# A Cutting Plane Method for Risk-constrained Traveling Salesman Problem with Random Arc Costs

Z.Huang*, Q.P.Zheng*, T. Zhang†, and V.Boginski*

*Department of Industrial Engineering and Management Systems, University of Central Florida, FL, USA
†Shanghai Key Laboratory of Financial Information Technology, Shanghai University of Finance and Economics, Shanghai, China

October 10, 2015

**Abstract**

This paper considers the risk-constrained stochastic traveling salesman problem with random arc costs. In the context of stochastic arc costs, the deterministic traveling salesman problem's optimal solutions would be ineffective because the selected route might be exposed to a greater risk where the actual cost can exceed the resource limit in extreme scenarios. We present a stochastic model of the traveling salesman problem that incorporates risk management. Value at Risk and Conditional Value at Risk are respectively applied to measure and control the risk experiencing overly costly scenarios. We propose a novel cutting plane method to find the minimum-cost route in the stochastic environment while the risk level of the route is controlled by bounding the risk measures. Computational experiments are conducted to demonstrate the properties of the proposed models and the performance of the proposed cutting plan algorithm.

## 1 Introduction

Traveling Salesman Problem (TSP) has been extensively studied in operations research because of its close connection to other optimization problems and wide practical applications. Given a list of cities, the objective of TSP is to find the minimum-cost Hamiltonian cycle of visiting each city exactly once. The history and early works on the TSP can be seen in [7, 5, 20, 13], while recent reviews are found in [17, 16, 1].

While a number of TSP's parameters such as node and arc failures have been modeled as random variables to address various uncertainties in real-world problems, we consider travel cost as the uncertain element in this paper. In the stochastic context, the random cost of each arc is assumed to follow a given/konwn probability distribution. The stochastic TSP, seeking the minimal expected total cost, is essentially a deterministic TSP which uses the expected cost for each arc because of the homogeneity and additivity of the expectation function. However, the expectation only tells one aspect of the total travel cost. For any two routes with the same expected total cost, the variabilities of travel costs can be very different. For example, it is better to take a road with a lower speed limit but few traffic jam other than a road that has a higher speed limit but is frequently subject to constructions and accidents, given that the two road has the same expected travel time. In the presence of stochastic travel costs with such different variabilities, simply minimizing the expected total travel cost or time would be inadequate in these cases. When the costs of one or more arcs have very large variances, it would increase the risk that the total cost exceeds the acceptable limit. A solution with a slightly higher expected cost but a much lower risk would be preferred other than the one with a low cost but much higher risk of being overcostly.

In this paper, we study a risk-constrained stochastic TSP model to help find the route with the least expected total cost while the risk of exceeding the budget is controlled at a certain level. We assume that a directed graph is available and the travel cost/time incurred to move from one node to another is an independent random variable following a distribution with fixed parameters, i.e., distinct mean and variance. All the other parameters in the graph are considered deterministic. In addition, we assume that the route is determined before the random travel costs are known with certainty and no re-route optimization is allowed due to time constraints or applicability; the *a priori* optimal route must be followed once selected. To be general and as the first step, from the various continuous distribution families (e.g., gamma, exponential, student's t, etc.), we choose to use normal distribution for the random arc cost because it has been the most widely applied of all distributions and developed as a standard of reference for many statistic problems. Moreover, we will discuss how our models and solutions algorithm are actually general and can be used to treat cases with other probability distributions.

For controlling the risk associated with the route, one way is to control the fluctuations of its total travel cost by modeling a variance-constrained TSP. However, the main disadvantage of this approach is that it cannot provide quantitative details on risk evaluations such as the risky value and the risky probability (or confidence level), etc., to perform the risk management. For example, the model cannot quantify how bad the cost or loss could be in a planned route. As the cost of each arc is independently and normally distributed, we will demonstrate in Section 3 that the total travel cost is also a random variable with a normal density and the risk management in the stochastic TSP can be achieved by controlling the possible heavy tail in the total cost distribution. With the ability of quantifying worst-cost scenarios for the outcomes of a random variable, value at risk (VaR) and conditional value at risk (CVaR) can be applied as another risk measures for the stochastic TSP. Both VaR and CVaR are commonly used in many engineering areas involving uncertainties, such as military, airspace, and finance [19]. Given a confidence level $\alpha \in (0, 1)$, the VaR at confidence level $\alpha$ is given by the smallest number $l$ such that the probability that the loss exceeds $l$ is at most $(1 - \alpha)$. By calculating the expectation/average of the outcomes that exceed VaR, CVaR provides a more conservative measure to manage the risk of having extreme loss or being overcostly.

We consider our main contributions as follows. We introduce a risk-constrained stochastic TSP with independent arc costs normally distributed, and address its advantages in real-world applications over deterministic TSP. We model the risk-constrained stochastic TSP to control the risk of planned routes by using explicit risk measures, i.e., VaR or CVaR. In addition, we study cutting plane methods for exact solution algorithms to solve the risk-constrained stochastic TSP, and compare its computational performance with other methods including linear reformulation and general branch and cut algorithms. In addition to the application in the TSP, the algorithm could be further generalized to benefit other stochastic network optimization problems in which some information regarding the arcs/links can be modeled as normally distributed random variables.

As follows is the structure of the manuscript. In Section 2 we review related works. Section 3 models the mathematical formulation of the STSP with a risk constraint included to bound the VaR or CVaR. Various exact algorithms are derived in Section 4 for solving the problem. In Section 5, we introduce our numerical experiment setup and report computational results. Final conclusions are made in the last section.

## 2 Literature Review

Researchers have studied the TSP under various uncertain parameters. [8] considers uncertain demand from the nodes, and allow the agent to skip nodes with no demand, and aims to find a priori tour with the minimal expected travel cost. [14] formulate this a priori TSP with uncertain demands as a two-stage stochastic program and solve it using a branch-and-cut approach.

The TSP with random arc cost have also been widely researched. [15] and [2] consider the distances between the nodes nondeterministic. [12] apply the random link TSP with large node size in statistical physics where each arc is weighted as a Boltzmann factor. [11] model vehicle routing problems in which both arc travel times and node service times are stochastic. A branch-and-cut scheme embedded in a Monte Carlo sampling-based procedure is proposed as their solution method. A dynamic TSP with stochastic arc costs are proposed by [23] to allow the salesman to observe outgoing arc realizations at each city before deciding what place to visit next. A probabilistic TSP with deadlines is studied in [3] and time constraints in the context of stochastic customer presence is addressed.

