Noname manuscript No.
(will be inserted by the editor)

Distributionally robust chance-constrained games: Existence and characterization of Nash equilibrium

Vikas Vikram Singh · Oualid Jouini · Abdel Lisser

Received: date / Accepted: date

Abstract We consider an n-player finite strategic game. The payoff vector of each player is a random vector whose distribution is not completely known. We assume that the distribution of a random payoff vector of each player belongs to a distributional uncertainty set. We define a distributionally robust chance-constrained game using worst-case chance constraint. We consider two types of distributional uncertainty sets. We show the existence of a mixed strategy Nash equilibrium of a distributionally robust chance-constrained game corresponding to both types of distributional uncertainty sets. For each case, we show a one-to-one correspondence between a Nash equilibrium of a game and a global maximum of a certain mathematical program.

Keywords Distributionally robust chance-constrained games · Chance constraints · Nash equilibrium · Semidefinite programming · Mathematical program.

1 Introduction

The work on the existence of an equilibrium in game theory started with the paper by John von Neumann [23]. He showed the existence of a mixed strategy

Vikas Vikram Singh
Laboratoire de Recherche en Informatique, Université Paris Sud XI,
Bât 650, 91405 Orsay Cedex France
E-mail: vikas.singh@lri.fr

Oualid Jouini
Laboratoire Génie Industriel, Ecole Centrale Paris,
Grande Voie des Vignes, 92 290 Châtenay-Malabry, France
E-mail: oualid.jouini@ecp.fr

Abdel Lisser
Laboratoire de Recherche en Informatique, Université Paris Sud XI,
Bât 650, 91405 Orsay Cedex France
E-mail: abdel.lisser@lri.fr

saddle point equilibrium for a two player zero sum matrix game. In 1950, John Nash [22] showed the existence of a mixed strategy Nash equilibrium for a finite strategic game. A saddle point equilibrium of a two player zero sum matrix game can be obtained from the optimal solutions of a primal-dual pair of linear programs [1, 11], and a Nash equilibrium of a two player bimatrix game can be obtained from a global maximum of a certain quadratic program [20]. Lemke and Howson [18] proposed a pivoting algorithm to compute a Nash equilibrium of a two player bimatrix game. The above mentioned papers considered the games where the payoffs of the players are deterministic. However, in many situations players' payoffs are better modeled by random variables due to uncertainty caused by various external factors. The electricity markets are good examples that capture this situation [10, 21, 33, 35]. One way to handle these games is by taking the expectation of the random variables and solve the corresponding deterministic game [33, 35]. Some recent papers on the games with random payoffs using expected payoff criterion include [12, 15, 27, 36].

The expected payoff criterion is more appropriate for the cases where the decision makers are risk neutral. The risk averse situation can be handled using chance constraint programming [5, 9, 26]. The payoff function of a player is defined using a chance constraint due to which these games are called chance-constrained games. Few papers on the zero sum chance-constrained games are available in the literature [3, 4, 6, 8, 32]. Recently, Singh et al. [30] considered an n-player finite strategic game where the payoff vector of each player is a random vector. They considered the cases where the components of the payoff vector of each player are independent normal/Cauchy random variables, and they also considered the case where the payoff vector of each player follow a multivariate elliptically symmetric distribution. They showed the existence of a mixed strategy Nash equilibrium, in all these cases, for the corresponding chance-constrained game.

The application of a chance-constrained game can be found in electricity markets [10, 21]. The players' action sets are not finite in [10, 21]. However, there are few papers on electricity market where the game between electricity firms is formulated as a finite strategic game [17, 31]. The games considered in [17] are based on Cournot and Bertrand models. The players' actions are electricity generation quantities in Cournot model and biding prices in Bertrand model. Using discretization, the players' action sets are assumed to be finite. In [31], a finite strategic electricity market auction game is studied under different pricing mechanisms. The payoffs of the players in [17, 31] are deterministic. However, the demands and costs appeared in the payoff functions considered in [17, 31] can be random due to [19, 34]. Then, the chance-constrained game framework can be useful to model such situations.

There are many situations where we do not have a complete knowledge of a distribution. The only information we have of a distribution is that it belongs to some distributional uncertainty set. In this paper, we consider an n-player finite strategic game with random payoffs whose distributions are not completely known. We assume that a distribution of the payoff vector of each player belongs to a distributional uncertainty set. We consider two types of distributional uncertainty sets depending on the partially available information about the mean and the covariance matrix of the random payoff vector. We consider a distributionally robust approach which is suitable for such cases. We define the payoff function of each player using worst-case chance constraint. We call this game a distributionally robust chance-constrained game. For each distributional uncertainty set we show the existence of a mixed strategy Nash equilibrium. We characterize a mixed strategy Nash equilibrium of these games using a global maximum of a certain mathematical program. In fact, we show a one-to-one correspondence between a Nash equilibrium of the game and a global maximum of a mathematical program.

In [14, 24], slightly related game models have been studied. Both [14, 24] considered the games where each player can neither evaluate his cost function exactly nor estimate his opponents' strategies accurately. The cost functions and strategies of the players belong to certain Euclidean uncertainty sets.

The rest of the paper is organized as follows. Section 2 contains the definition of a distributionally robust chance-constrained game. In Section 3, we show the existence of a mixed strategy Nash equilibrium for a distributionally robust chance-constrained game. Section 4 presents a mathematical programming formulation for a distributionally robust chance-constrained game. Section 5 shows some numerical results. We conclude our paper in Section 6.

2 The model

We consider an n-player strategic game with random payoffs. It is described by the tuple $(I, (A_i)_{i\in I}, (r_i)_{i\in I})$. The finite set $I = \{1, 2, \cdots, n\}$ denote the set of players. For each $i \in I$, A_i denote a finite action set of player i and a_i denotes its generic element. An action profile of the game is denoted by a vector $a = (a_1, a_2, \cdots, a_n)$. The set of all action profiles of the game is denoted by the product set $A = \times_{i=1}^{n} A_i$. Denote, $A_{-i} = \times_{j=1; j\neq i}^{n} A_j$, and $a_{-i} \in A_{-i}$ is a vector of actions a_j, $j \neq i$. An action a_i is also called a pure strategy of player i. A mixed strategy of a player is a probability distribution over his action set. We denote the set of all mixed strategies of player i by X_i. A mixed strategy $\tau_i \in X_i$ is represented by $\tau_i = (\tau_i(a_i))_{a_i\in A_i}$, where $\tau_i(a_i)$ is a probability with which player i chooses an action a_i and $\sum_{a_i\in A_i} \tau_i(a_i) = 1$. The set of all mixed strategy profiles is denoted by the product set $X = \times_{i=1}^{n} X_i$ and $\tau = (\tau_i)_{i\in I}$ is a generic element of X. Denote, $X_{-i} = \times_{j=1; j\neq i}^{n} X_j$, and $\tau_{-i} \in X_{-i}$ is a vector of mixed strategies τ_j, $j \neq i$. We define (ν_i, τ_{-i}) to be a strategy profile where, player i uses the strategy ν_i and each player j, $j \neq i$, uses the strategy τ_j. For each $i \in I$, $r_i = \big(r_i(a)\big)_{a\in A}$ is a random payoff vector of player i. That is, at action profile a the payoff of player i is given by a random variable $r_i(a)$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then, for each $i \in I$, $r_i : \Omega \to \mathbb{R}^{|A|}$, where $|A|$ denotes the cardinality of set A. Then, for a given strategy profile $\tau \in X$, the payoff of player i, $r_i(\tau)$ given by

$$r_i(\tau) = \sum_{a\in A} \prod_{j\in I} \tau_j(a_j) r_i(a), \tag{1}$$

would be a random variable. Given a $\tau \in X$, let $\eta^\tau = \big(\eta^\tau(a)\big)_{a\in A}$ be a vector, where $\eta^\tau(a) = \prod_{i\in I}\tau_i(a_i)$. Then, $r_i(\tau) = r_i^T\eta^\tau$, where T is transposition. Let $\mathcal{M}$ denote the set of all probability measures over the set of all measurable subsets of $\mathbb{R}^{|A|}$. A probability distribution F of r_i is a member of set $\mathcal{M}$. Under chance constraint programming based payoff criterion, at strategy profile $\tau \in X$, the payoff of a player is the highest level of his payoff that he can attain with at least a specified level of confidence. The confidence level of each player is given a priori and we assume that it is known to other players. Let $\alpha_i \in [0,1]$ be a confidence level of player i and $\alpha = (\alpha_i)_{i\in I}$ denotes a confidence level vector. For a given strategy profile $\tau \in X$, and a given confidence level vector α, the payoff of player i, $i \in I$, is given by

$$u_i^{\alpha_i}(\tau) = \sup\{v_i | P(r_i(\tau) \geq v_i) \geq \alpha_i\}.$$

When the distribution of r_i is known, e.g., if r_i follows a multivariate normal distribution with the mean vector μ_i and the positive definite covariance matrix Σ_i,

$$u_i^{\alpha_i}(\tau) = \sum_{a\in A}\prod_{j\in I}\tau_j(a_j)\mu_i(a) + ||\Sigma_i^{\frac{1}{2}}\eta^\tau||F_{Z_i}^{-1}(1-\alpha_i),$$

where Z_i follows a standard normal distribution and $F_{Z_i}^{-1}(\cdot)$ denotes a quantile function of a standard normal distribution, and $||\cdot||$ is the Euclidean norm. The chance-constrained game with some known distributions has been studied in [30]. We consider the case where we do not have the complete knowledge of the distributions of the players' payoff vectors. The only knowledge we have of a distribution is that it belongs to some uncertainty set. Such an uncertainty set is based on the partially available information about the distribution, it is called distributional uncertainty set. Let $\mathcal{D}_i$ denotes a distributional uncertainty set for player i, $i \in I$. We assume that the distributional uncertainty set of each player is known to all the players. We use a distributionally robust approach which is more suitable for such situations. The payoff of a player is defined using his worst-case chance constraint. That is, for a given strategy profile $\tau \in X$, and a given confidence level vector α, the payoff of player i, $i \in I$, is given by

$$u_i^{\alpha_i}(\tau) = \sup\left\{v_i \,\middle|\, \inf_{F\in\mathcal{D}_i} P_F(r_i(\tau) \geq v_i) \geq \alpha_i\right\}. \tag{2}$$

