \in \mathcal{K}_+$, there exists an invertible linear map $\mathcal{A}$ so that $\mathcal{A}(x) = s$ and $\mathcal{A}(\mathcal{K}) = \mathcal{K}$. The cone $\mathcal{K}$ is *symmetric* if it is both self-dual and homogeneous, see [23] for further details.

When symmetric cones are used in a conic optimization problem, then the conic linear optimization problem reduces to a symmetric conic optimization (SCO) problem. Mathematically speaking, the primal and dual problems of SCO are represented as follows

$$\begin{aligned}
&(P) \quad \min\{\langle c, x\rangle : \ \langle a_i, x\rangle = b_i, \ i = 1, ..., m, \ x \in \mathcal{K}\},\\
&(D) \quad \max\{b^T y : \ \sum_{i=1}^m y_i a_i + s = c, \ s \in \mathcal{K}, \ y \in \mathbb{R}^m\},
\end{aligned}$$

where $b \in \mathbb{R}^m$, and $c, a_i \in \mathcal{J}$ for $i = 1, ..., m$. Note that $\mathcal{K} = \mathcal{K}^*$ by the definition of a symmetric cone. The constraint $x \in \mathcal{K}$ ($s \in \mathcal{K}$) is also denoted by $0 \preceq_{\mathcal{K}} x$ ($0 \preceq_{\mathcal{K}} s$).

In recent years, IPMs have been effectively tailored to solve SCO problems. The first idea of IPMs goes back to the work of Frisch [7] who suggested using logarithmic barrier functions in LO. Then, IPMs were extensively studied for nonlinear optimization in the 1960's by Fiacco and McCormic [6]. Karmarkar revived the interest in IPMs by his polynomial-time algorithm for LO [11]. Later, Nesterov and Nemirovskii [15] proved that the theoretical efficiency of IPMs is maintained when a so-called self-scaled cone (which is identical to a symmetric cone [9]) replaces the nonnegative orthant.

The study of primal-dual IPMs for SCO problems was introduced by Nesterov and Todd [16,17] for LO problems over self-scaled cones. Faybusovich [4, 5] invoked Euclidean Jordan algebras to analyze a variety of search directions for SCO. Sturm [24] established the underlying theory of his SeDuMi software in the context of Euclidean Jordan algebras. Schmieta [20] and Schmieta and Alizadeh [21,22,23] used the Euclidean Jordan algberaic framework to extend the analysis of the Monteiro-Zhang family [25] to all symmetric cones. Rangarajan [19] and Gu et al. [8] applied the Euclidean Jordan algebras in their analysis of infeasible IPMs.

Dikin's affine scaling method is originally a primal (or dual) method, where each step aims for minimizing the objective function over an ellipsoid inscribed in the primal feasible region. The notion of affine scaling methods were extended to the primal-dual space by Monteiro et al. [14] with worst-case iteration complexity $\mathcal{O}(nL^2)$. In 1996, Jansen et al. [10] derived a primal-dual Dikin-type affine scaling method which at each iteration minimizes the duality gap over the so-called Dikin ellipsoid in the primal-dual space. This method not only has an improved $\mathcal{O}(nL)$ polynomial complexity but it also features both centering and reduction of the duality gap in contrast to the method of Monteiro et al. [14]. De Klerk et al. [12] generalized the methods of Monteiro et al. [14] and Jansen et al. [10] to SDO. Nevertheless, the extension of affine scaling methods from LO to SCO is not as straightforward as from LO to SDO, because SCO relies on a rather different algebra.

In this paper, we generalize the primal-dual Dikin-type affine scaling method of Jansen et al. [10] and de Klerk et al. [12] to SCO. For the sake of brevity, this method is referred to as Dikin-type algorithm from this point on. The Dikin-type algorithm has an $\mathcal{O}(\tau r L)$ iteration complexity, where r denotes the order of the symmetric cone, see Theorem A.1, and τ stands for the measure of proximity, see Section 3.3. The rest of this paper is organized as follows. Section 2 gives a concise review of the notion of the central path in SCO. Section 3 elaborates on the Dikin-type algorithm for SCO. Section 4 presents the complexity analysis of the Dikin-type algorithm for SCO problems, and Section 5 provides numerical results for some SOCO test instances.

2 The central path for SCO

The primal (dual) problem has a strictly feasible solution if there exists $\tilde{x}$ ($\tilde{s}$) belonging to the interior of $\mathcal{K}$, denoted by $0 \prec_{\mathcal{K}} x$ ($0 \prec_{\mathcal{K}} s$) which satisfies the primal (dual) constraints. By Theorem 2.4.1 in [2], strong duality holds, and the primal and dual problems are solvable if both (P) and (D) contain strictly feasible solutions. Hence, we assume with no loss of generality that the interior point condition holds. Let

$$Ax = \begin{pmatrix} \langle a_1, x \rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle a_m, x \rangle \end{pmatrix},$$

where Ax is assumed to be a surjective linear map. We define $x \circ s$ as a bilinear map, which is characterized by a symmetric matrix $L(x)$, see Definition A.1, as given below

$$x \circ s = L(x)s.$$

Then, it turns out that (x, y, s) is an optimal solution to the primal and dual problems (P) and (D) if and only if it satisfies [1,4]

$$\begin{aligned} Ax &= b, \quad 0 \preceq_{\mathcal{K}} x, \\ A^T y + s &= c, \quad 0 \preceq_{\mathcal{K}} s, \\ x \circ s &= 0, \end{aligned} \tag{1}$$

where $x \circ s = 0$ runs through the complementarity condition.

