New Formulation and Strong MISOCP Relaxations for AC Optimal Transmission Switching Problem

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Abstract

As the modern transmission control and relay technologies evolve, transmission line switching has become an important option in power system operators' toolkits to reduce operational cost and improve system reliability. Most recent research has relied on the DC approximation of the power flow model in the optimal transmission switching problem. However, it is known that DC approximation may lead to inaccurate flow solutions and also overlook stability issues. In this paper, we focus on the optimal transmission switching problem with the full AC power flow model, abbreviated as AC OTS. We propose a new exact formulation for AC OTS and its mixed-integer second-order conic programming (MISOCP) relaxation. We improve this relaxation via several types of strong valid inequalities inspired by the recent development for the closely related AC Optimal Power Flow (AC OPF) problem [27]. We also propose a practical algorithm to obtain high quality feasible solutions for the AC OTS problem. Extensive computational experiments show that the proposed formulation and algorithms efficiently solve IEEE standard and congested instances and lead to significant cost benefits with provably tight bounds.

1 Introduction

Transmission switching, as an emerging operational scheme, has gained considerable attention in both industry and academia in the recent years [38, 12, 16, 26, 17]. Switching on and off transmission lines, therefore, changing the network topology in the real-time operation, may bring several benefits that the traditional economic dispatch cannot offer, such as reducing the total operational cost[12, 15, 14], mitigating transmission congestion[43], clearing contingencies[25, 30], and improving do-not-exceed limits[31] .

Previous literature on OTS mainly relies on the DC approximation of the power flow model to avoid the mathematical complexity induced by the non-convexity of AC power flow equations (see e.g. [38, 12, 39, 40]). This DC version of the OTS problem can be modeled as a mixed-integer linear program (MILP), which is a computationally challenging problem and several heuristic method are proposed [4, 13, 47]. In a recent work [29], the authors propose a new formulation and a class of valid inequalities to exactly solve the MILP problem. However, even if this problem can be solved quickly, it has been recognized that the optimal topology obtained by solving DC transmission switching is not guaranteed to be AC feasible, also it may over-estimate cost improvements and overlook stability issues [18].

The AC optimal transmission switching problem (AC OTS) is much less explored. In [18], a convex relaxation of AC OTS is proposed based on trigonometric outer-approximation. The problem is formulated as a mixed integer nonlinear program (MINLP) and solved using the solver BONMIN to obtain upper bounds. In [42], a new ranking heuristic is proposed based on the economic dispatch solutions and the corresponding dual variables. In [5], DC OTS is solved first and then a heuristic correction mechanism is utilized to restore AC feasibility of the solutions. In this paper, we aim to push the control scheme for transmission switching closer to the real-world power system operation by proposing a new exact formulation and an efficient algorithm for the AC OTS problem.

There are several closely related problems in the literature, which involve line switching decisions, such as the network configuration problem [23, 11], transmission system planning [22], and intentional islanding [44]. The main ideas of these works are based on conic relaxations or piecewise linear approximations of the non-convex power flow equations.

Our study starts from the recent advances in a related fundamental problem in power system analysis, namely the AC Optimal Power Flow (AC OPF) problem, which minimizes the generation cost to satisfy load and various physical constraints represented in the AC power flow constraints, while the power network topology is kept unchanged. It is demonstrated by several authors that convex relaxations, especially semidefinite programming (SDP) relaxations, of the AC OPF problem provide tight lower bounds on standard IEEE test instances [3, 32, 35, 36]. However, the computational burden of solving large-scale SDP relaxations is still unwieldy. To solve for large-scale systems, one may need to turn to computationally less demanding alternatives such as quadratic convex [9, 18, 8] or linear programming relaxations [6].

In a recent work [27], we proposed several strong second-order cone programming (SOCP) relaxations for AC OPF, which produce extremely high quality feasible AC solutions (not dominated by the SDP relaxations) in a time that is an order of magnitude faster than solving the SDP relaxations. In this paper, we extend these new techniques to the more

challenging AC OTS problem. In particular, we first formulate the AC OTS problem as an MINLP problem. Then, we propose a mixed-integer second-order cone programming (MISOCP) relaxation, which relaxes the non-convex AC power flow constraints to a set of convex quadratic constraints, represented in the form of SOCP constraints. The paper then provides several techniques to strengthen this MISOCP relaxation by adding several types of valid inequalities. Some of these valid inequalities have demonstrated to have excellent performance for the AC OPF in [27], and some others are specifically developed for the AC OTS problem. Finally, we also propose practical algorithms that utilize the solutions from the MISOCP relaxation to obtain high quality feasible solutions for the AC OTS problem.

The rest of the paper is organized as follows: In Section 2 we formally define AC OPF and present two exact formulations. In Section 3, we present AC OTS as an MINLP problem and discuss its MISOCP relaxation. Then, we propose several valid inequalities in Section 4 and develop a practical algorithm to solve AC OTS in Section 5. We present the results of our extensive computational experiments in Section 6. Finally, some concluding remarks are given in Section 7.

2 AC Optimal Power Flow

Consider a power network $\mathcal{N} = (\mathcal{B}, \mathcal{L})$, where $\mathcal{B}$ and $\mathcal{L}$ respectively denote the set of buses and transmission lines. Generation units are connected to a subset of buses, denoted as $\mathcal{G} \subseteq \mathcal{B}$. The aim of the AC optimal power flow (OPF) problem is to satisfy demand at all buses with the minimum total production costs of generators such that the solution obeys the physical laws (e.g., Ohm's and Kirchoff's Law) and other operational restrictions (e.g., transmission line flow limit constraints).

Let $Y \in \mathbb{C}^{|\mathcal{B}| \times |\mathcal{B}|}$ denote the nodal admittance matrix, which has components $Y_{ij} = G_{ij} + \mathrm{i}B_{ij}$ for each line $(i,j) \in \mathcal{L}$, and $G_{ii} = g_{ii} - \sum_{j \neq i} G_{ij}, B_{ii} = b_{ii} - \sum_{j \neq i} B_{ij}$, where g_{ii} (resp. b_{ii}) is the shunt conductance (resp. susceptance) at bus $i \in \mathcal{B}$ and $\mathrm{i} = \sqrt{-1}$. Let p_i^g, q_i^g (resp. p_i^d, q_i^d) be the real and reactive power output of the generator (resp. load) at bus i. The complex voltage V_i at bus i can be expressed either in the rectangular form as $V_i = e_i + \mathrm{i}f_i$ or in the polar form as $V_i = |V_i|(\cos\theta_i + \mathrm{i}\sin\theta_i)$, where $|V_i| = \sqrt{e_i^2 + f_i^2}$ is the voltage magnitude and θ_i is the phase angle. Real and reactive power on line (i,j) are denoted by p_{ij} and q_{ij}, respectively and computed as follows:

$$
\begin{aligned}
p_{ij} &= -G_{ij}(e_i^2 + f_i^2) + G_{ij}(e_i e_j + f_i f_j) - B_{ij}(e_i f_j - e_j f_i) \\
q_{ij} &= B_{ij}(e_i^2 + f_i^2) - B_{ij}(e_i e_j + f_i f_j) - G_{ij}(e_i f_j - e_j f_i).
\end{aligned}
\tag{1}
$$

With the above notation, the AC OPF problem is given in the so-called rectangular formulation as follows:

$$\min \ \sum_{i \in \mathcal{G}} C_i(p_i^g) \tag{2a}$$

$$\text{s.t. } p_i^g - p_i^d = g_{ii}(e_i^2 + f_i^2) + \sum_{j \in \delta(i)} p_{ij} \qquad i \in \mathcal{B} \tag{2b}$$

$$q_i^g - q_i^d = -b_{ii}(e_i^2 + f_i^2) + \sum_{j \in \delta(i)} q_{ij} \qquad i \in \mathcal{B} \tag{2c}$$

$$\underline{V}_i^2 \le e_i^2 + f_i^2 \le \overline{V}_i^2 \qquad i \in \mathcal{B} \tag{2d}$$

$$p_{ij}^2 + q_{ij}^2 \le (S_{ij}^{\max})^2 \qquad (i,j) \in \mathcal{L} \tag{2e}$$

$$p_i^{\min} \le p_i^g \le p_i^{\max} \qquad i \in \mathcal{G} \tag{2f}$$

$$q_i^{\min} \le q_i^g \le q_i^{\max} \qquad i \in \mathcal{G}, \tag{2g}$$

$$(1).$$

The objective function $C_i(p_i^g)$ is typically linear or convex quadratic in the real power output p_i^g of generator i. Constraints (2b) and (2c) correspond to the conservation of active and reactive power flows at each bus, respectively. Here, $\delta(i)$ denotes the set of neighbor buses of bus i. Constraint (2d) restricts voltage magnitude at each bus. Constraint (2e) puts an upper bound on the apparent power on each line. Finally, constraints (2f) and (2g), respectively, limit the active and reactive power output of each generator to respect its physical capability.