If the worst-case chance constraint is not satisfied, $u_i^{\alpha_i}(\tau) = -\infty$. We call this game a distributionally robust chance-constrained game. For a given $\alpha \in [0,1]^n$, the payoff function of each player defined by (2) is known to all the players. Therefore, for an $\alpha \in [0,1]^n$, the distributionally robust chance-constrained game is a non-cooperative game with complete information. The set of best response strategies of player i, $i \in I$, against a given strategy profile τ_{-i} of the other players is given by

$$BR_i^{\alpha_i}(\tau_{-i}) = \left\{\bar{\tau}_i \in X_i | u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i}) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}),\ \forall\ \tau_i \in X_i\right\}.$$

Next, we give the definition of a Nash equilibrium.

Definition 1 (Nash equilibrium) A strategy profile $\tau^* \in X$ is said to be a Nash equilibrium for a given $\alpha \in [0,1]^n$, if for all $i \in I$, the following inequality holds,

$$u_i^{\alpha_i}(\tau_i^*, \tau_{-i}^*) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}^*),\ \forall\ \tau_i \in X_i.$$

That is, τ^* is a Nash equilibrium if and only if $\tau_i^* \in BR_i^{\alpha_i}(\tau_{-i}^*)$ for all $i \in I$.

3 Existence of Nash equilibrium

We have the following general result for the existence of a mixed strategy Nash equilibrium of a distributionally robust chance-constrained game defined in Section 2.

Theorem 1 *For a given confidence level vector $\alpha \in [0,1]^n$, suppose*

1. *for each $i \in I$, the payoff function of player i, $u_i^{\alpha_i} : X_i \times X_{-i} \to \mathbb{R}$, defined by* (2)*, is a continuous function,*
2. *for each $i \in I$, the payoff function $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of τ_i for fixed $\tau_{-i} \in X_{-i}$.*

Then, there always exists a mixed strategy Nash equilibrium of a distributionally robust chance-constrained game at confidence lever vector α.

Proof Let $\mathcal{P}(X)$ be a power set of X. Define a set valued map $G : X \to \mathcal{P}(X)$ such that $G(\tau) = \prod_{i\in I} BR_i^{\alpha_i}(\tau_{-i})$. A strategy profile $\tau \in X$ is said to be a fixed point of the set valued map G if $\tau \in G(\tau)$. It is easy to see that a fixed point of G is a Nash equilibrium. In order to show that G has a fixed point, we show that G satisfies all the following conditions of Kakutani fixed point theorem [16]:

- X is a non-empty, convex, and compact subset of a finite dimensional Euclidean space.
- $G(\tau)$ is non-empty and convex for all $\tau \in X$.
- $G(\cdot)$ has closed graph: If $(\tau_n, \bar{\tau}_n) \to (\tau, \bar{\tau})$ with $\bar{\tau}_n \in G(\tau_n)$ for all n then $\bar{\tau} \in G(\tau)$.

First condition follows from the definition of X. For each $i \in I$, $BR_i^{\alpha_i}(\tau_{-i})$ is non-empty because $u_i^{\alpha_i}(\cdot)$ is a continuous function and X_i is a compact set, and $BR_i^{\alpha_i}(\tau_{-i})$ is a convex set because $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of τ_i for fixed $\tau_{-i} \in X_{-i}$. This shows that $G(\tau)$ is non-empty and convex for all $\tau \in X$. The closed graph condition of $G(\cdot)$ can be proved using the continuity of functions $u_i^{\alpha_i}(\cdot)$, $i \in I$. For detailed proof of closed graph condition see Theorem 3.2 of [30]. The set valued map $G(\cdot)$ satisfies all the conditions of Kakutani fixed point theorem. Hence, there exists a strategy profile τ^* which is a Nash equilibrium of the game. □

If for a given strategy profile the worst-case chance constraint of a player does not hold, his payoff is $-\infty$. Then, he can increase his payoff by deviating to the strategies that give him finite payoff. Therefore, we consider the case where the worst-case chance constraint of each player holds at all the strategy profiles, so that the payoff function of each player is finite valued. We consider two types of distributional uncertainty sets. For each case, we show the existence of a mixed strategy Nash equilibrium using the sufficient conditions given in Theorem 1.

3.1 Distribution with polytopic uncertainty

We consider the case where for each $i \in I$, the mean vector μ_i and the covariance matrix Σ_i of the distribution of r_i are only known to belong to some polytopes described by their vertices. That is, $\mu_i \in \mathcal{U}_{\mu_i}$ and $\Sigma_i \in \mathcal{U}_{\Sigma_i}$, where

$$\mathcal{U}_{\mu_i} := \mathbf{Co}\{\mu_i^1, \mu_i^2, \cdots, \mu_i^{l_i}\}, \quad \mathcal{U}_{\Sigma_i} := \mathbf{Co}\{\Sigma_i^1, \Sigma_i^2, \cdots, \Sigma_i^{l_i}\}.$$

Co stands for convex hull. The vertices $(\mu_i^k)_{k=1}^{l_i}$ and $(\Sigma_i^k)_{k=1}^{l_i}$ are given and known to all the players. For a given matrix B, $B \succ 0$ (resp. $B \succeq 0$) means B is a positive definite (resp. semidefinite) matrix. We assume that $\Sigma_i^k \succ 0$ for all $k = 1, 2, \cdots, l_i$. Let $\mathcal{D}_i(\mu_i, \Sigma_i)$ denotes the set of all probability distributions that have the mean $\mu_i \in \mathcal{U}_{\mu_i}$ and the covariance matrix $\Sigma_i \in \mathcal{U}_{\Sigma_i}$, and otherwise the distribution is arbitrary. Such polytopic uncertainty is considered in [13]. For each $i \in I$, (2) can be equivalently written as,

$$u_i^{\alpha_i}(\tau) = -\inf\left\{u_i \,\middle|\, \sup_{F\in\mathcal{D}_i(\mu_i,\Sigma_i)} P_F(u_i \leq -r_i(\tau)) \leq 1-\alpha_i\right\}. \quad (3)$$

The problem $\inf\{u_i | \sup_{F\in\mathcal{D}_i(\mu_i,\Sigma_i)} P_F(u_i \leq -r_i(\tau)) \leq 1-\alpha_i\}$ is the same as the worst-case value-at-risk with polytopic moment uncertainty problem considered in [13]. It follows from [13] that, for each $i \in I$, and $\tau \in X$,

$$u_i^{\alpha_i}(\tau_i, \tau_{-i}) = \min_{1\leq k\leq l_i} (\mu_i^k)^T \eta^\tau - \max_{1\leq k\leq l_i} \sqrt{\frac{\alpha_i}{1-\alpha_i}} ||(\Sigma_i^k)^{\frac{1}{2}} \eta^\tau||. \quad (4)$$

Lemma 1 *For each $i \in I$, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ given by (4) is a concave function of τ_i for all $\alpha_i \in [0, 1)$.*

Proof Fix $i \in I$, $\alpha_i \in [0, 1)$ and $\tau_{-i} \in X_{-i}$. For each k, $k = 1, 2, \cdots, l_i$, $(\mu_i^k)^T \eta^\tau$ is a linear function of τ_i, hence it is a concave function of τ_i. The minimum of a set of concave functions is a concave function. Therefore, the first term of (4) is a concave function of τ_i. Let τ_i^1, $\tau_i^2 \in X_i$. Take $\lambda \in [0, 1]$. Then, for a strategy profile $(\lambda\tau_i^1 + (1-\lambda)\tau_i^2, \tau_{-i})$ we have $\eta^{(\lambda\tau_i^1+(1-\lambda)\tau_i^2,\tau_{-i})}(a) = \big(\lambda\tau_i^1(a_i) + (1-\lambda)\tau_i^2(a_i)\big) \prod_{j\in I; j\neq i} \tau_j(a_j)$ for each $a \in A$. Therefore, $\eta^{(\lambda\tau_i^1+(1-\lambda)\tau_i^2,\tau_{-i})} = \lambda\eta^{(\tau_i^1,\tau_{-i})} + (1-\lambda)\eta^{(\tau_i^2,\tau_{-i})}$. Then, for each k, it follows from the property of norm that $\sqrt{\frac{\alpha_i}{1-\alpha_i}}||(\Sigma_i^k)^{\frac{1}{2}}\eta^\tau||$ is a convex function of τ_i. The maximum of a set of convex functions is a convex function, and the negative of a convex function is a concave function. Then, the second term of (4) is also a concave function of τ_i. Hence, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of τ_i. □

