A composition projection method for convex feasibility problems

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Abstract

In this article, we propose a composition projection algorithm for solving feasibility problem in Hilbert space. The convergence of the proposed algorithm are established by using gap vector which does not involve the nonempty intersection assumption. Moreover, we provide the sufficient and necessary condition for the convergence of the proposed method.

Key words: Feasibility problem; Gap vector; Projection.
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1 Introduction

Throughout this paper, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle \text{ and norm } \|\cdot\|,$$

and that

$$A \text{ and } B \text{ are two nonempty closed and convex subsets of } X.$$

The *distance* between the subsets A, B of X by

$$d(A,B) := \inf\{\|w-l\| : w \in A, l \in B\}.$$

If B is empty, we set $\inf_{w\in A} d(x,B) = +\infty$.

Let $x \in X$. The *metric projection* of x onto a nonempty closed and convex subset B is defined by

$$P_B(x) = \text{argmin}_{y\in B}\|y-x\|.$$

It is well-known that P_B is single-valued and nonexpansive. For $x \in X$, the *metric projection* $P_B(x)$ is characterized by

Kolmogorov's criterion :

(1) $$P_B(x) \in B \text{ and } \langle y - P_B(x), x - P_B(x)\rangle \leq 0 \text{ for all } \quad y \in B.$$

We are interested in the following feasibility problem (shortly, (FP)):

(2) $$\text{Find } x \in A\bigcap B.$$

It is worth noting that many authors studied the common element for variational inequalities, equilibrium problem, maximal monotone operators and fixed points of nonlinear operators which can be

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considered as special cases of the problem (FP) (see, [2, 6, 7, 8, 10]). In many practical problems, the set $A \bigcap B$ is empty. A natural question arises: whether there exists a good substitute for $A \bigcap B$ when it is empty?

Bauschke and Borwein[1] introduced two good generalization of $A \bigcap B$:

$$E := \{a \in A : d(a, B) = d(A, B)\}, \tag{3}$$

$$F := \{b \in B : d(b, A) = d(B, A)\}. \tag{4}$$

Particularly, if $A \bigcap B \neq \emptyset$, then $E = F = A \bigcap B$.

For the reader's convenience, we recall the following well-known definitions and results.

Definition 1.1 Let a mapping $T : X \rightrightarrows X$ with graph $grT = \{(x, u) \in X \times X : u \in T(x)\}$. T is said to be:

(i) *monotone* if $\langle x - y, \xi - \zeta \rangle \geq 0$ for all $(x, \xi), (y, \zeta) \in grT$;

(ii) *maximal monotone* if T is monotone and no proper enlargement of grT is monotone.

We also denote the set of *fixed points* of T by $\mathrm{Fix}(T) = \{x \in X : x \in T(x)\}$, and the *resolvent* of T is defined as $J_T := (I + T)^{-1}$.

Definition 1.2 (See [1, 11]) Let $v \in X$. v is said to be a *gap vector* from A to B if, $v = P_{\overline{B-A}}(0)$.

It is easy to see that if v is a gap vector from A to B, then $-v$ is also a gap vector from B to A, and $-v = P_{\overline{A-B}}(0)$.

Fact 1.3 (See [1].) Let v be a gap vector from A to B, and E, F be defined by (3) and (4). Then

(i) $\|v\| = d(A, B), \quad E + v = F$;

(ii) $E = \mathrm{Fix}(P_A P_B) = A \bigcap (B - v), \quad F = \mathrm{Fix}(P_B P_A) = B \bigcap (A + v)$;

(iii) $P_B e = P_F e = e + v \quad (e \in E), \quad P_A f = P_E f = f - v \quad (f \in F)$.

For more information on the gap vector see, for instance, [1, 11] and the references therein.