Related studies to our work can be seen in [9], [22], and [4], which also considers a stochastic TSP with independent and normally distributed arc times. Their objective is to maximize the probability of completing the tour by a deadline, which would be inappropriate when it is unnecessary to achieve the lowest risk. We address in the following sections that the expected total cost could be largely increased as the probability increases a little. By using the risk management tool, VaR or CVaR, we can obtain the minimum expected cost route with the risk controlled at a desired level. Another related work is in [13] which considers vehicle routing problems with stochastic service and travel times. A chance constrained model is proposed to minimize the routing costs while ensuring that the probability of the total cost exceeding a given number is at most equal to a confidence level. A general branch and cut algorithm is described to solve the problem. Specifically, if a violation is detected at a candidate integer solution, i.e. the total cost exceeding budget, route or subtour elimination constraints are added to exclude the infeasible solution. Although the method can solve moderate problems to optimality, we demonstrate in the following sections that it would suffer from low efficiency of eliminating infeasible routes in large networks.

The pros and cons of VaR and CVaR for risk management are discussed by [19] regarding stability, simplicity, or specific problems, etc. [6] present an overview of VaR and its usage in finance and optimization. [18] optimize VaR and CVaR simultaneously. [21] present a two-stage stochastic integer program using CVaR. In this paper we model VaR or CVaR constraints to estimate and control the risk associated with the optimal route planned for the traveling salesman. Our model can reduce to deterministic TSP when the risk management constraint is relaxed.

## 3 Problem Description

Given a network $G = (N, A)$, where $N$ denotes the set of nodes indexed by $i$ or $j = 1, 2, ..., n$ and $A$ represents the set of arcs. In this paper we aim to study the risk-constrained TSP where the cost of each arc $(i, j)$ is independently and normally distributed and denoted by $C_{ij}$. The objective is to minimize the expected total cost as follows,

$$\mathbb{E}(Z_{\mathbf{x}}) = \mathbb{E}\left(\sum_{(i,j)\in A} C_{ij}x_{ij}\right) = \sum_{(i,j)\in A} \mathbb{E}(C_{ij})x_{ij} = \sum_{(i,j)\in A} e_{ij}x_{ij} \tag{1}$$

where $Z_{\mathbf{x}}$ is a random variable representing the random total cost given routing decisions $\mathbf{x} = [x_{ij}, (i,j) \in A]^{\mathsf{T}}$; $C_{ij}$ is the random variable normally distributed with mean $e_{ij}$ and variance $v_{ij}$, $\mathbb{E}$ denotes the expectation function of a random variable. Without the requirement of risk controlling, therefore, the stochastic TSP minimizing the expected total travel cost/time can be modeled by its deterministic counterpart as follows,

$$\textbf{[DTSP]:} \quad \min \sum_{(i,j)\in A} e_{ij}x_{ij} \tag{2a}$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in A} x_{ij} = 1, \quad i = 1,2,...,n \tag{2b}$$

$$\sum_{i:(i,j)\in A} x_{ij} = 1, \quad j = 1,2,...,n \tag{2c}$$

$$\sum_{i\in S, j\in S} x_{ij} \leq |S| - 1, \quad S \subsetneq N, |S| > 1 \tag{2d}$$

$$x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A \tag{2e}$$

where all parameters except the arc costs are assumed to be deterministic. Constraints (2b) and (2c) ensure that each node is visited exactly once and constraint (2d) is used to eliminate isolated subtours. Constraints (2e) require routing variables $x_{ij}$'s to be binary (with 1 meaning chosen and 0 otherwise).

Note that a feasible route for the TSP would include a set of arcs to form a Hamiltonian cycle. As described above, the costs of the arcs in the network are random variables which have normal probability densities with distinct means and variances. The probability distribution of the random total cost, $Z_{\mathbf{x}}$, can be obtained from the summation of costs of all the arcs included in the route. For independent normally distributed arc costs, we have the following remark regarding $Z_{\mathbf{x}}$.

*Remark.* In the stochastic TSP with arc costs independently and normally distributed, the total travel cost has a normal probability density with the mean and variance respectively equal to the summation of the expected value and variance of the costs incurred in all included arcs, i.e., that

$$Z_{\mathbf{x}} \sim \mathcal{N}\left(\sum_{(i,j)\in A} e_{ij}x_{ij}, \sum_{(i,j)\in A} v_{ij}x_{ij}\right)$$

Note that for normally distributed random variables, both VaR and CVaR are proportional to the standard deviation (see [18]). Given a TSP solution (a Hamiltonian cycle), we can express the the VaR and CVaR of its total travel cost in terms of means and variances on individual arcs as follows,

$$\text{VaR}_{\alpha}(Z_{\mathbf{x}}) = \min_{l}\{l|P(Z_{\mathbf{x}} \leq l) \geq \alpha\} = \sum_{(i,j)\in A} e_{ij}x_{ij} + k_1(\alpha)(\sum_{(i,j)\in A} v_{ij}x_{ij})^{\frac{1}{2}} \tag{3}$$

$$\text{CVaR}_{\alpha}(Z_{\mathbf{x}}) = E[Z_{\mathbf{x}}|Z_{\mathbf{x}} \geq \text{VaR}_{\alpha}(Z_{\mathbf{x}})] = \sum_{(i,j)\in A} e_{ij}x_{ij} + k_2(\alpha)(\sum_{(i,j)\in A} v_{ij}x_{ij})^{\frac{1}{2}} \tag{4}$$

where $\text{VaR}_\alpha(Z_\mathbf{x})$ denotes the $\alpha$-level VaR, i.e., the $\alpha$-quantile of the random variable $Z_\mathbf{x}$; $\text{CVaR}_\alpha(Z_\mathbf{x})$, i.e., the $\alpha$-level CVaR, equals the average of worst-case costs that exceeds the $\alpha$-level VaR. $k_1(\alpha)$ and $k_2(\alpha)$ are constants and their values are determined by following two equations,

$$\begin{aligned} k_1(\alpha) &= \sqrt{2}\text{erf}^{-1}(2\alpha-1) \\ k_2(\alpha) &= (\sqrt{2\pi}exp(\text{erf}^{-1}(2\alpha-1))^2(1-\alpha))^{-1} \end{aligned}$$

where $\text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\tau^2}d\tau$ is the error function based on normal distribution, and $erf^{-1}(t)$ is its corresponding inverse function.

In addition to the [DTSP], we would model VaR or CVaR contraints for risk-constrained stochastic TSP to fend off the risk of having extreme loss or being overcostly. Specifically, the VaR constraint, $\text{VaR}_\alpha(Z_\mathbf{x}) \leq L$, is modeled so that the probability of the total cost exceeding a certain cost threshold, $L$, is below the confidence level $\alpha$; the CVaR constraint, $\text{CVaR}_\alpha(Z_\mathbf{x}) \leq L$, is modeled to control the $\alpha$-level CVaR of total cost below the threshold $L$.