Theorem 2 *Consider an n-player finite strategic game where the payoff vector $r_i = (r_i(a))_{a\in A}$ of player i, $i \in I$, is a random vector. If for each $i \in I$, the mean-covariance pair (μ_i, Σ_i) of the distribution of r_i are such that $\mu_i \in \mathcal{U}_{\mu_i}$ and $\Sigma_i \in \mathcal{U}_{\Sigma_i}$, and otherwise the distribution is arbitrary, there always exists a mixed strategy Nash equilibrium of a distributionally robust chance-constrained game for all $\alpha \in [0, 1)^n$.*

Proof From Lemma 1, $u_i^{\alpha_i}(\cdot, \tau_{-i})$, $i \in I$, given by (4) is a concave function. From (4), $u_i^{\alpha_i}(\cdot)$, $i \in I$, is a continuous function of τ. That is, all the conditions of Theorem 1 are satisfied. Hence, there exists a mixed strategy Nash equilibrium from Theorem 1. □

Remark 1 The case where the mean vector and the covariance matrix of the payoff vector of each player is known exactly comes under a special case of polytopic uncertainty considered in Section 3.1. Hence, the existence of a Nash equilibrium for this case follows from Theorem 2.

3.2 Distribution with known mean and an upper bound on covariance matrix

We consider the case where the distributional uncertainty set for player i, $i \in I$, accounts for the information about the mean vector μ_i and an upper bound $\Sigma_i \succ 0$ on the covariance matrix of the random payoff vector r_i. We assume that, for each $i \in I$, the mean μ_i and the upper bound Σ_i on the covariance matrix are known to all the players. The distributional uncertainty set for player i, $i \in I$, is given by

$$\mathcal{D}_i(\mu_i, \Sigma_i) = \left\{ F \in \mathcal{M} \left| \begin{array}{c} \mathbb{E}_F[r_i] = \mu_i \\ \mathbb{E}_F[(r_i - \mu_i)(r_i - \mu_i)^T] \preceq \Sigma_i \end{array} \right. \right\}. \tag{5}$$

Such an uncertainty set is considered by Cheng et al. [7]. For given two matrices B and C, $B \succeq C$ means $B - C \succeq 0$.

To show the existence of a mixed strategy Nash equilibrium, first we get a closed form expression for the payoff function of each player defined by (2). For this we need to further simplify the worst-case chance constraint used in the definition of the payoff function. For each $i \in I$, the worst-case chance constraint from (2) can be written as

$$\sup_{F \in \mathcal{D}_i(\mu_i, \Sigma_i)} \mathbb{E}_F[\mathbb{1}_{\{r_i^T \eta^\tau \leq v_i\}}] \leq 1 - \alpha_i, \tag{6}$$

where $\mathbb{1}_{\{\cdot\}}$ is an indicator function that gives value one if the condition is true and zero otherwise. For each $i \in I$, we begin with the optimization problem

$$\left.\begin{aligned} &\sup_{F\in\mathcal{M}} \int_{\mathbb{R}^{|A|}} \mathbb{1}_{\{\bar{r}_i^T\eta^\tau \le v_i\}} dF(\bar{r}_i) \\ \text{s.t.} \\ &\int_{\mathbb{R}^{|A|}} \bar{r}_i dF(\bar{r}_i) = \mu_i, \\ &\int_{\mathbb{R}^{|A|}} (\bar{r}_i - \mu_i)(\bar{r}_i - \mu_i)^T dF(\bar{r}_i) \preceq \Sigma_i. \end{aligned}\right\}$$

Let $B \bullet C = \sum_{i,j} B_{ij} C_{ij}$ represents the Frobenius product between two given matrices B and C of the same dimensions. The dual of the above problem is

$$\left.\begin{aligned} &\min_{\bar{t}_i,\bar{q}_i,\bar{Q}_i} \bar{t}_i + 2\bar{q}_i^T\mu_i + (\Sigma_i + \mu_i\mu_i^T) \bullet \bar{Q}_i \\ \text{s.t.} \\ &-\mathbb{1}_{\{\bar{r}_i^T\eta^\tau \le v_i\}} + \bar{t}_i + 2\bar{r}_i^T\bar{q}_i + \bar{r}_i^T\bar{Q}_i\bar{r}_i \ge 0,\ \forall\ \bar{r}_i \in \mathbb{R}^{|A|}, \\ &\bar{Q}_i \succeq 0. \end{aligned}\right\}$$

For details about above duality formulation see [29]. The strong duality follows from [29] because Dirac measure δ_{μ_i} lies in the relative interior of the set $\mathcal{D}_i(\mu_i, \Sigma_i)$. Hence, constraint (6) can be reformulated as

$$\left.\begin{aligned} &\bar{t}_i + 2\bar{q}_i^T\mu_i + (\Sigma_i + \mu_i\mu_i^T) \bullet \bar{Q}_i \le 1 - \alpha_i, \\ &\bar{Q}_i \succeq 0, \\ &\bar{t}_i + 2\bar{r}_i^T\bar{q}_i + \bar{r}_i^T\bar{Q}_i\bar{r}_i \ge 0,\ \forall\ \bar{r}_i \in \mathbb{R}^{|A|}, \\ &-1 + \bar{t}_i + 2\bar{r}_i^T\bar{q}_i + \bar{r}_i^T\bar{Q}_i\bar{r}_i \ge 0,\ \forall\ \bar{r}_i \in \mathbb{R}^{|A|} \text{ such that } \bar{r}_i^T\eta^\tau - v_i \le 0. \end{aligned}\right\} \quad (7)$$

Given $\tau \in X$ and $v_i \in \mathbb{R}$ there always exists $\bar{r}^0 \in \mathbb{R}^{|A|}$ such that $\bar{r}_0^T\eta^\tau - v_i < 0$, i.e., Slater condition holds. This is possible because η^τ is a probability distribution over the set A of all the action profiles and hence it cannot be a zero vector. From Theorem 2.1 of [25], the last constraint from (7) is equivalent to:

$$\begin{gathered} -1 + \bar{t}_i + 2\bar{r}_i^T\bar{q}_i + \bar{r}_i^T\bar{Q}_i\bar{r}_i + 2\lambda_i(\bar{r}_i^T\eta^\tau - v_i) \ge 0,\ \forall\ \bar{r}_i \in \mathbb{R}^{|A|}, \\ \lambda_i \ge 0. \end{gathered}$$

So, the new set of constraints equivalent to (6) is

$$\left.\begin{aligned} &M_i \bullet \Gamma_i \le 1 - \alpha_i, \\ &M_i \succeq 0, \\ &M_i + \begin{bmatrix} \mathbf{0}_{|\mathbf{A}|\times|\mathbf{A}|} & \lambda_i\eta^\tau \\ \lambda_i(\eta^\tau)^T & -1 - 2\lambda_i v_i \end{bmatrix} \succeq 0, \\ &\lambda_i \ge 0, \end{aligned}\right\}$$

where, $\mathbf{0}_{|\mathbf{A}|\times|\mathbf{A}|}$ is the $|A| \times |A|$ zero matrix, $M_i = \begin{bmatrix} \bar{Q}_i & \bar{q}_i \\ \bar{q}_i^T & \bar{t}_i \end{bmatrix}$ and $\Gamma_i = \begin{bmatrix} \Sigma_i + \mu_i\mu_i^T & \bar{\mu}_i \\ \bar{\mu}_i^T & 1 \end{bmatrix}$. From (2), we have

$$\left.\begin{aligned} u_i^{\alpha_i}(\tau) = & \sup_{v_i, M_i, \lambda_i} v_i \\ & \text{s.t.} \\ & \quad M_i \bullet \Gamma_i \leq 1 - \alpha_i, \\ & \quad M_i \succeq 0, \\ & \quad M_i + \begin{bmatrix} 0 & \lambda_i \eta^\tau \\ \lambda_i (\eta^\tau)^T & -1 - 2\lambda_i v_i \end{bmatrix} \succeq 0, \\ & \quad \lambda_i \geq 0. \end{aligned}\right\} \tag{8}$$

The problem (8) is equivalent to

$$\left.\begin{aligned} u_i^{\alpha_i}(\tau) = & -\inf_{v_i, M_i, \lambda_i} v_i \\ & \text{s.t.} \\ & \quad M_i \bullet \Gamma_i \leq 1 - \alpha_i, \\ & \quad M_i \succeq 0, \\ & \quad M_i + \begin{bmatrix} 0 & \lambda_i \eta^\tau \\ \lambda_i (\eta^\tau)^T & -1 + 2\lambda_i v_i \end{bmatrix} \succeq 0, \\ & \quad \lambda_i \geq 0. \end{aligned}\right\} \tag{9}$$