Definition 1.4 (See [9]) Let $h : X \to (-\infty, +\infty]$ be a proper convex function. The *subdifferential* of h at x is defined by

$$\partial h(x) := \{\xi \in X : h(x + \tau) \geq h(x) + \langle \xi, \tau \rangle, \quad \forall \tau \in X\};$$

Definition 1.5 (See [3, 9]) Let Ω be a subset of X. The dual cone of Ω is

$$\Omega^* = \{\xi \in X : \langle \xi, x \rangle \geq 0, \quad \forall x \in \Omega\},$$

the polar cone of Ω is $\Omega^\circ = -\Omega^*$, the tangent cone of Ω at x is

$$T_\Omega(x) := \begin{cases} \overline{cone}(\Omega - x), & \text{if} \quad x \in \Omega, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Fact 1.6 (See [9, 11]) Let $\Omega \subseteq X$ and $h : X \to (-\infty, +\infty]$ be a proper convex function. Then

(i) the subdifferential operator $\partial h : X \rightrightarrows X$ is maximal monotone;

(ii) the proximal mapping of h, denoted by Prox_h, has a full domain and $\mathrm{Prox}_h := J_{\partial h}$.

Particularly, if $h = \iota_\Omega$, then $\text{Prox}_{\iota_\Omega} = P_\Omega$ and $\partial\iota_\Omega = N_\Omega$, where N_Ω is the normal cone operator, and ι_Ω is the indicator function of Ω defined by

$$N_\Omega(x) := \begin{cases} \{\xi \in X : \langle \xi, y - x\rangle \le 0, \quad \forall y \in \Omega\}, & \text{if} \quad x \in \Omega, \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$\iota_\Omega(x) = \begin{cases} 0, & \text{if} \quad x \in \Omega, \\ +\infty, & \text{otherwise.} \end{cases}$$

Fact 1.7 (See [3, 9]) Let Ω be a nonempty convex subset of X and let $x \in \Omega$. Then the following hold:

(i) $N_\Omega(x) = T^\circ_\Omega(x) = -T^*_\Omega(x)$ and $N^\circ_\Omega(x) = -N^*_\Omega(x) = T_\Omega(x)$;

(ii) $T_\Omega(x) = X \Leftrightarrow N_\Omega(x) = \{0\}$.

We now explore some properties of the gap vector.

Lemma 1.8 *Let $\bar{a} \in A$ and $\bar{b} \in B$ and $v = \bar{a} - \bar{b}$. Then the following statements are equivalent:*

(i) *v is a gap vector from B to A;*

(ii) *$\bar{a} = P_A(\bar{b})$ and $\bar{b} = P_B(\bar{a})$;*

(iii) *$v \in N_B(\bar{b})$ and $-v \in N_A(\bar{a})$;*

(iv) *$v \in T^\circ_B(\bar{b}) \bigcap T^*_A(\bar{a})$;*

(v) *$(\bar{a}, \bar{b})$ is a solution of the following optimization problem:*

$$\min_{(a,b)} \quad [1/2\|a - b\|^2 + \iota_{A\times B}(a, b)], \tag{5}$$

where $\iota_{A\times B}$ is the indicator function of $A \times B$.

Proof. "(i)$\Rightarrow$(ii)": Suppose that $v = \bar{a} - \bar{b}$ is a gap vector from B to A. By Definition 1.2, we have

$$v = P_{\overline{A-B}}(0).$$

Then

$$\langle y - v, -v\rangle \le 0, \quad \forall y \in \overline{A - B}.$$

That is,

$$\langle y - (\bar{a} - \bar{b}), \bar{b} - \bar{a}\rangle \le 0, \quad \forall y \in \overline{A - B}. \tag{6}$$

For any $x \in A$ and $z \in B$, $x - z \in \overline{A - B}$. It follows from (6) that

$$\langle x - z - (\bar{a} - \bar{b}), \bar{b} - \bar{a}\rangle \le 0, \quad \forall x \in A, z \in B.$$

Moreover, one has

$$\langle x - \bar{a}, \bar{b} - \bar{a}\rangle \le \langle z - \bar{b}, \bar{b} - \bar{a}\rangle, \quad \forall x \in A, z \in B. \tag{7}$$

Take $z = \bar{b}$ and $x = \bar{a}$ in (7), respectively, we have

$$\langle x - \bar{a}, \bar{b} - \bar{a} \rangle \leq 0, \quad \forall x \in A$$

and

$$\langle z - \bar{b}, \bar{a} - \bar{b} \rangle \leq 0, \quad \forall z \in B.$$

Therefore, from (1), we derive that $\bar{a} = P_A(\bar{b})$ and $\bar{b} = P_B(\bar{a})$.

