Noname manuscript No.
(will be inserted by the editor)

On Cournot-Nash-Walras equilibria and their computation

Jiří V. Outrata · Michael C. Ferris · Michal Červinka · Michal Outrata

Received: date / Accepted: date

Abstract This paper considers a model of Cournot-Nash-Walras (CNW) equilibrium where the Cournot-Nash concept is used to capture equilibrium of an oligopolistic market with non-cooperative players/firms who share a certain amount of a so-called rare resource needed for their production, and the Walras equilibrium determines the price of that rare resource. We prove the existence of CNW equilibria under reasonable conditions and examine various numerical approaches for their computation. Finally, we demonstrate remarkably stable behavior of CNW equilibria with respect to small perturbations of problem data including, for instance, the available amount of the rare resource.

Keywords Cournot-Nash-Walras equilibrium · Existence · Stationarity Conditions · Stability · Mathematical Program with Equilibrium Constraints · Implicit programming approach

This work was partially supported by the Grant Agency of the Czech Republic under Grant P402/12/1309 and 15-00735S, by the Grant Agency of the Charles University by Grant SFG 2567, by the Australian Research Council under grant DP-110102011 and in part by a grant from the Department of Energy and funding from the USDA.

Jiří V. Outrata
Institute of Information Theory and Automation, Czech Academy of Sciences, Prague, Czech Republic
Centre for Informatics and Applied Optimization, Federation University of Australia, Ballarat, Australia
E-mail: outrata@utia.cas.cz

Michael C. Ferris
Department of Computer Sciences, University of Wisconsin-Madison, Madison, WI, USA

Michal Červinka
Institute of Information Theory and Automation, Czech Academy of Sciences, Prague, Czech Republic
Faculty of Social Sciences, Charles University in Prague, Czech Republic

Michal Outrata
Faculty of Mathematics and Physics, Charles University in Prague, Czech Republic

1 Introduction and preliminaries

Recently, S.D. Flåm investigated markets where the players/firms behave non-cooperatively and some of their inputs are limited but transferable. These so-called rare resources are controlled by some national or international authority that typically provides each agent with some initial endowment of these resources. Examples of such rare resources include fish quotas or rights to water usage. In the same way one can also handle production allowances or pollution permits. Since these rare resources are transferable, after the initial allocation they may be bought or sold in a market. This eventually leads to a Walras equilibrium specifying the equilibrium price of a unit of the rare resources. This price is either nil, in the case when the available amount of rare resources exceeds the interests of the market, or it is nonnegative provided that the demand amounts exactly to the available quantity. In either case, the initial endowments can be reallocated, which leads to a joint improvement. In [10] the author speaks about Nash-Walras equilibria and divides the process of their finding into two phases. In the first one, the agents compute a Nash equilibrium corresponding to their initial endowments. In the second phase, the agents approach a Nash-Walras equilibrium step by step by bilateral exchanges of their shares of rare resources so that the overall amount of them remains unchanged. In this way the author attempts to model real processes leading to an equilibrium price of the rare resources.

In contrast to this approach, in this paper we look at this problem from a slightly different perspective. The authority controlling the rare resources might, in reality, be interested in computing a Nash-Walras equilibrium in one step in order to get a feedback about the influence of the initial allocation on the overall production and the price of the rare resources. Likewise a firm might wish to learn how an improved technology (leading to a smaller consumption of a rare resource) would influence his profit. So, in this paper, we suggest a procedure for computing a Nash-Walras equilibrium in one step, without any phases and evolutionary processes. Since our agents are firms and behave according to the Cournot-Nash concept, we prefer to use the terminology Cournot-Nash-Walras (CNW) equilibrium in the sequel.

The plan of the paper is as follows: In Section 2 we formulate the problem, collect the standing assumptions and analyze some elementary properties. Section 3 proves the existence of a CNW equilibrium. Section 4 is devoted to several numerical approaches enabling us to compute CNW equilibria using existing software. They are based either on the original formulation in Definition 2, or on the reduced formulation developed in Section 3. Alternatively, CNW equilibria can be computed via a suitable *mathematical program with equilibrium constraints* (MPEC) and this program again can be solved by various methods. As a test example we have used an adaptation of the five-firm oligopolistic market from [17]. Its description and the obtained numerical results are presented in Section 5. Finally, Section 6 deals with local stability analysis of the mapping which assigns the CNW equilibria to the problem data. It turns out that this mapping is locally single-valued and Lipschitz whenever the price of the rare resource is positive.

Our notation is basically standard. For a closed cone K with vertex at 0, K^0 denote its negative polar and for a set $A, \mathrm{dist}_A(x)$ stands for the distance of x to A. Given a multifunction $F[\mathbb{R}^n \rightrightarrows \mathbb{R}^m], \mathrm{Gr}F := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in F(x)\}$ is the graph of F and $\mathbb{B}$ denotes the unit ball.

We conclude the introductory section with the definitions of some basic notions from modern variational analysis which will be extensively used in this paper.

Consider a closed set $A \subset \mathbb{R}^n$ and $\bar{x} \in A$. We define the *contingent (Bouligand) cone* to A at $\bar{x}$ as the cone

$$\begin{aligned} T_A(\bar{x}) :=& \operatorname*{Lim\,sup}_{\tau \downarrow 0} \frac{A - \bar{x}}{\tau} \\ =& \{h \in \mathbb{R}^n | \exists h_k \to h, \lambda_k \searrow 0 \text{ such that } \bar{x} + \lambda_k h_k \in A \text{ for all } k\} \end{aligned}$$

and the *regular (Fréchet) normal cone* to A at $\bar{x}$ as $\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^0$. Moreover, the *limiting (Mordukhovich) normal* cone to A at $\bar{x}$ is defined by

$$\begin{aligned} N_A(\bar{x}) :=& \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} \widehat{N}(x) \\ =& \{x^* \in \mathbb{R}^n | \exists\, x_k \xrightarrow{A} \bar{x}, x_k^* \to x^* \text{ such that } x_k^* \in \widehat{N}_A(x_k) \text{ for all } k\}. \end{aligned}$$

We say that A is *(normally) regular* at $\bar{x}$ provided $N_A(\bar{x}) = \widehat{N}_A(\bar{x})$. Convex sets are regular at all points. Now consider a closed-graph multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ and a point $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, \Phi$.

The multifunction $D^*\Phi(\bar{x}, \bar{y})[\mathbb{R}^m \rightrightarrows \mathbb{R}^n]$ defined by

$$D^*\Phi(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n | (x^*, -y^*) \in N_{\mathrm{Gr}\Phi}(\bar{x}, \bar{y})\}$$

is the *limiting (Mordukhovich) coderivative* of Φ at $(\bar{x}, \bar{y})$. Let $f[\mathbb{R}^n \to \bar{\mathbb{R}}]$ be *lsc* and $\bar{x} \in$ dom f. The set

$$\hat{\partial} f(\bar{x}) := \{v \in \mathbb{R}^n | (v, -1) \in \widehat{N}_{\mathrm{epi} f}(\bar{x}, f(\bar{x}))\}$$

is called the *regular (Fréchet) subdifferential* of f at $\bar{x}$ and its elements are termed the *regular (Fréchet) subgradients* of f at $\bar{x}$. Analogously, the set

$$\partial f(\bar{x}) := \{v \in \mathbb{R}^n | (v, -1) \in N_{\mathrm{epi} f}(\bar{x}, f(\bar{x}))\}$$

is called the *limiting (Mordukhovich) subdifferential* of f at $\bar{x}$ and its elements are termed the *limiting (Mordukhovich) subgradients* of f at $\bar{x}$.

We say that $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ is *calm* at $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, \Phi$, provided there are neighborhoods $\mathcal{U}$ of $\bar{x}$, $\mathcal{V}$ of $\bar{y}$ and a real $L \geq 0$ such that

$$\Phi(x) \cap \mathcal{V} \subset \Phi(\bar{x}) + L\|x - \bar{x}\|\mathbb{B} \quad \text{for all} \quad x \in \mathcal{U}.$$

In Section 4 we will employ the following result.