From [13] it follows that λ_i-components of the optimal solutions of (9) are uniformly bounded from below by a positive number. So, we can divide by λ_i in the matrix inequalities above, and replace $\frac{1}{\lambda_i}$ by λ_i, and $\frac{M_i}{\lambda_i}$ by M_i. Now, we have the following semidefinite programming (SDP) problem,

$$\left.\begin{aligned} u_i^{\alpha_i}(\tau) = & -\inf_{v_i, M_i, \lambda_i} v_i \\ & \text{s.t.} \\ & \quad M_i \bullet \Gamma_i \leq \lambda_i(1 - \alpha_i), \\ & \quad M_i \succeq 0, \\ & \quad M_i + \begin{bmatrix} 0 & \eta^\tau \\ (\eta^\tau)^T & -\lambda_i + 2v_i \end{bmatrix} \succeq 0, \\ & \quad \lambda_i \geq 0. \end{aligned}\right\} \tag{10}$$

The SDP problem (10) is the same as the SDP problem (17) used in the proof of Theorem 1 of El-Ghaoui et al. [13] except the negative sign before inf. Then, from the proof given in [13], the explicit expression for the payoff function of player i, $i \in I$, is given by

$$u_i^{\alpha_i}(\tau_i, \tau_{-i}) = \mu_i^T \eta^\tau - \sqrt{\frac{\alpha_i}{1-\alpha_i}} ||\Sigma_i^{\frac{1}{2}} \eta^\tau||, \ \forall \ \tau \in X. \tag{11}$$

Lemma 2 *For each $i \in I$, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ given by (11) is a concave function of τ_i for all $\alpha_i \in [0, 1)$.*

Proof Fix $i \in I$, $\alpha_i \in [0, 1)$ and $\tau_{-i} \in X_{-i}$. The first term $(\mu_i^k)^T \eta^\tau$ of (11) is a linear function of τ_i, hence it is a concave function of τ_i. From the same arguments used in Lemma 1, the second term $\sqrt{\frac{\alpha_i}{1-\alpha_i}} ||\Sigma_i^{\frac{1}{2}} \eta^\tau||$ is a convex function of τ_i and its negative would be a concave function of τ_i. Hence, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of τ_i.

Theorem 3 *Consider an n-player finite strategic game where the payoff vector $r_i = (r_i(a))_{a \in A}$ of player i, $i \in I$, is a random vector whose distribution is not completely known. If it belongs to an uncertainty set $\mathcal{D}_i(\mu_i, \Sigma_i)$ defined by (5), where Σ_i is a positive definite matrix, there always exists a mixed strategy Nash equilibrium of a distributionally robust chance-constrained game for all $\alpha \in [0, 1)^n$.*

Proof From Lemma 2, for each $i \in I$, $u_i^{\alpha_i}(\cdot, \tau_{-i})$, given by (11) is a concave function of τ_i. From (11), $u_i^{\alpha_i}(\cdot)$, $i \in I$, is a continuous function of τ. Then, the proof follows from Theorem 1. □

4 Mathematical programming formulation

We formulate distributionally robust chance-constrained games considered in Section 3 as equivalent mathematical programs. We show a one-to-one correspondence between a Nash equilibrium of a distributionally robust chance-constrained game and a global maximum of a certain mathematical program.

4.1 Distributionally robust chance-constrained game for polytopic uncertainty

We consider the distributionally robust chance-constrained game defined in Section 3.1. For each $i \in I$, the payoff function of player i defined by (4) is a concave function of τ_i for a fixed $\tau_{-i} \in X_{-i}$. Therefore, the best response of player i against a fixed strategy profile of other players can be obtained by solving a convex optimization problem which can be further reformulated as a second order cone programming problem. For fixed τ_{-i}, a best response of player i, $i \in I$, can be obtained from an optimal solution of the following second order cone programming problem:

$$[\text{P1}] \qquad \min_{\beta_i^1,\beta_i^2,\tau_i} \beta_i^1 - \beta_i^2$$

s.t.

$$
\begin{aligned}
&(i)\ \sqrt{\frac{\alpha_i}{1-\alpha_i}}||(\Sigma_i^k)^{\frac{1}{2}}\eta^\tau|| - \beta_i^1 \le 0,\ \forall\ k = 1,2,\cdots,l_i,\\
&(ii)\ \beta_i^2 - (\mu_i^k)^T\eta^\tau \le 0,\ \forall\ k = 1,2,\cdots,l_i,\\
&(iii)\ \sum_{a_i\in A_i} \tau_i(a_i) = 1,\\
&(iv)\ \tau_i(a_i) \ge 0,\ \forall\ a_i \in A_i.
\end{aligned}
$$

Denote, $X_i^+ = \{\tau_i = (\tau_i(a_i))_{a_i\in A_i} | \tau_i(a_i) \ge 0,\ \forall\ a_i \in A_i\}$. Let the Lagrange multipliers corresponding to constraints (i), (ii), and (iii) of [P1] be $\delta_i^1 = (\delta_{i,k}^1)_{k=1}^{l_i}$, $\delta_i^2 = (\delta_{i,k}^2)_{k=1}^{l_i}$, and λ_i respectively. For a given vector y, $y \ge 0$ implies the component-wise non-negativity. The Lagrangian dual problem of the primal problem [P1] is,

$$
\max_{\delta_i^1\ge 0,\delta_i^2\ge 0,\lambda_i} \left\{ \min_{\beta_i^1,\beta_i^2,\tau_i\in X_i^+} \left\{ \beta_i^1 - \beta_i^2 + \sum_{k=1}^{l_i} \delta_{i,k}^1 \left(\sqrt{\frac{\alpha_i}{1-\alpha_i}}||(\Sigma_i^k)^{\frac{1}{2}}\eta^\tau|| - \beta_i^1 \right) + \sum_{k=1}^{l_i} \delta_{i,k}^2 \left(\beta_i^2 - (\mu_i^k)^T\eta^\tau\right) + \lambda_i \left(1 - \sum_{a_i\in A_i}\tau_i(a_i)\right) \right\} \right\}.
$$

For given $\delta_i^1 \ge 0, \delta_i^2 \ge 0$, and $\lambda_i \in \mathbb{R}$, we have,

$$
\begin{aligned}
&\min_{\beta_i^1,\beta_i^2,\tau_i\in X_i^+} \left\{ \beta_i^1 - \beta_i^2 + \sum_{k=1}^{l_i} \delta_{i,k}^1 \left(\sqrt{\frac{\alpha_i}{1-\alpha_i}}||(\Sigma_i^k)^{\frac{1}{2}}\eta^\tau|| - \beta_i^1 \right) + \sum_{k=1}^{l_i} \delta_{i,k}^2 \left(\beta_i^2 - (\mu_i^k)^T\eta^\tau\right) + \lambda_i \left(1 - \sum_{a_i\in A_i}\tau_i(a_i)\right) \right\}\\
&= \min_{\beta_i^1,\beta_i^2,\tau_i\in X_i^+} \max_{\substack{v_i^k\in\mathbb{R}^{|A|}, ||v_i^k||\le 1\\ k=1,2,\cdots,l_i}} \left\{ \beta_i^1 \left(1 - \sum_{k=1}^{l_i}\delta_{i,k}^1\right) + \beta_i^2 \left(\sum_{k=1}^{l_i}\delta_{i,k}^2 - 1\right) + \sqrt{\frac{\alpha_i}{1-\alpha_i}} \sum_{k=1}^{l_i} \delta_{i,k}^1 \left((\Sigma_i^k)^{\frac{1}{2}}\eta^\tau\right)^T v_i^k - \lambda_i \sum_{a_i\in A_i}\tau_i(a_i) + \lambda_i \right.\\
&\qquad \left. - \sum_{k=1}^{l_i}\delta_{i,k}^2 \sum_{a_i\in A_i}\tau_i(a_i) \sum_{a_{-i}\in A_{-i}} \prod_{j\in I; j\ne i} \tau_j(a_j)\mu_i^k(a_i,a_{-i}) \right\}
\end{aligned}
$$

$$= \max_{\substack{v_i^k \in \mathbb{R}^{|A|}, ||v_i^k|| \le 1 \\ k=1,2,\cdots,l_i}} \min_{\beta_i^1, \beta_i^2, \tau_i \in X_i^+} \Bigg\{ \beta_i^1 \left(1 - \sum_{k=1}^{l_i} \delta_{i,k}^1\right) + \beta_i^2 \left(\sum_{k=1}^{l_i} \delta_{i,k}^2 - 1\right)$$
$$+ \sum_{a_i \in A_i} \tau_i(a_i) \Bigg[\sum_{k=1}^{l_i} \sum_{a_{-i} \in A_{-i}} \prod_{j \in I; j \neq i} \tau_j(a_j) \Bigg(\sqrt{\frac{\alpha_i}{1-\alpha_i}} \left((\Sigma_i^k)^{\frac{1}{2}} v_i^k \right)_{(a_i, a_{-i})} \delta_{i,k}^1$$
$$- \mu_i^k(a_i, a_{-i}) \delta_{i,k}^2 \Bigg) - \lambda_i \Bigg] \Bigg\} + \lambda_i,$$