A nonlinear programming model, [STSP-NLP], is then formed to find the minimum-expected-cost route with the risk of loss being managed by the either VaR or CVaR constraint.

$$\textbf{[STSP-NLP]} \quad \min \sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij}x_{ij} \tag{5a}$$

$$\text{s.t.} \quad (2b) - (2e) \tag{5b}$$

$$\sum_{(i,j)\in A} e_{ij}x_{ij} + k(\alpha)\left(\sum_{(i,j)\in A} v_{ij}x_{ij}\right)^{\frac{1}{2}} \leq L \tag{5c}$$

The objective function (5a) in this formulation also seeks to minimize the expected total cost for the problem. The constraints (5b), which are equivalent to that in the [DTSP], are used to ensure that a Hamiltonian cycle is planned for the traveling salesman. The risk management constraint (5c) is included where the parameter $k(\alpha)$ is equal to $k_1(\alpha)$ when the VaR technique is applied and $k_2(\alpha)$ for CVaR, respectively.

# 4 Algorithms for solving the STSP

We note that the difficulty of solving the proposed [STSP-NLP] mainly comes from the risk management constraint (5c), without which the problem would reduce to the deterministic TSP. In this manuscript we will propose and discuss various strategies to deal with this constraint in order to address the difficulty. First we consider a special case for the constraint, that is, the variances of arc costs are proportional to the means and propose Theorem 4.1 which indicates that in this case the problem becomes a mixed integer linear program (MILP).

**Theorem 4.1.** *Let* $f(\mathbf{x}) = \sum_{(i,j)\in A} e_{ij}x_{ij} + k(\alpha)\left(\sum_{(i,j)\in A} v_{ij}x_{ij}\right)^{\frac{1}{2}} - L : \mathbf{x} \in \mathbb{R}$. *Then* $f(\mathbf{x}) = 0$ *defines a hyperplane in the affine space of* $\mathbf{x} \in \mathbb{R}$, *if* $\frac{v_{ij}}{e_{ij}} = h$, *where* $h$ *is a scalar constant.*

*Proof.* Substitute $v_{ij}$ with $he_{ij}$ and we have, $f(\mathbf{x}) = \sum_{(i,j)\in A} e_{ij}x_{ij} + k(\alpha)\sqrt{h}\left(\sum_{(i,j)\in A} e_{ij}x_{ij}\right)^{\frac{1}{2}} - L$. Solve $f(\mathbf{x}) = 0$ and we can obtain

$$\sum_{(i,j)\in A} e_{ij}x_{ij} = L + [k(\alpha)^2h - k(\alpha)\sqrt{4Lh + k(\alpha)^2h^2}]/2 \tag{6}$$

which defines a hyperplane. □

It can be observed that the hyperplane (6) defined by the risk management constraint is a contour of the objective function in our proposed risk-constrained stochastic TSP. Therefore, the optimal solution of [DTSP] would be optimal to the [STSP-NLP] if the solution is within the half space below the hyperplane, and otherwise the [STSP-NLP] would be infeasible.

When the special case is not applied, on the other hand, the feasible region of the [STSP-NLP] LP relaxation is then neither concave nor convex as indicated by Theorem 4.2. It is then unlikely to adopt the idea of approximating a convex set with a joint half-space combination, as used in the Kelley's cutting plane method [10], to solve the problem.

**Theorem 4.2.** *$f(\mathbf{x}) = \sum_{(i,j)\in A} e_{ij}x_{ij} + k(\alpha)\left(\sum_{(i,j)\in A} v_{ij}x_{ij}\right)^{\frac{1}{2}} - L : \mathbf{x} \in \tilde{P}$ is a concave function, where $\tilde{P}$ is the feasible region of [DTSP] LP relaxation.*

*Proof.* Let $\mathbf{v} = [v_{ij}, (i,j) \in A]^{\mathsf{T}}$. The Hessian matrix of the function $f(x)$ is,

$$H(\mathbf{x}) = -\frac{k(\alpha)}{4}\left(\sum_{(i,j)\in A} v_{ij}x_{ij}\right)^{-\frac{3}{2}} \mathbf{v}\mathbf{v}^{\mathsf{T}}$$

the following observation is straightforward:

$$\mathbf{x}^{\mathsf{T}} H(\mathbf{x})\mathbf{x} < 0, \quad \forall \mathbf{x} \in \tilde{P}$$

which indicates that $H(\mathbf{x})$ is negative definite and thus $f(\mathbf{x})$ is concave. □

In the rest of this paper we will consider that the variances of arc costs are not proportional to the means and propose and compare various strategies to address the difficulty of the proposed problem.

## 4.1 A Linearization Approach

The first method we present here is to develop and solve a mixed integer linear programming (MILP) model by linearizing the risk management constraint in the [STSP-NLP]. Since we only have binary variables, we can take advantage of properties of bilinear and quadratic functions of binary variables. First, we move the expectation on the left of nonlinear constraint (5c) to the right. Second, we square both sides as they are all nonnegative. These two steps will create bilinear and quadratic terms, i.e., $x_{ij}x_{i'j'}$ and $x_{ij}^2$ respectively. The bilinear term can be linearized by introducing additional binary variables $y_{ij,i'j'}$ while enforcing constraints (7d)–(7e). The quadratic term is equivalent to the binary variable itself. Hence, we can transform the [STSP-NLP] to an equivalent mixed integer linear program, [STSP-MILP], formulated as follows,

$$\textbf{[STSP-MILP]} \quad \min \sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij}x_{ij} \tag{7a}$$

$$\text{s.t.} \quad (2b)-(2e) \tag{7b}$$

$$\sum_{(i,j)\in A} a_{ij}x_{ij} - 2\sum_{(i,j)\in A}\sum_{(i'j')\in A'} e_{ij}e_{i'j'}y_{iji'j'} \le L^2 \tag{7c}$$

$$x_{ij} - y_{iji'j'} \ge 0 \quad \forall (i,j)\in A, (i',j')\in A' \tag{7d}$$

$$x_{i'j'} - y_{iji'j'} \ge 0 \quad \forall (i,j)\in A, (i',j')\in A' \tag{7e}$$

$$\sum_{(i,j)\in A} e_{ij}x_{ij} \le L \tag{7f}$$

$$y_{iji'j'} \ge 0 \quad \forall (i,j)\in A, (i',j')\in A' \tag{7g}$$

where $a_{ij} = k^2(\alpha)v_{ij} + 2Le_{ij} - e_{ij}^2$ is a constant determined by the arc $(i,j)$, cost threshold or budget limit $L$ and the confidence level $\alpha$, and $A' = A\backslash\{(i,j)\}$ denotes a set of arcs so that $A'\bigcup\{(i,j)\} = A$ and $A'\bigcap\{(i,j)\} = \emptyset$. The risk management constraint (5c) in the [STSP-NLP] is linearized and replaced by the constraints (7c)-(7g) with a new set of continuous variables $y$ included. Note that we add constraint (7f) to ensure the right side (when we take the sqaure) is a nonnegative value. This is in accordance with the original model, because any solution with $\sum_{(i,j)\in A} e_{ij}x_{ij} > L$ would violate the risk constraint (5c).