Note that the rectangular formulation (2) is a non-convex quadratic optimization problem. However, we note that all the nonlinearity and non-convexity comes from one of the following three forms: (1) $e_i^2 + f_i^2 = |V_i|^2$, (2) $e_i e_j + f_i f_j = |V_i||V_j| \cos(\theta_i - \theta_j)$, (3) $e_i f_j - f_i e_j = -|V_i||V_j| \sin(\theta_i - \theta_j)$. We define new variables c_{ii}, c_{ij} and s_{ij} for each bus i and each transmission line (i,j) to capture the non-convexity. In particular, we define for each $i \in \mathcal{B}$ and $(i,j) \in \mathcal{L}$,

$$c_{ii} := e_i^2 + f_i^2, \quad c_{ij} := e_i e_j + f_i f_j, \quad s_{ij} := e_i f_j - e_j f_i. \tag{3}$$

Now, we introduce an equivalent, alternative formulation of the OPF problem as follows:

$$\min \ \sum_{i \in \mathcal{G}} C_i(p_i^g) \tag{4a}$$

$$\text{s.t. } p_i^g - p_i^d = g_{ii} c_{ii} + \sum_{j \in \delta(i)} p_{ij} \qquad i \in \mathcal{B} \tag{4b}$$

$$q_i^g - q_i^d = -b_{ii}c_{ii} + \sum_{j \in \delta(i)} q_{ij} \qquad\qquad i \in \mathcal{B} \qquad\qquad \text{(4c)}$$

$$p_{ij} = -G_{ij}c_{ii} + G_{ij}c_{ij} - B_{ij}s_{ij} \qquad\qquad (i,j) \in \mathcal{L} \qquad\qquad \text{(4d)}$$

$$q_{ij} = \quad B_{ij}c_{ii} - B_{ij}c_{ij} - G_{ij}s_{ij} \qquad\qquad (i,j) \in \mathcal{L} \qquad\qquad \text{(4e)}$$

$$\underline{V}_i^2 \le c_{ii} \le \overline{V}_i^2 \qquad\qquad i \in \mathcal{B} \qquad\qquad \text{(4f)}$$

$$c_{ij} = c_{ji}, \quad s_{ij} = -s_{ji} \qquad\qquad (i,j) \in \mathcal{L} \qquad\qquad \text{(4g)}$$

$$c_{ij}^2 + s_{ij}^2 = c_{ii}c_{jj} \qquad\qquad (i,j) \in \mathcal{L} \qquad\qquad \text{(4h)}$$

$$\theta_j - \theta_i = \text{atan2}(s_{ij}, c_{ij}) \qquad\qquad (i,j) \in \mathcal{L}, \qquad\qquad \text{(4i)}$$

$$\text{(2e)-(2g)}.$$

A variant of this formulation without (4i) was previously proposed in [10] and [19] for radial networks (also see [28]) while it was later adapted to general networks in [20, 21].

3   AC Optimal Transmission Switching

AC Optimal Transmission Switching (AC OTS) is a variant of the AC OPF problem in which transmission lines are allowed to be switched on and off to reduce the total cost of dispatch. AC OTS can be formulated as an optimization problem, which aims to find a topology with the least cost while achieving feasible AC power flow solutions. In this section, we first formulate AC OTS as an MINLP and then, propose an MISOCP relaxation to obtain lower bounds. We will use OTS (resp. OPF) to denote AC OTS (resp. AC OPF) for brevity, unless stated otherwise.

3.1   MINLP Formulation

Mathematical programming formulation of OTS can be stated with the same variables as used in OPF with the addition of a set of binary variables, denoted by x_{ij}, for each line. The variable x_{ij} takes the value one if the corresponding line (i,j) is switched on, and zero otherwise. Then, OTS is formulated as the following MINLP problem:

$$\min \sum_{i \in \mathcal{G}} C_i(p_i^g) \qquad\qquad\qquad\qquad\qquad \text{(5a)}$$

$$\text{s.t. } p_{ij} = (-G_{ij}c_{ii} + G_{ij}c_{ij} - B_{ij}s_{ij})x_{ij} \qquad\qquad (i,j) \in \mathcal{L} \qquad\qquad \text{(5b)}$$

$$q_{ij} = \quad (B_{ij}c_{ii} - B_{ij}c_{ij} - G_{ij}s_{ij})x_{ij} \qquad\qquad (i,j) \in \mathcal{L} \qquad\qquad \text{(5c)}$$

$$(c_{ij}^2 + s_{ij}^2 - c_{ii}c_{jj})x_{ij} = 0 \qquad\qquad (i,j) \in \mathcal{L} \qquad\qquad \text{(5d)}$$

5

$$(\theta_j - \theta_i - \text{atan2}(s_{ij}, c_{ij}))x_{ij} = 0 \qquad (i,j) \in \mathcal{L} \qquad (5e)$$

$$x_{ij} \in \{0,1\} \qquad (i,j) \in \mathcal{L}, \qquad (5f)$$

$$(2e)\text{-}(2g), (4b)\text{-}(4c), (4f)\text{-}(4g).$$

Here, constraints (5b) and (5c) guarantee that real and reactive flow on every line takes the associated values if the line is switched on and zero otherwise. Similarly, constraints (5d) and (5e) are active only when the corresponding binary variable takes the value one.

We also note that the model (5) can be appropriately modified to include circuit breakers between bus bars [45].

3.2 MISOCP Relaxation

Now, we propose an MISOCP relaxation of OTS (5). For notational convenience, let $\underline{c}_{ii} = \underline{V}_i^2$ and $\bar{c}_{ii} = \bar{V}_i^2$. Here, we extend the definition of variables c_{ij} and s_{ij}, which now take the values as before when the corresponding line is switched on and zero otherwise. We also denote lower and upper bounds of c_{ij} (resp. s_{ij}) as $\underline{c}_{ij}$ (resp. $\underline{s}_{ij}$) and $\bar{c}_{ij}$ (resp. $\bar{s}_{ij}$), respectively, when the line is switched on. Next, we define new variables $c_{ii}^j := c_{ii}x_{ij}$. Using this notation, we present an MISOCP relaxation as follows:

$$\min \sum_{i \in \mathcal{G}} C_i(p_i^g) \qquad (6a)$$

$$\text{s.t. } p_{ij} = -G_{ij}c_{ii}^j + G_{ij}c_{ij} - B_{ij}s_{ij} \qquad (i,j) \in \mathcal{L} \qquad (6b)$$

$$q_{ij} = \quad B_{ij}c_{ii}^j - B_{ij}c_{ij} - G_{ij}s_{ij} \qquad (i,j) \in \mathcal{L} \qquad (6c)$$

$$\underline{c}_{ij}x_{ij} \leq c_{ij} \leq \bar{c}_{ij}x_{ij} \qquad (i,j) \in \mathcal{L} \qquad (6d)$$

$$\underline{s}_{ij}x_{ij} \leq s_{ij} \leq \bar{s}_{ij}x_{ij} \qquad (i,j) \in \mathcal{L} \qquad (6e)$$

$$\underline{c}_{ii}x_{ij} \leq c_{ii}^j \leq \bar{c}_{ii}x_{ij} \qquad (i,j) \in \mathcal{L} \qquad (6f)$$

$$c_{ii} - \bar{c}_{ii}(1 - x_{ij}) \leq c_{ii}^j \qquad (i,j) \in \mathcal{L} \qquad (6g)$$

$$c_{ii}^j \leq c_{ii} - \underline{c}_{ii}(1 - x_{ij}) \qquad (i,j) \in \mathcal{L} \qquad (6h)$$

$$c_{ij}^2 + s_{ij}^2 \leq c_{ii}^j c_{jj}^i \qquad (i,j) \in \mathcal{L}, \qquad (6i)$$

$$(2e)\text{-}(2g), (4b)\text{-}(4c), (4f)\text{-}(4g), (5f).$$

Here, constraints (6b) and (6c) again guarantee that flow variables takes the correct value when the line is switched on and zero otherwise, due to constraints (6d)-(6f). On the other hand, (6g)-(6h) restrict that c_{ii}^j takes value c_{ii} when line in switched on. We note that constraints (6f)-(6h) are precisely the McCormicks envelopes [34] applied to $c_{ii}^j = c_{ii}x_{ij}$.