where $\left((\Sigma_i^k)^{\frac{1}{2}} v_i^k \right)_{(a_i, a_{-i})}$ represents the a^{th} element of vector $(\Sigma_i^k)^{\frac{1}{2}} v_i^k$. The first equality is obtained by using Cauchy-Schwartz inequality, and the second equality follows from Corollary 37.3.2 of [28]. The minimum in the second equality is $-\infty$, unless

$$\sum_{k=1}^{l_i} \delta_{i,k}^1 = 1, \quad \sum_{k=1}^{l_i} \delta_{i,k}^2 = 1,$$
$$\lambda_i \le \sum_{k=1}^{l_i} \sum_{a_{-i} \in A_{-i}} \prod_{j \in I; j \neq i} \tau_j(a_j) \Bigg(\sqrt{\frac{\alpha_i}{1-\alpha_i}} \left((\Sigma_i^k)^{\frac{1}{2}} v_i^k \right)_{(a_i, a_{-i})} \delta_{i,k}^1$$
$$- \mu_i^k(a_i, a_{-i}) \delta_{i,k}^2 \Bigg), \ \forall \ a_i \in A_i.$$

Hence, the dual of [P1] is

[D1] $$\max_{\lambda_i, \delta_i^1, \delta_i^2, (v_i^k)_{k=1}^{l_i}} \lambda_i$$

s.t.

(i) $$\lambda_i \le \sum_{k=1}^{l_i} \sum_{a_{-i} \in A_{-i}} \prod_{j \in I; j \neq i} \tau_j(a_j) \Bigg(\sqrt{\frac{\alpha_i}{1-\alpha_i}} \left((\Sigma_i^k)^{\frac{1}{2}} v_i^k \right)_{(a_i, a_{-i})} \delta_{i,k}^1 - \mu_i^k(a_i, a_{-i}) \delta_{i,k}^2 \Bigg), \ \forall \ a_i \in A_i,$$

(ii) $\sum_{k=1}^{l_i} \delta_{i,k}^1 = 1,$

(iii) $\sum_{k=1}^{l_i} \delta_{i,k}^2 = 1,$

(iv) $||v_i^k|| \le 1, \ \forall \ k = 1, 2, \cdots, l_i,$

(v) $\delta_{i,k}^1 \ge 0, \ \forall \ k = 1, 2, \cdots, l_i,$

(vi) $\delta_{i,k}^2 \ge 0, \ \forall \ k = 1, 2, \cdots, l_i.$

4.1.1 Mathematical program

We denote the decision variables and the objective function of mathematical program [MP1] by $\zeta = (\beta_i^1, \beta_i^2, \lambda_i, \delta_i^1, \delta_i^2, (v_i^k)_{k=1}^{l_i}, \tau_i)_{i\in I}$ and $\psi(\cdot)$ respectively. Then, using all the n primal-dual pairs (one for each player) [P1]-[D1] of convex programs we have the following characterization for Nash equilibrium.

Theorem 4 *Consider an n-player finite strategic game where the payoff vector $r_i = (r_i(a))_{a\in A}$ of player i, $i \in I$, is a random vector. For each $i \in I$, the mean-covariance pair (μ_i, Σ_i) of the distribution of r_i are such that $\mu_i \in \mathcal{U}_{\mu_i}$ and $\Sigma_i \in \mathcal{U}_{\Sigma_i}$, and otherwise the distribution is arbitrary. Consider a point $\zeta^* = (\beta_i^{1*}, \beta_i^{2*}, \lambda_i^*, \delta_i^{1*}, \delta_i^{2*}, (v_i^{k*})_{k=1}^{l_i}, \tau_i^*)_{i\in I}$. Then, for a given $\alpha \in [0,1)^n$, a strategy part τ^* of ζ^* is a Nash equilibrium of the distributionally robust chance-constrained game if and only if ζ^* is a global maximum of the mathematical program [MP1] given below*

$$\text{[MP1]} \quad \max_{\zeta} \sum_{i\in I} [\lambda_i - (\beta_i^1 - \beta_i^2)]$$

s.t.

$$(i)\ \lambda_i \le \sum_{k=1}^{l_i} \sum_{a_{-i}\in A_{-i}} \prod_{j\in I; j\neq i} \tau_j(a_j) \left(\sqrt{\frac{\alpha_i}{1-\alpha_i}} \left((\Sigma_i^k)^{\frac{1}{2}} v_i^k \right)_{(a_i,a_{-i})} \delta_{i,k}^1 - \mu_i^k(a_i,a_{-i})\delta_{i,k}^2 \right),\ \forall\, a_i \in A_i,\ i \in I,$$

$$(ii)\ \sqrt{\frac{\alpha_i}{1-\alpha_i}} ||(\Sigma_i^k)^{\frac{1}{2}} \eta^{\tau}|| - \beta_i^1 \le 0,\ \forall\, k = 1,2,\cdots,l_i,\ i \in I,$$

$$(iii)\ \beta_i^2 - (\mu_i^k)^T \eta^{\tau} \le 0,\ \forall\, k = 1,2,\cdots,l_i,\ i \in I,$$

$$(iv)\ ||v_i^k|| \le 1,\ \forall\, k = 1,2,\cdots,l_i,\ i \in I,$$

$$(v)\ \sum_{a_i\in A_i} \tau_i(a_i) = 1,\ \forall\, i \in I,$$

$$(vi)\ \sum_{k=1}^{l_i} \delta_{i,k}^1 = 1,\ \forall\, i \in I,$$

$$(vii)\ \sum_{k=1}^{l_i} \delta_{i,k}^2 = 1,\ \forall\, i \in I,$$

$$(viii)\ \tau_i(a_i) \ge 0,\ \forall\, a_i \in A_i,\ i \in I,$$

$$(ix)\ \delta_{i,k}^1 \ge 0,\ \forall\, k = 1,2,\cdots,l_i,\ i \in I,$$

$$(x)\ \delta_{i,k}^2 \ge 0,\ \forall\, k = 1,2,\cdots,l_i,\ i \in I,$$

with objective function value $\psi(\zeta^*) = 0$.

Proof Let τ^* be a Nash equilibrium of a distributionally robust chance-constrained game. For each $i \in I$, τ_i^* would be a best response of τ_{-i}^*. Then, there exist β_i^{1*} and β_i^{2*} such that $(\beta_i^{1*}, \beta_i^{2*}, \tau_i^*)$ is an optimal solution of $[P1]$ for fixed τ_{-i}^*. The second order cone program [P1] satisfies all the conditions of strong duality Theorem 6.2.4 of [2]. Hence, there exists an optimal solution $(\lambda_i^*, \delta_i^{1*}, \delta_i^{2*}, (v_i^{k*})_{k=1}^{l_i})$ of dual problem [D1] such that all the constraints (i)-(x) of [MP1] are satisfied at $\zeta^* = (\beta_i^{1*}, \beta_i^{2*}, \lambda_i^*, \delta_i^{1*}, \delta_i^{2*}, (v_i^{k*})_{k=1}^{l_i}, \tau_i^*)_{i\in I}$, and

$$\lambda_i^* = \beta_i^{1*} - \beta_i^{2*}, \ \forall\, i \in I.$$

Hence, ζ^* is a feasible point of [MP1] and the objective function value $\psi(\zeta^*) = 0$. Now, we show that ζ^* is a global maximum of [MP1]. Let ζ be a feasible point of [MP1]. For each $i \in I$, multiply each constraint (i) corresponding to a_i by $\tau_i(a_i)$ and then add over all $a_i \in A_i$. Then, by using the Cauchy-Schwartz inequality and the constraints (ii)-(x), we have

$$\lambda_i \leq \beta_i^1 - \beta_i^2, \ \forall\, i \in I. \tag{12}$$

That is, $\psi(\zeta) \leq 0$ for all feasible point ζ of [MP1]. Hence, ζ^* is a global maximum of mathematical program [MP1].