"(ii)⇒(iii)": Note that

$$\begin{cases} \bar{a} = P_A(\bar{b}), \\ \bar{b} = P_B(\bar{a}), \end{cases} \Leftrightarrow \begin{cases} \bar{a} = (I + N_A)^{-1}(\bar{b}), \\ \bar{b} = (I + N_B)^{-1}(\bar{a}), \end{cases}$$
$$\Leftrightarrow \begin{cases} \bar{a} \in (I + N_B)(\bar{b}), \\ \bar{b} \in (I + N_A)(\bar{a}), \end{cases}$$
$$\Leftrightarrow \begin{cases} \bar{a} - \bar{b} \in N_B(\bar{b}), \\ \bar{b} - \bar{a} \in N_A(\bar{a}), \end{cases}$$
$$\Leftrightarrow \begin{cases} v \in N_B(\bar{b}), \\ -v \in N_A(\bar{a}). \end{cases}$$

"(iv)⇔(iii)": It directly follows from Fact 1.7.

"(v)⇔(iii)": Let $f(a, b) = 1/2\|a - b\|^2 + \iota_{A \times B}(a, b)$ for all $(a, b) \in X \times X$. It is well-known that $(\bar{a}, \bar{b})$ is a solution of the problem (5) if and only if $(0, 0) \in \partial f(\bar{a}, \bar{b})$. Note that

$$\partial f(\bar{a}, \bar{b}) = (\bar{a} - \bar{b} + \partial \iota_A(\bar{a}), \bar{b} - \bar{a} + \partial \iota_B(\bar{b})) = (\bar{a} - \bar{b} + N_A(\bar{a}), \bar{b} - \bar{a} + N_B(\bar{b})).$$

Then

$$(0, 0) \in \partial f(\bar{a}, \bar{b}) \quad \Leftrightarrow \begin{cases} 0 \in \bar{a} - \bar{b} + N_A(\bar{a}), \\ 0 \in \bar{b} - \bar{a} + N_B(\bar{b}), \end{cases}$$
$$\Leftrightarrow \begin{cases} -v \in N_A(\bar{a}), \\ v \in N_B(\bar{b}). \end{cases}$$

"(iii)⇒(i)": Suppose that $v \in N_B(\bar{b})$ and $-v \in N_A(\bar{a})$. Then

$$\langle v, z - \bar{b} \rangle \leq 0, \ \langle -v, x - \bar{a} \rangle \leq 0, \ \forall x \in A, z \in B.$$

Moreover, we obtain that

$$\langle x - z - v, 0 - v \rangle \leq 0, \ \forall x \in A, z \in B.$$

Hence, we get

(8) $$\langle \omega - v, 0 - v \rangle \leq 0, \ \forall \omega \in A - B.$$

Claim. $v = P_{\overline{A-B}}(0)$. Suppose to the contrary that there exists $y \in \overline{A - B}$ such that

(9) $$\langle y - v, 0 - v \rangle > 0.$$

Then there exists a sequence $y_n \in A - B$ such that $y_n \to y$. By (8), we have

$$\langle y_n - v, 0 - v \rangle \leq 0.$$

Taking the limit in the above inequality, one has

$$\langle y - v, 0 - v \rangle = \lim_{n \to \infty} \langle y_n - v, 0 - v \rangle \leq 0,$$

which contradicts (9). This completes the proof. ■

Lemma 1.9 *Let $\bar{a} \in A$ and $\bar{b} \in B$. Then*

$$\bar{a} = P_A(\bar{b}), \bar{b} = P_B(\bar{a}) \Leftrightarrow \bar{a} = P_A P_B(\bar{a}), \bar{b} = P_B P_A(\bar{b}).$$