It can be easily observed that in the [STSP-MILP], the total number of constraints (7d) and (7e) equals $|A| * (|A| - 1)$ and would largely increase with the number of arcs. For small-sized networks, many off-the-shelf commercial MILP optimization solvers (e.g., CPLEX) can efficiently solve the model to optimality. When the problem size keeps growing, however, a huge number of constraints would be generated, and it would take an unacceptable computational cost even just to obtain a feasible route.

## 4.2 The Branch and Cut Algorithm

A general branch and cut algorithm is introduced in [13] to solve chance constraint vehicle routing problems with stochastic travel times. The problems are firstly solved with the chance constraint relaxed. Integer solutions are evaluated, and marked as illegal routes while the relaxed constraint is violated. A cut is then added to the problem to eliminate the illegal solution. Here we utilize the same idea to iteratively generate route elimination constraints by separating TSP and risk evaluation. Specifically, the [STSP-NLP] is first relaxed by removing the risk management constraint (5c) to form the relaxed master problem [RMP] as follows,

$$\textbf{[RMP]} \quad \min \sum_{(i,j)\in A} e_{ij}x_{ij} \tag{8a}$$

$$\text{s.t.} \quad (2b)-(2e) \tag{8b}$$

$$\sum_{(i,j)\in A} b_{ij}x_{ij} \le b_0 \tag{8c}$$

The [RMP] is almost identical to the deterministic TSP [DTSP] except the additional route elimination constraints (8c), which can be solved by the off-the-shelf commercial solvers. Given the solution from a [RMP], $\hat{\mathbf{x}} = \{\hat{x}_{ij} | (i,j) \in A\}$, the risk evaluation is performed with the route defined by the solution. Note that when the TSP solution or a Hamiltonian route is known, the VaR and CVaR of the random total cost $Z_{\mathbf{x}}$ can be easily obtained by (3) and (4), respectively. When the VaR or CVaR exceeds the cost threshold, the route is marked to be infeasible and we generate the following combinatorial cut for the purpose of route elimination:

$$\sum_{(i,j)\in\bar{A}} x_{ij} \leq |N| - 1 \tag{9}$$

where $\bar{A}$ represents the set of arcs included in the route and $|N|$ is the number of nodes in the input graph. The validity of the cut (9) is straightforward. Specifically, $|N|$ arcs would be included in any TSP route to form a Hamiltonian cycle; when the route is infeasible according to the risk management constraint, the arcs in the set $\bar{A}$ are not allowed to be simultaneously included in one Hamiltonian route in the presence of (9); any other solutions with at least one of $x_{ij}$'s ($(i,j) \in \bar{A}$) equal to 0, would not be eliminated by the combinatorial cut.

A branch and cut algorithm can then be applied to solve the STSP. Specifically, in the branch-and-cut scheme of solving the [RMP], every candidate solution in the search tree is evaluated according to the VaR or CVaR requirement. The feasible one with the objective value better than that of the current incumbent is considered as the new incumbent and produces a new upper bound (UB). Otherwise, the combinatorial cut (9) is added should the VaR or CVaR constraint be violated, giving the problem a new LB. The convergence of the LB and UB would then lead to the optimal solution. The algorithm is summarized as follows,

**Algorithm 1** ALG($\mathcal{B}nC$): Branch and Cut algorithm.

Set UB $= +\infty$, LB $= -\infty$ and the optimal route $\mathbf{x}^* = null$
**while** (UB $-$ LB $> 0$) **do**
  Solve the [RMP], and get a candidate integer solution $\hat{\mathbf{x}} = \{\hat{x_{ij}} : (i,j) \in A\}$ and its objective function value $\hat{z}$.
  Compute VaR or CVaR by the (3) or (4), respectively.
  **if** $\text{VaR}_\alpha(Z_{\hat{\mathbf{x}}}) \leq L$ or $\text{CVaR}_\alpha(Z_{\hat{\mathbf{x}}}) \leq L$ **then**
    Accept $(\hat{x}, \hat{z})$ as the new incumbent and set UB $\leftarrow \hat{z}, \mathbf{x}^* \leftarrow \hat{\mathbf{x}}$.
  **else**
    Reject the incumbent and add the combinatorial cut (9) to the model.
    Update LB by solving the updated [RMP].
  **end if**
**end while**

This branch and cut algorithm could be expected to solve small-sized and some moderately-sized network problems to optimality by iteratively removing the infeasible nodes in the branching tree. While in large-sized networks, there could be a huge number of infeasible routes needed to be assessed and eliminated before reaching the optimal route with desired reliability, which renders the method very low efficiency.

### 4.3 A Cutting Plane Algorithm

In this paper we mainly propose a cutting plane (CP) algorithm to more efficiently remove those "risky" routes that violate the risk management constraint by taking advantage of the structure of the constraint. Given an arbitrary lower bound of the original problem which is denoted as $\bar{z}$, a valid cut can be generated as described in the following proposition.

**Proposition 4.3.**

$$\sum_{(i,j)\in A} v_{ij}x_{ij} \leq [(L-\bar{z})/k(\alpha)]^2 \tag{10}$$

*is a valid cut to the STSP.*

*Proof.* Denote the optimal objective value of STSP as $z^*$ and apparently we have $z^* = \sum_{(i,j)\in A} e_{ij}x_{ij} \geq \bar{z}$. The cut (10) is more relaxed than the constraint (5c) and is then valid to be included in our problem. □

It can be observed that the strength of cutting plane (10) increases as its right hand side decreases (i.e., the lower bound $\bar{z}$ increases). The cut is the strongest when $\bar{z} = z^*$ but invalid when $\bar{z} > z^*$. A bisection method can therefore be considered to find the target right hand side so that the strongest and valid cut is achieved. Specifically, a window $[l, r]$ is first initialized with $l = [(L-\bar{z})/k(\alpha)]^2$ and $r = [(L-\hat{z})/k(\alpha)]^2$ to define the interval for the binary search, where $\bar{z}$ is the optimal cost and $\hat{z}$ is the highest cost of the deterministic TSP, respectively. Assume that the [DTSP] feasible region is close and bounded, then both $\bar{z}$ and $\hat{z}$ are finite numbers. The cut with the right hand side equal to the middle value within the window is then evaluated, that is, $\sum_{(i,j)\in A} v_{ij}x_{ij} \leq (l+r)/2$, by solving the [RMP] with this cut being added. If a feasible solution is obtained from the [RMP] and satisfies the risk management constraint, then the search continues on the left half of the window and the solution provides an UB for the STSP; The right half of the window is also eliminated when the [RMP] is infeasible; The search continues on the right half and a LB is obtained, on the other hand, should the solution obtained from [RMP] violates the VaR or CVaR requirement. This process is repeatedly performed until UB$-$LB $< \varepsilon$.