Proposition 1 *Let $F[\mathbb{R}^n \to \mathbb{R}^m]$ be Lipschitz around $\bar{x}$ and $f[\mathbb{R}^m \to \mathbb{R}]$ be Lipschitz around $F(\bar{x})$. Then for the composition $g = f \circ F$ one has that*

$$\partial g(\bar{x}) \subset \bigcup_{\xi \in \partial f(\bar{y})} D^* F(\bar{x}, \bar{y})(\xi)$$

with $\bar{y} = F(\bar{x})$.

If f is continuously differentiable around $\bar{y}$, then

$$\partial g(\bar{x}) = D^* F(\bar{x}, \bar{y})(\nabla f(\bar{y})).$$

The first statement follows from [24, Theorem 10.49], the second one from [16, Theorem 1.110(ii)]. An interested reader may find a full account of properties of the above notions for example in the monographs [24] and [16].

2 Problem formulation

Consider an oligopolistic market with m firms, each of which produce a homogeneous commodity. As mentioned in the introduction, they each need a certain amount of a *rare resource* for this production, that is dependent on the technology that is used. It follows that each firm optimizes his profit by using two strategies: his production and the amount of the rare resource that he intends to purchase or to sell. Consequently, the ith firm solves the optimization problem

$$\begin{array}{ll} \text{minimize} & c_i(y_i) + \pi x_i - p(T) y_i \\ \text{subject to} & \\ & (y_i, x_i) \in (A_i \times \mathbb{R}) \cap \mathcal{B}_i, \end{array} \tag{1}$$

where y_i is the production, x_i is the amount of the rare resource that is purchased (or sold), $c_i[\mathbb{R}_+ \to \mathbb{R}_+]$ specifies the production costs, π is the price of the rare resource, $T = \sum_{i=1}^m y_i$ signifies the overall amount of the produced commodity in the market and $A_i = [a_i, b_i]$ specifies the production bounds. The function $p[\text{int } \mathbb{R}_+ \to \mathbb{R}_+]$ assigns each amount T the price at which (price-taking) consumers are willing to demand. It is usually called the *inverse demand curve*. The relationship between y_i and the required amount of the rare resource is reflected via the set

$$\mathcal{B}_i = \{(y_i, x_i) | q_i(y_i) \leq x_i + e_i\},$$

where e_i is the initial endowment of the rare resource and $q_i[\mathbb{R}_+ \to \mathbb{R}_+]$ is a (technological) function assigning each production value the corresponding amount required of the rare resource. Observe that in problem (1) the variables $y_j, j \neq i$, and π play the role of parameters.

Throughout the whole paper we will impose the following assumptions:

A1: All functions c_i can be extended to open intervals containing the sets A_i. These extensions are convex and twice continuously differentiable.
A2: p is strictly convex and twice continuously differentiable on int $\mathbb{R}_+$.

A3: $\alpha p(\alpha)$ is a concave function of α.
A4: For all i one has $0 \leq a_i < b_i$ and there is an index i_0 such that $a_{i_0} > 0$.
A5: All functions q_i satisfy $q_i(0) = 0$ and can be extended to open intervals containing the sets A_i. These extensions are convex, increasing and twice continuously differentiable.
A6: For all i, $q_i(a_i) \leq e_i$ and there is an index i_0 such that $q_{i_0}(a_{i_0}) < e_{i_0}$.
A7: $\pi \geq 0$.

The assumptions A1 - A3 are not too restrictive and arise in a similar form in most treatments of oligopolistic markets, cf. [17], [19]. They ensure in particular that the objective in (1) is convex for all i. Assumption A4 ensures that $p(T)$ is well-defined. Assumptions A5 and A6 are related to the rare resource and play an important role both in the existence proof in the next section as well as in one of the numerical approaches discussed in Section 4. The economic interpretation of A6 says that the initial endowment of the rare resource enables each firm to produce more than its lower production bound. Finally, A7 is natural.

Denote by $J_i, i = 1, 2, \dots, m$, the objective in (1) and by Ξ the overall available amount of the rare resource so that

$$\Xi \geq \sum_{i=1}^{m} e_i. \tag{2}$$

Further, to simplify the notation, $y = (y_1, y_2, \dots, y_m)$ and $x = (x_1, x_2, \dots, x_m)$ stand for the vectors of cumulative strategies y_i, x_i of all firms. To introduce the CNW equilibrium, we define first the Cournot-Nash equilibrium generated by problems (1).

Definition 1 The strategy pair $(\bar{y}, \bar{x})$ is a *Cournot-Nash equilibrium* in the considered market for a given $\pi \geq 0$ provided for all i one has

$$J_i(\pi, \bar{y}, \bar{x}_i) = \min_{(y_i, x_i) \in (A_i \times \mathbb{R}) \cap \mathcal{B}_i} J_i(\pi, \bar{y}_i, \bar{y}_2, \dots, \bar{y}_{i-1}, y_i, \bar{y}_{i+1}, \dots, \bar{y}_m, x_i) \tag{3}$$

Remark 1 If the constraint (2) is neglected and all endowments e_i vanish (so that we do not consider a “rare” resource), then we may put $x_i = q(y_i)$, the production costs become $c_i(y_i) + \pi q_i(y_i)$ and the constraint set in (1) can be simplified to $y_i \in A_i$. Definition 1 then amounts to the classical notion of Cournot (or Cournot-Nash) equilibrium from 1838, cf. [17]. For this reason we use the terminology Cournot-Nash equilibrium also in our slightly more complex case reflecting the above described mechanism of trading with the rare resource.

Remark 2 In [10] the author assumes that the production cost functions c_i also depend on x_i.

Definition 2 *(Flåm)* The triple $(\bar{\pi}, \bar{y}, \bar{x})$ is a *Cournot-Nash-Walras* (CNW) equilibrium in the considered market provided that

(i) $(\bar{y}, \bar{x})$ is a Cournot-Nash equilibrium for $\pi = \bar{\pi}$, and
(ii) one has

$$\bar{\pi} \geq 0,\ \Xi - \sum_{i=1}^{m}(e_i + \bar{x}_i) \geq 0,\ \bar{\pi} \cdot (\Xi - \sum_{i=1}^{m}(e_i + \bar{x}_i)) = 0.$$

Clearly, the conditions in (ii) characterize a Walras equilibrium with respect to the rare resource which determines a price $\bar{\pi}$ under which the (secondary) market with the rare resource is cleared. From the point of view of the firms, the computation of $\bar{\pi}$ is a dynamical process starting after the initial allocation has been conducted. From the point of view of the authority controlling the rare resource, however, the whole problem can be solved in one step. The results provide the authority with information about the influence of the initial allocation on the CNW equilibrium.

The Cournot-Nash equilibrium from Definition 1 can easily be characterized via standard stationarity/optimality conditions. For the readers' convenience we state this result here with a proof.