Proof. The necessity is obvious. We only need to prove the sufficiency. Assume that $\bar{a} = P_A P_B(\bar{a})$ and $\bar{b} = P_B P_A(\bar{b})$. Then

$$\bar{a} = P_A P_B(\bar{a}) = P_A P_B P_A(\bar{a}) = P_A(\bar{b})$$

and

$$\bar{b} = P_B P_A(\bar{b}) = P_B P_A P_B(\bar{b}) = P_B(\bar{a}).$$

Consequently, $\bar{a} = P_A(\bar{b}), \bar{b} = P_B(\bar{a})$. This completes the proof. ■

Fact 1.10 (See [3, 5]) Let Ω be a nonempty closed and convex subset of X, let $T : \Omega \to X$ be nonexpansive, let $(x_n)_{n \in N}$ be a sequence in Ω, and $x \in X$. Suppose that $x_n \rightharpoonup x$ and that $x_n - T(x_n) \to 0$. Then $x \in \mathrm{Fix}(T)$.

2 Main results

In this section, we propose a composition projection algorithm for solving feasibility problem in Hilbert space. The asymptotic behaviors of the proposed algorithm are established by using gap vector which does not involve the nonempty intersection assumption. Moreover, we provide the sufficient and necessary condition for the convergence of the proposed algorithm.

Theorem 2.1 Let A, B be two nonempty closed convex subsets of a Hilbert space X, and let the sequence (x_n) be generated by the following algorithm:

$$\begin{cases} x_1 \in X \quad \text{arbitrarily}, \\ y_n = P_A P_B(x_n), \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{10}$$

where $C_1 = X$. Assume that there exist $a \in A$ and $b \in B$ such that $d(A, B) = \|a - b\|$. Then the following statements hold:

(i) the sequence (x_n) generated by Algorithm (10) strongly converges to the point p, where $p = P_E(x_1)$ and $E = \mathrm{Fix}(P_A P_B)$;

(ii) $p - P_B(p) = P_{\overline{A-B}}(0)$;

(iii) $\lim_{n\to\infty} \|x_n - P_B(x_n)\| = \lim_{n\to\infty} \|y_n - P_B(x_n)\| = \|p - P_B(p)\| = d(A, B)$;

(iv) $p - P_B(p) \in N_B(P_B(p))$ and $P_B(p) - p \in N_A(p)$.

Proof. (i) We proceed in several steps.

Step 1. $E = \mathrm{Fix}(P_A P_B)$ is nonempty closed and convex.

To this aim, we divide into two cases:

$(a1)$ Let $A \cap B \neq \emptyset$. Since A and B are two nonempty closed convex subsets of a Hilbert space X, from Fact 1.3(ii), $E = \mathrm{Fix}(P_A P_B) = A \cap B \neq \emptyset$.

$(b1)$ Let $A \cap B = \emptyset$. Since there exist $a \in A$ and $b \in B$ such that $d(A, B) = \|a - b\|$, then $b - a$ is a gap vector from A to B. Moreover, we conclude that $d(A, B) = \|a - b\| > 0$ and

$$E = \text{Fix}(P_A P_B) = A \cap (B - v) \neq \emptyset,$$

where $v = P_{\overline{B-A}}(0) = b - a$ and $\|v\| = d(A, B)$.

Combining (a1) and (b1) yield that $E = \text{Fix}(P_A P_B) \neq \emptyset$. Since P_A and P_B are nonexpansive, for any $x, y \in X$,

$$\|P_A P_B(x) - P_A P_B(y)\| \leq \|P_B(x) - P_B(y)\| \leq \|x - y\|.$$

This means that $P_A P_B$ is also nonexpansive on X. Thus, $E = \text{Fix}(P_A P_B)$ is closed and convex.

Step 2. $E \subseteq C_n$ for all $n \geq 1$.

For any given $u \in E = \text{Fix}(P_A P_B)$, we have

$$\|u - y_1\| = \|P_A P_B(u) - P_A P_B(x_1)\| \leq \|u - x_1\|.$$

So, $E = \text{Fix}(P_A P_B) \subseteq C_1$. Moreover, one has

$$\|u - y_n\| = \|P_A P_B(u) - P_A P_B(x_n)\| \leq \|u - x_n\|, \quad \forall n \in N,$$

that is, $E \subseteq C_{n+1}$ for all $n \geq 1$. Therefore, $E \subseteq C_n$ for all $n \geq 1$.