Essentially, primal-dual path-following IPMs deal with a relaxation of (1) by replacing the complementary condition by $x \circ s = \mu e$ with $\mu > 0$ as given below

$$\begin{aligned} Ax &= b, \quad 0 \prec_{\mathcal{K}} x, \\ A^T y + s &= c, \quad 0 \prec_{\mathcal{K}} s, \\ x \circ s &= \mu e, \end{aligned} \tag{2}$$

where e denotes the identity element, see the begining part of the Appendix. Assuming that the interior point condition holds, and A is surjective, for all $\mu > 0$ this system of equations has a unique solution $(x(\mu), y(\mu), s(\mu))$, which is called the μ-center of (P) and (D). The trajectory of the μ-centers is known as the central path of the primal and dual problems [4]. Notice that at the μ-center, we have

$$\langle x(\mu), s(\mu) \rangle = \mathrm{tr}(x(\mu) \circ s(\mu)) = \mathrm{tr}(\mu e) = r\mu,$$

where r denotes the order of the symmetric cone $\mathcal{K}$, see Theorem A.1. Applying the Newton method to the system (2) leads to

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ x \circ \Delta s + s \circ \Delta x &= \mu e - x \circ s. \end{aligned} \tag{3}$$

Note that even if $0 \prec_{\mathcal{K}} x$ and $0 \prec_{\mathcal{K}} s$, the system (3) is not necessarily well-defined. For instance, the coefficient matrix in SOCO case might be singular [18]. An effective way to get around this problem is to scale (P) and (D) to project x and s on the same point. This is known as the NT scaling scheme [16] which is as follows. For strictly feasible solutions x and s, there exists a unique $0 \prec_{\mathcal{K}} w$ so that [5]

$$v = P(w)^{-\frac{1}{2}} x = P(w)^{\frac{1}{2}} s, \tag{4}$$

where $P(w) = 2L(w^2) - L(w)^2$ is called the quadratic representation of w, see the begining part of the Appendix, and w itself is defined as

$$w = \left[P(s^{-\frac{1}{2}}) (P(s^{\frac{1}{2}}) x)^{\frac{1}{2}} \right]^{-\frac{1}{2}} = \left[P(x^{\frac{1}{2}}) (P(x^{\frac{1}{2}}) s)^{-\frac{1}{2}} \right]^{-\frac{1}{2}}. \tag{5}$$

It follows from Eq.(5) and Lemma A.5 that $0 \prec_{\mathcal{K}} x, s$ implies $0 \prec_{\mathcal{K}} w$, and the latter implies non-singularity of $P(w)$. Thus, Lemma A.5 implies that $0 \prec_{\mathcal{K}} v$.

Now, by Part 2 of Lemma A.1 and using simple algebraic manipulations, we can verify that the scaled Newton system is given by

$$\begin{aligned} \bar{A} d_x &= 0, \\ \bar{A}^T \Delta y + d_s &= 0, \\ d_x + d_s &= \mu v^{-1} - v, \end{aligned} \tag{6}$$

where

$$\begin{aligned} d_x &= P(w)^{-\frac{1}{2}} \Delta x, \\ d_s &= P(w)^{\frac{1}{2}} \Delta s, \\ \bar{A} &= A P(w)^{\frac{1}{2}}. \end{aligned} \tag{7}$$

Note that d_x and d_s belong to the null space and row space of $\bar{A}$, respectively. All this implies that that the Newton system (6) uniquely determines d_x and d_s as the orthogonal components of $\mu v^{-1} - v$. The duality gap for the scaled problem is given by

$$\langle x, s \rangle = \operatorname{tr}(x \circ s) = \operatorname{tr}(P(w)^{\frac{1}{2}} v \circ P(w)^{-\frac{1}{2}} v) = \operatorname{tr}(v^2) = \|v\|_F^2,$$

where $\|.\|_F$ denotes the Frobenious norm, see Definition A.3.

3 Dikin-type algorithm for SCO

By orthogonality of d_x and d_s, a feasible primal-dual step along the search directions arrives at the duality gap

$$\operatorname{tr}((v + d_x) \circ (v + d_s)) = \operatorname{tr}(v^2 + v \circ d_v), \tag{8}$$

where $d_v = d_x + d_s$ stands for the Newton step in the scaled primal-dual space, referred to as the v-space. The Dikin-type algorithm aims for minimizing the duality gap (8) over a suitable ellipsoid in the v-space which is given by

$$\|v^{-1} \circ d_v\|_F \leq 1. \tag{9}$$

Instead of following the central path, the Dikin-type algorithm chooses the scaled primal-dual solutions from the ellipsoid (9). Ellipsoid (9) in the original space can be given as

$$\|P(w)^{\frac{1}{2}} x^{-1} \circ P(w)^{-\frac{1}{2}} \Delta x + P(w)^{-\frac{1}{2}} s^{-1} \circ P(w)^{\frac{1}{2}} \Delta s\|_F \leq 1.$$

It can be easily verified that this ellipsoid is indeed a generalization of the suitable ellipsoid introduced for LO in [10].

It is worth mentioning that a word-for-word generalization of the Dikin ellipsoid is written as

$$\mathcal{E}ll(x,s) = \{(\Delta x, \Delta s) \mid \|x^{-1} \circ \Delta x + s^{-1} \circ \Delta s\|_F \leq 1\}.$$

Notice that $\mathcal{E}ll(x,s)$ intersected with

$$A\Delta x = 0, \quad A^T \Delta y + \Delta s = 0$$

is not necessarily bounded because the ellipsoid $\mathcal{E}ll(x,s)$ contains the affine space $x^{-1} \circ \Delta x + s^{-1} \circ \Delta s = 0$. Thus, the system

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ x^{-1} \circ \Delta x + s^{-1} \Delta s &= 0, \end{aligned}$$

does not necessarily have a unique solution as $L^{-1}(x^{-1})$ and $L(s^{-1})$ do not commute in general.

3.1 Minimizing the duality gap over the ellipsoid

The Dikin-type seach directions are derived by minimizing the duality gap (8) over the ellipsoid (9)

$$\begin{aligned} \min \quad & \mathrm{tr}(v^2 + v \circ d_v) \\ \text{s.t.} \quad & \|v^{-1} \circ d_v\|_F \leq 1. \end{aligned} \tag{10}$$

Recall that $0 \prec_{\mathcal{K}} x, s$ implies $0 \prec_{\mathcal{K}} v$. Then, we can realize that $\mathrm{tr}(v \circ d_v) = \mathrm{tr}(v^2 \circ (v^{-1} \circ d_v))$, where $v^2 = v \circ v$. Now, letting $\xi = v^{-1} \circ d_v$, optimization problem (10) can be written as

$$\begin{aligned} \min \quad & \mathrm{tr}(v^2 + v^2 \circ \xi) \\ \text{s.t.} \quad & \|\xi\|_F \leq 1. \end{aligned} \tag{11}$$

It is easy to show that the optimal solution of (11) is given by $\xi^* = -\frac{v^2}{\|v^2\|_F}$, and the optimal objective value is $\|v\|_F^2 - \|v^2\|_F$. Therefore, we have

$$d_v^* = L(v^{-1})^{-1} \xi^* = -\frac{v^3}{\|v^2\|_F}, \tag{12}$$

where we have used the fact that v and v^{-1} have the same Jordan frame, and $L(v^{-1})$ is invertible, see Theorem A.1 and Lemma A.5. Consequently, the Dikin-type search directions are obtained by solving

$$\begin{aligned} \bar{A} d_x &= 0, \\ \bar{A}^T \Delta y + d_s &= 0, \\ d_x + d_s &= -\frac{v^3}{\|v^2\|_F}. \end{aligned} \tag{13}$$