Proposition 2 *Given a price $\pi \geq 0$, under the posed assumptions, a pair $(\bar{y}, \bar{x})$ is a Cournot-Nash equilibrium in the sense of Definition 1 if and only if it fulfills the relations*

$$0 \in \begin{bmatrix} \nabla c_1(y_1) - \bar{y}_1 \nabla p(\overline{T}) - p(T) \\ \vdots \\ \nabla c_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + \pi \begin{bmatrix} \nabla q_1(y_1) \\ \vdots \\ \nabla q_m(y_m) \end{bmatrix} + \bigtimes_{i=1}^{m} N_{A_i}(y_i) \tag{4}$$

$$\begin{aligned} &\pi \cdot (q_i(y_i) - x_i - e_i) = 0, \quad i = 1, 2, \ldots, m. \\ &q_i(y_i) \leq e_i + x_i, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{5}$$

Proof The constraints in (3) satisfy the linear independence constraint qualification (LICQ) due to A4. Moreover, the standing assumptions ensure that the functions J_i are jointly convex in (x_i, y_i). The Cournot-Nash equilibria are henceforth characterized by the standard first-order optimality conditions for the single optimization problems (3). Putting them together, we obtain the *generalized equation* (GE)

$$0 \in \begin{bmatrix} \nabla c_1(y_1) - \bar{y}_1 \nabla p(T) - p(T) \\ \pi \\ \vdots \\ \nabla c_m(y_m) - \bar{y}_m \nabla p(T) - p(T) \\ \pi \end{bmatrix} + \bigtimes_{i=1}^{m} N_{A_i \times \mathbb{R}}(y_i, x_i) + \begin{bmatrix} \lambda_1 \nabla q_1(y_1) \\ -\lambda_1 \\ \vdots \\ \lambda_m \nabla q_m(y_m) \\ -\lambda_m \end{bmatrix}, \tag{6}$$

where $\lambda_1, \ldots, \lambda_m$ are nonnegative Lagrange multipliers associated with the inequalities defining the sets $\mathcal{B}_i$. They must fulfill the complementarity conditions

$$\lambda_i(q_i(y_i) - x_i - e_i) = 0 \quad \text{for all} \quad i = 1, 2, \ldots, m. \tag{7}$$

Since $N_{\mathbb{R}}(x_i) = \{0\}$ for all i, we immediately conclude that

$$\pi = \lambda_1 = \lambda_2 = \ldots = \lambda_m. \tag{8}$$

In this way we arrive at the simplified (but equivalent) conditions (4), (5) in which only the partial derivatives $\nabla_{y_i} J_i$ arise.

3 Existence of CNW equilibria

Let us associate with the ith firm, instead of (1), a different problem, namely

$$\begin{array}{ll} \text{minimize} & c_i(y_i) + \pi(q_i(y_i) - e_i) - p(T)y_i \\ \text{subject to} & \\ & y_i \in A_i \end{array} \tag{9}$$

solely in the variable y_i. It corresponds to replacing the inequality

$$q_i(y_i) \leq x_i + e_i$$

by an equality so that variable x_i can be completely eliminated. If we replace the functions J_i in Definition 1 by the objectives from (9), we obtain a different non-cooperative equilibrium characterized by the GE

$$0 \in \begin{bmatrix} \nabla c_1(y_1) + \pi \nabla q_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) + \pi \nabla q_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + \bigtimes_{i=1}^{m} N_{A_i}(y_i). \tag{10}$$

Lemma 1 *Let $\bar{y}$ satisfy condition (10). Then the pair $(\bar{y}, \bar{x})$ with $\bar{x}_i = q_i(\bar{y}_i) - e_i$ for all i is a Cournot-Nash equilibrium in the sense of Definition 1. Conversely, for each solution $(\bar{y}, \bar{x})$ of system (4), (5), the component $\bar{y}$ fulfills GE (10) whenever $\pi > 0$.*

The proof follows immediately from the comparison of GE (10) with the conditions (4), (5).

Denote by $S[\mathbb{R}_+ \rightrightarrows \mathbb{R}^m]$ the mapping which assigns each $\pi \geq 0$ the set of solutions to GE (10). The statement of Lemma 1 can then be written down as follows:

(i) For any $\pi \geq 0$ one has the implication

$$y \in S(\pi), x_i = q_i(y_i) - e_i \text{ for all } i \ \Rightarrow \ (y, x) \text{ fulfills conditions (4), (5).}$$

(ii) For $\pi > 0$ the above implication becomes equivalence.

Remark 3 It follows from Lemma 1 that for $\pi \geq 0$ the initial endowment $e_1, \ldots, e_m$ does not influence the component y of the Cournot-Nash equilibrium pair (y, x).

Lemma 2 *There is a positive real L such that in all CNW equilibria one has $\pi \leq L$.*

Proof Assume that $\bar{\pi} > 0$ is so large that

$$\min_{i=1,\ldots,m} \left\{ \nabla c_i(a_i) - a_i \nabla p\left(\sum_{i=1}^m a_i\right) - p\left(\sum_{i=1}^m a_i\right) + \bar{\pi}\nabla q_i(a_i) \right\} > 0. \qquad (11)$$

By virtue of (11) it follows that the stationarity condition (4) can be fulfilled only in the case when $y_i = a_i$ for all i. Indeed, since the functions J_i are convex in variables (y_i, x_i), their partial derivatives with respect to y_i are nondecreasing, and so for $y_i \geq a_i$ the quantities $\nabla c_i(y_i) - y_i \nabla p(T) - p(T) + \bar{\pi}\nabla q_i(y_i)$ are positive as well. It follows that $y_i = a_i$ for all i in order to bring the normal cones to A_i into play. By (7) and (8) the respective values of x_i are given by $x_i = q_i(a_i) - e_i$ and thus, thanks to assumption A6, all of them are nonpositive and at least one of them is negative. This implies, however, that the corresponding excess demand $\sum_{i=1}^m (e_i + x_i) - \Xi$ is negative as well, which contradicts the complementarity condition of the Walras equilibrium. As L we can thus choose any positive real satisfying inequality (11) with $\bar{\pi}$ replaced by L.

On the basis of Lemmas 1 and 2 we are now able to state our main existence result.

Theorem 1 *Under the posed assumptions there is a CNW equilibrium.*

Proof Define the mapping $Q[\mathbb{R}^m \to \mathbb{R}]$ by

$$Q(y) := \sum_{i=1}^n q_i(y_i).$$

By virtue of Lemma 1 it suffices to show the existence of a pair $(\bar{\pi}, \bar{y})$ which solves the (aggregated) GE

$$\left.\begin{array}{l} 0 \in \Xi - Q(y) + N_{\mathbb{R}_+}(\pi) \\ 0 \in \begin{bmatrix} \nabla c_1(y_1) + \pi\nabla q_1(y_1) - y_1\nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) + \pi\nabla q_m(y_m) - y_m\nabla p(T) - p(T) \end{bmatrix} + \bigtimes_{i=1}^m N_{A_i}(y_i) \end{array}\right\} \qquad (12)$$

in variables (π, y). Thanks to Lemma 2, $\mathbb{R}_+$ in the first line of (12) can be replaced by a bounded interval $[0, L]$. In this way, once obtains a variational inequality with a bounded constraints set which possesses a solution $(\bar{\pi}, \bar{y})$ as a consequence of the Brouwer Fixed Point Theorem. It follows that $(\bar{\pi}, \bar{y}, \bar{x})$ with $\bar{x}_i = q_i(\bar{y}_i) - e_i$ is a CNW equilibrium.

If the functions q_i are not convex, then the whole above argumentation remains valid provided that in Lemma 2 we replace the expression on the left-hand side of (11) by

$$\min_{i=1,\ldots,m} \min_{y_i \in A_i} \left\{ \nabla c_i(y_i) - y_i \nabla p\left(\sum_{i=1}^m y_i\right) - p\left(\sum_{i=1}^m y_i\right) + \bar{\pi}\nabla q_i(y_i) \right\} > 0. \qquad (13)$$

Note that the second minimum on the left-hand side of (13) is attained by the boundedness of intervals A_i and by assumptions A1 and A2. Moreover, by increasing π, the validity of inequality (13) can be ensured due to positivity of $\nabla q_i(y_i)$ for all i.

Unfortunately, in case of nonconvex functions q_i, GE (10) is not a characterization but only a stationarity condition for the Cournot-Nash equilibria generated by problems (9). In Theorem 1 we thus do not prove the existence of CNW equilibria, but only the existence of points satisfying a stationarity condition for CNW equilibria. In the numerical approach, presented in the first part of the next section, the convexity of $q_i, i = 1, \ldots, m$, is not needed and so we compute (in the non-convex case) stationary triples (π, y, x) for CNW equilibria in the sense specified above.

The last statement of this section will be important for the development in Section 6.

