New Exact Approaches to Row Layout Problems

Anja Fischer*, Frank Fischer†, Philipp Hungerländer‡

November 3, 2015

Abstract. Given a set of departments, a number of rows and pairwise connectivities between these departments, the multi-row facility layout problem (MRFLP) looks for a non-overlapping arrangement of these departments in the rows such that the weighted sum of the center-to-center distances is minimized. As even small instances of the (MRFLP) are rather challenging, several special cases have been considered in the literature. In this paper we present new mixed-integer linear programming formulations for the (space-free) multi-row facility layout problem with given assignment of the departments to the rows that combine distance and betweenness variables. Using these formulations instances with up to 25 departments can be solved to optimality (within at most six hours) for the first time. Furthermore we are able to reduce the running times for instances with up to 23 departments significantly in comparison to the literature. Later on we use these formulations in an enumeration scheme for solving the (space-free) double-row facility layout problem. In particular, we test all possible row assignments, where some assignments are excluded due to our new combinatorial investigations. For the first time this approach enables us to solve instances with up to 16 departments to optimality in reasonable time.
Keywords: Double-Row Layout Problem, Facility Layout, Integer Programming, Combinatorial Bounds

1 Introduction

In this paper, we focus on mathematical programming approaches that can certify global optimality of solutions for *Multi-Row Facility Layout Problems* (MRFLP) and variants thereof. We start with an introduction of the problems considered in this paper and of layout problems related to them.

The Multi-Row Facility Layout Problem. An instance of the (MRFLP) consists of n departments $\{1, \dots, n\} =: [n]$ with given positive lengths $\ell_i > 0, i \in [n]$, pairwise non-negative connectivities $w_{ij} \geq 0, i, j \in [n], i < j$, and a set $\mathcal{R} := [m]$ of rows available for placing the departments. For sake of simplicity we assume that

- each department can be assigned to any of the given rows,
- inter-row distances between the departments are neglected.

*Georg-August-Universität Göttingen, Institut für Numerische und Angewandte Mathematik, anja.fischer@mathematik.uni-goettingen.de

†Universität Kassel, Institut für Mathematik, frank.fischer@uni-kassel.de

‡Alpen-Adria Universität Klagenfurt, Institut für Mathematik, philipp.hungerlaender@aau.at

Now the task of the (MRFLP) is finding an assignment $r_i \in \mathcal{R}, i \in [n]$, of departments to rows, and feasible horizontal positions for the centers of the departments within the assigned rows, i. e., positions

$$p_i \in \mathbb{R}, i \in [n], \text{ satisfying } \tfrac{1}{2}(\ell_i + \ell_j) \leq |p_i - p_j| \text{ if } r_i = r_j \text{ for a } j \in [n] \setminus \{i\}, \tag{1}$$

such that the total weighted sum of the center-to-center distances between all pairs of departments is minimized. Hence the (MRFLP) can be formulated as follows:

$$\begin{aligned} \min \quad & \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} |p_i - p_j| \\ \text{s. t.} \quad & (r_i = r_j) \;\Rightarrow\; \tfrac{1}{2}(\ell_i + \ell_j) \leq |p_i - p_j|, \qquad i, j \in [n], i < j, \\ & r \in \mathcal{R}^n, p \in \mathbb{R}^n. \end{aligned}$$

The (MRFLP) is of special interest in the design of flexible manufacturing systems because the layout of the machines highly influences the throughput, the material handling time and other performance characteristics of factories [17]. Apart from manufacturing planning it has several applications, see e. g., [10, 14, 28, 31] and [24] for more detailed lists. Despite its broad applicability the general (MRFLP) received only limited attention in the literature. Looking at exact algorithms for the (MRFLP) we are aware of the following papers. In the 1980s, Heragu and Kusiak [18, 19] obtained locally optimal solutions for the *Single-Row Facility Layout Problem* (SRFLP) and the *Double-Row Facility Layout Problem* (DRFLP) using a non-linear programming model. The (SRFLP) is a special case of the (MRFLP) with m equal to 1. The (DRFLP) is another special (MRFLP) with $m = 2$, i. e., the departments are arranged above and below a single path. This variant is especially relevant in the planning of factory layouts. Chung and Tanchoco [11] (see also Zhang and Murray [33]) presented a mixed integer linear programming (MILP) formulation for the (DRFLP) in 2010. With this approach instances with up to 10 departments could be solved to optimality. Later on Amaral [4], again concentrating on the (DRFLP), proposed an improved MILP formulation and was able to solve instances with up to 12 departments to optimality. Recently, Hungerländer and Anjos [24] suggested a semidefinite programming (SDP) approach for the general (MRFLP) that yields tight global bounds for instances with up to 12 departments.