Akin to the Newton system (6), this system of equations has a unique solution. Taking a Dikin step along the search directions d_x and d_s, the new iterate in the v-space is obtained as

$$\begin{aligned} v_x^\alpha &= v + \alpha d_x, \\ v_s^\alpha &= v + \alpha d_s, \end{aligned} \tag{14}$$

and in the original space as

$$\begin{aligned} x^\alpha &= x + \alpha \Delta x = x + \alpha P(w)^{\frac{1}{2}} d_x, \\ s^\alpha &= s + \alpha \Delta s = x + \alpha P(w)^{-\frac{1}{2}} d_s. \end{aligned}$$

3.2 Proximity to the central path and feasibility

While reducing the duality gap, the Dikin-type algorithm keeps the iterates in a predefined neighborhood of the central path. The proximity measure given in [10] is generalized to

$$\kappa(v^2) = \frac{\lambda_{\max}(v^2)}{\lambda_{\min}(v^2)}, \tag{15}$$

where $\lambda_{\min}(v^2)$ and $\lambda_{\max}(v^2)$ denote the smallest and largest eigenvalues of v^2, respectively. Note that $\kappa(v^2) \geq 1$, and equality holds only when $x \circ s = \mu e$, see Lemma 28 in [23]. Further

$$x \circ s = \mu e \quad \Longleftrightarrow \quad v^2 = \mu e.$$

3.3 The Dikin-type algorithm

The outline of the Dikin-type algorithm is described in Algorithm 1. The Dikin-type algorithm starts with a strictly feasible primal-dual solution (x^0, y^0, s^0) which is close enough to the central path in terms of the proximity measure κ. The algorithm uses the default steplength $\frac{1}{\tau\sqrt{r}}$ which, after each Dikin step, maintains feasibility and proximity to the central path. Dikin steps are taken until the duality gap decreases below the accuracy parameter ϵ.

4 Complexity analysis of the Dikin-type algorithm

In this section, we provide technical results to show that the default steplength $\alpha = \frac{1}{\tau\sqrt{r}}$ leads to a strictly feasible primal-dual solution which also stays in a close proximity to the central path. We also prove that the Dikin-type

Algorithm 1 Dikin-type algorithm

Input
  A strictly feasible solution (x^0, y^0, s^0)
Parameters
  Proximity measure $\tau > 1$ so that $\kappa(x^0 \circ s^0) \le \tau$
  Steplength α with default value $\frac{1}{\tau\sqrt{r}}$
  Accuracy parameter ϵ
$x := x^0, \;\; s := s^0$
repeat
  Obtain $(\Delta x, \Delta s)$ by solving (13) and then using (7)
  Set $x := x + \alpha\Delta x$
  Set $s := s + \alpha\Delta s$
until $\mathrm{tr}(x \circ s) \le \epsilon$

algorithm arrives at a strictly feasible ϵ-optimal solution in $\mathcal{O}(\tau r L)$ iterations, where $L = \log\left(\dfrac{\mathrm{tr}(x^0 \circ s^0)}{\epsilon}\right)$.

The next lemma provides a sufficient condition for the steplength α, by which the Dikin step gives a strictly feasible primal-dual solution.

Lemma 1 *Let $\alpha \ge 0$, and assume that $0 \prec_{\mathcal{K}} v$. Then, the steplength $\bar{\alpha}$ is feasible if*

$$0 \prec_{\mathcal{K}} v_x^\alpha \circ v_s^\alpha, \quad \forall \;\; 0 \le \alpha \le \bar{\alpha},$$

where $v_x^0 = v$, $v_s^0 = v$, and v_x^α and v_s^α are defined by (14).

Proof By Lemma A.3, $0 \prec_{\mathcal{K}} v_x^\alpha \circ v_s^\alpha$ implies that $\det(v_x^\alpha) \neq 0$ and $\det(v_s^\alpha) \neq 0$ for $0 \le \alpha \le \bar{\alpha}$. Since the eigenvalues of v_x^α and v_s^α are continuous functions of α and $0 \prec_{\mathcal{K}} v$ holds, then the eigenvalues of v_x^α and v_s^α do not vanish and remain positive on $[0, \bar{\alpha}]$. □

Lemma 2 *Let $v^\alpha = P(w^\alpha)^{-\frac{1}{2}} x^\alpha = P(w^\alpha)^{\frac{1}{2}} s^\alpha$, where w^α denotes the scaling point of x^α and s^α, where*

$$x^\alpha = x + \alpha\Delta x,$$
$$s^\alpha = s + \alpha\Delta s.$$

Then, we have

$$\kappa((v^\alpha)^2) \le \kappa(v_x^\alpha \circ v_s^\alpha),$$
$$\kappa((v^\alpha)^2) \le \kappa(x^\alpha \circ s^\alpha).$$

Proof Since $0 \prec_{\mathcal{K}} w$ and $0 \prec_{\mathcal{K}} s$, it follows from part 1 of Lemma A.1 and part 2 of Lemma A.6 that

$$v^\alpha = P(w^\alpha)^{\frac{1}{2}} s^\alpha \sim P((s^\alpha)^{\frac{1}{2}}) w^\alpha = P((s^\alpha)^{\frac{1}{2}}) P((s^\alpha)^{-\frac{1}{2}}) \left[P((s^\alpha)^{\frac{1}{2}}) x^\alpha \right]^{\frac{1}{2}}$$
$$= (P((s^\alpha)^{\frac{1}{2}}) x^\alpha)^{\frac{1}{2}} \sim (P((x^\alpha)^{\frac{1}{2}}) s^\alpha)^{\frac{1}{2}},$$

where $\sim$ denotes the similarity of the eigenvalues. Thus, according to Theorem A.1, we get

$$(v^\alpha)^2 \sim P((x^\alpha)^{\frac{1}{2}})s^\alpha,$$

where

$$\begin{aligned}P((x^\alpha)^{\frac{1}{2}})s^\alpha &= P(P(w^\alpha)^{\frac{1}{2}}(v+\alpha d_x))^{\frac{1}{2}}(P(w^\alpha)^{-\frac{1}{2}}(v+\alpha d_s))\\ &\sim P(v+\alpha d_x)^{\frac{1}{2}}(v+\alpha d_s)\end{aligned}$$