Proposition 3 *The set of solutions to (12) is closed and convex.*

Proof By virtue of [24, Example 12.48] it suffices to prove that the single-valued part of GE (12) is a monotone operator relative to $\mathbb{R}_+ \times \mathsf{X}_{i=1}^m A_i$. By invoking [18, Theorem 5.4.3 (a)] this is ensured provided the symmetric matrix $\frac{1}{2}[D(\pi, y) + (D(\pi, y))^T]$ with

$$D(\pi, y) := \begin{bmatrix} 0 & -\nabla q_1(y_1) & \ldots & -\nabla q_m(y_m) \\ \nabla q_1(y_1) & & & \\ \vdots & & \nabla_y H(\pi, y) & \\ \nabla q_m(y_m) & & & \end{bmatrix} \tag{14}$$

is positive semidefinite over $\mathbb{R}_+ \times \mathsf{X}_{i=1}^m A_i$. In (14), H stands for the mapping defined by

$$H(\pi, y) := G(y) + \pi \begin{bmatrix} \nabla q_1(y_1) \\ \vdots \\ \nabla q_m(y_m) \end{bmatrix} \tag{15}$$

with

$$G(y) := \begin{bmatrix} \nabla c_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix}.$$

Clearly,

$$\frac{1}{2}[D(\pi, y) + (D(\pi, y))^T] = \begin{bmatrix} 0 \; 0 & \ldots & 0 \\ 0 & & \\ \vdots & \frac{1}{2}[\nabla_y H(\pi, y) + (\nabla_y H(\pi, y))^T] & \\ 0 & & \end{bmatrix}.$$

The matrix $\frac{1}{2}[\nabla G(y) + (\nabla G(y))^T]$ is positive definite due to [19, Lemma 12.2]. Under the posed assumptions the second matrix in (15) is symmetric positive semidefinite and so the proof is complete.

4 Computation of CNW equilibria

4.1 Nonlinear programming approaches

We describe several approaches that can be used for the computation of CNW equilibria, based on standard nonlinear solvers that incorporate complementarity constraints. These approaches are available within modeling systems such as GAMS [2] and AMPL [8].

The first approach corresponds directly to the model given in Definition 2. This is an example of a MOPEC (multiple optimization problems with equilibrium constraints) [1] and has the general form:

$$\min_{y_i \in Y_i} \theta_i(y, \pi), \forall i$$

solved in a Nash sense, coupled with the equilibrium constraints

$$0 \leq \mathcal{H}(y, \pi) \perp \pi \geq 0.$$

Within a modeling system, such problems are defined using standard constructs to specify θ_i and $\mathcal{H}$, and the structure of the model is given via an additional EMP [7] info file:

```
equilibrium
min theta(1) y(1) defY1
min theta(2) y(2) defY2
...
min theta(m) y(m) defYm
vi H pi
```

One approach (the default within GAMS) for solving this problem is to convert the MOPEC into a complementarity model, precisely as was carried out above, resulting in a standard mixed complementarity problem (4) for $\pi = \bar{\pi}$ and Definition 2(ii). This model can then be solved using the PATH solver [3, 9] for example.

The second approach also solves a MOPEC, but in this case it uses the results of Section 3 in which the variables x_i are eliminated from the problem to generate the model (9). We also modify the rare resource pricing relationship to reflect this elimination of variables, using the following complementarity relationship instead of Definition 2(ii):

$$\begin{aligned} &\pi \cdot (\Xi - Q(y)) = 0, \\ &\pi \geq 0,\ \Xi - Q(y) \geq 0. \end{aligned} \tag{16}$$

This modified set of optimization problems gives rise to a smaller MOPEC, for which the problem (9) is replaced by the conditions (10), generating a complementarity problem amenable for solution by PATH.

4.2 Solution via an MPEC

Consider the MPEC

$$
\begin{array}{ll}
\text{minimize} & \pi \cdot (\Xi - Q(y)) \\
\text{subject to} & \\
& 0 \in \begin{bmatrix} \nabla c_1(y_1) + \pi \nabla q_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) + \pi \nabla q_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + \bigtimes_{i=1}^{m} N_{A_i}(y_i) \\
& \pi \geq 0 \\
& \Xi - Q(y) \geq 0,
\end{array}
\tag{17}
$$

where π is the *control* and y is the *state* variable. Let $(\bar{\pi}, \bar{y})$ be a solution of (17) such that the corresponding (optimal) objective value vanishes. Then it is easy to see that the triple $(\bar{\pi}, \bar{y}, \bar{x})$ with $\bar{x}_i = q_i(\bar{y}_i) - e_i$ for all i is a CNW equilibrium. Indeed, the GE in (17) ensures condition (i) in Definition 2 and the complementarity relations

$$\pi \geq 0, \;\; \Xi - Q(y) \geq 0, \;\; \pi \cdot (\Xi - Q(y)) = 0$$

imply the satisfaction of condition (ii). Conversely, if $(\bar{\pi}, \bar{y})$ is a solution of GE (12), then it is an admissible pair with respect to the constraints in (17). Since the objective in (17) is nonnegative over all feasible pairs (π, y), the complementarity

$$\bar{\pi} \cdot (\Xi - Q(\bar{y})) = 0$$

implies that $(\bar{\pi}, \bar{y})$ is even a global optimizer in (17). So, instead of solving GE (12), one may attempt to compute global solutions of (17).

Note that the objective in (17) amounts to the so-called primal gap function which is frequently used in connection with complementarity problems, cf. [6]. Modeling systems like GAMS include model reformulation tools to convert MPEC (17) into a sequence of nonlinear programs (the NLPEC solver), or standard nonlinear programming solvers that can process these constraints directly (such as KNITRO or BARON). In one such reformulation, the original MPEC is replaced by a sequence of NLPs that depend on a certain regularization parameter t with the property that as $t \to 0$ their solutions approach the solution (or at least a stationary point of certain type) of the original problem.

Apart from these universal approaches, MPEC (17) can be solved also via the *implicit programming approach* (ImP) which proved its efficiency in a number of various MPECs. The terminology "ImP" has been coined in [12] and this approach has been deeply investigated e.g. in the monographs [15] and [19]. Since the application of ImP is closely connected with important properties of considered equilibria, we will analyze this approach now in a more detailed way.

Lemma 3 *The mapping S is single-valued and locally Lipschitz over $\mathbb{R}_+$.*

Proof Clearly, the single-valued part of GE (10) amounts to $H(\pi, y)$. By the arguments mentioned in the proof of Proposition 3 at any pair $(\pi, y) \in \mathbb{R}_+ \times \bigtimes_{i=1}^{m} A_i$ the (symmetrized) partial Jacobian of H with respect to y is positive definite. We deduce from this property (by virtue of [19, Theorem 5.8]) that GE (10) is strongly regular (in the sense of Robinson) at (π, y) and so S admits a single-valued and Lipschitz localization around (π, y), i.e., there are neighborhoods $\mathcal{U}$ of π, $\mathcal{V}$ of y and a Lipschitz single-valued mapping $\sigma[\mathcal{U} \mapsto \mathbb{R}^m]$ such that

$$y = \sigma(\pi) \text{ and } S(\alpha) \cap \mathcal{V} = \{\sigma(\alpha)\} \text{ for all } \alpha \in \mathcal{U}.$$

Further this property implies that, for $\pi \geq 0$, $H(\pi, \cdot)$ is strictly monotone over $\bigtimes_{i=1}^{m} A_i$ ([18, Theorem 5.4.3]) so that S is single-valued over $\mathbb{R}_+$ ([19, Theorem 4.4(i)]). A combination of these two facts implies the statement of the lemma.

Thanks to Lemma 3, MPEC (17) can be written down as the optimization problem

$$\begin{array}{ll} \text{minimize} & \pi \cdot (\Xi - Q \circ S(\pi)) \\ \text{subject to} & \\ & \pi \geq 0 \\ & \Xi - Q \circ S(\pi) \geq 0 \end{array} \tag{18}$$

in variable π only. Unfortunately, the last nonsmooth inequality corresponding to the state constraint in (17) is difficult to handle. In [22] the authors have suggested to augment inequality state constraints to the objective and this technique can be applied in case of (18) as well.