Finally, (6i) is the SOCP relaxation of (5d).

We note that the non-convex constraint (5e) is dropped altogether to obtain the MISOCP relaxation (6). In the next section, we propose three ways to incorporate the constraint (5e) back into the MISOCP relaxation.

4 Valid Inequalities

In this section, we propose three methods to strengthen the MISOCP relaxation (6). They are based on the strengthening methods we recently proposed for the SOCP relaxation of the AC OPF problem in [27], which are combined with integer programming techniques. In Section 4.1, we construct a polyhedral envelope for the arctangent constraint (5e) in 3-dimension. In Section 4.2, we propose a disjunctive cut generation scheme that separates a given SOCP solution from the SDP cones. In Section 4.3, we propose another disjunctive cut generation scheme that separates a given SOCP solution from a newly-proposed cycle based McCormick relaxation of the OPF problem. Finally, in Section 4.4, we propose variable bounding techniques that provide tight variable bounds, which is essential for the success of the proposed approach.

4.1 Arctangent Envelopes

First, we propose a convex outer-approximation of the angle condition (5e) to the MISOCP relaxation. Our construction uses four linear inequalities to approximate the convex envelope for the following set defined by the arctangent constraint (5e) for each line $(i, j) \in \mathcal{L}$,

$$\mathcal{AT} := \left\{ (c, s, \theta) \in \mathbb{R}^3 : \theta = \arctan\left(\frac{s}{c}\right), (c, s) \in B \right\}, \tag{7}$$

where we denote $\theta = \theta_j - \theta_i$ and drop (i, j) indices for brevity and define the box $B := [\underline{c}, \overline{c}] \times [\underline{s}, \overline{s}]$. We also assume $\underline{c} > 0$. The four corners of the box correspond to four points in the (c, s, θ) space:

$$\begin{aligned}
z^1 &= (\underline{c}, \overline{s}, \arctan(\overline{s}/\underline{c})), & z^2 &= (\overline{c}, \overline{s}, \arctan(\overline{s}/\overline{c})), \\
z^3 &= (\overline{c}, \underline{s}, \arctan(\underline{s}/\overline{c})), & z^4 &= (\underline{c}, \underline{s}, \arctan(\underline{s}/\underline{c})).
\end{aligned} \tag{8}$$

Let us first focus on the upper envelopes. Proposition 4.1 is adapted from [27] to the case of OTS:

Proposition 4.1. *Let $\theta = \gamma_1 + \alpha_1 c + \beta_1 s$ and $\theta = \gamma_2 + \alpha_2 c + \beta_2 s$ be the planes passing*

through points $\{z^1, z^2, z^3\}$, and $\{z^1, z^3, z^4\}$, respectively. Then, for $k = 1, 2$, we have

$$\gamma'_k + \alpha_k c + \beta_k s + (2\pi - \gamma'_k)(1 - x) \geq \arctan\left(\frac{s}{c}\right) \tag{9}$$

for all $(c, s) \in B$ with $\gamma'_k = \gamma_k + \Delta\gamma_k$ where

$$\Delta\gamma_k = \max_{(c,s)\in B} \left\{\arctan\left(\frac{s}{c}\right) - (\gamma_k + \alpha_k c + \beta_k s)\right\}. \tag{10}$$

The nonconvex optimization problem (10) can be solved by enumerating all possible Karush-Kuhn-Tucker (KKT) points. A similar argument can be used to construct lower envelopes as well. See [27] for details.

4.2 SDP Disjunction

In the second method to strengthen the MISOCP relaxation (6), we propose a cutting plane approach to separate a given SOCP relaxation solution from the feasible region of the SDP relaxation of cycles. To start with, let us consider a cycle with the set of lines C and the set of buses $\mathcal{B}_C$. Let $v \in \mathbb{R}^{2|C|}$ be a vector of bus voltages defined as $v = [e; f]$ such that $v_i = e_i$ for $i \in \mathcal{B}$ and $v_{i'} = f_i$ for $i' = i + |C|$. Observe that if we have a set of c, s variables satisfying the definitions in (3) and a matrix variable $W = vv^T$, then the following relationship holds between c, s, x and W,

$$
\begin{align}
c_{ij} &= (W_{ij} + W_{i'j'})x_{ij} & (i, j) &\in C \tag{11a}\\
s_{ij} &= (W_{ij'} - W_{ji'})x_{ij} & (i, j) &\in C \tag{11b}\\
c_{ii} &= W_{ii} + W_{i'i'} & i &\in \mathcal{B}_C \tag{11c}\\
\underline{c}_{ij}x_{ij} &\leq c_{ij} \leq \overline{c}_{ij}x_{ij} & (i, j) &\in C \tag{11d}\\
\underline{s}_{ij}x_{ij} &\leq s_{ij} \leq \overline{s}_{ij}x_{ij} & (i, j) &\in C \tag{11e}\\
\underline{c}_{ii} &\leq c_{ii} \leq \overline{c}_{ii} & i &\in \mathcal{B}_C \tag{11f}\\
c_{ii}^j &= c_{ii}x_{ij} & (i, j) &\in C \tag{11g}\\
x_{ij} &\in \{0, 1\} & (i, j) &\in C \tag{11h}\\
W &\succeq 0. & & \tag{11i}
\end{align}
$$

Let us define $\mathcal{S} := \{(c, s, x) : \exists W : (11)\}$. Clearly, any feasible solution to the OTS formulation (5) must also satisfy (11). Therefore, any valid inequality for $\mathcal{S}$ is also valid for the formulation (5).

Note that $\mathcal{S}$ is a mixed-integer set. Ideally, one would be interested in finding $\text{conv}(\mathcal{S})$ to

generate strong valid inequalities. However, this is a quite computationally challenging task, no easier than solving the original MINLP. Instead, we outer-approximate $\mathrm{conv}(\mathcal{S})$ and obtain cutting planes by utilizing a simple disjunction for a cycle C: Either every line is active, that is $\sum_{(i,j)\in C} x_{ij} = |C|$, or at least one line is disconnected, that is $\sum_{(i,j)\in C} x_{ij} \leq |C| - 1$. Below, we approximate these two disjunctions.

Disjunction 1: In the first disjunction, we have $x_{ij} = 1$ for all $(i,j) \in C$. Let us consider the following constraints

$$
\begin{align}
c_{ij} &= W_{ij} + W_{i'j'} & (i,j) \in C \tag{12a}\\
s_{ij} &= W_{ij'} - W_{ji'} & (i,j) \in C \tag{12b}\\
c_{ii} &= c_{ii}^{j} & (i,j) \in C \tag{12c}\\
x_{ij} &= 1 & (i,j) \in C, \tag{12d}
\end{align}
$$

and define $\mathcal{S}_1 := \{(c,s,x) : \exists W : (12), (11c) - (11f), (11i)\}$.

Disjunction 0: In the second disjunction, $x_{ij} = 0$ for some $(i,j) \in C$. Let us consider the following constraints

$$
\begin{align}
c_{ij}^2 + s_{ij}^2 &\leq c_{ii}^{j} c_{jj}^{i} & (i,j) \in C \tag{13a}\\
\underline{c}_{ii} x_{ij} \leq c_{ii}^{j} &\leq \overline{c}_{ii} x_{ij} & (i,j) \in C \tag{13b}\\
c_{ii} - \overline{c}_{ii}(1 - x_{ij}) &\leq c_{ii}^{j} & (i,j) \in C \tag{13c}\\
c_{ii}^{j} &\leq c_{ii} - \underline{c}_{ii}(1 - x_{ij}) & (i,j) \in C \tag{13d}\\
0 \leq x_{ij} &\leq 1 & (i,j) \in C \tag{13e}\\
\sum_{(i,j)\in C} x_{ij} &\leq |C| - 1, && \tag{13f}
\end{align}
$$

and define $\mathcal{S}_0 := \{(c,s,x) : (13), (11d)\text{-}(11f)\}$.

We note that both $\mathcal{S}_1$ and $\mathcal{S}_0$ are conic representable. In particular, these bounded sets are respectively semidefinite and second-order cone representable. Therefore, $\mathrm{conv}(\mathcal{S}_1 \cup \mathcal{S}_0)$ is also conic representable (see Appendix A on how to obtain a representation as an extended formulation), and by construction, contains $\mathcal{S}$.