Now, considering Lemma A.7, we can conclude that

$$\begin{aligned}\kappa((v^\alpha)^2) &= \kappa(P(v+\alpha d_x)^{\frac{1}{2}}(v+\alpha d_s)) \le \kappa(v_x^\alpha \circ v_s^\alpha),\\ \kappa((v^\alpha)^2) &= \kappa(P((x^\alpha)^{\frac{1}{2}})s^\alpha) \le \kappa(x^\alpha \circ s^\alpha),\end{aligned}$$

which completes the proof. □

Assume that a strictly feasible solution (x, y, s) is given satisfying $\kappa(v^2) \le \tau$, where $\tau \ge 1$. Then, it follows from (15) that there exists $\tau_1, \tau_2 > 0$ with $\tau_2 = \tau\tau_1$ so that

$$\tau_1 e \preceq_{\mathcal{K}} v^2 \preceq_{\mathcal{K}} \tau_2 e. \tag{16}$$

Now, Lemma 3 establishes a bound for the steplength α which guarantees feasibility and proximity to the central path after a Dikin step.

Lemma 3 *The steps x^α and s^α are strictly feasible and $\kappa((v^\alpha)^2) \le \tau$ if*

$$\alpha \le \min\left\{\frac{\|v^2\|_F}{2\tau_2}, \frac{4\tau_1}{\|v^2\|_F}\right\}.$$

Proof Recall from (4) that $P(w)^{\frac{1}{2}}$ and $P(w)^{-\frac{1}{2}}$ are invertible maps from $\mathcal{K}_+$ to $\mathcal{K}_+$. Therefore, $0 \prec_{\mathcal{K}} x^\alpha, s^\alpha$ if and only if $0 \prec_{\mathcal{K}} v_x^\alpha, v_s^\alpha$. Hence, by considering Lemma 1, we only need to show that $0 \prec_{\mathcal{K}} v_x^\alpha \circ v_s^\alpha$, where

$$v_x^\alpha \circ v_s^\alpha = (v+\alpha d_x)\circ(v+\alpha d_s) = v^2 - \alpha\frac{v^4}{\|v^2\|_F} + \alpha^2 d_x \circ d_s.$$

Note that v^2 and v^4 share the same Jordan frame by Theorem A.1. Hence, $\lambda_k(v^2) - \alpha\frac{\lambda_k(v^2)^2}{\|v^2\|_F}$ serves as the eigenvalue of $\phi(v^2)$, where

$$\phi(t) = t - \alpha\frac{t^2}{\|v^2\|_F},$$

and $\lambda_k(v^2)$ denotes the eigenvalue of v^2 for $k = 1, \ldots, r$. For $\alpha \le \frac{\|v^2\|_F}{2\tau_2}$, function $\phi(t)$ is monotonically increasing on $[0, \tau_2]$. All this means that

$$\phi(\tau_1)e \preceq_{\mathcal{K}} v^2 - \alpha\frac{v^4}{\|v^2\|_F} \preceq_{\mathcal{K}} \phi(\tau_2)e,$$

and thus

$$\phi(\tau_1)e + \alpha^2 d_x \circ d_s \preceq_{\mathcal{K}} v_x^\alpha \circ v_s^\alpha \preceq_{\mathcal{K}} \phi(\tau_2)e + \alpha^2 d_x \circ d_s.$$

As long as the Dikin step is feasible, i.e., $0 \prec_{\mathcal{K}} \phi(\tau_1)e + \alpha^2 d_x \circ d_s$, we will have $\kappa(v_x^\alpha \circ v_s^\alpha) \leq \tau$ if

$$\phi(\tau_2)e + \alpha^2 d_x \circ d_s \prec_{\mathcal{K}} \tau(\phi(\tau_1)e + \alpha^2 d_x \circ d_s),$$

which can be further simplified to

$$0 \prec_{\mathcal{K}} \frac{\tau_1\tau_2}{\|v^2\|_F} e + \alpha d_x \circ d_s. \tag{17}$$

By Lemma A.2, $\frac{1}{4}\|d_x + d_s\|_F^2$ gives an upper bound for the eigenvalues of $d_x \circ d_s$. Thus, by Lemma A.4 and (**??**), we get

$$d_x \circ d_s \preceq_{\mathcal{K}} \tfrac{1}{4}\|d_x + d_s\|_F^2 \, e = \tfrac{1}{4}\left\|\frac{v^3}{\|v^2\|_F}\right\|_F^2 e = \tfrac{1}{4}\frac{\operatorname{tr}(v^6)}{\|v^2\|_F^2} e \preceq_{\mathcal{K}} \tfrac{1}{4}\lambda_{\max}(v^2)e \preceq_{\mathcal{K}} \tfrac{1}{4}\tau_2 e.$$

Consequently, condition (17) is satisfied if

$$0 \prec_{\mathcal{K}} \left(\frac{\tau_1\tau_2}{\|v^2\|_F} - \frac{1}{4}\alpha\tau_2\right)e,$$

which in turn implies that $\alpha < \frac{4\tau_1}{\|v^2\|_F}$. Thus, considering Lemma 2, the result follows. □

The next lemma shows that after a feasible Dikin step, the duality gap is reduced by at least a factor of $\left(1 - \frac{\alpha}{\sqrt{r}}\right)$.

Lemma 4 *Let (x, y, s) be a feasible primal-dual solution. Then, after a feasible Dikin step, we get*

$$\operatorname{tr}(x^\alpha \circ s^\alpha) \leq \left(1 - \frac{\alpha}{\sqrt{r}}\right) \operatorname{tr}(x \circ s).$$

Proof Since $\operatorname{tr}(d_x \circ d_s) = 0$, it follows that

$$\operatorname{tr}((v + \alpha d_x) \circ (v + \alpha d_s)) = \operatorname{tr}\left(v^2 - \alpha\frac{v^4}{\|v^2\|_F}\right) = \|v\|_F^2 - \alpha\|v^2\|_F.$$

Using the Cauchy-Schwarz inequality, a lower bound of $\|v\|_F^2$ is given by

$$\|v\|_F^2 = \operatorname{tr}(v^2) = \operatorname{tr}(v^2 \circ e) \leq \|v^2\|_F \|e\|_F = \sqrt{r}\|v^2\|_F.$$