Step 3. (x_n) is well defined.

From Steps 1,2 and Algorithm (10), it is easy to see that C_n is nonempty closed and convex for all $n \geq 1$. Consequently, (x_n) is well defined.

Step 4. The sequence (x_n) is bounded and $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Note that $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subseteq C_n$. For $x_n = P_{C_n} x_1 (n > 1)$, one has

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|. \tag{11}$$

For any given $u \in E$, from Step 2, $u \in C_n$ for all $n \geq 1$. It follows from $x_n = P_{C_n} x_1$ that

$$\langle u - x_n, x_1 - x_n \rangle \leq 0.$$

Note that

$$\begin{aligned}
& \|u - P_{C_n} x_1\|^2 + \|P_{C_n} x_1 - x_1\|^2 \\
= \;& \|u - x_n\|^2 + \|x_n - x_1\|^2 \\
= \;& \|u\|^2 - 2\langle u, x_n \rangle + 2\|x_n\|^2 - 2\langle x_n, x_1 \rangle + \|x_1\|^2 \\
= \;& \|u - x_1\|^2 + 2\langle u, x_1 \rangle - 2\langle u, x_n \rangle + 2\|x_n\|^2 - 2\langle x_n, x_1 \rangle \\
= \;& \|u - x_1\|^2 + 2\langle u - x_n, x_1 \rangle + 2\langle x_n - u, x_n \rangle \\
= \;& \|u - x_1\|^2 + 2\langle u - x_n, x_1 - x_n \rangle \\
\leq \;& \|u - x_1\|^2.
\end{aligned}$$

i.e.,

$$\|u - x_n\|^2 + \|x_n - x_1\|^2 \leq \|u - x_1\|^2.$$

Therefore, one has

$$\|x_n\| - \|x_1\| \leq \|x_n - x_1\| \leq \|u - x_1\|$$

and so, $\|x_n\| \leq \|u - x_1\| + \|x_1\|$. These show that the sequences (x_n) and $(x_n - x_1)$ are bounded. It follows from (11) that $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Step 5. The sequence (x_n) is a Cauchy sequence.

For any positive integer numbers m, n and $m > n$, one has $x_m \in C_m \subseteq C_n$. Again from $x_n = P_{C_n} x_1$, we have

$$\langle x_m - x_n, x_1 - x_n \rangle \leq 0. \tag{12}$$

Taking into account $\|x_m - x_n\|^2 + \|x_n - x_1\|^2 = \|x_m - x_1\|^2 + 2\langle x_m - x_n, x_1 - x_n\rangle$, from (12), we have

$$\|x_m - x_n\|^2 + \|x_n - x_1\|^2 \leq \|x_m - x_1\|^2.$$

Then

$$\begin{aligned} \|x_m - x_n\|^2 &\leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2 \\ &= (\|x_m - x_1\| + \|x_n - x_1\|)(\|x_m - x_1\| - \|x_n - x_1\|). \end{aligned}$$

Therefore, (x_n) is a Cauchy sequence and so, $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Without loss of generality, let $x_n \to p \in X$.

Step 6. $p \in E = \mathrm{Fix}(P_A P_B)$.

Since $x_{n+1} = P_{C_{n+1}} x_1$, $x_{n+1} \in C_{n+1}$. By the definition of C_{n+1}, one has

$$\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|. \tag{13}$$

It follows from (13) and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ that $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Noticing that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

We have that $\|x_n - y_n\| \to 0$ as $n \to \infty$. Consequently, $y_n \to p$ and $x_n - P_A P_B x_n \to 0$ as $n \to \infty$. By the nonexpansiveness of $P_A P_B$ and Corollary 4.18 ([3],p64), we conclude that $p \in \mathrm{Fix}(P_A P_B) = E$.

Last step . $p = P_E x_1$.