Further Relevant Row Layout Problems. Recent surveys of applications of and global optimization approaches to the (SRFLP) can, e. g., be found in [7, 25, 27]. Note that at least one optimal solution of the (SRFLP) has no empty spaces between neighboring departments. Hence the (SRFLP) consists of finding a permutation of the departments that minimizes the total weighted sum of all center-to-center distances. The *Single-Row Equidistant Facility Layout Problem* (SREFLP) is the special case of the (SRFLP) with all departments equal in shape. For both (SRFLP) and (SREFLP) the largest instances solved to optimality consist of 42 departments [21, 25]. Let us also mention that the *Linear Arrangement Problem*, where nodes of a graph are assigned to positions on the real number line minimizing the sum of the pairwise distances between adjacent nodes, is a special (SREFLP) where all connectivities are binary. It is already an NP-hard problem [13], even if the underlying graph is bipartite [12].

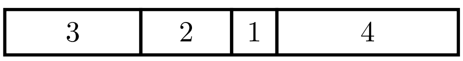
As the (MRFLP) (and the (DRFLP)) is a rather challenging problem and only small instances can be solved to optimality several simplifications have been studied in the literature. In the *Multi-Row Equidistant Facility Layout Problem* (MREFLP) all departments have the same length. Recently Anjos et al. [9] proposed specially tailored ILP and SDP models for this problem. They could prove global optimality for some instances with up to 25 departments and achieved optimality gaps smaller than 1 % for instances with up to 50 departments using a semidefinite programming approach.

The *Space-Free Multi-Row Facility Layout Problem* (SF-MRFLP) is a restricted version of the (MRFLP) in which all the rows have a common left origin and no empty space is allowed between the departments (in the same row). If we restrict the (SF-MRFLP) to two rows we obtain the *Space-Free Double-Row Facility Layout Problem* (SF-DRFLP), also denoted as *Corridor Allocation Problem* [3], as a special case. An MILP formulation for the (SF-DRFLP) is presented in [3]. With this model instances with up to 13 departments can be solved to optimality. Additionally, a semidefinite optimization approach is proposed in [23]. It provides high-quality global bounds for (SF-DRFLP) instances with up to 15 departments and for (SF-MRFLP) instances with 3 to 5 rows and 11 departments.

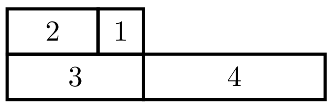
Additionally fixing the row assignment in advance leads to the *k-Parallel Row Ordering Problem* (kPROP) as a simplified (SF-MRFLP) version. It asks for an optimal arrangement of the departments along multiple rows, without space between departments in the same row, but where the row of each department is fixed. Hence the (kPROP) asks for a permutation of the departments within each row so that the total weighted sum of the center-to-center distances between all pairs of departments (with a common left origin) is minimized. If the (kPROP) is restricted to two rows we simply call it (PROP). Amaral [5] suggested an MILP formulation for the (PROP) that allowed to solve instances with up to 23 departments to optimality. Furthermore Hungerländer [20] proposed an SDP approach that yields reasonable global bounds for (kPROP) instances with up to 100 departments.

Illustration. The following examples are designed to illustrate the differences between the (SRFLP), the (kPROP), the (SF-MRFLP) and the (MRFLP). We consider the following instance consisting of 4 departments with lengths $\ell_i = i, i = 1, \dots, 4$, and pairwise connectivities $w_{12} = w_{34} = 1,\ w_{14} = w_{23} = 2$. Figure 1 illustrates optimal layouts for the four different problems:

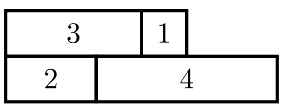
- Figure 1a shows an optimal layout for the (SRFLP) with total cost $1.5 \cdot 1 + 6.5 \cdot 1 + 2.5 \cdot 2 + 2.5 \cdot 2 = 18$.
- Figure 1b depicts an optimal layout for the (PROP) when departments 1 and 2 are assigned to row 1 and departments 3 and 4 are assigned to row 2, i. e. $r_1 = r_2 = 1$, $r_3 = r_4 = 2$. The corresponding total cost is $1.5 \cdot 1 + 3.5 \cdot 1 + 2.5 \cdot 2 + 0.5 \cdot 2 = 11$.
- Figure 1c shows an optimal layout for the (SF-DRFLP) with total cost $2.5 \cdot 1 + 2.5 \cdot 1 + 0.5 \cdot 2 + 0.5 \cdot 2 = 7$.
- Finally, Figure 1d depicts an optimal layout for the (DRFLP). The corresponding total cost is $3 \cdot 1 + 3 \cdot 1 = 6$. This solution shows that it might be advantageous to have space between neighboring departments in the same row and to allow different starting points (the leftmost point of the leftmost department in a row) in different rows.



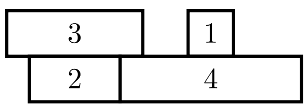
(a) (SRFLP)



(b) (PROP)



(c) (SF-DRFLP)



(d) (DRFLP)

Figure 1: We consider an instance with $\ell_i = i$, $i = 1, \dots, 4$, $w_{12} = w_{34} = 1$, $w_{14} = w_{23} = 2$ and depict optimal layouts for four different row layout problems.

Contributions and Outline. The contributions of this paper are new exact approaches for the (DRFLP), the (SF-DRFLP) and the (kPROP) that clearly outperform all exact approaches from the literature for these problems. For the (kPROP) we transform and extend the semidefinite model from [20] to an MILP model that yields cheap and strong global bounds compared to other existing models. For the (SF-DRFLP) and (DRFLP) we adapt the idea of enumerating over all possible row assignments from the semidefinite approach in [23] to appropriate linear models. To further improve the efficiency of the enumeration scheme we prove some combinatorial properties of optimal (PROP) and double-row layouts.

The paper is structured as follows. First, we shortly describe our enumeration scheme. In Section 3 we suggest a new MILP formulation for the (kPROP) that can be incorporated in an enumerative scheme for solving the (SF-DRFLP). Section 4 contains a model for the (DRFLP) with fixed row assignment that can also be integrated in our enumeration framework. In Section 5 we study the structure of optimal solutions of some of the layout problems. These results are later exploited in the computational experiments. Detailed computational results are reported in Section 6. Section 7 concludes the paper.

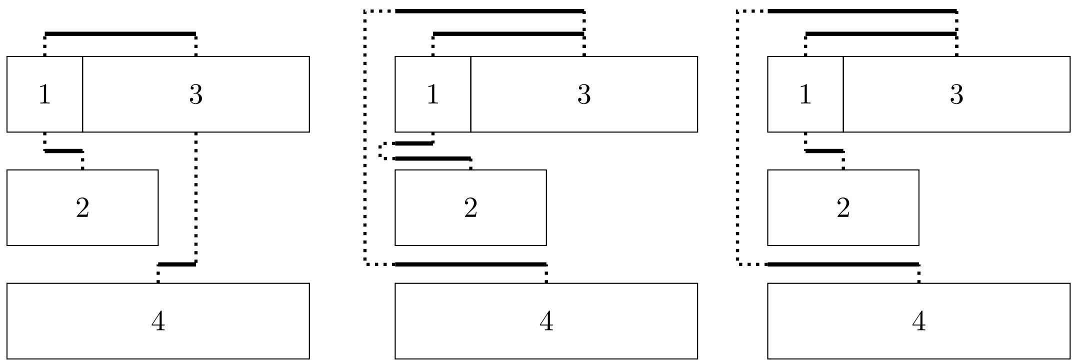
2 An Enumeration Scheme for Solving Row Layout Problems

In this section we shortly describe our enumeration scheme for solving row layout problems like (MRFLP) and (SF-MRFLP), respectively, exactly. Given an instance of the respective row layout problem, we consider for each possible row assignment the restricted version of the row layout problem, in which the assignment is fixed. Hopefully, these restricted problems are easier to solve because of less degree of freedom. We then enumerate over all possible row assignments in a branch-and-cut approach. Clearly, we can stop a single solution step if the lower bound in branch-and-cut exceeds the currently best solution of the general problem. Our enumeration scheme is shown in Algorithm 1. Note that this approach works well as long as the subproblems with fixed row assignment can be solved quickly. In the following sections we present MILP formulations for the (SF-MRFLP) and the (MRFLP) with fixed row assignments to be applied in this enumeration scheme.