Thus, we can conclude that

$$\operatorname{tr}(x^\alpha \circ s^\alpha) = \operatorname{tr}((v + \alpha d_x) \circ (v + \alpha d_s)) \leq \left(1 - \frac{\alpha}{\sqrt{r}}\right) \operatorname{tr}(v^2),$$

which completes the proof. □

Theorem 1 *Let $\epsilon > 0$, $\alpha = \frac{1}{\tau\sqrt{r}}$ and $\tau > 1$ so that $\kappa(x^0 \circ s^0) \leq \tau$. Then, the Dikin-type algorithm terminates after at most $\lceil \tau r \log \frac{\mathrm{tr}(x^0 \circ s^0)}{\epsilon} \rceil$ iterations yielding a feasible solution (x, s) such that $\kappa(v^2) \leq \tau$ and $\mathrm{tr}(x \circ s) \leq \epsilon$.*

Proof By the left hand side inequality in (16), we have $\|v^2\|_F \geq \|\tau_1 e\|$ and thus

$$\alpha = \frac{1}{\tau\sqrt{r}} = \frac{\tau_1}{\tau_2\sqrt{r}} \leq \frac{\tau_1\sqrt{r}}{2\tau_2} = \frac{\|\tau_1 e\|_F}{2\tau_2} \leq \frac{\|v^2\|_F}{2\tau_2}.$$

By the right hand side inequality in (16) we have $\|v^2\|_F \leq \tau_2\sqrt{r}$, and thus

$$\frac{4\tau_1}{\|v^2\|_F} \geq \frac{4\tau_1}{\tau_2\sqrt{r}} = \frac{4}{\tau\sqrt{r}} > \alpha.$$

Therefore, the default value of α satisfies the conditions in Lemma 3.

As Lemma 4 proves, each Dikin step with the default value of α reduces the duality gap by a factor of $\left(1 - \frac{1}{\tau r}\right)$. Consequently, the duality gap reduces below ϵ after k iterations if

$$\left(1 - \frac{1}{\tau r}\right)^k \mathrm{tr}(x^0 \circ s^0) \leq \epsilon.$$

Taking the logarithm of both sides gives

$$k \log\left(1 - \frac{1}{\tau r}\right) + \log(\mathrm{tr}(x^0 \circ s^0)) \leq \log(\epsilon).$$

This inequality is satisfied if

$$\frac{k}{\tau r} \geq \log(\mathrm{tr}(x^0 \circ s^0)) - \log(\epsilon) = \log\left(\frac{\mathrm{tr}(x^0 \circ s^0)}{\epsilon}\right),$$

where we have used the fact that $-\log\left(1 - \frac{1}{\tau r}\right) \geq \frac{1}{\tau r}$. This completes the proof. □

5 Numerical results

In this section, we provide a set of SOCO problems to demonstrate the perfomance of the Dikin-type algorithm. Toward this end, a set of 13 SOCO problems are chosen from the DIMACS library as listed in Table 1. The test problems are of minimization type, and their optimal solutions are provided by Mittelmann in [13].

All the test problems are embedded in a self-dual embedding format, see [18] for details. We adopt SeDuMi as a competing method for comparison purposes. Both SeDuMi and the Dikin-type algorithm are implemented in MATLAB 8.5 under Arch Linux operating system.

Table 1: The specifications of the SOCO problems

Name	#Rows	#Lorentz Cones	#Linear Variables	Optimum
nql30new	3680	[900; 900x3]	3602	-0.946028
nql30old	3601	[900; 900x3]	5560	0.946028
nql60new	14560	[3600;3600x3]	14402	-0.935423
nql60old	14401	[3600; 3600x3]	21920	0.935423
nql180new	130080	[32400;32400x3]	129602	-0.927717
nb	123	[793; 793x 3]	4	-0.050703
nb-L1	915	[793; 793x 3]	797	-13.01227
nb-L2	123	[839; 1x1677,838x3]	4	-1.628972
nb-L2-Bessel	123	[839; 1x 123, 838x 3]	4	-0.102571
qssp30new	3691	[1891; 1891x 4]	2	-6.496675
qssp30old	5674	[1891; 1891x 4]	3600	6.496675
qssp60new	14581	[7381; 7381x 4]	2	-6.562696
qssp60old	22144	[7381; 7381x 4]	14400	6.562696

In practice, the theoretical steplength $\alpha = \frac{1}{\tau\sqrt{r}}$ is nearly zero for large values of r, which prevents taking a long Dikin-step. To remove this drawback, we apply a decreasing sequence of steplenghts in $(0, \alpha_{\max})$ and take the largest value of α which satisfies

$$\kappa(v_x^\alpha \circ v_s^\alpha) \leq \tau.$$

In our experiments, τ is fixed at 4.2, and $\alpha_{\max}$ is set to the threshold value for the boundary of the cone. The Dikin-type algorithm terminates if the primal and dual infeasibility along with the duality gap drops below the SeDuMi's default threshold (10^{-8}), or if it gains less then 0.1% improvement in duality gap, whichever comes first.

A safeguard procedure is considered in the Dikin-type algorithm for the case when the Dikin-step provides no significant improvement in the duality gap (i.e., the relative improvement is less than 5%). In this situation, we skip the Dikin-step and take a centering step which is obtained by solving

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T \Delta y + d_s &= 0, \\ d_x + d_s &= \mu v^{-1} - v, \end{aligned}$$

where $\mu = \frac{\mathrm{tr}(v^2)}{r}$. Doing so, we indeed improve the centrality but keep the duality gap constant. Depending on the amount of improvement in the centrality, we might need multiple centering steps.

Tables 2 and 3 illustrate the results of SeDuMi and the Dikin-type algorithm, in which the best primal objective (Primal), duality gap (Gap), primal infeasibility (pinf), dual infeasibility (dinf), computational time (Time), the number of iterations (#Iter), and the number of centering steps (#Centering) are provided.

As illustrated by the results, both the Dikin-type algorithm and SeDuMi perform equally well on the test instances in terms of the obejctive value even

Table 2: Numerical results in terms of the optimum solution and feasibility.