Without loss of generality, let $q = P_E x_1 = P_{\mathrm{Fix}(P_A P_B)} x_1$. Then $q \in E \subseteq C_n$ for all $n \geq 1$. Since $x_{n+1} = P_{C_{n+1}} x_1$, $\|x_{n+1} - x_1\| \leq \|z - x_1\|$ for all $z \in C_{n+1}$. From this and $E \subseteq C_{n+1}$, we have

$$\|x_{n+1} - x_1\| \leq \|q - x_1\|. \tag{14}$$

In view of $x_{n+1} \to p \in E$. Take $n \to \infty$ in (14), one has

$$\|p - x_1\| \leq \|q - x_1\|.$$

This, together with $q = P_E x_1$, shows that $p = q = P_E x_1$.

(ii) Let us prove that $p - P_B(p) = P_{\overline{A-B}}(0)$.

Indeed, since $p \in E = \mathrm{Fix}(P_A P_B)$, $p = P_A P_B(p)$. Therefore, by Fact 1.3 and Lemma 1.8, we have

$$v = p - P_B(p) = P_{\overline{A-B}}(0). \tag{15}$$

(iii) Let us prove that

$$\lim_{n\to\infty} \|x_n - P_B(x_n)\| = \lim_{n\to\infty} \|y_n - P_B(x_n)\| = \|p - P_B(p)\| = d(A, B).$$

Since $x_n, y_n \to p$ and from the continuity of P_B, one has

$$\lim_{n\to\infty} \|x_n - P_B(x_n)\| = \lim_{n\to\infty} \|y_n - P_B(x_n)\| = \|p - P_B(p)\|.$$

It follows from (15) that $p - P_B(p)$ is a gap vector from B to A. This, together with Facts 1.3, shows that

$$\|v\| = \|p - P_B(p)\| = d(A, B).$$

As a consequence, we derive that

$$\lim_{n\to\infty} \|x_n - P_B(x_n)\| = \lim_{n\to\infty} \|y_n - P_B(x_n)\| = \|p - P_B(p)\| = d(A, B).$$

(iv) It follows from (15) and Lemma 1.8 that

$$p - P_B(p) \in N_B(P_B(p)), \quad P_B(p) - p \in N_A(p).$$

This completes the proof. ■

The next corollary shows the Algorithm (10) to solve a convex feasibility problem.

Corollary 2.2 *Let A, B be two nonempty closed convex subsets of a Hilbert space X such that $A \cap B \neq \emptyset$. Then the sequence (x_n) generated by Algorithm (10) strongly converges to some point p of $A \cap B$, moreover, $p = P_{A\cap B}(x_1)$.*

Particularly, if A and B are two closed affine subspaces of X, we have the following result.

Corollary 2.3 *Let A, B be two closed affine subspaces of a Hilbert space X. Then the sequence (x_n) generated by Algorithm (10) strongly converges to some point p of $A \cap B$, moreover, $p = P_{A\cap B}(x_1)$.*

If $A \cap B = \emptyset$, then we can find the distance between A and B from Algorithm (10).

Corollary 2.4 *Let A, B be two nonempty closed convex subsets of a Hilbert space X such that $A \cap B = \emptyset$, and let the sequences (x_n) and (y_n) be generated by Algorithm (10). Assume that there exist $a \in A$ and $b \in B$ such that $d(A, B) = \|a - b\|$. Then*

(i) $d(A, B) = \lim_{n\to\infty} \|x_n - P_B(x_n)\| = \lim_{n\to\infty} \|y_n - P_B(x_n)\| = \|p - P_B(p)\| > 0$;

(ii) $p - P_B(p) = P_{\overline{A-B}}(0)$, *where* $p = P_{Fix(P_A P_B)}(x_1)$.

Remark 2.5 (i) The assumption "there exist $a \in A$ and $b \in B$ such that $d(A, B) = \|a - b\|$" is reasonable. On the one hand, from the computational viewpoint, we, in general, can only obtain approximate solutions (ϵ-optimal solutions) of nonlinear and linear problems by using the algorithms proposed in the literature, where ϵ is the tolerance. So, we can view the assumption "there exist $a \in A$ and $b \in B$ such that $d(A, B) = \|a - b\|$" as the terminative condition or the tolerance $\epsilon = d(A, B) = \|a - b\|$ of Algorithm (10) in the numerical experimentation. On the other hand, if $A \bigcap B = \emptyset$, and A or B is bounded, we know that the assumption "there exist $a \in A$ and $b \in B$ such that $d(A, B) = \|a - b\|$" holds.