Let ζ^* be a global maximum of [MP1] such that $\psi(\zeta^*) = 0$. Since, ζ^* is a feasible point of [MP1], then (12) also holds at ζ^*. This together with $\psi(\zeta^*) = 0$ implies that at ζ^* (12) is equality. From constraint (ii) and (iii) of [MP1], we have

$$\beta_i^{1*} - \beta_i^{2*} \geq \max_{1\leq k\leq l_i} \sqrt{\frac{\alpha_i}{1-\alpha_i}} ||(\Sigma_i^k)^{\frac{1}{2}} \eta^{\tau^*}|| - \min_{1\leq k\leq l_i} (\mu_i^k)^T \eta^{\tau^*}, \ \forall\, i \in I. \tag{13}$$

At ζ^*, by multiplying the constraint (i) of [MP1] corresponding to a_i by $\tau_i(a_i)$ and then by adding over all $a_i \in A_i$, and using Cauchy-Schwartz inequality we have

$$\lambda_i^* \leq \sum_{k=1}^{l_i} \delta_{i,k}^{1*} \sqrt{\frac{\alpha_i}{1-\alpha_i}} ||(\Sigma_i^k)^{\frac{1}{2}} \eta^{(\tau_i, \tau_{-i}^*)}|| - \sum_{k=1}^{l_i} \delta_{i,k}^{2*} (\mu_i^k)^T \eta^{(\tau_i, \tau_{-i}^*)}, \ \forall\, \tau_i \in X_i, \ i \in I.$$

Since, $(\delta_{i,k}^{1*})_{k=1}^{l_i}$ and $(\delta_{i,k}^{2*})_{k=1}^{l_i}$ are probability vectors, then we can write

$$\lambda_i^* \leq \max_{1\leq k\leq l_i} \sqrt{\frac{\alpha_i}{1-\alpha_i}} ||(\Sigma_i^k)^{\frac{1}{2}} \eta^{(\tau_i, \tau_{-i}^*)}|| - \min_{1\leq k\leq l_i} (\mu_i^k)^T \eta^{(\tau_i, \tau_{-i}^*)}, \ \forall\, \tau_i \in X_i, \ i \in I.$$

Using (13) and the equality of (12) at ζ^*, we have

$$\begin{aligned} &\max_{1\leq k\leq l_i} \sqrt{\frac{\alpha_i}{1-\alpha_i}} ||(\Sigma_i^k)^{\frac{1}{2}} \eta^{\tau^*}|| - \min_{1\leq k\leq l_i} (\mu_i^k)^T \eta^{\tau^*} \\ &\leq \max_{1\leq k\leq l_i} \sqrt{\frac{\alpha_i}{1-\alpha_i}} ||(\Sigma_i^k)^{\frac{1}{2}} \eta^{(\tau_i, \tau_{-i}^*)}|| - \min_{1\leq k\leq l_i} (\mu_i^k)^T \eta^{(\tau_i, \tau_{-i}^*)}, \ \forall\, \tau_i \in X_i, \ i \in I. \end{aligned}$$

This implies that, for each $i \in I$,

$$u_i^{\alpha_i}(\tau_i^*, \tau_{-i}^*) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}^*), \ \forall\, \tau_i \in X_i.$$

Hence, τ^* is a Nash equilibrium.

4.2 Distributionally robust chance-constrained game for known mean and an upper bound on covariance matrix

We consider the distributionally robust chance-constrained game defined in Section 3.2. For each $i \in I$, a best response of player i, for a fixed τ_{-i}, can be obtained by solving the following convex program,

$$
\begin{aligned}
[\text{P2}] \qquad & \min_{\tau_i} \sqrt{\frac{\alpha_i}{1-\alpha_i}}||\Sigma_i^{\frac{1}{2}}\eta^\tau|| - \mu_i^T\eta^\tau \\
\text{s.t.} \quad & \\
& (i) \sum_{a_i \in A_i} \tau_i(a_i) = 1, \\
& (ii)\ \tau_i(a_i) \geq 0,\ \forall\ a_i \in A_i.
\end{aligned}
$$

From the similar arguments used in Section 4.1, the dual of [P2] is,

$$
\begin{aligned}
[\text{D2}] \qquad & \max_{\lambda_i, v_i} \lambda_i \\
\text{s.t.} \quad & \\
& (i)\ \lambda_i \leq \sum_{a_{-i} \in A_{-i}} \prod_{j \in I; j \neq i} \tau_j(a_j) \left[\sqrt{\frac{\alpha_i}{1-\alpha_i}} \left(\Sigma_i^{\frac{1}{2}} v_i\right)_{(a_i, a_{-i})} - \mu_i(a_i, a_{-i})\right], \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\ a_i \in A_i, \\
& (ii)\ ||v_i|| \leq 1.
\end{aligned}
$$

4.2.1 Mathematical program

Similar to the previous case, by using the n primal-dual pairs [P2]-[D2] of convex programs we have the following characterization for Nash equilibrium.

Theorem 5 *Consider an n-player finite strategic game where the payoff vector $r_i = (r_i(a))_{a \in A}$ of player i, $i \in I$, is a random vector whose distribution belongs to an uncertainty set $\mathcal{D}_i(\mu_i, \Sigma_i)$ defined by (5), where Σ_i is a positive definite matrix. Consider a point $\zeta^* = (\lambda_i^*, v_i^*, \tau_i^*)_{i \in I}$. Then, for a given $\alpha \in [0,1)^n$, a strategy part τ^* of ζ^* is a Nash equilibrium of the distributionally robust chance-constrained game if and only if ζ^* is a global maximum of the mathematical program [MP2] given below,*

$$
[\text{MP2}] \qquad \max_{\zeta} \sum_{i \in I} \left[\lambda_i - \left(\sqrt{\frac{\alpha_i}{1-\alpha_i}}||\Sigma_i^{\frac{1}{2}}\eta^\tau|| - \mu_i^T\eta^\tau\right)\right]
$$

s.t.

$$(i)\ \lambda_i \leq \sum_{a_{-i}\in A_{-i}} \prod_{j\in I; j\neq i} \tau_j(a_j) \left[\sqrt{\frac{\alpha_i}{1-\alpha_i}} \left(\Sigma_i^{\frac{1}{2}} v_i \right)_{(a_i,a_{-i})} - \mu_i(a_i,a_{-i}) \right], \quad \forall\ a_i \in A_i, i \in I,$$

$$(ii)\ ||v_i|| \leq 1,\ \forall\ i \in I,$$

$$(iii)\ \sum_{a_i\in A_i} \tau_i(a_i) = 1,\ \forall\ i \in I,$$

$$(iv)\ \tau_i(a_i) \geq 0,\ \forall\ a_i \in A_i, i \in I,$$

with objective function value $\psi(\zeta^*) = 0$.

Proof The proof follows from the similar arguments used in Theorem 4. □

5 Numerical results

We illustrate our theoretical results from Section 4 using some randomly generated examples. We compute the Nash equilibria, of distributionally robust chance-constrained games corresponding to both distributional uncertainty sets, by solving respective mathematical programs [MP1] and [MP2]. Our numerical experiments were carried out on an Intel(R) 32-bit core(TM) i3-3110M CPU @ 2.40GHz×4 and 3.8 GiB of RAM machine. We use **fmincon** in MATLAB optimization toolbox to solve the corresponding optimization problems.

Example 1 We consider a distributionally robust chance-constrained game as defined in Section 3.1, where $I = \{1, 2\}$, $A_1 = \{1, 2, 3\}$ and $A_2 = \{1, 2, 3\}$. We assume that the mean vectors $\mu_1 \in \mathcal{U}_{\mu_1}$ and $\mu_2 \in \mathcal{U}_{\mu_2}$, and the covariance matrices $\Sigma_1 \in \mathcal{U}_{\Sigma_1}$ and $\Sigma_2 \in \mathcal{U}_{\Sigma_2}$, where $\mathcal{U}_{\mu_i} = \mathbf{Co}\{\mu_i^1, \mu_i^2, \mu_i^3\}$ and $\mathcal{U}_{\Sigma_i} = \mathbf{Co}\{\Sigma_i^1, \Sigma_i^2, \Sigma_i^3\}$, $i = 1, 2$. The data describing the sets $\mathcal{U}_{\mu_1}$ and $\mathcal{U}_{\mu_2}$ are as follows: $\mu_1^1 = (8, 10, 8, 9, 8, 10, 10, 8, 8)^T$, $\mu_1^2 = (10, 8, 10, 8, 9, 8, 10, 10, 8)^T$, $\mu_1^3 = (8, 10, 10, 10, 10, 9, 8, 9, 10)^T$, $\mu_2^1 = (9, 8, 10, 9, 9, 8, 8, 9, 10)^T$, $\mu_2^2 = (10, 10, 8, 10, 10, 9, 9, 10, 10)^T$, $\mu_2^3 = (9, 8, 10, 8, 9, 10, 9, 9, 9)^T$. The data describing the sets $\mathcal{U}_{\Sigma_1}$ and $\mathcal{U}_{\Sigma_2}$ are as follows:

$$\Sigma_1^1 = \begin{pmatrix} 7&3&3&4&3&4&2&4&4\\ 3&7&4&3&3&2&2&2&3\\ 3&4&7&3&4&4&2&2&2\\ 4&3&3&7&2&4&2&3&2\\ 3&3&4&2&5&4&2&2&3\\ 4&2&4&4&4&7&4&3&2\\ 2&2&2&2&2&4&7&4&3\\ 4&2&2&3&2&3&4&5&3\\ 4&3&2&2&3&2&3&3&7 \end{pmatrix}, \quad \Sigma_1^2 = \begin{pmatrix} 8&3&3&2&3&4&2&4&3\\ 3&8&2&3&4&3&3&3&3\\ 3&2&8&4&2&3&2&2&3\\ 2&3&4&6&3&4&3&4&3\\ 3&4&2&3&6&3&4&3&2\\ 4&3&3&4&3&6&3&4&3\\ 2&3&2&3&4&3&6&2&2\\ 4&3&2&4&3&4&2&6&4\\ 3&3&3&3&2&3&2&4&6 \end{pmatrix}, \quad \Sigma_1^3 = \begin{pmatrix} 5&2&4&3&3&3&3&3&3\\ 2&7&4&4&3&3&3&3&3\\ 4&4&7&3&2&2&2&4&3\\ 3&4&3&7&2&2&3&4&3\\ 3&3&2&2&5&3&3&2&3\\ 3&3&2&2&3&7&3&3&3\\ 3&3&2&3&3&3&7&4&3\\ 3&3&4&4&2&3&4&7&2\\ 3&3&3&3&3&3&3&2&5 \end{pmatrix},$$