Theorem 2 *[22, Proposition 3] Let $(\bar{\pi}, \bar{y})$ be a solution of MPEC (17) and assume that the* perturbation mapping $\mathcal{M}[\mathbb{R} \rightrightarrows \mathbb{R}_+]$ *defined by*

$$\mathcal{M}(d) = \{\pi \in \mathbb{R}_+ | \Xi - Q \circ S(\pi) \geq d\}$$

is calm at $(0, \bar{\pi})$. Then there is a positive real R such that $\bar{\pi}$ is a solution of the (augmented or penalized) program

$$\begin{array}{ll} \textit{minimize} & \pi \cdot (\Xi - Q \circ S(\pi)) + R[(Q \circ S(\pi) - \Xi)_+] \\ \textit{subject to} & \\ & \pi \geq 0 \end{array} \tag{19}$$

in variable π only.

The verification of the above required calmness property is, however, a nontrivial task. One possibility consists in the application of a result from [13] and the interested reader will find the respective statement in the Appendix. Since the objective in (19) is locally Lipschitz, this optimization problem can be solved by a bundle method of nonsmooth optimization. These methods typically require the user to be able to compute at each $\pi \geq 0$ the corresponding value of the objective and one arbitrary element of its Clarke or limiting subdifferential. We will construct this element on the basis of the limiting coderivative of S and Proposition 1.

Proposition 4 *Let $\bar{y} = S(\bar{\pi})$. Under the posed assumptions for any $y^* \in \mathbb{R}^m$ one has*

$$D^*S(\bar{\pi},\bar{y})(y^*) \subset$$

$$\left\{ \langle \nabla Q(\bar{y}), u\rangle \,\middle|\, 0 \in y^* + (\nabla_y H(\bar{\pi},\bar{y}))^T u + \begin{bmatrix} D^*N_{A_1}(\bar{y}_1, -H_1(\bar{\pi},\bar{y}))(u_1) \\ \vdots \\ D^*N_{A_m}(\bar{y}_m, -H_m(\bar{\pi},\bar{y}))(u_m) \end{bmatrix} \right\}. \tag{20}$$

Proof It suffices to apply [16, Corollary 4.48] and to take into account that the respective qualification condition is fulfilled due to the Lemma 2 and [21, Proposition 3.2]. The coderivative $D^*N_{\times_{i=1}^m A_i}(\bar{y}, -H(\bar{\pi},\bar{y}))$ can be expressed in the form arising in (20) thanks to [24, Proposition 6.41].

Since all the sets A_i are non-degenerate intervals, the coderivatives $D^*N_{A_i}$ can easily be computed, cf. [20, Lemma 2.2] and [4, Section 4].

Specifically, as a subgradient of the objective in (19) we will use a vector ξ computed in the following two steps.

1° Given a pair $(\hat{\pi},\hat{y})$ with $\hat{y} = S(\hat{\pi})$ and a penalty parameter $R > 0$, solve the "adjoint" equation system

$$\begin{aligned} &0 \in y^* + (\nabla_y H(\hat{\pi},\hat{y}))^T u + w \\ &(w_i, -u_i) \in L_i, \quad i = 1,2,\ldots,m \end{aligned} \tag{21}$$

in variables u, w, where L_i are linear subspaces of the cones $N_{\mathrm{Gr}N_{A_i}}(\hat{y}_i, -H(\hat{\pi},\hat{y}))$ and

$$y^* = (-\hat{\pi} + v)\begin{bmatrix} \nabla q_1(\hat{y}_1) \\ \vdots \\ \nabla q_2(\hat{y}_2) \end{bmatrix} \text{ with } v = \begin{cases} 0 & \text{if } \ \Xi - Q(\hat{y}) \geq 0 \\ R & \text{otherwise;} \end{cases}$$

2° Put

$$\xi := \Xi - Q(\hat{y}) + \langle \nabla Q(\hat{y}), u\rangle.$$

System (21) allows us to compute a solution of the GE arising as the right-hand side of (20). Indeed, the cones $N_{\mathrm{Gr}N_{A_i}}$ are of a very simple nature ([20, Section 2], [4, Section 4]) and always contain suitable linear subspaces L_i. In this way the computation of ξ is simplified considerably. ξ is a correct subgradient whenever the sets $\mathrm{Gr}N_{A_i}$ are regular at $(\hat{y}_i, -H_i(\hat{\pi},\hat{y}))$ for all i ([24, Theorem 6.14]). The computational experience with the application of ImP to MPECs shows that during the iteration process we typically always meet this type of regularity. But even an occasional occurrence of a point where the subgradient information could be incorrect does not usually cause the bundle method to collapse.

To summarize, the ImP approach essentially performs a decomposition of GE (12) with respect to variables π and y. This decomposition is efficient,

Table 1 Parameter specification

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
q_i	1.63	1.5	1.48	1.5	1.4
c_i	10	8	6	4	2
K_i	5	5	5	5	5
β_i	1.2	1.1	1.0	0.9	0.8

provided we use a fast method for the solution of GE (10) for a fixed π. In problems of low dimension, however, such a decomposition is not needed and one can compute CNW equilibria easily by the methods described in Section 4.1.

The numerical results, obtained via the above described techniques, are presented in the next section.

5 Numerical results

In this section, to test the performance of the solution techniques proposed in the previous section we modify correspondingly the oligopolistic market example from [17], see also [19, Section 12.1] and [22, Example 3].

Consider an example of five producers/firms supplying a quantity $y_i \in \mathbb{R}_+, i = 1, \ldots, 5$, of some homogeneous product on the market with the inverse demand function

$$p(T) = 5000^{\frac{1}{\gamma}} T^{-\frac{1}{\gamma}},$$

where γ is a positive parameter termed *demand elasticity.*

Let the production cost functions be of the form

$$c_i(y_i) = c_i y_i + \frac{\beta_i}{1+\beta_i} K_i^{-\frac{1}{\beta_i}} (y_i)^{\frac{1+\beta_i}{\beta_i}},$$

where c_i, K_i and $\beta_i,\ i = 1, \ldots, 5$, are positive parameters. Suppose the technological functions q_i are linear and in the form $q_i(y_i) = q_i y_i, i = 1, \ldots, 5$. Table 1 specifies values of parameters q_i, c_i, K_i and $\beta_i, i = 1, \ldots, 5$. Further, let the demand elasticity $\gamma = 1.3$, assume initial endowments of the rare resource $e_i = 25$ for each firm i, $i = 1, \ldots, 5$, and consider production bounds $A_i = [0, 30], i = 1, \ldots, 4$ and $A_5 = [1, 30]$.

Each production cost function is convex and twice continuously differentiable on some open set containing the feasible set of strategies of a corresponding player. The inverse demand curve is twice continuously differentiable on int $\mathbb{R}_+$, strictly decreasing, and convex. Observe that the so-called *industry revenue curve*

$$Tp(T) = 5000^{\frac{1}{\gamma}} T^{\frac{\gamma-1}{\gamma}}$$

is concave on int $\mathbb{R}_+$ for $\gamma \geq 1$. Thus, all assumptions A1 - A7 are satisfied.

For the numerical tests, we have first considered a situation where there is no additional amount of the rare resource available in the market, i.e. $\varXi = 125$, (case A). Furthermore, we will consider the following modifications of this example:

- An additional 10 units of the rare resource is available (case B);
- Firm 1 has better production technology such that $c_1 = 5$ (case C);
- upper bounds on production are increased to 35 and initial endowments of the rare resource are increased to 45 (case D)
- upper bounds on production are lowered to 23 (case E);
- Producer 1 has worse technological function such that $q_1 = 4$ (case F);
- Producer 1 has worse technological function such that $q_1 = 4$ and upper bounds on production are lowered to 23 (case G).

The following results were obtained by all the methods outlined in the previous section.

For all the GAMS models, the solution times for each model are less than 0.1 seconds on a standard laptop, and solutions are found to default tolerances on each of the solvers. The PATH solver finds the solutions of all the complementarity problems generated by EMP in no more than 5 (pure) Newton steps. Both KNITRO and NLPEC solve the MPEC models easily, NLPEC using CONOPT as the NLP subsolver. If we use a non-default option file for NLPEC, namely

```
initmu 1
finalmu 1e-6
```

then both CONOPT and IPOPT can solve the sequence (of 2) nonlinear programs (where the orthogonality condition is relaxed to be no greater than μ for the two given values) and thus solve (to high accuracy) each MPEC generated. We believe this quite conclusively demonstrates that each of these problems is very easy for standard off-the-shelf complementarity solvers.