Now, suppose a point (c^*, s^*, x^*) is given. We want to decide whether this point belongs to $\mathrm{conv}(\mathcal{S}_1 \cup \mathcal{S}_0)$ or otherwise, find a separating hyperplane. Given that we have an extended semidefinite representation for $\mathrm{conv}(\mathcal{S}_1 \cup \mathcal{S}_0)$, we can solve an SDP separation problem to achieve this. See Appendix B.

4.3 McCormick Disjunction

The last method to strengthen the MISOCP relaxation (6) is based on a new cycle-based OPF formulation we propose in [27]. The key observation is as follows: instead of satisfying the angle condition (5e) for each $(i, j) \in \mathcal{L}$, we guarantee that angle differences sum up to 0 modulo 2π over every cycle C in the power network if all the lines of the cycle C are switched on, i.e.

$$\Big(\sum_{(i,j) \in C} \theta_{ij} - 2\pi k \Big) \prod_{(i,j) \in C} x_{ij} = 0, \quad \text{for some } k \in \mathbb{Z}, \tag{14}$$

where $\theta_{ij} := \theta_j - \theta_i$.

Next, we consider

$$\Big[\cos\Big(\sum_{(i,j) \in C} \theta_{ij} \Big) - 1 \Big] \prod_{(i,j) \in C} x_{ij} = 0 \tag{15a}$$

$$c_{ij} = \sqrt{c_{ii} c_{jj}} \cos \theta_{ij} x_{ij} \qquad (i, j) \in C \tag{15b}$$

$$s_{ij} = \sqrt{c_{ii} c_{jj}} \sin \theta_{ij} x_{ij} \qquad (i, j) \in C, \tag{15c}$$

$$(11d) - (11h).$$

Here, (15a) is equivalent to (14) and (15b)-(15c) follow from the definition of c, s variables. Let us define $\mathcal{M} := \{(c, s, x) : \exists \theta : (15), (11d) - (11h)\}$. Again, observe that any feasible solution to the OTS formulation (5) must also satisfy (15). Therefore, any valid inequality for $\mathcal{M}$ is also valid for the formulation (5).

We again follow a similar procedure to the previous section and consider two disjunctions for a cycle C.

<u>Disjunction 1</u>: In the first disjunction, we have $x_{ij} = 1$ for all $(i, j) \in C$. Note that (15a) reduces to

$$\cos\Big(\sum_{(i,j) \in C} \theta_{ij} \Big) = 1.$$

Now, we can expand the cosine appropriately and replace $\cos(\theta_{ij})$'s and $\sin(\theta_{ij})$'s in terms of c, s variables following (15b)-(15c). This transformation yields a homogeneous polynomial, denoted by p_C, in terms of only c, s variables, and an equivalent constraint $p_C = 0$. However, p_C can have up to $2^{|C|-1} + 1$ monomials and each monomial of degree $|C|$. In [27], we propose a method, which is used to "bilinearize" this high degree polynomial by decomposing larger cycles into smaller ones by the addition of artificial lines and corresponding variables. We refer the reader to [27] for details.

Using the proposed decomposition scheme, we obtain a set of bilinear polynomials $q_k(c, s, \tilde{c}, \tilde{s}) = 0$, $k \in \mathcal{K}_C$, for a given cycle C. Here, $\tilde{c}, \tilde{s}$ denote the extra variables used in the construction.

Finally, we use McCormick envelopes for each bilinear constraint to linearize the system of polynomials. For a given cycle C, consider the McCormick relaxation of the bilinear cycle constraints, which can be written compactly as follows:

$$Az + \tilde{A}\tilde{z} + By \leq c \tag{16a}$$

$$Ey = 0. \tag{16b}$$

Here, z is a vector composed of the c, s variables, $\tilde{z}$ is a vector composed of the additional $\tilde{c}, \tilde{s}$ variables introduced in the cycle decomposition, and y is a vector of new variables defined to linearize the bilinear terms in the cycle constraints. Constraint (16a) contains the McCormick envelopes of the bilinear terms and bounds on the c, s variables, while (16b) includes the linearized cycle equality constraints. Finally, we define the set $\mathcal{M}_1 := \{(c, s, x) : \exists(\tilde{c}, \tilde{s}) : (16), (11d)\text{-}(11f), (12c)\text{-}(12d)\}$.

Disjunction 0: In the second disjunction, $x_{ij} = 0$ for some $(i, j) \in C$. We take $\mathcal{M}_0 := \mathcal{S}_0$.

We note that both $\mathcal{M}_1$ and $\mathcal{M}_0$ are conic representable. In particular, these bounded sets are respectively polyhedral and second-order cone representable. Therefore, $\text{conv}(\mathcal{M}_1 \cup \mathcal{M}_0)$ is also conic representable, and by construction, contains $\mathcal{M}$.

Now, suppose a point (c^*, s^*, x^*) is given. We want to decide whether this point belongs to $\text{conv}(\mathcal{M}_1 \cup \mathcal{M}_0)$ or otherwise, find a separating hyperplane. Given that we have an extended second-order cone representation for $\text{conv}(\mathcal{M}_1 \cup \mathcal{M}_0)$, we can solve an SOCP separation problem.

In our computations, we observed that stronger cuts are obtained by combining SDP and McCormick Disjunction. In particular, we separate cutting planes from $\text{conv}((\mathcal{S}_1 \cap \mathcal{M}_1) \cup \mathcal{S}_0)$ by solving SDP separation problems.

4.4 Obtaining Variable Bounds

Note that the arctangent envelopes and the McCormick relaxations are more effective when tight variable upper/lower bounds are available for the c and s variables. Now, we explain how we obtain good bounds for these variables, which is the key ingredient in the success of our proposed methods.

Observe that c_{ij} and s_{ij} do not have explicit variable bounds except the implied bounds due to (4f) and (4h) as

$$-\overline{V}_i\overline{V}_j \leq c_{ij}, s_{ij} \leq \overline{V}_i\overline{V}_j \quad (i, j) \in \mathcal{L}.$$

However, these bounds may be quite loose, especially when the phase angle differences are

small, implying $c_{ij} \approx 1$ and $s_{ij} \approx 0$ when the corresponding line is switched on. Therefore, one should try to improve these bounds.

We adapt the procedure proposed in [27] (which dealt only with OPF) to the case of OTS in order to obtain variable bounds, that is, we solve a reduced version of the full MISOCP relaxation to efficiently compute bounds. In particular, to find variable bounds for c_{kl} and s_{kl} for some $(k, l) \in \mathcal{L}$, consider the buses which can be reached from either k or l in at most r steps. Denote this set of buses as $\mathcal{B}_{kl}(r)$. For instance, $\mathcal{B}_{kl}(0) = \{k, l\}$, $\mathcal{B}_{kl}(1) = \delta(k) \cup \delta(l)$, etc. We also define $\mathcal{G}_{kl}(r) = \mathcal{B}_{kl}(r) \cap \mathcal{G}$ and $\mathcal{L}_{kl}(r) = \{(i, j) \in \mathcal{L} : i \in \mathcal{B}_{kl}(r) \text{ or } j \in \mathcal{B}_{kl}(r)\}$. Then, we consider the following SOCP relaxation:

$$p_i^g - p_i^d = g_{ii}c_{ii} + \sum_{j \in \delta(i)} p_{ij} \qquad i \in \mathcal{B}_{kl}(r) \tag{17a}$$

$$q_i^g - q_i^d = -b_{ii}c_{ii} + \sum_{j \in \delta(i)} q_{ij} \qquad i \in \mathcal{B}_{kl}(r) \tag{17b}$$

$$p_{ij} = -G_{ij}c_{ii}^j + G_{ij}c_{ij} - B_{ij}s_{ij} \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17c}$$

$$q_{ij} = B_{ij}c_{ii}^j - B_{ij}c_{ij} - G_{ij}s_{ij} \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17d}$$

$$p_{ij}^2 + q_{ij}^2 \leq (S_{ij}^{\max})^2 \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17e}$$

$$\underline{V}_i^2 \leq c_{ii} \leq \overline{V}_i^2 \qquad i \in \mathcal{B}_{kl}(r + 1) \tag{17f}$$

$$p_i^{\min} \leq p_i^g \leq p_i^{\max} \qquad i \in \mathcal{G}_{kl}(r) \tag{17g}$$