Algorithm 1: Enumeration scheme for problem $P \in \{$(MRFLP), (SF-MRFLP)$\}$

Input : instance of P with departments $[n]$, rows $\mathcal{R}$ and connectivities w
Output : optimal value v^* of P

1 $v^* \leftarrow \infty$
2 **for** $r = (r_1, \ldots, r_n) \in \mathcal{R}^n$ `// test all row assignments` **do**
    if *row assignment* r *can be neglected* **then**
        continue
    Compute a lower bound $\underline{v}_r$ of P.
    if $\underline{v}_r \geq v^*$ **then**
        continue
    Determine optimal value v_r^* of P with fixed row assignment r.
    if $v_r^* < v^*$ **then**
        $v^* \leftarrow v_r^*$
3 **return** v^*



(a) Type 1 (b) Type 2 (c) Type 3

Figure 2: Visualization of the three distance calculation types. We consider an instance with 4 departments and $\ell_i = i$, $i = 1, \dots, 4$, row assignments $r_1 = r_3 = 1, r_2 = 2, r_4 = 3$, and connectivities $w_{12} = 1, w_{13} = 2$ and $w_{34} = 3$. The objective value for the displayed solution equals $1 \cdot 0.5 + 2 \cdot 2 + 3 \cdot 0.5 = 6$ for distance-calculation type 1, $1 \cdot (0.5 + 1) + 2 \cdot 2 + 3 \cdot (2.5 + 2) = 19$ for type 2, and $1 \cdot 0.5 + 2 \cdot 2 + 3 \cdot (2.5 + 2) = 18$ for type 3.

3 An MILP Model for the k-Parallel Row Ordering Problem

In the following we suggest MILP models for the `(kPROP)`, where we consider different variants for determining the distances of departments in different rows. We distinguish between the following three variants, where d_{ij} denotes the distance between departments $i, j \in [n], i < j$, and p_i denotes the x-coordinate of the center of department $i \in [n]$:

- Direct variant (this is the standard variant, see, e. g., [5]) – type 1: $d_{ij} = |p_i - p_j|$.
- Border variant – type 2: $d_{ij} = \begin{cases} p_i + p_j, & \text{if } r_i \neq r_j, \\ |p_i - p_j|, & \text{otherwise.} \end{cases}$
- Combined variant – type 3: $d_{ij} = \begin{cases} p_i + p_j, & \text{if } |r_i - r_j| \geq 2, \\ |p_i - p_j|, & \text{otherwise.} \end{cases}$

In variant 1 only the distances in x-direction are considered. In variants 2 and 3 this still holds true for departments in the same row. Considering type 2 we sum up the distances to the left border for departments in different rows. This can be interpreted in the following way. The means of transport can only move in x-direction and so, for going from one to another row, it first has to go from one of the departments to the left border, then it goes to the other row (for fixed row assignments this length is fixed and so it can be neglected) and then it moves to the other department. The only difference between type 2 and type 3 is that for type 3 we allow direct connections between neighboring rows. Figure 2 shows an illustration.

To simplify the presentation of the models we define the sets $R_h, h \in \mathcal{R}$, that contain the indices of the departments assigned to row h, i. e., $j \in R_h \Leftrightarrow r_j = h$. Further we introduce two additional dummy departments $n+1$ and $n+2$, to be placed at the left and the right boundary of each row, respectively, with $\ell_{n+1} = \ell_{n+2} = 0$ as well as $w_{ij} = 0$ if $i, j \in [n+2], \{i, j\} \cap \{n+1, n+2\} \neq \emptyset$. We set $\tilde{R}_h := R_h \cup \{n+1, n+2\}$ for $h \in \mathcal{R}$.