Instance	Dikin-type affine scaling				SeDuMi			
	Primal	Gap	pinf	dinf	Primal	Gap	pinf	dinf
nql30new	-0.94602799	2.20E-09	1.19E-07	3.87E-09	-0.94602788	3.22E-07	5.41E-08	4.94E-09
nql30old	0.94604297	1.27E-05	1.49E-08	1.09E-08	0.94603446	3.86E-09	1.85E-09	2.98E-08
nql60new	-0.93505294	2.29E-07	1.52E-02	1.12E-11	-0.93505119	2.68E-06	7.14E-08	7.59E-09
nql60old	0.93513085	6.56E-05	3.23E-08	1.41E-08	0.93516151	1.62E-08	2.24E-09	1.58E-07
nql180new	-0.92772536	5.56E-05	1.43E-06	1.12E-08	-0.92772365	9.77E-06	3.73E-07	6.28E-09
nb	-0.05070309	5.90E-11	9.45E-06	3.14E-10	-0.05070309	0.00E+00	4.50E-10	1.01E-10
nb_L1	-13.01226955	4.56E-07	2.09E-07	7.02E-09	-13.01227003	3.00E-09	9.49E-08	5.68E-09
nb_L2	-1.62896959	4.20E-06	8.25E-05	6.68E-09	-1.62897196	7.00E-10	1.14E-09	5.55E-08
nb_L2_Bessel	-0.10256951	2.63E-09	1.75E-10	1.89E-10	-0.10256950	1.33E-08	8.77E-10	2.42E-10
qssp30new	-6.49667283	6.25E-08	2.46E-07	8.33E-09	-6.49666910	6.50E+00	8.99E-08	2.01E-08
qssp30old	6.50278614	2.07E-02	0.0032	6.21E-10	6.58927911	2.15E-01	0.0223	1.71E-04
qssp60new	-6.56268516	5.33E-07	9.71E-06	3.60E-06	-6.56269721	1.16E-08	7.19E-08	7.18E-09
qssp60old	6.59462727	1.07E-01	0.0067	9.54E-10	6.56552293	1.28E-02	0.004	2.32E-07
Average		9.84E-03	1.94E-03	2.82E-07		5.17E-01	2.02E-03	1.32E-05

Table 3: Numerical results in terms of computational time and the number of iterations.

Instance	Dikin-type affine scaling			SeDuMi	
	Time(s)	#Iter	#Centering	Time(s)	#Iter
nql30new	4.6	34	0	1.1	15
nql30old	8.8	41	1	4.8	19
nql60new	21.4	46	0	3.3	14
nql60old	52	61	3	21.2	22
nql180new	380.5	70	2	56	16
nb	7.4	48	1	2.1	20
nb-L1	6.1	32	0	2.8	18
nb-L2	13.9	48	4	3.8	16
nb-L2-Bessel	5.1	34	0	2.1	16
qssp30new	15.8	79	12	1.4	20
qssp30old	46.2	75	11	9.2	15
qssp60new	103.1	125	30	8.3	27
qssp60old	349.4	109	27	88	29
Average	78.023	62	7	15.7	19

though SeDuMi performs quite faster. In 7 out of 13 instances, the Dikin-type algorithm arrives at as good solutions as the optimal values in [13] with 10^{-5} precision while this value is 9 for SeDuMi. The Dikin-type algorithm obtains the average duality gap 9.84E-03 within the average of 62 iterations while SeDuMi ends up with 5.17E-01 in 19 iterations. The Dikin-type algorithm also outperforms SeDuMi in terms of primal and dual infeasibility. As demonstrated by the entries, the average of pinf and dinf over the test instances are 1.94E-03 and 2.82E-07, respectively for the Dikin-type algorithm and 2.02E-03 and 1.32E-05, respectively for SeDuMi.

It is worth mentioning that the Dikin-type algorithm takes only a few centering steps to get the iterates back to the vicinity of the central path. To be more precise, in 9 out of 13 instances, the Dikin-type algorithm uses less than 10% of the iterations for centering and more than 90% of iterations for reducing the duality gap. Further, no centering was used for the rest of problems. The worst case belongs to the test instances“qssp60new” and “qssp60old“[1], on which 24% and 24.77% of the iterations are spent on the centering.

6 Conclusions

In this paper, we generalized the Dikin-type affine scaling method of Jansen et al. [10] to SCO using the notion of Euclidean Jordan algebras. The method starts with an interior feasible solution which is not necessarily centered. In contrast to the primal-dual affine scaling method of Monteiro et al. [14], the method features simultaneously centering and reducing the duality gap. This generalization has an $\mathcal{O}(\tau r L)$ iteration complexity, where τ and r denotes the measure of proximity and the order of the symmetric cone, respectively. The method was tested againt SeDuMi on a set of 13 SOCO test problems. The numerical experiments showed that the method is viable and robust even though it is outperformed by SeDuMi in terms of the computational time.

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References

1. Alizadeh, F., Schmieta, S.H.: Optimization with semidefinite, quadratic and linear constraints. Technical report RRR 23-97, RUTCOR, Rutgers University, USA (1997).
2. Ben-Tal, A., Nemirovski, A.: Lectures on Modern Convex Optimization. MPS/SIAM Series on Optimization, SIAM, Philadelphia, PA, USA (2001).
3. Faraut, J., Korányi, A.: Analysis on Symmetric Cones. Oxford University Press, Oxford, UK (1994).
4. Faybusovich, L.: Euclidean Jordan algebras and interior point algorithms. Positivity, 1(4), 331-357 (1997).
5. Faybusovich, L.: A Jordan algebraic approach to potential-reduction algorithms. Mathematische Zeitschrift, 239(1), 117-129 (2002).
6. Fiacco, A.V., McCormick, G.P.: Nonlinear Programming: Sequential Unconstrained Minimization Techniques. SIAM Classics in Applied Mathematics, SIAM Publications, Philadelphia, PA, USA (1990).
7. Frisch, K.R.: The logarithmic potential method for solving linear programming problems. Memorandum, University Institute of Economics, Oslo, Norway (1955).
8. Gu, G., Zangiabadi, M., Roos, C.: Full Nesterov-Todd step infeasible interior-point method for symmetric optimization. European Journal of Operational Research, 214(3), 473-484 (2011).
9. Güler, O.: Barrier functions in interior point methods. Mathematics of Operations Research, 21(4), 860-885 (1996).
10. Jansen, B., Roos, C., Terlaky, T.: A Polynomial primal-dual Dikin-type algorithm for linear programming. Mathematics of Operations Research, 21(2), 341-353 (1996).
11. Karmarkar, N.: A new polynomial-time algorithm for linear programming. Combinatorica, 4(4), 373-395 (1984).
12. de Klerk, E., Roos, C., Terlaky, T.: Polynomial primal-dual affine scaling algorithms in semidefinite programming. Journal of Combinatorial Optimization, 2(1), 51-69 (1998).

[1] When run under Windows operating system, SeDuMi fails in qssp60old after 3 iterations.