$$q_i^{\min} \leq q_i^g \leq q_i^{\max} \qquad i \in \mathcal{G}_{kl}(r) \tag{17h}$$

$$\underline{c}_{ij}x_{ij} \leq c_{ij} \leq \overline{c}_{ij}x_{ij} \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17i}$$

$$\underline{s}_{ij}x_{ij} \leq s_{ij} \leq \overline{s}_{ij}x_{ij} \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17j}$$

$$\underline{c}_{ii}x_{ij} \leq c_{ii}^j \leq \overline{c}_{ii}x_{ij} \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17k}$$

$$c_{ii} - \overline{c}_{ii}(1 - x_{ij}) \leq c_{ii}^j \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17l}$$

$$c_{ii}^j \leq c_{ii} - \underline{c}_{ii}(1 - x_{ij}) \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17m}$$

$$c_{ij} = c_{ji}, \quad s_{ij} = -s_{ji} \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17n}$$

$$c_{ij}^2 + s_{ij}^2 \leq c_{ii}^j c_{jj}^i \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17o}$$

$$0 \leq x_{ij} \leq 1 \qquad (i, j) \in \mathcal{L}_{kl}(r) \tag{17p}$$

$$x_{kl} = 1. \tag{17q}$$

Essentially, (17) is the continuous relaxation of MISOCP relaxation applied to the part of the power network within r steps of the buses k and l. c_{kl} and s_{kl} can be minimized and maximized subject to (17) for each edge (k, l) to obtain lower and upper bounds, respectively. These SOCPs can be solved in parallel, since they are independent of each other. It is

observed that a good tradeoff between accuracy and speed is to select $r = 2$[27]. Constraint (17q) may seem to restrict the feasible region, however, the way we defined c_{kl} and s_{kl} variables, they are the values for cosine and sine components when $x_{kl} = 1$ (otherwise, they are 0). Therefore, it is enough for the bounds to be valid for $x_{kl} = 1$ only.

Bounds on an artificial edge (i, j) used in the construction of McCormick envelopes are chosen as follows:

$$\overline{c}_{ij} = -\underline{c}_{ij} = \overline{s}_{ij} = -\underline{s}_{ij} = \overline{V}_i \overline{V}_j. \tag{18}$$

A similar idea can be used to fix some of the binary variables as well. In particular, we can minimize x_{kl} over (17) after omitting (17q). If the optimal value turns out to be strictly positive, then x_{kl} can be fixed to one.

5 Algorithm

In this section, we propose an algorithm to solve OTS. The algorithm has two phases. The first phase involves solving a sequence of SOCPs obtained by relaxing integrality restriction of the binary variables in MISOCP (6), and incorporates cycle inequalities generated from the extended SDP and McCormick relaxations in Section 4.2 and 4.3. In this phase, the aim is to strengthen the lower bound on the MISOCP relaxation. The second phase involves solving a sequence of MISOCP relaxations strengthened by cycle inequalities. The aim in this phase is to obtain high quality feasible solutions for OTS. In particular, this is achieved by solving OPF subproblems with fixed topologies obtained from the integral solutions found during the branch-and-cut process of solving the MISOCP (6). This procedure is repeated by "forbidding" the topologies already considered in order to obtain different network configurations in the subsequent iterations.

Now we formally define the ingredients of the algorithm. First, let $SOCP(\mathcal{V})$ be the continuous relaxation of MISOCP (6) with a set of valid inequalities $\mathcal{V}$ obtained from cycle inequalities using extended SDP and McCormick relaxations. The set $\mathcal{V}$ is dynamically updated T_1 times. Similarly, we define $MISOCP(\mathcal{V}, \mathcal{F})$ as the MISOCP relaxation of OTS with a set of valid cycle inequalities $\mathcal{V}$ and forbidden topologies $\mathcal{F}$. Here, we forbid a topology $x^* \in \mathcal{F}$ by adding the following "no-good" cut (see [2] for generalizations) to the formulation:

$$\sum_{(i,j):x_{ij}^*=1} (1 - x_{ij}) + \sum_{(i,j):x_{ij}^*=0} x_{ij} \geq 1. \tag{19}$$

We denote by LB_t as the optimal value of $MISOCP(\mathcal{V}, \mathcal{F})$ and $\mathcal{P}_t$ as the set of all integral solutions found by the solver at the t-th iteration. For instance, CPLEX offers this option

called *solution pool*. In a practical implementation, this part is repeated T_2 times.

Let $OPF(x)$ denote the value of a feasible solution to OPF problem (4) for the fixed topology induced by the integral vector x. Finally, UB is the best upper bound on OTS. Now, we present Algorithm 1.

Algorithm 1 OTS algorithm.

Input: T_1, T_2, ϵ.
Phase I: Set $\mathcal{V} \leftarrow \emptyset$, $\mathcal{F} \leftarrow \emptyset$, $UB \leftarrow \infty$.
for $\tau = 1, \ldots, T_1$ **do**
   Solve $SOCP(\mathcal{V})$.
   Separate cycle inequalities for each cycle in a cycle basis to obtain a set of valid inequalities $\mathcal{V}_\tau$.
   Update $\mathcal{V} \leftarrow \mathcal{V} \cup \mathcal{V}_\tau$.
end for
Phase II: Set $t \leftarrow 0$.
repeat
   $t \leftarrow t + 1$
   Solve $MISOCP(\mathcal{V}, \mathcal{F})$ to obtain a pool of integral solutions $\mathcal{P}_t$ and record the optimal cost as LB_t.
   for all $x \in \mathcal{P}_t$ **do**
     if $OPF(x) < UB$ **then**
       $UB \leftarrow OPF(x)$
     end if
   end for
   Update $\mathcal{F} \leftarrow \mathcal{F} \cup \mathcal{P}_t$.
until $LB_t \geq (1 - \epsilon)UB$ or $t \geq T_2$

Observation 5.1. *If $OPF(x)$ returns globally optimal solution for every topology x, $\epsilon = 0$ and $T_2 = \infty$, then Algorithm 1 converges to the optimal solution of OTS in finitely many iterations.*

Observation 5.1 follows from the fact that there are finitely many topologies and by the hypothesis that $OPF(x)$ can be solved globally, which is possible for some IEEE instances using moment/sum-of-squares relaxations [24]. Although Observation 5.1 states that Algorithm 1 can be used to solve OTS to global optimality in finitely many iterations, the requirement of solving $OPF(x)$ to global optimality may not be satisfied always. In practice, we can solve OPF subproblems using local solver, in which case we have Observation 5.2.

Observation 5.2. *If $OPF(x)$ is solved by a local solution method, then we have $LB_1 \leq z^* \leq UB$ upon termination of Algorithm 1, where z^* is the optimal value of OTS.*

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6 Computational Experiments

In this section, we present the results of our extensive computational experiments on standard IEEE instances available from MATPOWER [48] and instances from NESTA 0.3.0 archive with congested operating conditions [7]. The code is written in the C# language with Visual Studio 2010 as the compiler. For all experiments, we used a 64-bit computer with Intel Core i5 CPU 2.50GHz processor and 16 GB RAM. Time is measured in seconds. We use three different solvers:

- CPLEX 12.6 [1] to solve MISOCPs.

- Conic interior point solver MOSEK 7.1 [37] to solve SDP separation problems.

- Nonlinear interior point solver IPOPT [46] to find local optimal solutions to $OPF(x)$.

We use a Gaussian elimination based approach to construct a cycle basis proposed in [29] and use this set of cycles in the separation phase.

6.1 Methods

We report the results of three algorithmic settings:

- SOCP: MISOCP formulation (6) in Phase II without Phase I (i.e. $T_1 = 0$).

- SOCPA: SOCP strengthened by the arctangent envelopes introduced in Section 4.1.

- SOCPA_Disj: SOCPA strengthened further by dynamically generating linear valid inequalities obtained from separating an SOCP feasible solution from the SDP and McCormick relaxation over cycles using a disjunctive argument T_1 times. In particular, a separation oracle is used to separate a given point from $\text{conv}((\mathcal{S}_1 \cap \mathcal{M}_1) \cup \mathcal{S}_0)$.

The following four performance measures are used to assess the accuracy and the efficiency of the proposed methods:

- "%OG" is the percentage optimality gap proven by our algorithm calculated as $100 \times (1 - \widetilde{LB}_1/UB)$. Here, $\widetilde{LB}_1$ is the lower bound proven, which may be strictly smaller than LB_1 due to optimality gap tolerance and time limit.