$$\Sigma_2^1 = \begin{pmatrix} 6&4&3&4&2&2&3&4&3\\ 4&8&3&2&3&4&2&2&3\\ 3&3&8&4&4&2&3&4&4\\ 4&2&4&8&3&4&3&4&4\\ 2&3&4&3&6&3&3&3&3\\ 2&4&2&4&3&6&3&4&3\\ 3&2&3&3&3&3&6&3&3\\ 4&2&4&4&3&4&3&8&4\\ 3&3&4&4&3&3&3&4&8 \end{pmatrix}, \quad \Sigma_2^2 = \begin{pmatrix} 8&3&4&4&3&4&3&2&3\\ 3&8&4&4&3&4&3&3&3\\ 4&4&6&4&3&2&2&4&4\\ 4&4&4&8&2&2&3&3&3\\ 3&3&3&2&6&3&3&4&2\\ 4&4&2&2&3&8&3&3&4\\ 3&3&2&3&3&3&8&3&3\\ 2&3&4&3&4&3&3&8&3\\ 3&3&4&3&2&4&3&3&6 \end{pmatrix}, \quad \Sigma_2^3 = \begin{pmatrix} 8&2&4&2&2&4&3&2&3\\ 2&8&4&3&4&3&2&3&3\\ 4&4&6&4&2&4&3&4&2\\ 2&3&4&6&3&3&3&3&2\\ 2&4&2&3&6&4&4&4&3\\ 4&3&4&3&4&8&4&3&3\\ 3&2&3&3&4&4&8&3&2\\ 2&3&4&3&4&3&3&8&2\\ 3&3&2&2&3&3&2&2&6 \end{pmatrix}.$$

The matrices Σ_i^k, $k = 1, 2, 3$, $i = 1, 2$ are positive definite. The data given here is randomly generated. We order the random payoff vectors of player 1 and player 2 as follows: $r_1^T = ((r_1(a_1, a_2))_{a_2=1}^3)_{a_1=1}^3$, $r_2^T = ((r_2(a_1, a_2))_{a_2=1}^3)_{a_1=1}^3$. For example, the mean of random payoff $r_1(2, 1)$ is a convex combination of $\mu_1^1(4) = 9$, $\mu_1^2(4) = 8$ and $\mu_1^3(4) = 10$, and variance of $r_1(2, 1)$ is a convex combination of $\Sigma_1^1(4, 4) = 7$, $\Sigma_1^2(4, 4) = 6$ and $\Sigma_1^3(4, 4) = 7$, and covariance of $r_1(2, 1)$ and $r_1(1, 2)$ is a convex combination of $\Sigma_1^1(4, 2) = 3$, $\Sigma_1^2(4, 2) = 3$ and $\Sigma_1^3(4, 2) = 4$. The mean, variance and covariance for other random payoffs are defined similarly. We compute a global maximum of the mathematical program [MP1] corresponding to the given data. The mathematical program [MP1] has 78 variables and 48 constraints. The strategy part of a global maximum is a Nash equilibrium of the game. We solve the equivalent minimization problem using **fmincon** in MATLAB optimization toolbox. Table 1 summarizes the Nash equilibria for various values of α.

Table 1: Nash equilibria for various values of α

α		Nash Equilibrium	
α_1	α_2	x^*	y^*
0.6	0.6	$\left(\frac{4130}{10000}, \frac{4494}{10000}, \frac{1376}{10000}\right)$	$\left(\frac{167}{1000}, \frac{82}{1000}, \frac{751}{1000}\right)$
0.7	0.7	$\left(\frac{4263}{10000}, \frac{4395}{10000}, \frac{1342}{10000}\right)$	$\left(\frac{1001}{10000}, \frac{1477}{10000}, \frac{7522}{10000}\right)$
0.8	0.8	$\left(\frac{1527}{10000}, \frac{1879}{10000}, \frac{6594}{10000}\right)$	$\left(\frac{3755}{10000}, 0, \frac{6245}{10000}\right)$

Example 2 We consider a distributionally robust chance-constrained game as defined in Section 3.2, where $I = \{1, 2\}$, $A_1 = \{1, 2, 3\}$ and $A_2 = \{1, 2, 3\}$. The mean vectors for both the players are $\mu_1 = (10, 9, 11, 8, 12, 10, 7, 8, 13)^T$, $\mu_2 = (9, 7, 8, 9, 10, 10, 10, 9, 8)^T$ and the upper bounds on the covariance matrices for both the players are

$$\Sigma_1 = \begin{pmatrix} 6 & 4 & 3 & 3 & 2 & 3 & 4 & 2 & 4 \\ 4 & 6 & 3 & 4 & 3 & 3 & 3 & 2 & 3 \\ 3 & 3 & 8 & 4 & 2 & 3 & 3 & 2 & 4 \\ 3 & 4 & 4 & 6 & 2 & 3 & 3 & 3 & 2 \\ 2 & 3 & 2 & 2 & 6 & 2 & 4 & 3 & 3 \\ 3 & 3 & 3 & 3 & 2 & 6 & 3 & 3 & 4 \\ 4 & 3 & 3 & 3 & 4 & 3 & 8 & 4 & 3 \\ 2 & 2 & 2 & 3 & 3 & 3 & 4 & 6 & 4 \\ 4 & 3 & 4 & 2 & 3 & 4 & 3 & 4 & 8 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 6 & 3 & 3 & 3 & 3 & 2 & 4 & 3 & 2 \\ 3 & 6 & 3 & 3 & 2 & 2 & 3 & 3 & 4 \\ 3 & 3 & 6 & 3 & 3 & 3 & 4 & 3 & 4 \\ 3 & 3 & 3 & 6 & 3 & 2 & 2 & 3 & 3 \\ 3 & 2 & 3 & 3 & 6 & 4 & 2 & 2 & 3 \\ 2 & 2 & 3 & 2 & 4 & 6 & 3 & 3 & 4 \\ 4 & 3 & 4 & 2 & 2 & 3 & 6 & 3 & 2 \\ 3 & 3 & 3 & 3 & 2 & 3 & 3 & 6 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 2 & 3 & 6 \end{pmatrix}.$$

The matrices Σ_1 and Σ_2 are positive definite. We compute a global maximum of the mathematical program [MP2] corresponding to the given data. The mathematical program [MP2] has 26 variables and 16 constraints. Table 2 summarizes the Nash equilibria of distributionally robust chance-constrained game for various values of α.

Table 2: Nash equilibria for various values of α

α		Nash Equilibrium	
α_1	α_2	x^*	y^*
0.6	0.6	$\left(\frac{2777}{10000}, \frac{6583}{10000}, \frac{640}{10000}\right)$	$\left(\frac{217}{1000}, \frac{245}{1000}, \frac{538}{1000}\right)$
0.7	0.7	$\left(\frac{2978}{10000}, \frac{6732}{10000}, \frac{290}{10000}\right)$	$\left(\frac{2168}{10000}, \frac{2308}{10000}, \frac{5524}{10000}\right)$
0.8	0.8	$\left(\frac{3256}{10000}, \frac{6744}{10000}, 0\right)$	$\left(\frac{3279}{10000}, \frac{2347}{10000}, \frac{4374}{10000}\right)$

We also perform numerical experiments by considering various random instances with different sizes for both the cases. Let $I = \{1, 2\}$, $A_1 = \{1, 2, \cdots, m_1\}$ and $A_2 = \{1, 2, \cdots, m_2\}$. We first consider the games from Section 3.1. We consider the case where $l_1 = 3$ and $l_2 = 3$. We use the integer random generator **randi** to generate the data. We take $\mu_i^k = \textbf{randi}\left([m_1 + m_2, m_1 + m_2 + 2], m_1 m_2, 1\right)$ for all $k = 1, 2, 3$ and $i = 1, 2$. It generates $m_1 m_2 \times 1$ integer random vector within interval $[m_1 + m_2, m_1 + m_2 + 2]$. We generate the positive definite matrices Σ_i^k, $k = 1, 2, 3,\ i = 1, 2$, by setting $\Sigma_i^k = B + B^T + \theta \cdot I_{m_1 m_2 \times m_1 m_2}$, where $B = \textbf{randi}(2, m_1 m_2)$ is an $m_1 m_2 \times m_1 m_2$ integer random matrix with entries not more than 2, and θ is sufficiently large so that Σ_i^k is positive definite, and $I_{m_1 m_2 \times m_1 m_2}$ is an $m_1 m_2 \times m_1 m_2$ identity matrix. In our experiments, we take $\theta = m_1 + m_2$. We take $\alpha = (0.6, 0.6)$. We solve the mathematical program [MP1] for various random instances with different sizes. For these games the mathematical program [MP1] has $18 + 6m_1 m_2 + m_1 + m_2$ variables and $36{+}2m_1{+}2m_2$ constraints. Table 3 summarizes the average time for computing Nash equilibrium for different sizes of games considered in Section 3.1.