13. Mittelmann, H.D.: An independent benchmarking of SDP and SOCP solvers. Mathematical Programming, 95(2):407-430 (2003).
14. Monteiro, R.D., Adler, I., Resende, M.G.: A polynomial-time primal-dual affine scaling algorithm for linear and convex quadratic programming and its power series extension. Mathematics of Operations Research, 15(2), 191-214 (1990).
15. Nesterov, Y.E., Nemirovskii, A.: Interior Point Polynomial Methods in Convex Programming: Theory and Applications. SIAM, Philadelphia, PA, USA (1994).
16. Nesterov, Y.E., Todd, M.J.: Self-scaled barriers and interior point methods for convex programming. Mathematics of Operations Research, 22(1), 1-42 (1997).
17. Nesterov, Y.E., Todd, M.J.: Primal-dual interior point methods for self-scaled cones. SIAM Journal on Optimization, 8(2), 324-364 (1998).
18. Peng, J., Roos, C., Terlaky, T.: Self-Regularity: A New Paradigm for Primal-Dual Interior Point Algorithms. Princeton Series in Applied Mathematics (2002).
19. Rangarajan, B.K.: Polynomial convergence of infeasible-interior point methods over symmetric cones. SIAM Journal on Optimization, 16(4), 1211-1229 (2006).
20. Schmieta, S.H.: Application of Jordan Algebras to the design and analysis of interior-point algorithms for linear, quadratically constrained quadratic, and semidefinite programming. PhD Thesis, RUTCOR, Rutgers University, USA (1999).
21. Schmieta, S.H., Alizadeh, F.: Extension of commutative class of primal-dual interior-point algorithms to symmetric cones. Technical Report RRR 13-99, RUTCOR, Rutgers University, USA (1999).
22. Schmieta, S.H., Alizadeh, F.: Associative and Jordan algebras, and polynomial time interior-point algorithms for symmetric cones. Mathematics of Operations Research, 26(3), 543-564 (2001).
23. Schmieta, S.H., Alizadeh, F.: Extension of polynomial time interior-point algorithms to symmetric cones. Mathematical Programming, 96(3), 409-438 (2003).
24. Sturm, J.F.: Similarity and other spectral relations for symmetric cones. Linear Algebra and Its Applications, 312(1), 135-154 (2000).
25. Zhang, Y.: On extending primal-dual interior-point algorithms from linear programming to semidefinite programming. SIAM Journal on Optimization, 8(2), 356-386 (1998).

Appendices

A Euclidean Jordan algebras

This section gives a brief review of the basic properties of the Euclidean Jordan algebras. For the sake of simplicity, we only provide the necessary concepts which will be required in this paper. For detailed studies of Euclidean Jordan algebras the reader can consult [3] and [23].

Definition A.1 Let $\mathcal{J}$ be an n-dimensional vector space over the field of real numbers with a bilinear map $(x, s) \to x \circ s$. Then, $(\mathcal{J}, \circ)$ is referred to as the Euclidean Jordan algebra if for all $x, s, z \in \mathcal{J}$

1. $x \circ s = s \circ x$,
2. $x \circ (x^2 \circ s) = x^2 \circ (x \circ s)$,
3. $\langle x, x\rangle > 0$ for all $x \neq 0$,

where $x^2 = x \circ x$, and $\langle ., .\rangle$ denotes an inner product defined over $\mathcal{J}$.

An identity element is defined for a Euclidean Jordan algebra $\mathcal{J}$, if there exists a unique element e, such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$.

Roughly speaking, a Euclidean Jordan algebra is a commutative algebra over the field of real numbers which is not necessarily associative. Nevertheless, the Euclidean Jordan algebra is power associative; that is $x^{p+q} = x^p \circ x^q$.

For all $x, s \in \mathcal{J}$, the bilinear map $(x, s) \to x \circ s$ is characterized by

$$L(x)s = x \circ s,$$

where $L(x)$ denotes a symmetric matrix. In particular, $L(x)e = x$ and $L(x)x = x^2$. The quadratic representation of x is defined as

$$P(x) = 2L^2(x) - L(x^2).$$

Definition A.2 The cone of squares of a Euclidean Jordan algebra $\mathcal{J}$ is defined as

$$\mathcal{K}(\mathcal{J}) = \{x^2 : x \in \mathcal{J}\},$$

where $x^2 = x \circ x$, and $\mathcal{K}(\mathcal{J})$ is a closed pointed convex cone with nonempty interior.

Example A.1 Let $\mathcal{J}$ be an n-dimensional vector space over the field of real numbers, where the identity element e and $L(x)$ are defined as

$$e_{n\times 1} = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad L(x) := \begin{bmatrix} x_0 & x_{1:n-1}^T \\ x_{1:n-1} & x_0 I_{n-1} \end{bmatrix}.$$

The vector space $\mathcal{J}$ endowed with the bilinear map characterized by $L(x)$ is a Euclidean Jordan algebra. Further, $\mathcal{K}(\mathcal{J}) \equiv \mathcal{L}^n$, where

$$\mathcal{L}^n = \{x \in \mathbb{R}^n : \; x_0 \geq \|x_{1:n-1}\|\},$$

where$\|.\|$ denotes the l^2-norm. The cone $\mathcal{L}^n$ is referred to as Lorentz cone or the second-order cone. In this algebra, the quadratic representation of $x \in \mathcal{K}(\mathcal{J})$ is given by

$$P(x) = \begin{bmatrix} \|x\|^2 & 2x_0 x_{1:n-1}^T \\ 2x_0 x_{1:n-1} & \gamma(x) I_{n-1} + 2x_{1:n-1} x_{1:n-1}^T \end{bmatrix},$$

where $\gamma(x) = x_0^2 - \|x_{1:n-1}\|^2$, and I_{n-1} denotes an identity matrix of size $n-1$. □

The following lemma specifies some important properties of the quadratic representation. The interested reader is referred to [3] for more details.

Lemma A.1 (Proposition II.3.1 in [3], Lemma 8 in [23]) *For an invertible $x \in \mathcal{J}$ and integer value t, we have*

1. $P(x^{-1}) = P(x)^{-1}$ *and in general* $P(x^t) = P(x)^t$,
2. $P(x)x^{-1} = x$,
3. $P(x)e = x^2$.

We now introduce the concept of eigenvalue and spectral decomposition in Euclidean Jordan algebras. Let r be the smallest integer such that the set $\{e, x, x^2, \ldots, x^r\}$ is linearly dependent for $x \in \mathcal{J}$. Then, r is denoted as the degree of x, $\deg(x)$. The rank of $\mathcal{J}$ is defined as the largest value of $\deg(x)$ over $x \in \mathcal{J}$.