- "%CB" is the percentage cost benefit obtained by line switching calculated as $100 \times (1 - UB/OPF(e))$, where e is the vector of ones so that $OPF(e)$ corresponds to the OPF solution with the initial topology.

- "#off" is the number of lines switched off in the topology which gives UB.

- "TT" is the total time in seconds, including preprocessing (bound tightening), solution of $T_1 = 5$ rounds of SOCPs to improve lower bound and separation problems to generate cutting planes (in the case of SOCPA_Disj), solution of T_2 rounds of MISOCPs and several calls to local solver IPOPT with given topologies. MISOCPs are solved under 720 seconds time limit so that 5 iterations take about 1 hour (optimality gap for integer programs is 0.01%). Preprocessing and separation subproblems are parallelized.

We choose parameter $T_2 = 5$ and pre-terminate Algorithm 1 if 0.1% optimality gap is proven.

6.2 Results

The results of our computational experiments are presented in Tables 1 and 2 for standard IEEE and NESTA instances, respectively. We considered instances up to 300-bus since Phase II of the Algorithm 1 does not scale up well for larger instances. Let us start with the former: IEEE instances are a relatively easy set since transmission line limits are generally not binding. Therefore, cost benefits obtained by switching are also limited. The largest cost reduction is obtained for case30Q with 2.24%. Among the three methods, the most successful one is SOCPA_Disj, on average proving 0.05% optimality gap and providing 0.31% cost savings. In terms of computational time, SOCP is the fastest, however, its performance is not as good as the other two. Quite interestingly, SOCPA_Disj is faster than SOCPA, on average, for this set of instances. In terms of comparison with other methods, unfortunately, there is limited literature for this purpose. In [18], nine of these instances (except for cases 9Q and 30Q) are considered and a quadratic convex (QC) relaxation based approach is used. On average, their approach proves 0.14% optimality gap, which is worse than any of our methods over the same nine instances. The only instance QC approach is better is 118ieee with 0.11% optimality gap, while it is worse than our methods for case300 with a 0.47% optimality gap.

Table 1: Results summary for standard IEEE Instances.

case	SOCP				SOCPA				SOCPA_Disj			
	%OG	%CB	#off	TT(s)	%OG	%CB	#off	TT(s)	%OG	%CB	#off	TT(s)
6ww	0.16	0.48	2	1.29	0.02	0.48	2	0.67	0.01	0.48	2	1.28
9	0.00	0.00	0	0.26	0.00	0.00	0	0.22	0.00	0.00	0	0.55
9Q	0.04	0.00	0	0.42	0.04	0.00	0	0.33	0.04	0.00	0	0.97
14	0.08	0.00	0	0.66	0.09	0.00	0	0.70	0.01	0.00	1	1.81
ieee30	0.05	0.00	1	1.95	0.05	0.00	0	1.67	0.02	0.00	1	3.84
30	0.07	0.52	1	4.60	0.06	0.52	2	5.01	0.03	0.51	2	9.39
30Q	0.44	2.05	2	24.43	0.43	2.03	5	25.80	0.13	2.24	5	44.16
39	0.03	0.00	0	2.53	0.01	0.02	1	3.17	0.01	0.02	1	4.48
57	0.07	0.02	4	6.18	0.07	0.02	4	8.72	0.08	0.01	1	13.59
118	0.19	0.08	4	3065.64	0.15	0.12	10	2553.59	0.17	0.08	16	3174.01
300	0.16	0.02	9	2318.89	0.15	0.03	12	3624.12	0.10	0.05	15	2803.31
Avg.	**0.12**	**0.29**	**2.1**	**493.35**	**0.10**	**0.29**	**3.3**	**565.82**	**0.05**	**0.31**	**4.0**	**550.67**

Table 2: Results summary for NESTA Instances from Congested Operating Conditions.

case	SOCP				SOCPA				SOCPA_Disj			
	%OG	%CB	#off	TT(s)	%OG	%CB	#off	TT(s)	%OG	%CB	#off	TT(s)
3lmbd	3.30	0.00	0	0.14	2.00	0.00	0	0.14	1.17	0.00	0	0.30
4gs	0.65	0.00	0	0.11	0.16	0.00	0	0.13	0.00	0.00	0	0.27
5pjm	0.18	0.27	1	0.61	0.01	0.27	1	0.41	0.02	0.27	1	0.89
6ww	6.06	7.74	1	1.23	1.34	7.74	1	1.64	1.05	7.74	1	1.97
9wscc	0.00	0.00	0	0.19	0.00	0.00	0	0.20	0.00	0.00	0	0.30
14ieee	1.02	0.33	1	2.86	0.89	0.45	2	3.48	0.41	0.45	2	4.49
29edin	0.43	0.00	2	12.79	0.24	0.18	13	299.82	0.33	0.08	21	181.74
30as	1.81	3.13	2	14.82	0.35	3.30	5	19.52	0.34	3.30	5	24.93
30fsr	3.24	44.20	2	9.72	0.05	44.98	2	4.76	0.03	44.98	3	6.97
30ieee	0.54	0.46	1	12.28	0.40	0.48	2	10.61	0.15	0.48	2	13.37
39epri	1.92	1.10	1	11.56	0.80	1.41	2	13.20	0.70	1.52	2	12.65
57ieee	0.12	0.10	3	41.48	0.12	0.10	2	58.97	0.09	0.10	3	29.86
118ieee	41.67	4.33	3	225.57	21.51	27.98	30	3838.62	7.50	39.09	21	3856.76
162ieee	0.57	1.05	9	3675.75	0.63	1.00	15	3861.29	0.60	1.00	15	3855.50
189edin	5.31	1.10	3	540.02	4.81	0.13	2	2194.80	5.58	0.00	0	3634.02
300ieee	1.00	0.10	12	3655.10	0.65	0.37	21	3640.14	0.61	0.35	21	3651.95
Avg.	**4.24**	**3.99**	**2.6**	**512.76**	**2.12**	**5.52**	**6.1**	**871.73**	**1.16**	**6.21**	**6.1**	**954.75**

Now let us consider NESTA instances with congested operating conditions. This set is particularly suited for line switching as more stringent transmission line limits are imposed.

In fact, large cost improvements are observed for some test cases. For instance, about 45% and 39% cost reductions are possible for cases 30fsr and 118ieee, respectively. Other instances with sizable cost reductions include cases 6ww and 30as. SOCPA_Disj is again the most successful method if we look at averages of optimality gap (1.16%) and cost savings (6.21%). It certifies that the best topology is within 1.17% of the optimal for all the cases except for 118ieee and 189edin. In terms of computational time, SOCP is again the fastest, however, its performance is significantly worse than the other two. We also note that SOCPA improves quite a bit over SOCP in terms of optimality and cost benefits with 70% increase in computational time. SOCPA_Disj takes about only 10% more time than SOCPA. As we go from SOCP to SOCPA_Disj, problems get more complicated and sometimes, MISOCPs are not solved to optimality within time limit. Consequently, for cases 189edin and 300ieee, the optimality gaps proven and cost benefits obtained by SOCPA_Disj can be slightly worse.

Finally, we note that that optimality gaps can be explained by two non-convexities: 1) integrality, 2) power flow equations. For instance, in case 3lmbd, the optimality gap can only be explained by the non-convexity of power flow equation since all the relevant topologies are considered. Similarly, at least some portion of the relatively large optimality gaps for cases 118ieee and 189edin may be attributed to non-convexity of power flow equations. Consequently, any future improvements on strengthening the convex relaxations of OPF problem can be useful in closing more gaps in OTS as well.

6.3 Discussion

In this section, we take a closer look at some of the instances with large cost benefits and try to gain some insight as to 1) how these large savings are obtained, and 2) how simple heuristics may fail to produce comparable results. Firstly, using a small example, we illustrate how large cost savings can be obtained. Secondly, we compare the results of our algorithm with a commonly used heuristic based on switching the best line and demonstrate how different the solution quality can be.

To address the first issue, let us concentrate on a small instance, namely case6ww from NESTA archive. This instance has the same topology and line characteristics as the standard IEEE test case but load and generation parameters are slightly different. In particular, $p_i^d = 78.24$, $q_i^d = 70$, $\underline{V}_i = 0.95$ and $\overline{V}_i = 1.05$ for the load buses $i = 4, 5, 6$ while the data for generation buses 1, 2 and 3 is summarized in Table 3. With this topology, the local optimal solution obtained using IPOPT with objective value of 273.76 is given in Figure 1. We note that the lines $(1, 5)$, $(2, 4)$ and $(3, 6)$ are congested in this configuration. On the other hand, if the line $(1, 2)$ is switched off, then the objective value reduces to 252.57, corresponding to a

Table 3: Generator data for NESTA case6ww test case.