Analogously to [19] and [22], to obtain numerical results via ImP, we have applied the Bundle-Trustregion code BT described in [27]. In order to solve the generalized equation (10), we have used an idea of Smith [28] and solved the resulting convex optimization problems via the sequential quadratic programming code NLPQL [25]. All ImP results have been computed in Fortran 77 with the accuracy $\epsilon = 1.0 \times 10^{-12}$ in BT and $\epsilon = 1.0 \times 10^{-14}$ in the sequential quadratic programming code NLPQL and the starting point has been set to 50 for variable π and upper production bounds for variables $y_i, i = 1, \ldots, 5$.

In the light of recent results from [14], we have applied also a MATLAB implementation of the regularization method by Scholtes [26] to the same example as above.

In Table 2 we present the productions, profits and purchased rare resource of each firm along with price of the rare resource and value of the objective of (18) in each of the above specified variants of the example. Since the obtained results from each of the method applied differ from those reported in Table 2 only up to chosen computational accuracy, we do not report them in a separate tables.

Table 2 Production, profits and purchased rare resources

		Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
case A	$\pi = 6.375$					
	production	6.651	14.018	18.600	21.347	23.988
	profit	172.491	217.617	266.497	311.268	374.633
	purchased rare resource	-14.159	-3.973	2.528	7.021	8.584
case B	$\pi = 5.437$					
	production	7.824	15.371	20.094	22.837	25.139
	profit	152.309	199.547	250.310	296.515	356.919
	purchased rare resource	-12.246	-1.944	4.739	9.256	10.195
case C	$\pi = 7.323$					
	production	16.789	10.711	15.545	18.607	21.894
	profit	264.428	217.855	259.032	299.450	362.854
	purchased rare resource	2.366	-8.934	-1.993	2.910	5.651
case D	$\pi = 0$					
	production	21.218	28.081	32.345	33.790	32.664
	profit	67.210	125.581	186.056	237.492	272.578
	purchased rare resource	-6.407	-2.878	2.870	5.685	0.729
case E	$\pi = 6.324$					
	production	6.919	14.256	18.811	21.531	23.000
	profit	172.283	218.386	267.803	312.850	370.267
	purchased rare resource	-13.722	-3.615	2.841	7.297	7.200
case F	$\pi = 5.764$					
	production	0	16.215	20.608	23.132	25.342
	profit	144.097	220.921	274.314	321.432	383.849
	purchased rare resource	-25.000	-0.677	5.500	9.699	10.479
case G	$\pi = 5.473$					
	production	0	17.448	21.708	23.000	23.000
	profit	136.816	225.780	281.586	324.237	373.337
	purchased rare resource	-25.000	1.172	7.128	9.500	7.200

Note that case D corresponds to a situation when the available amount of rare resources exceeds the interests of the market, in particular $\sum_{i=1}^{5}(e_i+x_i) = \Xi - 4.008$, and thus this price vanishes. The firms 1 and 2 would like to sell a certain amount of the rare resource but since the remaining firms are not willing to buy this whole amount for any positive price, the resulting $\pi = 0$. Cases E, F and G illuminate the effects of attaining the lower or upper production bounds.

6 Local stability of CNW equilibria

Consider the multifunction $\Psi[\mathbb{R}^{m+1} \rightrightarrows \mathbb{R}^{m+1}]$ defined by

$$\Psi(\pi, y) := \begin{bmatrix} \Xi - Q(y) \\ \nabla c_1(y_1) + \pi \nabla q_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) + \pi \nabla q_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + N_{\mathbb{R}_+ \times \bigtimes_{i=1}^{m} A_i}(\pi, y), \tag{22}$$

and denote by Σ its inverse. Clearly, for $(u, v) \in \mathbb{R} \times \mathbb{R}^m, \Sigma(u, v)$ amounts to the set of solutions to GE (12), where $(0, 0) \in \mathbb{R} \times \mathbb{R}^m$ on the left-hand side is replaced by (u, v). In this section we will examine local behavior of Σ around the reference point $(0_{\mathbb{R}^{m+1}}, \bar{\pi}, \bar{y})$ where $\bar{\pi} \geq 0$ and $\bar{y} = S(\bar{\pi})$. More precisely, we will concentrate on the strong metric regularity and subregularity of Ψ at $(\bar{\pi}, \bar{y}, 0_{\mathbb{R}^{m+1}})$, which imply valuable local stability properties of Σ with important consequences. A multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^l]$ is called *strongly metrically regular* at $(\bar{b}, \bar{a}) \in \operatorname{Gr} \Phi$, provided Φ^{-1} has a Lipschitz single-valued localization s around $(\bar{a}, \bar{b})$, i.e., there are neighborhoods $\mathcal{U}$ of $\bar{a}$, $\mathcal{V}$ of $\bar{b}$ and a Lipschitz single valued mapping $s[\mathcal{U} \to \mathbb{R}^m]$ such that

$$\bar{b} = s(\bar{a}) \text{ and } \Phi^{-1}(a) \cap \mathcal{V} = \{s(a)\} \text{ for all } a \in \mathcal{U}.$$

Theorem 3 *Let $(\bar{\pi}, \bar{y})$ be a solution of GE (12) and assume that $\bar{\pi} > 0$ and $\bar{y}_i \in \operatorname{int} A_i$ for at least one $i \in \{1, 2, \ldots, m\}$. Then Ψ is strongly metrically regular at $(\bar{\pi}, \bar{y}, 0_{\mathbb{R}^{m+1}})$, i.e., Σ has a Lipschitz single-valued localization around $(0_{\mathbb{R}^{m+1}}, \bar{\pi}, \bar{y})$.*

Proof By combining the results in [5, Theorem 3G4] and [4, Theorem 1], and applying the Mordukhovich criterion to ensure the metric regularity of Ψ at $(\bar{\pi}, \bar{y}, 0_{\mathbb{R}^{m+1}})$ [16, Corollary 4.61], it suffices to prove that the GE

$$0 \in (D(\bar{\pi}, \bar{y}))^T \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_m \end{bmatrix} + \begin{bmatrix} D^* N_{\mathbb{R}_+}(\bar{\pi}, P_0(\bar{\pi}, \bar{y}))(z_0) \\ D^* N_{A_1}(\bar{y}_1, P_1(\bar{\pi}, \bar{y}))(z_1) \\ \vdots \\ D^* N_{A_m}(\bar{y}_m, P_m(\bar{\pi}, \bar{y}))(z_m) \end{bmatrix} \tag{23}$$

has only the trivial solution $(z_0, z_1, \ldots, z_m) = 0$. In (23), $P(\pi, y) := (P_0(\pi, y), P_1(\pi, y), \ldots, P_m(\pi, y))$ denotes the single-valued mapping on the right-hand side of (22). It follows from $\bar{\pi} > 0$ that the first component of the multi-valued part of (23) vanishes so that, with $\tilde{z} := (z_1, \ldots, z_m)$, GE (23) amounts to the system

$$0 = \langle \nabla Q(\bar{y}), \tilde{z} \rangle \tag{24}$$

$$0 \in \nabla Q(\bar{y}) z_0 + (\nabla_y H(\bar{\pi}, \bar{y}))^T \tilde{z} + D^* N_{\times_{i=1}^m A_i}(\bar{y}, -\tilde{P}(\bar{\pi}, \bar{y}))(\tilde{z}), \tag{25}$$

where $\tilde{P}(\pi, y) = (P_1(\pi, y), \ldots, P_m(\pi, y))$.