While the convergence of UB and LB is expectable from the bisection method by iteratively eliminating half interval of the searching window, the performance of the method is limited because a large number of iterations would be required when the window is considerably large. To address this limitation, we develop a CP method to iteratively add a route elimination constraint to (8c), which continuously decreases the size of the [RMP] feasible region until its optimal solution satisfies the VaR or CVaR requirement, and obtain the following proposition.

**Proposition 4.4.** *The STSP can be solved within a finite number of iterations by iteratively solving the [RMP] and adding the cut* (10)*, with $\bar{z}$ equal to the optimal objective function value of [RMP], as a route elimination constraint in* (8c)*.*

*Proof.* Let $\bar{\mathbf{x}}^\nu$ denote the optimal solution of the [RMP] in the $\nu^{th}$ iteration and $\bar{z}^\nu$ be the corresponding objective function value, providing an LB for the STSP. Apparently the solution is also optimal to the STSP should it satisfy the constraint (5c). Otherwise, the [RMP] is updated with the cut $\sum_{(i,j)\in A} v_{ij}x_{ij} \leq [(L-\bar{z}^\nu)/k(\alpha)]^2$ being added, which would cut off the solution $\bar{\mathbf{x}}^\nu$ as well as other points that violate the cut from the [RMP] feasible region. Because of the closeness and boundedness of the [RMP] feasible region, a finite number of cuts need to be generated to continuously reduce the size of the feasible region until the optimal solution to the STSP is achieved. □

As the route elimination constraint (10) is invalid if $\bar{z}$ is replaced with an upper bound, we are unable to incorporate the cut in the branch and cut scheme. Instead of evaluating the candidate integer solutions in the branch and cut algorithm, the CP algorithm iteratively measure the risk of [RMP] optimal solutions by computing their VaRs or CVaRs. The solution is optimal to the STSP should the VaR or CVaR requirement be satisfied; otherwise, the cut (10) is generated and added to the [RMP] to be solved in the next iteration. The method is summarized as follows,

**Algorithm 2** ALG($\mathcal{C}P$): Cutting Plane Method.

**while** (1) **do**
  Solve the [RMP], and get the optimal solution $\bar{\mathbf{x}} = \{\bar{x_{ij}} : (i, j) \in A\}$ and its objective function value $\bar{z}$.
  Compute VaR or CVaR by the equations (3) or (4), respectively
  **if** $\text{VaR}_\alpha(Z_{\bar{\mathbf{x}}}) \leq L$ or $\text{CVaR}_\alpha(Z_{\bar{\mathbf{x}}}) \leq L$ **then**
    The current solution is optimal and break the loop.
  **else**
    Generate the cut (10) and add it to the [RMP].
  **end if**
**end while**

We note that the cut (10) used in the CP method has a higher dimension ($dim = |A| - 1$) than the combinatorial cut (9)($dim = |N| - 1$). In addition, it eliminates from solution space not only the current [RMP] optimal solution with unsatisfying risk level but also those with the cost variance exceeding its right hand side value. The strength of the proposed cutting plane is therefore expected to be improved compared to the combinatorial cut. Let $conv(P)$ represent the convex hull of the [RMP] feasible region and $\bar{\mathbf{x}}$ be the optimal solution violating the risk management constraint with the objective function value $\bar{z}$. Then the developed cut is added as illustrated in Figure 1. A larger lower bound $\bar{z}'$ can be obtained in the next iteration, resulting in a stronger cut ($10'$) as shown in the figure. We repeatedly solve the [RMP] and generate the cuts until achieving the optimal route $\mathbf{x}^*$ which satisfies the VaR or CVaR requirement.

It can be observed that the CP algorithm seeks to find the optimal route with the risk well managed by controlling the variance of the random total travel cost. However, the method differs from modeling a variance-constrained TSP, that is, simply including a variance constraint (i.e., $\text{Var}(Z_{\mathbf{x}}) \leq v^*$) to the deterministic TSP, in both measuring and controlling the risk. Specifically, the variance-constrained TSP measures the risk by evaluating the cost fluctuations or how much the cost deviates from its expected outcome, but what decision makers are really concerned would be how bad the worst-cost scenarios could be, which on the other hand can be quantified by the VaR or CVaR technique in the CP algorithm. While in both methods the risk associated with the route is controlled by bounding the variance of the travel cost, the threshold, $v^*$, cannot be appropriately determined in the variance-constrained model according to the information on risk evaluations. The CP method, however, can well analyze the risk evaluation details, including the confidence level, the mean and variance of total cost of the route evaluated, and cost limit in each iteration, and repeatedly updates and reduces the threshold until the optimal route is eventually found with the $\alpha$-VaR or $\alpha$-CVaR of its total cost controlled below the cost limit.

Compared to the cut (10) controlling only the variance for the purpose of risk management, here we propose another cut to manage the risk by taking into account both the expected value and the variance of total travel cost, as stated in the following proposition.

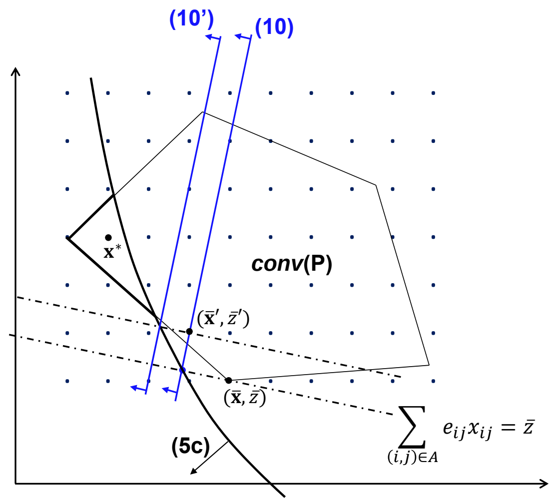


Figure 1: Developed cutting plane. Note: $conv(P)$ is the convex hull of the feasible region of [RMP]; (5c) represents the concave set defined by the risk management constraint. $\bar{\mathbf{x}}$ is the optimal solution of the current [RMP] and $\bar{z}$ is the corresponding cost; The cut (10) is generated and added to the [RMP], which is then solved to obtain a new optimal solution $\bar{\mathbf{x}}'$ with a larger cost $\bar{z}'$, leading to a stronger cut (10′).