	$p_i^{\min}$	$p_i^{\max}$	$q_i^{\min}$	$q_i^{\max}$	$\underline{V}_i = \overline{V}_i$	cost
1	25	200	-100	100	1.05	1.276311
2	18.75	106	-100	100	1.05	0.586272
3	22.5	93	-100	100	1.07	1.29111

7.74% cost saving over the initial topology. The difference is due to the fact that the outputs of generators 1 and 2 are now changed to (85.56, 32.74) and (84.25, 63.26), respectively. Notice that with the new topology, the cheaper generator 2 is used more, which results in the cost reduction. In the initial topology, this is not possible since the voltage magnitudes of the generators are fixed, and lines $(1, 5)$ and $(2, 4)$ are congested.

Figure 1: Flow diagram for the solution of NESTA case6ww without any line switching. The numbers above each generator node respectively represent the active and reactive power output. Similarly, the numbers near each edge respectively represent the active and reactive power flow of the line in the direction from the small indexed bus to the large indexed one. The figure is generated by modifying [33].
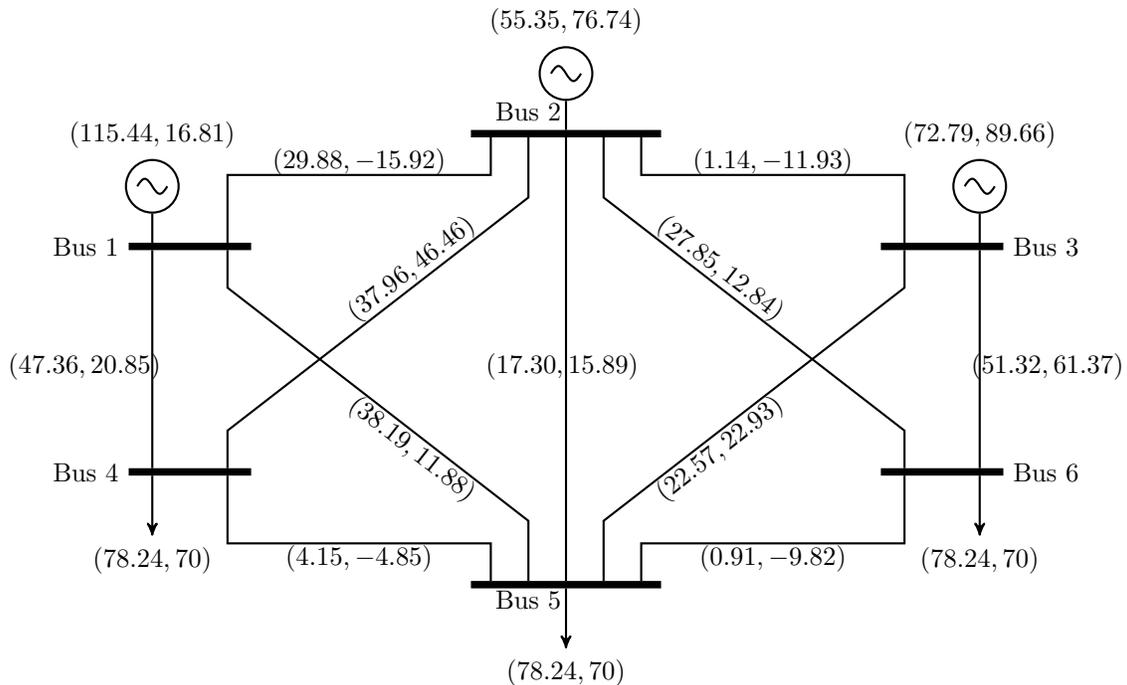
Now, let us consider the second issue. Due to the combinatorial nature of OTS problem, heuristics are frequently used to obtain suboptimal solutions. A commonly used one is to switch off a single line to obtain cost benefits [41, 31]. Although this heuristic idea is easy

to implement and works well in some instances, there are no guarantees on its accuracy. For example, in case6ww, the best line to switch off is, in fact, $(1, 2)$ suggested by both the best line heuristic and our algorithm. However, for other problems with large cost benefits, this is not always the case. For instance, in case30as and case30fsr, the best line heuristic reduces the overall cost to 2.99% and 44.02% respectively, compared to 3.30% and 44.98% obtained from our algorithm. For case118ieee, the cost reduction is dramatically different. The best line heuristic reduces the cost only by 19.52% while our algorithm provides a topology with 39.09% saving. Moreover, the best line heuristic does not provide any guarantee on how good the solution is while our algorithm gives optimality guarantees by construction. Therefore, adapting Algorithm 1 in real-time operations can yield significant savings over simple heuristics.

7   Conclusions

In this paper, we proposed a systematic approach to solve the AC OTS problem. In particular, we presented an alternative formulation for OTS and constructed a MISOCP relaxation. We improved the strength of this relaxation by the addition of arctangent envelopes and cutting planes obtained using disjunctive techniques. The use of these disjunctive cuts help in closing gap significantly. Our experiments on standard and congested instances suggest that the proposed methods are effective in obtaining strong lower bounds and producing provably good feasible solutions.

 We remind the reader that AC OTS is a challenging problem since it embodies two types of non-convexities due to AC power flow constraints and integrality of variables. We hope that the methodology developed in this paper can eventually be further improved to solve AC OTS problem in real life operations. As a future work, we would like to pursue finding ways to improve the solution time of MISOCPs as this step is the bottleneck in Algorithm 1. Also, decomposition methods can be sought to solve large-scale problems more efficiently, which could make the proposed approach adaptable to real life instances.

A   Convex Hull of Union of Two Conic Representable Sets

Let S_1 and S_2 be two bounded, conic representable sets

$$S_i = \{x : \exists u^i : A_i x + B_i u^i \succeq_{K_i} b_i\} \quad i = 1, 2.$$

Here, K_i's are regular (closed, convex, pointed with non-empty interior) cones. Then, a conic representation for $\text{conv}(S_1 \cup S_2)$ is given as follows:

$$x = x^1 + x^2, \; \lambda_1 + \lambda_2 = 1, \; \lambda_1, \lambda_2 \geq 0$$
$$A_i x^i + B_i u^i \succeq_{K_i} b_i \lambda_i \quad i = 1, 2.$$

B Separation from an Extended Conic Representable Set

Let S be a conic representable set $S = \{x : \exists u : Ax + Bu \succeq_K b\}$. Here, K is a regular cone. Suppose we want to decide if a given point x^* belongs to S and find a separating hyperplane $\alpha^\top x \geq \beta$ if $x^* \notin S$. This problem can be formulated as $\max_{\alpha, \beta} \left\{ \beta - \alpha^\top x^* : \alpha^\top x \geq \beta \; \forall x \in S \right\}$, where the constraint can be further dualized as

$$Z^* := \max_{\alpha, \beta, \mu} \{ \beta - \alpha^\top x^* : b^\top \mu \geq \beta, A^\top \mu = \alpha, B^\top \mu = 0,$$

$$\mu \in K^*, -e \leq \alpha \leq e, -1 \leq \beta \leq 1 \},$$

where K^* is the dual cone of K. If $Z^* \leq 0$, then $x^* \in S$, otherwise, the optimal α, β from the above program gives the desired separating hyperplane. For details, please see [27].

References

[1] *User's Manual for CPLEX Version 12.6*. IBM, 2014.

[2] Gustavo Angulo, Shabbir Ahmed, Santanu S Dey, and Volker Kaibel. Forbidden vertices. *Mathematics of Operations Research*, 40(2):350–360, 2014.

[3] X. Bai, H. Wei, K. Fujisawa, and Y. Wang. Semidefinite programming for optimal powerflow problems. *International Journal on Electric Power Energy Systems*, 30(6-7):383–392, 2008.

[4] C. Barrows, S. Blumsack, and R. Bent. Computationally efficient optimal transmission switching: Solution space reduction. In *Power and Energy Society General Meeting, 2012 IEEE*, pages 1–8, July 2012.

[5] C. Barrows, S. Blumsack, and P. Hines. Correcting optimal transmission switching for AC power flows. In *47th Hawaii International Conference on System Sciences (HICSS)*, page 2374–2379, 2014.

[6] D. Bienstock and G. Munoz. On linear relaxations of OPF problems. *arXiv preprint arXiv:1411.1120*, 2014.