Table 3: Average time for computing Nash equilibrium

Number of instances	Number of actions		Size of [MP1]		Average time (seconds)
	Player 1	Player 2	Variables	Constraints	
10	5	5	178	56	10.44
10	10	10	638	76	229.8
10	15	15	1398	96	2761.2

Next, we consider the games from Section 3.2. For each i, $i = 1, 2$, we take the mean vector $\mu_i = \textbf{randi}\left([m_1 + m_2, m_1 + m_2 + 2], m_1 m_2, 1\right)$ and the upper bound on covariance matrix as $\Sigma_i = B + B^T + \theta \cdot I_{m_1 m_2 \times m_1 m_2}$. We solve the mathematical program [MP2] for various random instances with different sizes. For these games the mathematical program [MP2] has $2 + 2m_1 m_2 + m_1 + m_2$ variables and $4{+}2m_1 + 2m_2$ constraints. Table 4 summarizes the average time for computing Nash equilibrium for different sizes of games considered in Section 3.2.

Table 4: Average time for computing Nash equilibrium

Number of instances	Number of actions		Size of [MP2]		Average time (seconds)
	Player 1	Player 2	Variables	Constraints	
10	5	5	62	24	4.29
10	10	10	222	44	33.23
10	15	15	482	64	238.2
10	20	20	842	84	850.8

6 Conclusions

We consider an n-player finite strategic game where the payoff vector of each player is a random vector whose distribution is not completely known. It belongs to a certain distributional uncertainty set. We define a distributionally robust chance-constrained game using worst-case chance constraint. We consider two types of distributional uncertainty sets based on the partially available information about mean and covariance matrix of random payoff vector. For each case, we show that there always exists a mixed strategy Nash equilibrium. We characterize the Nash equilibria of the game by proposing an equivalent mathematical program. For each case, we show a one-to-one correspondence between a Nash equilibrium of a game and a global maximum of a certain mathematical program. For illustration purpose, we consider various random instances of games with different sizes. We compute the Nash equilibria of the game by solving equivalent mathematical program. We use **fmincon** in MATLAB optimization toolbox to solve the mathematical programs. The proposed mathematical programs have nice structure, e.g., the objective function value is non-positive for all feasible points and a global maximum is attained when the objective function value is zero. Due to this property these mathematical programs are not very difficult to solve using known optimization solvers.

Acknowledgements

This research was supported by Fondation DIGITEO, SUN grant No. 2014-0822D.

References

1. Adler, I.: The equivalence of linear programs and zero-sum games. International Journal of Game Theory **42**(1), 165–177 (2013)
2. Bazaraa, M., Sherali, H., Shetty, C.: Nonlinear Programming Theory and Algorithms. John Wiley & Sons, Inc., U.S.A, Third ed. (2006)
3. Blau, R.A.: Random-payoff two person zero-sum games. Operations Research **22**(6), 1243–1251 (1974)
4. Cassidy, R.G., Field, C.A., Kirby, M.J.L.: Solution of a satisficing model for random payoff games. Management Science **19**(3), 266–271 (1972)
5. Charnes, A., Cooper, W.W.: Deterministic equivalents for optimizing and satisficing under chance constraints. Operations Research **11**(1), 18–39 (1963)
6. Charnes, A., Kirby, M.J.L., Raike, W.M.: Zero-zero chance-constrained games. Theory of Probability and its Applications **13**(4), 628–646 (1968)
7. Cheng, J., Delage, E., Lisser, A.: Distributionally robust stochastic knapsack problem. SIAM Journal on Optimization **24**(3), 1485–1506 (2014)
8. Cheng, J., Leung, J., Lisser, A.: Random-payoff two-person zero-sum game with joint chance constraints. European Journal of Operational Research (2015). DOI http://dx.doi.org/10.1016/j.ejor.2015.12.024
9. Cheng, J., Lisser, A.: A second-order cone programming approach for linear programs with joint probabilistic constraints. Operations Research Letters **40**(5), 325–328 (2012)

10. Couchman, P., Kouvaritakis, B., Cannon, M., Prashad, F.: Gaming strategy for electric power with random demand. IEEE Transactions on Power Systems **20**(3), 1283–1292 (2005)
11. Dantzig, G.B.: A proof of the equivalence of the programming problem and the game problem. In: T.C. Koopmans (ed.) Activity analysis of production and allocation, pp. 330–335. John Wiley & Sons, New York (1951)
12. DeMiguel, V., Xu, H.: A stochastic multiple leader Stackelberg model: Analysis, computation, and application. Operations Research **57**(5), 1220–1235 (2009)
13. El-Ghaoui, L., Oks, M., Oustry, F.: Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. Operations Research **51**(4), 543–556 (2003)
14. Hayashi, S., Yamashita, N., Fukushima, M.: Robust Nash equilibria and second-order cone complementarity problems. Journal of Nonlinear and Convex Analysis **6**(2), 283–296 (2005)
15. Jadamba, B., Raciti, F.: Variational inequality approach to stochastic Nash equilibrium problems with an application to Cournot oligopoly. Journal of Optimization Theory and Application **165**(3), 1050–1070 (2015)
16. Kakutani, S.: A generalization of brouwer's fixed point theorem. Duke Mathematical Journal **8**(3), 457–459 (1941)
17. Lee, K.H., Baldick, R.: Solving three-player games by the matrix approach with application to an electric power market. IEE Transactions on Power Systems **18**(4), 1573–1580 (2003)
18. Lemke, C.E., Howson, J.T.: Equilibrium points of bimatrix games. Journal of the Society for Industrial and Applied Mathematics **12**(2), 413–423 (1964)
19. Lepore, J.J.: Cournot outcomes under Bertrand-Edgeworth competition with demand uncertainty. Journal of Mathematical Economics **48**(3), 177–186 (2012)
20. Mangasarian, O.L., Stone, H.: Two-person nonzero-sum games and quadratic programming. Journal of Mathematical Analysis and Applications **9**(3), 348–355 (1964)
21. Mazadi, M., Rosehart, W.D., Zareipour, H., Malik, O.P., Oloomi, M.: Impact of wind integration on electricity markets: A chance-constrained Nash Cournot model. International Transactions on Electrical Energy Systems **23**(1), 83–96 (2013)
22. Nash, J.F.: Equilibrium points in n-person games. Proceedings of the National Academy of Sciences **36**(1), 48–49 (1950)
23. Neumann, J.V.: Zur theorie der gesellschaftsspiele. Math. Annalen **100**(1), 295–320 (1928)
24. Nishimura, R., Hayashi, S., Fukushima, M.: Semidefinite complementarity reformulation for robust Nash equilibrium problems with Euclidean uncertainty sets. Journal of Global Optimization **53**(1), 107–120 (2012)

25. Polik, I., Terlaky, T.: A survey of the S-lemma. SIAM Review **49**(3), 371–418 (2007)
26. Prékopa, A.: Stochastic Programming. Springer, Netherlands (1995)
27. Ravat, U., Shanbhag, U.V.: On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games. Siam Journal on Optimization **21**(3), 1168–1199 (2011)
28. Rockafellar, R.T.: Convex Analysis. Princeton University Press (1970)
29. Shapiro, A.: Semi-Infinite Programming: Recent Advances, chap. On duality theory of conic linear problems, pp. 135–165. Springer US, Boston, MA (2001)
30. Singh, V.V., Jouini, O., Lisser, A.: Existence of Nash equilibrium for chance-constrained games `http://www.optimization-online.org/DB_HTML/2015/06/4977.html` (2015)
31. Son, Y.S., Baldick, R., Lee, K.H., Siddiqi, S.: Short-term electricity market auction game analysis: Uniform and pay-as-bid pricing. IEEE Transactions on Power Systems **19**(4), 1990–1998 (2004)
32. Song, T.: Systems and Management Science by Extremal Methods, chap. On random payoff matrix games, pp. 291–308. Springer US, Boston, MA (1992)
33. Valenzuela, J., Mazumdar, M.: Cournot prices considering generator availability and demand uncertainty. IEEE Transactions on Power Systems **22**(1), 116–125 (2007)
34. Wambach, A.: Bertrand competition under cost uncertainty. International Journal of Industrial Organization **17**(7), 941–951 (1999)
35. Wolf, D.D., Smeers, Y.: A stochastic version of a Stackelberg-Nash-Cournot equilibrium model. Management Science **43**(2), 190–197 (1997)
36. Xu, H., Zhang, D.: Stochastic Nash equilibrium problems: Sample average approximation and applications. Computational Optimization and Applications **55**(3), 597–645 (2013)