**Proposition 4.5.** *Let $\bar{z}$ denote the optimal cost of the [RMP], then the cut*

$$\sum_{(i,j)\in A} [e_{ij} + k(\alpha)^2 v_{ij}/(L - \bar{z})]x_{ij} \leq L \tag{11}$$

*is valid and stronger than the cut* (10)*.*

*Proof.* The risk management constraint (5c) can be reformulated to

$$\sum_{(i,j)\in A} e_{ij}x_{ij} + k(\alpha)\left(\sum_{(i,j)\in A} v_{ij}x_{ij}\right)\left(\sum_{(i,j)\in A} v_{ij}x_{ij}\right)^{-\frac{1}{2}} \leq L$$

According to (10), the validity of the inequality (11) can be confirmed by substituting the second $\sum_{(i,j)\in A} v_{ij}x_{ij}$ above with $[(L-\bar{z})/k(\alpha)]^2$. Assume that an arbitrary point $\tilde{\mathbf{x}} \in conv(P)$ is removed from solution space by the cut (10), which indicates that $\sum_{(i,j)\in A} v_{ij}\tilde{x}_{ij} > [(L-\bar{z})/k(\alpha)]^2$. It is straightforward that $\sum_{(i,j)\in A} e_{ij}\tilde{x}_{ij} \geq \bar{z}$ since $\bar{z}$ is the optimal cost within $conv(P)$. Therefore the solution $\tilde{\mathbf{x}}$ also violates the inequality (11), proving that the cut (11) is stronger than (10). □

Although a stronger route elimination constraint can be generated from the cut (11), the updated [RMP] still provides a LB for the STSP because of the validity of the constraint to the proposed problem. Therefore Proposition 4.4 and Algorithm 2 are both valid when (11) is used in place of the weaker cut (10), while the better performance of the CP method can be expected.

# 5 Numerical Experiments and Results

Numerical experiments and results of different methods are presented in this section on solving the STSP. The algorithms are coded in Microsoft Visual C++ linked with CPLEX 12.5. All the programs are run in Microsoft Windows 7 Professional operating system on a Dell Desktop with Intel Core i7-2600 CPU 3.40GHz and 8GB RAM.

## 5.1 Input data generation.

We consider different settings to precisely evaluate the performance of each algorithm. The STSP are modelled in a set of networks which are generated with different numbers of nodes and arcs. For each size of 10-,50-,100- and 200-nodes, we respectively produce the networks with low, medium and high densities. In order to do so, we randomly generate an integer value from three different value ranges, as shown in 1 and assign it as the outgoing degree to each node. To provide an initially connected graph, we manually generate a cycle route that uses all nodes in $N$.

Table 1: **Input graphs.**

| Nodes | Density of outgoing degrees | | |
|---|---|---|---|
| | Low | Medium | High |
| 10 | 4-6 | 6-8 | 8-9 |
| 50 | 5-15 | 20-30 | 35-45 |
| 100 | 10-20 | 45-55 | 80-90 |
| 200 | 20-30 | 90-110 | 160-180 |

The mean and variance of arc costs in the graphs are produced to define their normal distributions. Specifically, the mean cost of each arc is randomly generated in the range of [30,40] and the variance in [1,1600]. The levels of confidence are set as $\alpha = 0.90$, 0.95 and 0.99, respectively.

## 5.2 Computational results.

The computational results obtained by using different methods on the STSP are shown in Table 2, where computational times are recorded in seconds. We would note that the only difference between the STSP with VaR and CVaR requirements regards to the constant $k(\alpha)$, whose value would not significantly affect the performance of each algorithm. Therefore, in Table 2 we only report the results of solving the problem with the CVaR constraint included to compare the algorithms present in Section 4.

In the table, columns "Node" and "Density" describe the size of each input graph for the STSP and column "L" gives each problem the cost threshold which is arbitrarily determined. Column "Linearize" shows the results when we call CPLEX to solve the linearized model [STSP-MILP]. For the small-sized networks with 10 nodes and various densities, the number of arcs is at most 90 according to Table 1. We can solve the problems to optimality at accepted computational costs. When the network grows to 50 nodes or more, hundreds or thousands of arcs would be included and the number of constraints (7d) and (7e) would be enormous. As a consequence, the problem size would become too large to be handled by our computer.

Table 2: **Solution times, in CPU seconds, for solving the STSP with different algorithms.**

| Node | L | Density | $\alpha$ | Linearize | B&B&C | CP(I) | | CP(II) | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | Itn | time(sec.) | Itn | time(sec.) |
| 10 | 450 | Low | 0.9 | 2.762 | 0.203 | 2 | 0.318 | 2 | 0.312 |
| | | | 0.95 | 5.882 | 5.819 | 3 | 0.487 | 3 | 0.375 |
| | | | 0.99 | 5.569 | 5.959 | 2 | 1.344 | 2 | 0.281 |
| | | Medium | 0.9 | 2.964 | 0.109 | 1 | 0.135 | 1 | 0.14 |
| | | | 0.95 | 5.912 | 0.14 | 2 | 0.281 | 2 | 0.296 |
| | | | 0.99 | 138.461 | 1178.2 | 3 | 0.398 | 2 | 0.296 |
| | | High | 0.9 | 9.079 | 0.156 | 1 | 0.638 | 1 | 0.296 |
| | | | 0.95 | 631.244 | - | 5 | 1.255 | 3 | 0.608 |
| | | | 0.99 | 628.198 | - | 3 | 0.661 | 2 | 0.375 |
| 50 | 1800 | Low | 0.9 | - | - | 5 | 4.852 | 2 | 0.767 |
| | | | 0.95 | - | - | 3 | 3.083 | 2 | 0.768 |
| | | | 0.99 | - | - | 2 | 1.344 | 2 | 0.568 |
| | | Medium | 0.9 | - | 4.555 | 2 | 4.375 | 2 | 4.496 |
| | | | 0.95 | - | - | 4 | 10.897 | 4 | 9.139 |
| | | | 0.99 | - | - | 11 | 31.56 | 6 | 17.463 |
| | | High | 0.9 | - | 0.811 | 1 | 2.02 | 1 | 2.589 |
| | | | 0.95 | - | - | 2 | 4.162 | 2 | 4.94 |
| | | | 0.99 | - | - | 6 | 31.766 | 6 | 27.581 |
| 100 | 3300 | Low | 0.9 | - | - | 3 | 4.995 | 2 | 6.505 |
| | | | 0.95 | - | - | 2 | 6.061 | 2 | 7.285 |
| | | | 0.99 | - | - | 2 | 3.836 | 2 | 4.165 |
| | | Medium | 0.9 | - | - | 7 | 186.582 | 7 | 102.539 |
| | | | 0.95 | - | - | 6 | 90.691 | 3 | 28.439 |
| | | | 0.99 | - | - | 4 | 48.683 | 2 | 10.826 |
| | | High | 0.9 | - | - | 2 | 19.484 | 2 | 17.503 |
| | | | 0.95 | - | - | 6 | 164.509 | 3 | 64.631 |
| | | | 0.99 | - | - | 10 | 274.526 | 5 | 115.676 |
| 200 | 6500 | Low | 0.9 | - | - | 5 | 73.742 | 2 | 19.032 |
| | | | 0.95 | - | - | 4 | 49.948 | 2 | 15.803 |
| | | | 0.99 | - | - | 3 | 19.299 | 2 | 21.118 |
| | | Medium | 0.9 | - | - | 2 | 42.369 | 2 | 71.979 |
| | | | 0.95 | - | - | 2 | 170.582 | 2 | 47.284 |
| | | | 0.99 | - | - | 6 | 563.774 | 4 | 170.889 |
| | | High | 0.9 | - | - | 2 | 101.782 | 2 | 136.383 |
| | | | 0.95 | - | - | 2 | 103.124 | 2 | 97.61 |
| | | | 0.99 | - | - | 2 | 244.25 | 2 | 368.985 |