[7] C. Coffrin, D. Gordon, and P. Scott. NESTA, The NICTA energy system test case archive. *arXiv preprint arXiv:1411.0359*, 2014.

[8] C. Coffrin, H. L. Hijazi, and P. Van Hentenryck. The QC relaxation: Theoretical and computational results on optimal power flow. *arXiv preprint arXiv:1502.07847*, 2015.

[9] C. Coffrin and P. Van Hentenryck. A linear-programming approximation of AC power flows. *INFORMS Journal on Computing*, 26(4):718–734, 2014.

[10] A. G. Expósito and E. R. Ramos. Reliable load flow technique for radial distribution networks. *IEEE Trans. on Power Syst.*, 14(3):1063 – 1069, 1999.

[11] R.S. Ferreira, C.L.T. Borges, and M.V.F. Pereira. A flexible mixed-integer linear programming approach to the AC optimal power flow in distribution systems. *IEEE Trans. on Power Syst.*, 29(5):2447–2459, Sept 2014.

[12] E.B. Fisher, R.P. O'Neill, and M.C. Ferris. Optimal transmission switching. *IEEE Trans. on Power Syst.*, pages 1346–1355, 2008.

[13] J.D. Fuller, R. Ramasra, and A. Cha. Fast heuristics for transmission-line switching. *IEEE Trans. on Power Syst.*, 27(3):1377–1386, Aug 2012.

[14] J. Han and A. Papavasiliou. The impacts of transmission topology control on the European electricity network. *IEEE Trans. on Power Syst.*, 31(1):496–507, Jan 2016.

[15] K.W. Hedman, M.C. Ferris, R.P. O'Neill, E.B. Fisher, and S.S. Oren. Co-optimization of generation unit commitment and transmission switching with N-1 reliability. *IEEE Trans. on Power Syst.*, 25(2):1052–1063, May 2010.

[16] K.W. Hedman, R.P. O'Neill, E.B. Fisher, and S.S. Oren. Optimal transmission switching with contingency analysis. *IEEE Trans. on Power Syst.*, 24(3):1577–1586, Aug 2009.

[17] K.W. Hedman, S.S. Oren, and R.P. O'Neill. A review of transmission switching and network topology optimization. In *Power and Energy Society General Meeting, 2011 IEEE*, pages 1–7, July 2011.

[18] HL Hijazi, C Coffrin, and P Van Hentenryck. Convex quadratic relaxations of mixed-integer nonlinear programs in power systems. Technical report, NICTA, Canberra, ACT Australia, 2013.

[19] R. A. Jabr. Radial distribution load flow using conic programming. *IEEE Trans. on Power Syst.*, 21(3):1458–1459, 2006.

[20] R. A. Jabr. A conic quadratic format for the load flow equations of meshed networks. *IEEE Trans. on Power Syst.*, 22(4):2285–2286, 2007.

[21] R. A. Jabr. Optimal power flow using an extended conic quadratic formulation. *IEEE Trans. on Power Syst.*, 23(3):1000–1008, 2008.

[22] R.A. Jabr. Optimization of AC transmission system planning. *IEEE Trans. on Power Syst.*, 28(3):2779–2787, Aug 2013.

[23] R.A. Jabr, R. Singh, and B.C. Pal. Minimum loss network reconfiguration using mixed-integer convex programming. *IEEE Trans. on Power Syst.*, 27(2):1106–1115, May 2012.

[24] Cédric Josz and Daniel K. Molzahn. Moment/sum-of-squares hierarchy for complex polynomial optimization. *arXiv:1508.02068*, 2015.

[25] M. Khanabadi, H. Ghasemi, and M. Doostizadeh. Optimal transmission switching considering voltage security and N-1 contingency analysis. *IEEE Trans. on Power Syst.*, 28(1):542–550, Feb 2013.

[26] A. Khodaei, M. Shahidehpour, and S. Kamalinia. Transmission switching in expansion planning. *IEEE Trans. on Power Syst.*, 25(3):1722–1733, Aug 2010.

[27] B. Kocuk, S. S. Dey, and X. A. Sun. Strong SOCP relaxations for the optimal power flow problem. *to appear in Operations Research*, 2016.

[28] B. Kocuk, S.S. Dey, and X.A. Sun. Inexactness of sdp relaxation and valid inequalities for optimal power flow. *Power Systems, IEEE Transactions on*, 31(1):642–651, Jan 2016.

[29] B. Kocuk, H. Jeon, S. S. Dey, J. Linderoth, J. Luedtke, and X. A. Sun. A cycle-based formulation and valid inequalities for DC power transmission problems with switching. *to appear in Operations Research*, 2016.

[30] A.S. Korad and K.W. Hedman. Robust corrective topology control for system reliability. *IEEE Trans. on Power Syst.*, 28(4):4042–4051, Nov 2013.

[31] A.S. Korad and K.W. Hedman. Enhancement of do-not-exceed limits with robust corrective topology control. *to appear in IEEE Trans. on Power Syst.*, 2015.

[32] J. Lavaei and S. H. Low. Zero duality gap in optimal power flow problem. *IEEE Trans. on Power Syst.*, 27(1):92–107, 2012.

[33] Richard Lincoln. Learning to trade power. `https://github.com/rwl/thesis/blob/master/tikz/case6ww.tex`, 2010.

[34] G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I – convex underestimating problems. *Math. Prog.*, 10(1):147–175, 1976.

[35] D. K. Molzahn, J. T. Holzer, B. C. Lesieutre, and C. L DeMarco. Implementation of a large-scale optimal power flow solver based on semidefinite programming. *IEEE Trans. on Power Syst.*, 28(4):3987–3998, 2013.

[36] D.K. Molzahn and I.A. Hiskens. Sparsity-exploiting moment-based relaxations of the optimal power flow problem. *To appear in IEEE Trans. on Power Syst.*, 2015.

[37] MOSEK. *MOSEK Modeling Manual.* MOSEK ApS, 2013.

[38] R. O' Neill, R. Baldick, U. Helman, M. Rothkopf, and J. Stewart. Dispatchable transmission in RTO markets. *IEEE Trans. on Power Syst.*, 20(1):171–179, 2005.

[39] P. Ruiz, J. M. Forster, A. Rudkevich, and M. C. Caramanis. On fast transmission topology control heuristics. *IEEE Power and Energy Society General Meeting*, pages 1–8, 2011.

[40] P. Ruiz, J. M. Forster, A. Rudkevich, and M. C. Caramanis. Tractable transmission topology control using sensitivity analysis. *IEEE Trans. on Power Syst.*, 27(3):1550–1559, 2012.

[41] M. Sahraei-Ardakani, A. Korad, K.W. Hedman, P. Lipka, and S. Oren. Performance of AC and DC based transmission switching heuristics on a large-scale polish system. In *PES General Meeting — Conference Exposition, 2014 IEEE*, pages 1–5, July 2014.

[42] M. Soroush and J.D. Fuller. Accuracies of optimal transmission switching heuristics based on DCOPF and ACOPF. *IEEE Trans. on Power Syst.*, 29(2):924–932, March 2014.

[43] C. Thompson, K. McIntyre, S. Nuthalapati, A. Garcia, and E.A. Villanueva. Real-time contingency analysis methods to mitigate congestion in the ERCOT region. In *Power Energy Society General Meeting, 2009. PES '09. IEEE*, pages 1–7, July 2009.

[44] P.A. Trodden, W.A. Bukhsh, A. Grothey, and K.I.M. McKinnon. Optimization-based islanding of power networks using piecewise linear AC power flow. *IEEE Trans. on Power Syst.*, 29(3):1212–1220, May 2014.

[45] T. Van Acker, D. Van Hertem, D. Bekaert, K. Karoui, and C. Merckx. Implementation of bus bar switching and short circuit constraints in optimal power flow problems. In *PowerTech, 2015 IEEE Eindhoven*, pages 1–6, June 2015.

[46] A. Wächter and L. T. Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Math. Prog.*, 106(1):25–57, 2006.

[47] J. Wu and K.W. Cheung. On selection of transmission line candidates for optimal transmission switching in large power networks. In *Power and Energy Society General Meeting (PES), 2013 IEEE*, pages 1–5, July 2013.

[48] R.D. Zimmerman, C.E. Murillo-Sanchez, and R.J. Thomas. MATPOWER: Steady-state operations, planning, and analysis tools for power systems research and education. *IEEE Trans. Power Syst.*, 26(1):12–19, Feb 2011.