Premultiplying GE (25) by $\tilde{z}^T$, we obtain that

$$0 = \langle \tilde{z}, \nabla Q(\bar{y}) z_0 \rangle + \langle \nabla_y H(\bar{\pi}, \bar{y}) \tilde{z}, \tilde{z} \rangle + \langle \tilde{z}, d \rangle \tag{26}$$

$$d \in D^* N_{\times_{i=1}^m A_i}(\bar{y}, -\tilde{P}(\bar{\pi}, \bar{y}))(\tilde{z}).$$

The first term on the right-hand side of (26) amounts to zero due to (24). Further we note that $\langle \tilde{z}, d \rangle \geq 0$ which follows from the well-known result in [23, Theorem 2.1] because of the maximal monotonicity of the normal-cone mapping to a convex set. Since $\nabla_y H(\bar{\pi}, \bar{y})$ is positive definite by virtue of [19, Lemma 12.2] and by assumption A5, we conclude that $\tilde{z} = 0$ and (25) reduces thus to

$$0 = \nabla Q(\bar{y})z_0 + \sum_{i=1}^{m} D^* N_{A_i}(\bar{y}_i, -\tilde{P}_i(\bar{\pi}, \bar{y}))(0).$$

By the assumption there is an index $i_0 \in \{1, 2, \ldots, m\}$ such that $\bar{y}_{i_0} \in \text{int}A_{i_0}$ and, consequently, $\tilde{P}_{i_0}(\bar{\pi}, \bar{y}) = 0$. It follows that $D^* N_{A_{i_0}}(\bar{y}_{i_0}, -\tilde{P}_i(\bar{\pi}, \bar{y}))(0) = \{0\}$ as well and, since $\nabla q_0(\bar{y}_{i_0}) > 0$, one has that $z_0 = 0$. The statement has been established.

If $\bar{\pi} = 0$, then Ψ enjoys a somewhat weaker stability property. A multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^l]$ is called *strongly metrically subregular* at $(\bar{b}, \bar{a}) \in \text{Gr}\, \Phi$, provided Φ^{-1} has the *isolated calmness property* at $(\bar{a}, \bar{b})$, i.e., there are neighborhoods $\mathcal{U}$ of $\bar{a}$, $\mathcal{V}$ of $\bar{b}$ and a modulus $\ell \geq 0$ such that

$$\Phi^{-1}(a) \cap \mathcal{V} \subset \{\bar{b}\} + \ell ||a - \bar{a}|| \mathbb{B}_{\mathbb{R}^n} \text{ for all } a \in \mathcal{U}.$$

Theorem 4 *Let $(\bar{\pi}, \bar{y})$ be a solution of GE (12) and assume that $\bar{\pi} = 0$ and either $\Xi - Q(\bar{y}) > 0$ or $\bar{y}_i$ does not attain its production bounds for at least one $i \in \{1, 2, \ldots, m\}$. Then Ψ is strongly metrically subregular at $(\bar{\pi}, \bar{y}, 0_{\mathbb{R}^{m+1}})$, i.e., Σ has the isolated calmness property at $(0_{\mathbb{R}^{m+1}}, \bar{\pi}, \bar{y})$.*

Proof By applying the criterion from [4, Theorem 4E.1] it suffices to prove that the GE

$$0 \in D(\bar{\pi}, \bar{y}) \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_m \end{bmatrix} + \begin{bmatrix} N_{K_0}(z_0) \\ N_{K_1}(z_1) \\ \vdots \\ N_{K_m}(z_m) \end{bmatrix} \tag{27}$$

has only the trivial solution $(z_0, \tilde{z}) := (z_0, z_1, \ldots, z_m) = 0$. In (27),

$$K_0 = T_{\mathbb{R}_+}(0) \cap \{P_0(\bar{\pi}, \bar{y})\}^\perp, K_i = T_{A_i}(\bar{y}_i) \cap \{P_i(\bar{\pi}, \bar{y})\}^\perp, i = 1, \ldots, m.$$

If $\Xi - Q(\bar{y}) > 0$, then $K_0 = \{0\}$ and so (27) amounts to the GE

$$0 \in \nabla_y H(\bar{\pi}, \bar{y})\tilde{z} + \sum_{i=1}^{m} N_{K_i}(z_i). \tag{28}$$

Premultiplying GE (28) by $\tilde{z}^\top$, we obtain that

$$\langle \tilde{z}, \nabla_y H(\bar{\pi}, \bar{y})\tilde{z} \rangle = 0,$$

because for all z_i and $d_i \in N_{K_i}(z_i), i = 1, \ldots, m$, one has $\langle z_i, d_i \rangle = 0$. Since $\nabla_y H(\bar{\pi}, \bar{y}) = \nabla G(\bar{y})$ is positive definite ([19, Lemma 12.2]), we conclude that $\tilde{z} = 0$ and the statement holds true.

If $\Xi - Q(\bar{y}) = 0$, then $K_0 = \mathbb{R}_+$. If $z_0 = 0$, we can proceed exactly as in the preceding case. So, let us assume that $z_0 > 0$. GE (27) amounts then to the system

$$\begin{aligned} \langle \nabla Q(\bar{y}), \tilde{z} \rangle &= 0 \\ 0 \in \nabla Q(\bar{y})z_0 + \nabla_y H(\bar{\pi}, \bar{y})\tilde{z} &+ \sum_{i=1}^{m} N_{K_i}(z_i). \end{aligned} \tag{29}$$

By the same argumentation as in the proof of the preceding case we detect that $\tilde{z}$ must vanish so that the 2nd line in (29) reduces to

$$0 \in \nabla Q(\bar{y}) z_0 + \sum_{i=1}^{m} N_{K_i}(0).$$

Now it follows from the posed assumption that $K_i = \mathbb{R}$ for some i and, consequently,

$$\nabla q_i(\bar{y}) z_0 = 0.$$

By virtue of A5 this contradicts the positivity of z_0 and so the statement has been established.

It follows from Theorem 3 combined with [4, Theorem 3G.4] and directly from Theorem 4 that the above specified Lipschitzian behavior of Σ at $(0, 0, \bar{\pi}, \bar{y})$ is inherited by all mappings which assign (π, y) to any scalar- or vector-valued parameter on which P depends in a continuously differentiable way. This could be, e.g., Ξ or any parameter arising in the functions p, c_i or q_i.

This fact, together with Proposition 3, implies that these mappings are in fact truly single-valued (not only locally single-valued) as long as we consider CNW equilibria with $\pi > 0$ and at least one y_i not attaining the respective production bounds. This enables the authority, controlling the rare resources, for instance, to optimize the choice of Ξ via the MPEC

$$\begin{array}{ll} \text{minimize} & \mathcal{J}(\Xi, \pi, y) \\ \text{subject to} & \text{GE (12)} \end{array} \tag{30}$$

by using the ImP approach described in Section 4. By a suitable choice of the objective $\mathcal{J}$ this authority may thus correlate the choice of Ξ with the corresponding price π and production y. The solution of (30) goes, however, beyond the scope of this paper.

Acknowledgement

The authors would like to express their gratitude to S.D. Flåm for a helpful discussion about the considered model and to A. Schwartz for providing her MATLAB codes implementing a family of relaxation methods for solving MPECs.