Note: The dash in the table indicates that no feasible solution is obtained within 3600 sec.

Column “Branch and cut” shows the computational time of using the branch and cut algorithm as described in Section 4.2. No significant improvement is observed compared to calling CPLEX to solve the [STSP-MILP]. The algorithm can solve small-sized networks at low computational costs but is limited while the network size or the level of confidence increases. The reason is that, along the process of branch and bound to solve the relaxed problem, the generated combinatorial cut is not efficient enough to prune infeasible solutions in the search tree. Our computational results show that in most networks even a feasible solution cannot be found after exhaustive search (after 3600 seconds).

The performance of the cutting plane algorithm is shown in columns "CP(I)" and "CP(II)" in the table with the cuts (10) and (11) being respectively applied. The number of iterations as well as the computational time of the method are shown in the columns "Itn" and "time". Compared to the branch and cut algorithm that is expected to obtain the optimal solution by visiting the search tree only once, the cutting plane method would visit the tree for more times before finding the optimal solution. The generated cutting planes, however, is much more efficient than the combinatorial cut and can quickly remove infeasible solutions. Our computational results show that only a few iterations are needed to find the optimal solution and the problem in each generated network with up to 200 nodes can be solved at a reasonable cost. The better computational performance obtained from "CP(II)" for most instances confirms the higher strength of the cut (11).

The process of using the CP method is well demonstrated in Figure 2. The optimal objective values (OFV) of the [RMP] in each iteration is shown when the VaR and CVaR are respectively used for the risk management. The network with 50 nodes and medium density is applied and the level of confidence is set as 99%. The CVaR technique is known to be more conservative than VaR; this means that at the same confidence level, the higher total travel cost would be expected to select the route that satisfies the CVaR requirement than VaR. In addition, the more conservative method (i.e., CVaR) would expect more iterations in the CP algorithm before reaching optimality as shown in the figure. However, this is not the necessary scenario as the route elimination cuts (10) and (11) generated in each iteration of the algorithm is stronger when the CVaR is applied.

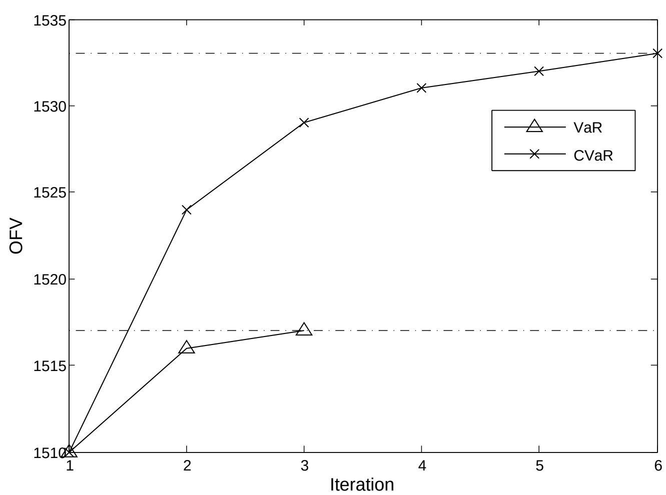


Figure 2: **The change in objective function values of updated [RMP].** Input graph: 50 nodes with high density.

Figure 3 shows the change in the probability distributions of total travel cost when the VaR or CVaR risk management is applied to the STSP, respectively. It can be observed that the route solved from the deterministic TSP [DTSP] is highly exposed to the risk that the actual cost can exceed the cost limit. The risk is well managed as the stochastic model is applied while expected total cost is not significantly increased as shown in the figure. The probability that the total travel cost exceeds the cost limit is controlled below the confidence level $\alpha$ by the VaR requirement. When the CVaR is applied, the risky level is further reduced so that the average of the percentage, $1-\alpha$, of worst cases is bounded by the cost limit. Along the process of running the CP algorithm, the change in the standard deviations (square root of variances) of total costs is represented by the dash lines in Figure 3(right) while the VaR and CVaR techniques are respectively applied. Each solid line in the figure shows the decrease of VaR or CVaR achieved from the CP method until the cost limit (1800) is crossed.