A nonzero element $q \in \mathcal{J}$ is called idempotent if $q^2 = q$. Furthermore, an idempotent is primitive if it is not the sum of two other idempotents. In light of these definition, a Jordan frame is defined as a set of primitive idempotents $\{q_1, \ldots, q_r\}$, where $q_i \circ q_j = 0$ for all $i \neq j$ and $q_1 + \ldots + q_r = e$.

Theorem A.1 (Theorem III.1.2 in [3]) *Let $\mathcal{J}$ be a Euclidean Jordan algebra with rank r. Then each $x \in \mathcal{J}$ can be represented as*

$$x = \lambda_1 q_1 + \ldots + \lambda_r q_r,$$

where $\{q_1, \ldots, q_r\}$ denotes a Jordan frame, and λ_i stands for the eigenvalues of x.

Example A.2 Let $x \in \mathcal{L}^n$. It can be easily shown that

$$x^2 - 2x_0 x + (x_0^2 - \|x_{1:n-1}\|^2)e = 0.$$

This implies that $r = 2$ for this Euclidean Jordan algebra. Hence, the spectral decomposition for an element $x \in \mathcal{L}^n$ is given by

$$x = \lambda_1 q_1 + \lambda_2 q_2,$$

where

$$\lambda_1 = x_0 - \|x_{1:n-1}\|, \text{ and } \lambda_2 = x_0 + \|x_{1:n-1}\|,$$
$$q_1 = \tfrac{1}{2}\begin{pmatrix} 1 \\ -\frac{x_{1:n-1}}{\|x_{1:n-1}\|} \end{pmatrix}, \text{ and } q_2 = \tfrac{1}{2}\begin{pmatrix} 1 \\ \frac{x_{1:n-1}}{\|x_{1:n-1}\|} \end{pmatrix}.$$

□

We can now extend the definition of any real valued function $f(.)$ to elements of the Euclidean Jordan algebras by

$$f(x) = f(\lambda_1)q_1 + \cdots + f(\lambda_r)q_r.$$

In particular, we have

$$x^{\frac{1}{2}} = \lambda_1^{\frac{1}{2}} q_1 + \cdots + \lambda_r^{\frac{1}{2}} q_r,$$
$$x^{-1} = \lambda_1^{-1} q_1 + \cdots + \lambda_r^{-1} q_r.$$

Note that $x^{-1} \circ x = e$. Further, x is called invertible if all the eigenvalues of x are nonzeros.

In light of the definitions given so far, trace, determinant and norms of x are formally defined as follows.

Definition A.3 Let $x \in \mathcal{J}$ and $\lambda_1, \cdots, \lambda_r$ be the eigenvalues of x. Then,

1. $\mathrm{tr}(x) = \lambda_1 + \lambda_2 + \cdots + \lambda_r$,
2. $\det(x) = \lambda_1 \lambda_2 \cdots \lambda_r$,
3. $\langle x, y\rangle = \mathrm{tr}(x \circ y)$,
4. $\|x\|_F = \sqrt{\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_r^2}$,
5. $\|x\|_2 = \max_i |\lambda_i|$.

We now review some technical lemmas (without their proofs) which are necessary for the complexity analysis of the Dikin-type algorithm. As stated at the begining part of this paper, it is assumed that $\mathcal{K}$ is a symmetric cone with nonempty interior $\mathcal{K}_+$.

Lemma A.2 (Lemma 2.13 in [8]) *Assume that $x, s \in \mathcal{J}$ and $\mathrm{tr}(x \circ s) = 0$. Then, we have*

$$-\tfrac{1}{4}\|x+s\|_F^2\, e \preceq_{\mathcal{K}} x \circ s \preceq_{\mathcal{K}} \tfrac{1}{4}\|x+s\|_F^2\, e.$$

Lemma A.3 (Lemma 2.15 in [8]) *Assume that $0 \prec_{\mathcal{K}} x \circ s$, where $x, s \in \mathcal{J}$. Then, $\det(x) \neq 0$.*

Lemma A.4 (Lemma 2.17 in [8]) *Let $x \in \mathcal{J}$ and $0 \prec_{\mathcal{K}} s$. Then, we have*

$$\lambda_{\min}(x)\,\mathrm{tr}(s) \le \mathrm{tr}(x \circ s) \le \lambda_{\max}(x)\,\mathrm{tr}(s).$$

The following lemma points out a nice property of the quadratic representation.

Lemma A.5 (Theorem III.2.1 and Proposition III.2.2 in [3]) *Let $x \in \mathcal{J}$. Then, $L(x)$ is positive definite (semidefinite) if and only if $0 \prec_{\mathcal{K}} x$ ($0 \preceq_{\mathcal{K}} x$). Further, $P(x)\mathcal{K}_+ = \mathcal{K}_+$ if x is invertible.*

In fact, Lemma A.5 states that for each interior solution $0 \prec_{\mathcal{K}} x$ and $0 \prec_{\mathcal{K}} s$, $P(x)s$ is an invertible linear map from $\mathcal{K}_+$ to $\mathcal{K}_+$. All this hints that the NT search directions obtained from (13) are well-defined.

Lemma A.6 (Proposition 21 in [23]) *Let $0 \prec_{\mathcal{K}} x$ and $0 \prec_{\mathcal{K}} s$, and w be the scaling point of x and s as defined in (5). Then,*

1. $\mathrm{tr}(P(w)^{-\frac{1}{2}} x \circ P(w)^{\frac{1}{2}} s) = \mathrm{tr}(x \circ s)$,
2. $P(x^{\frac{1}{2}})s \sim P(s^{\frac{1}{2}})x$,
3. $P(\tilde{x}^{\frac{1}{2}})\tilde{s} \sim P(x^{\frac{1}{2}})s$,

where $\tilde{x} = P(w)^{-\frac{1}{2}} x$ and $\tilde{s} = P(w)^{\frac{1}{2}} s$.

Lemma A.7 (Lemma 30 in [23]) *Let $0 \prec_{\mathcal{K}} x$ and $0 \prec_{\mathcal{K}} s$. Then, we have*

1. $\lambda_{\min}(P(x)^{\frac{1}{2}} s) \ge \lambda_{\min}(x \circ s)$,
2. $\lambda_{\max}(P(x)^{\frac{1}{2}} s) \le \lambda_{\max}(x \circ s)$.