(ii) The assumption "there exist $a \in A$ and $b \in B$ such that $d(A, B) = \|a - b\|$" is essential in Theorem 2.1 and Corollary 2.4.

Example 2.6 Let $A = \{(x, y) \in R^2 : y \leq 0\}$ and $\Upsilon(x) = e^x$ for all $x \in (-\infty, +\infty)$. Then the graph of Υ, denoted by $B, B = \{(x, y) \in R^2 : e^x \leq y\}$ is a nonempty closed and convex subset of $\mathbb{R}^2$. It is easy to check that $A \bigcap B = \emptyset$. But there does not exist $a \in A$ and $b \in B$ such that $d(A, B) = \|a - b\|$". Indeed, since the distance $d(A, B) = 0$ and $E = \text{Fix}(P_A P_B) = \emptyset$.

Remark 2.7 Theorem 2.1 and Corollaries 2.2 and 2.3 develop and improve Corollaries 5.23, 5.25 and 5.28 of (Bauschke and Combettes [3], pages 84-85) in the following aspects:

(i) Theorem 2.1 and Corollaries 2.2 and 2.3 do not involve the assumptions $\mathrm{Fix}(P_A P_B) \neq \emptyset$ and $A \bigcap B \neq \emptyset$;

(ii) The sequence (x_n) generated by Algorithm (10) can be guaranteed the strong convergence under the assumptions of Theorem 2.1 and Corollaries 2.2 and 2.3;

(iii) Compared with the Algorithms 6.1 and 6.2 of Kassay, Reich and Sabach [6] and Algorithm 6.1 of Sabach [10], the step $Q_{n+1} = \{z \in A \bigcap B : \langle x_1 - x_{n+1}, z - x_{n+1} \rangle \leq 0\}$ is removed.

In the proof of Theorem 2.1, we observe that

$$\exists a \in A, b \in B \text{ such that } d(A,B) = \|a - b\| \Rightarrow \mathrm{Fix}(P_A P_B) \neq \emptyset. \tag{16}$$

Naturally, a question arises: whether the converse of (16) is true?

The next proposition presents some sufficient and necessary conditions for $\mathrm{Fix}(P_A P_B) \neq \emptyset$ as well as $\mathrm{Fix}(P_A P_B) = \emptyset$.

Proposition 2.8 (i) *$\exists a \in A, b \in B$ such that $d(A,B) = \|a - b\| \Leftrightarrow \mathrm{Fix}(P_A P_B) \neq \emptyset$;*

(ii) *for any $a \in A, b \in B$ such that $d(A,B) < \|a - b\| \Leftrightarrow \mathrm{Fix}(P_A P_B) = \emptyset$.*

Proof. (i) By the proof of Theorem 2.1, we only need prove the sufficiency of (i).
Suppose that $\mathrm{Fix}(P_A P_B) \neq \emptyset$. We divide into two cases:
(a) If $A \bigcap B \neq \emptyset$, then (i) holds;
(b) If $A \bigcap B = \emptyset$. Since $\mathrm{Fix}(P_A P_B) \neq \emptyset$, for $f^* \in \mathrm{Fix}(P_A P_B)$, one has $f^* = P_A P_B(f^*) \in A$ and $P_B(f^*) = P_B P_A(P_B(f^*)) \in B$. By Lemmas 1.8 and 1.9, $f^* - P_B(f^*)$ is a gap vector from B to A, that is, $f^* - P_B(f^*) = P_{\overline{A-B}}(0)$. This shows that $\|f^* - P_B(f^*)\| = d(A,B)$, as required.
(ii) It directly follows from (i). This completes the proof. ■

We now propose another question: what will happen of Algorithm (10) when $\mathrm{Fix}(P_A P_B) = \emptyset$?

Theorem 2.9 Let A and B be two nonempty closed and convex subsets of a Hilbert space X such that $E = \mathrm{Fix}(P_A P_B) = \emptyset$. Assume that the sequence (x_n) is generated by Algorithm (10). Then exactly one of the following alternatives holds:

(i) $\|x_n\| \to +\infty$ as $n \to \infty$ whenever $C_n \neq \emptyset$ for all $n \geq 1$;

(ii) Algorithm (10) stops at finite iteration $n \geq 1$ whenever $C_n = \emptyset$ for some $n \geq 1$.