References

1. W. Britz, M.C. Ferris and A. Kuhn: Modeling Water Allocating Institutions based on Multiple Optimization Problems with Equilibrium Constraints. Environmental Modelling and Software, 46, 196–207 (2013)
2. A. Brooke, D. Kendrick and A. Meeraus: GAMS: A User's Guide, The Scientific Press, South San Francisco, California (1988)
3. S.P. Dirkse and M.C. Ferris: The PATH Solver: A Non-Monotone Stabilization Scheme for Mixed Complementarity Problems, Optimization Methods and Software, 5, 123–156 (1995)

4. A.L. Dontchev and R.T. Rockafellar: Characterizations of strong regularity for variational inequalities over polyhedral convex sets, SIAM J. Optimization, 7, 1087-1105 (1996)
5. A.L. Dontchev and R.T. Rockafellar: Implicit Functions and Solution Mappings, Springer (2009)
6. F. Facchinei and J.-S. Pang: Finite-Dimensional Variational Inequalaities and Complementarity Problems, Springer, New York (2003)
7. M.C. Ferris, S.P. Dirkse, J.-H. Jagla and A. Meeraus: An Extended Mathematical Programming Framework, Computers and Chemical Engineering, 33, 1973–1982 (2009)
8. M.C. Ferris, R. Fourer and D.M. Gay: Expressing Complementarity Problems and Communicating them to Solvers, SIAM Journal on Optimization, 9, 991–1009 (1999)
9. M.C. Ferris and T.S. Munson: Interfaces to PATH 3.0: Design, Implementation and Usage, Computational Optimization and Applications, 12, 207–227 (1999)
10. S.D. Flåm: Noncooperative games, coupling constraints and partial efficiency. Preprint, 2013.
11. S.D. Flåm: Reaching equilibrium in emission and resource markets. Preprint, 2013.
12. P.T. Harker and J.-S. Pang: On the existence of optimal solutions to mathematical program with equilibrium constraints, Oper. Res. Letters, 7, 61–64, (1988)
13. A.D. Ioffe and J.V. Outrata: On metric and calmness qualification conditions in subdifferential calculus, Set-Valued Anal, 16, 199–227 (2008)
14. Ch. Kanzow and A. Schwartz: The price of inexactness: convergence properties of relaxation methods for mathematical programs with equilibrium constraints revisited, to appear in Math. Oper. Res. 2014.
15. Z.Q.Luo, J.-S. Pang and D. Ralph: Mathematical Programs with Equilibrium Costraints, Cambridge Univ. Press, Cambridge (1996)
16. B.S. Mordukhovich: Variational Analysis and Generalized Differentiation. Basic Theory, vol.1, Springer, Berlin (2006)
17. F.H. Murphy, H,D. Sherali and A.L. Soyster: A mathematical programming approach for determining oligopolistic market equilibrium. Mathematical Programming, 24, 92–106 (1982)
18. J.M. Ortega and W.C. Rheinbolt: Iterative Solutions of Nonlinear Equations in Several Variables, Acad. Press, New York (1970)
19. J.V. Outrata, M. Kočvara and J. Zowe: Nonsmooth Approach to Optimization Problems with Equilibrium Constraints, Kluwer Academic Publishers, Dordrecht (1998)
20. J.V. Outrata: Optimality conditions for a class of mathematical programs with equilibrium constraints, Math. Oper. Res, 24, 627–644 (1999)
21. J.V. Outrata: Generalized mathematical program with equilibrium constraints, SIAM J. Control Optim, 38, 1623–1638 (2000)
22. J.V. Outrata and M. Červinka: On the implicit programming approach in a class of mathematical programs with equilibrium constraints, Control and Cybernetics, 38, 1557–1574 (2009)
23. R.A. Poliquin and R.T. Rockafellar: Tilt stability of a local minimum, SIAM J. Optim, 8, 287–299 (1998)
24. R.T. Rockafellar and R. J.-B. Wets: Variational Analysis, Springer, Berlin (1998)
25. K. Schittkowski: NLPQL: A Fortran subroutine solving constrained nonlinear programming problems, Annals of Oper. Res, 5, 485–500 (1985)
26. S. Scholtes: Convergence properties of a regularization scheme for mathematical programs with complementarity constraints, SIAM J. Optim, 11, 918–936 (2001)
27. H. Schramm and J. Zowe: A version of the bundle idea for minimizing a nonsmooth function: Conceptual idea, convergence analysis, numerical results, SIAM J. Optim, 2, 121–152 (1992)
28. M.J. Smith: A decent algorithm for solving monotone variational inequalities and monotone complementarity problems, Journal of Optimization Theory and Applications, 44, 485–496 (1984)

Appendix: Calmness of $\mathcal{M}$ at $(0, \bar{\pi})$

In the proof of Theorem 5 below, a crucial role is played by the following notion.

Definition 3 Let $f[\mathbb{R}^n \to \bar{\mathbb{R}}]$ be lsc and $\bar{x} \in \mathrm{dom} f$. The set

$$\partial^{>} f(\bar{x}) := \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ f(x) \to f(\bar{x}) \\ f(x) > f(\bar{x})}} \hat{\partial} f(x)$$

is called the *limiting outer subdifferential* of f at $\bar{x}$.

On the basis of the theory from [13] we arrive now at the next statement.

Theorem 5 *Assume that* $\bar{y} = S(\bar{\pi})$ *and*

$$0 \notin D^* S(\bar{\pi}, \bar{y})(\nabla Q(\bar{y})). \tag{31}$$

Then $\mathcal{M}$ *is calm at* $(0, \bar{\pi})$.

Proof It is well-known that the calmness of $\mathcal{M}$ at $(0, \bar{\pi})$ is equivalent with the *local error bound property* of the function

$$\beta(\pi) := \begin{cases} (Q \circ S(\pi) - \Xi)_+ & \text{if } \pi \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

at $\bar{\pi}$. This means that for $\Lambda := \{\pi \in \mathbb{R} | \beta(\pi) = 0\}$ there is a real $c > 0$ such that

$$\mathrm{dist}_\Lambda(\pi) \leq c\beta(\pi) \quad \text{for all} \quad \pi \geq 0 \quad \text{close to} \quad \bar{\pi}.$$

By virtue of [13, Theorem 2.1] the latter property is ensured by the subdifferential condition

$$0 \notin \partial^{>} \beta(\bar{\pi})$$

and so it remains to show that $\partial^{>} \beta(\bar{\pi})$ is included in the set on the right-hand side of (31). Since $\beta(\bar{\pi}) = 0$ and $\beta(\pi) \to \beta(\bar{\pi})$ for $\pi \xrightarrow{\mathbb{R}_+} \bar{\pi}$ by Lemma 3, one has that

$$\partial^{>} \beta(\bar{\pi}) = \operatorname*{Lim\,sup}_{\substack{\pi \xrightarrow{\mathbb{R}_+} \bar{\pi} \\ \beta(\pi) > 0}} \hat{\partial} \beta(\pi) = \operatorname*{Lim\,sup}_{\substack{\pi \xrightarrow{\mathbb{R}_+} \bar{\pi} \\ \beta(\pi) > 0}} \partial \beta(\pi).$$

If $\beta(\pi) > 0$ then β is continuously differentiable near π and according to Proposition 1

$$\partial \beta(\pi) = D^* S(\pi, y) \left(\begin{bmatrix} \nabla q_1(y_1) \\ \vdots \\ \nabla q_m(y_m) \end{bmatrix} \right) = D^* S(\pi, y)(\nabla Q(y))$$

with $y = S(\pi)$. Consider now an arbitrary sequence $\{\pi_i\} \subset \mathbb{R}_+$ converging to $\bar{\pi}$. Since $y_i = S(\pi_i)$ converges to $\bar{y} = S(\bar{\pi})$ and $\nabla Q(y_i)$ converges to $\nabla Q(\bar{y}) = (\nabla q_1(\bar{y}_1), \ldots, \nabla Q_m(\bar{y}_m))^T$, it follows that

$$\operatorname*{Lim\,sup}_{i \to \infty} D^* S(\pi_i, y_i)(\nabla Q(y_i)) \subset D^* S(\bar{\pi}, \bar{y})(\nabla Q(\bar{y}))$$

by the definition of the limiting coderivative and we are done.

Taking into account Proposition 4, we may easily conclude that the calmness of $\mathcal{M}$ at $(0, \bar{\pi})$ is ensured by the implication

$$0 = \begin{bmatrix} \nabla q_1(\bar{y}_1) \\ \vdots \\ \nabla q_m(\bar{y}_m) \end{bmatrix} + (\nabla_y H(\bar{\pi}, \bar{y}))^T u \;\Rightarrow\; \langle \nabla Q(\bar{y}), u \rangle \neq 0, \tag{32}$$

whenever $\bar{y}_i \in (a_i, b_i)$ for all i. However, condition (32) holds true by virtue of A5 and the positive definiteness of $\nabla_y H(\bar{\pi}, \bar{y})$. This is, of course, just a particular case when the calmness of $\mathcal{M}$ at $(0, \bar{\pi})$ can easily be detected on the basis of Theorem 5.