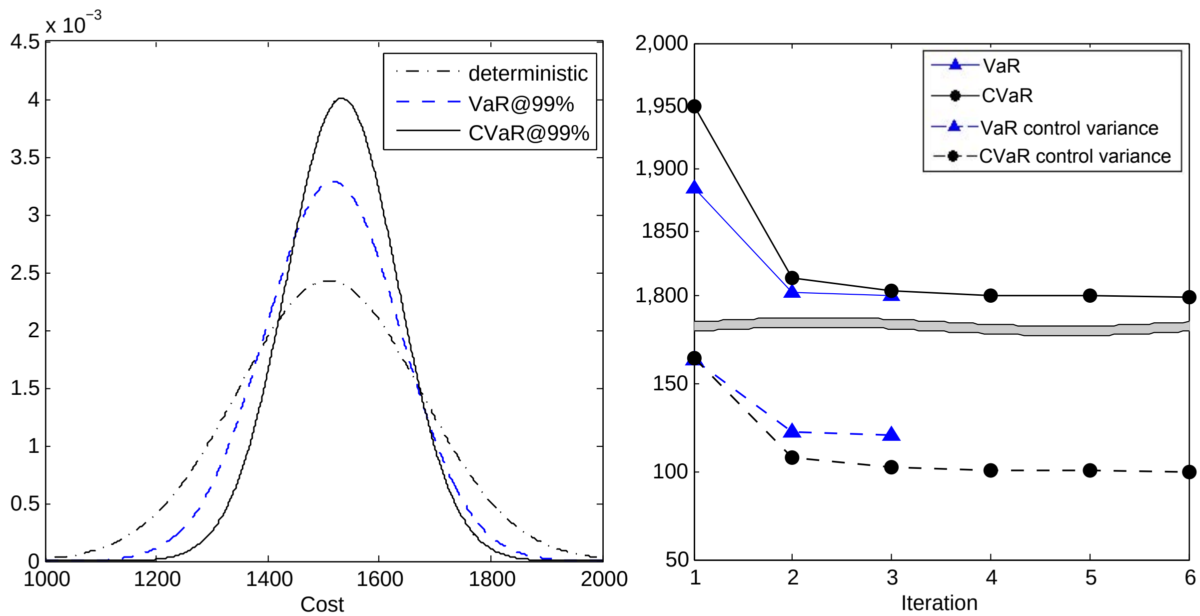


Figure 3: **Probability distribution of total optimal cost as VaR or CVaR method is used (left); the process of VaR or CVaR controlling cost variances for the purpose of risk management in the CP algorithm (right).** Note: The network with 50 nodes and high density is applied and the confidence level is 99%.

Figure 4 shows the relation between the optimal expected cost and confidence level in the generated network with 200 nodes and medium density. Higher cost would be expected to investigate on the routes with higher reliability. The problem turns out to be infeasible when it is impossible to reach the required reliability with the available resource. The expected total cost is largely increased as the probability increases a little from 97% to 98%, which indicates that it would be inappropriate to maximize the probability of completing the tour by a deadline (or a cost limit) in our case.

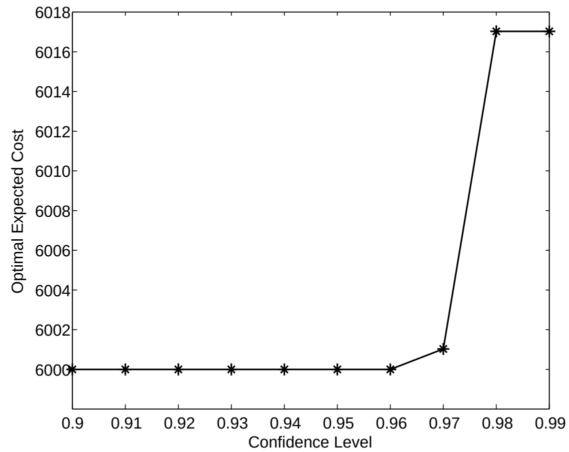


Figure 4: **The change in the optimal expected costs as the confidence level increases.** Note: The network with 200 nodes and medium density is applied.

# 6 Conclusions

In this paper, we proposed a STSP with the travel cost of each arc independently and normally distributed. A nonlinear programming model is formulated for the STSP to find the route with minimum expected cost while the risk is controlled at a confidence level. Both VaR and CVaR were respectively applied as the risk measures to evaluate the reliability of selected routes. A CP algorithm was developed to address the computational difficulty of the problem, and our computational results demonstrated that it can solve the STSPs in all the generated networks at reasonable computational costs and has overwhelming performance beyond the linearization method and the branch and cut algorithm.

# References

[1] D.L. Applegate, R.E. Bixby, V. Chvatal, and Cook W.J. *The traveling salesman problem: A computational study.* Princeton University Press, Princeton, New Jersey, 2006.

[2] O. Berman. The traveling salesman location problem on stochastic networks. *Transportation Science*, 23(1):54–57, 1989.

[3] A.M. Campbell and B.W. Thomas. Probabilistic traveling salesman problem with deadlines. *Transportation Science*, 42(1):1 – 21, 2008.

[4] R.L. Carraway, T.L. Morin, and H. Moskowit. Generalizedd ynamic programming for stochastic combinatorial optimization. *Operations Research*, 37(5):819–829, 1989.

[5] G. Dantzig, Fulkerson R., and S. Johnson. Solution of a large-scale traveling-salesman problem. *Operations Research*, 2:393–410, 1954.

[6] D. Duffie and J. Pan. An overview of value at risk. *The Journal of derivatives*, 4(3):7–49, 1997.

[7] M.M. Flood. The traveling-salesman problem. *Operations Research*, 4(1):61–75, 1956.

[8] P. Jaillet. A priori solution of a traveling salesman problem in which a random subset of the customers are visited. *Operations Research*, pages 929–936, 1988.

[9] E.P.C. Kao. A preference order dynamic program for a stochastic traveling salesman problem. *Operations Research*, 26(6):1033–1045, 1978.

[10] JEJ Kelley. The cutting-plane method for solving convex programs. *Journal of the Society for Industrial and Applied Mathematics*, pages 703–712, 1960.

[11] A.S. Kenyon and D.P. Morton. Stochastic vehicle routing with random travel times. *Transportation Science*, 37(1):69–82, 2003.

[12] W. Krauth and M. Mézard. The cavity method and the traveling-salesman problem. *Europhysics Letters*, 8(3):213–218, 1989.

[13] G. Laporte. The traveling salesman problem: An overview of exact and approximate algorithms. *European Journal of Operational Research*, 59:231–247, 1992.

[14] G. Laporte, F.V. Louveaux, and H. Ercure. A priori optimization of the probabilistic traveling salesman problem. *Operations Research*, 42(3):543–549, 1994.

[15] T. Leipälä. On the solutions of stochastic traveling salesman problems. *Eur. J. Oper. Res.*, 2:291–297, 1978.

[16] R. Matai, S. Singh, and M.L. Mittal. *Traveling Salesman Problem: an Overview of Applications, Formulations, and Solution Approaches*, chapter 1. Traveling Salesman Problem, Theory and Applications, 2010.

[17] A.J. Orman and H.P. Williams. *A Survey of Different Integer Programming Formulations of the Travelling Salesman Problem*, volume 9, pages 91–104. Optimisation, Econometric and Financial Analysis, Springer Berlin Heidelberg, 2007.

[18] R.T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *In Journal of Risk*, 2(3):21–41, 2000.

[19] Uryasev S. Sarykalin S., Serraino G. Value-at-risk vs. conditional value-at-risk in risk management and optimization.tutorials in operations research. *INFORMS*, pages 270–294, 2008.

[20] A. Schrijver. *On the history of combinatorial optimization till 1960*, volume 12. Elsevier, 2005.

[21] R. Schultz and S. Tiedemann. Conditional value-at-risk in stochastic programs with mixed-integer recourse. *Mathematical Programming*, 105(2):365–386, 2006.

[22] M. Sniedovich. Analysis of a preference order traveling salesman problem. *Operations Research*, 29(6):1234–1237, 1981.

[23] A. Toriello, W.B. Haskell, and M. Poremba. A dynamic traveling salesman problem with stochastic arc costs. 2013.