Proof. We only need to prove that (i) holds. Suppose that $\|x_n\| \not\to +\infty$ as $n \to \infty$. That is, for some $M > 0$, there exists a subsequence $(x_{n_k})_{k=1}^\infty$ of (x_n) such that $\|x_{n_k}\| \leq M$. In other word, the subsequence $(x_{n_k})_{k=1}^\infty$ is bounded. Then $x_{n_k} \rightharpoonup v \in X$ (here we may take a subsequence $(x_{n_{k_l}})$ of (x_{n_k}) if necessary). Since C_n is closed and convex for all $n \geq 1$, $v \in \bigcap_{k \geq 1} C_{n_k}$. Take into account $C_{n+1} \subseteq C_n$ for all $n \geq 1$, one has $\bigcap_{n \geq 1} C_n = \bigcap_{k \geq 1} C_{n_k}$. Moreover, $v \in \bigcap_{n \geq 1} C_n \neq \emptyset$. Without loss of generality, let $n_{(k+1)} \geq n_k + 1$. In view of $x_{n_{(k+1)}} = P_{C_{n_{(k+1)}}} x_1 \in C_{n_{(k+1)}} \subseteq C_{n_k}$. It follows from $x_{n_k} = P_{C_{n_k}} x_1$ that

$$\|x_{n_k} - x_1\| \leq \|v - x_1\| \tag{17}$$

and

$$\|x_{n_k} - x_1\| \leq \|x_{n_{(k+1)}} - x_1\|. \tag{18}$$

Both (17) and (18) imply that $\lim_{k \to \infty} \|x_{n_k} - x_1\|$ exists. Again, from $x_{n_{(k+1)}} \in C_{n_k}$ and $x_{n_k} = P_{C_{n_k}} x_1$, one has

$$\langle x_{n_{(k+1)}} - x_{n_k}, x_1 - x_{n_k} \rangle \leq 0. \tag{19}$$

Owing to $\|x_{n_{(k+1)}} - x_{n_k}\|^2 + \|x_{n_k} - x_1\|^2 = \|x_{n_{(k+1)}} - x_1\|^2 + 2\langle x_{n_{(k+1)}} - x_{n_k}, x_1 - x_{n_k}\rangle$. This, together with (19), shows that

$$\|x_{n_{(k+1)}} - x_{n_k}\|^2 + \|x_{n_k} - x_1\|^2 \leq \|x_{n_{(k+1)}} - x_1\|^2.$$

Furthermore, one has

$$\|x_{n_{(k+1)}} - x_{n_k}\|^2 \leq \|x_{n_{(k+1)}} - x_1\|^2 - \|x_{n_k} - x_1\|^2.$$

This implies that $\|x_{n_{(k+1)}} - x_{n_k}\| \to 0$ as $k \to \infty$. Since $x_{n_{(k+1)}} \in C_{n_{(k+1)}} \subseteq C_{n_k+1}$, by the definition of C_{n_k+1}, we have

$$\|x_{n_{(k+1)}} - y_{n_k}\| \leq \|x_{n_{(k+1)}} - x_{n_k}\|.$$

Then $\|x_{n_{(k+1)}} - y_{n_k}\| \to 0$ as $k \to \infty$. In the light of $\|x_{n_k} - y_{n_k}\| \leq \|x_{n_k} - x_{n_{(k+1)}}\| + \|x_{n_{(k+1)}} - y_{n_k}\|$, we conclude that $\|x_{n_k} - y_{n_k}\| \to 0$ as $k \to \infty$. That is,

$$\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = \lim_{k\to\infty} \|x_{n_k} - P_A P_B(x_{n_k})\| = 0.$$

By Fact 1.10, we derived that $v = P_A P_B(v)$ and so, $v \in \text{Fix}(P_A P_B) \neq \emptyset$, which contradicts $E = \text{Fix}(P_A P_B) = \emptyset$. This completes the proof. ■

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