Our MILP formulations use betweenness variables

$$x_{ikj} = x_{jki} = \begin{cases} 1, & \text{if } k \text{ lies between } i \text{ and } j \text{ (in the same row)}, \\ 0, & \text{otherwise,} \end{cases}$$

for $l \in \mathcal{R}, i,j,k \in \tilde{R}_l, |\{i,j,k\}| = 3, i < j$, and additionally distance variables $d_{ij} = d_{ji} \geq 0, i,j \in [n+1], i < j$.

Using these variables we can formulate the (kPROP) as follows. For all three distance calculation types we use the basic model

$$
\begin{aligned}
\min \quad & \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij} && \\
\text{s.t.} \quad & x_{ijk} + x_{ikj} + x_{jik} = 1, && l \in \mathcal{R}, i,j,k \in \tilde{R}_l, i < j < k, && (2)\\
& x_{(n+1)i(n+2)} = 1, && i \in [n], && (3)\\
& x_{i(n+1)j} = 0, && l \in \mathcal{R}, i,j \in R_l \cup \{n+2\}, i < j, && (4)\\
& x_{i(n+2)j} = 0, && l \in \mathcal{R}, i,j \in R_l \cup \{n+1\}, i < j, && (5)\\
& x_{(n+1)ij} = x_{ij(n+2)}, && l \in \mathcal{R}, i,j \in R_l, i \neq j, && (6)\\
& + x_{ihj} + x_{ihk} + x_{jhk} \leq 2, && l \in \mathcal{R}, i,j,k,h \in \tilde{R}_l, |\{i,j,k,h\}| = 4, i < j < k, && (7)\\
& - x_{ihj} + x_{ihk} + x_{jhk} \geq 0, && l \in \mathcal{R}, i,j,k,h \in \tilde{R}_l, |\{i,j,k,h\}| = 4, i < j < k, && (8)\\
& + x_{ihj} - x_{ihk} + x_{jhk} \geq 0, && l \in \mathcal{R}, i,j,k,h \in \tilde{R}_l, |\{i,j,k,h\}| = 4, i < j < k, && (9)\\
& + x_{ihj} + x_{ihk} - x_{jhk} \geq 0, && l \in \mathcal{R}, i,j,k,h \in \tilde{R}_l, |\{i,j,k,h\}| = 4, i < j < k, && (10)\\
& d_{ij} = \sum_{k \in R_l \setminus \{i,j\}} \ell_k x_{ikj} + \tfrac{1}{2}(\ell_i + \ell_j), && l \in \mathcal{R}, i,j \in R_l \cup \{n+1\}, i < j, && (11)\\
& x_{ijk} \in \{0,1\}, && l \in \mathcal{R}, i,j,k \in \tilde{R}_l, |\{i,j,k\}| = 3, i < k, && (12)\\
& d_{ij} \geq 0, && i,j \in [n+1], i < j. && (13)
\end{aligned}
$$

Equations (2) and inequalities (7)–(10) express that the departments in each row do not overlap. Indeed, these constraints were originally introduced by Amaral in a formulation of the (SRFLP) [2]. If $i,j,k \in [n], |\{i,j,k\}| = 3$, lie in the same row, exactly one of them lies in the middle (see (2)) and then certain transitivity constraints have to be fulfilled. Furthermore we ensure by (3)–(6) that in each row all departments lie between the dummy departments $n+1$ (the left border of the layout) and $n+2$ (the right border in each row). The distance of two departments in the same row equals the sum of the lengths of all departments between them plus half the length of both departments, see (11).

Remark 1. *In our model the variables $x_{(n+1)ij}, i,j \in [n], i \neq j$, (or $x_{ij(n+2)}$) comply with ordering variables that indicate whether a department lies right or left to a second department, i. e.,*

$$x_{(n+1)ij} = x_{ij(n+2)} = \begin{cases} 1, & \text{if } i \text{ lies left to } j \text{ in row } r_i = r_j, \\ 0, & \text{otherwise,} \end{cases}$$

see also models for various row layout problems or the linear ordering problem, e. g., [11, 15]. The constraints (2) and (4) imply that $x_{(n+1)ij} = 1 - x_{(n+1)ji}$, which is used in the formulations of classical ordering problems to half the number of variables, see, e. g., [16]. Furthermore note that the 3-cycle inequalities

$$0 \leq x_{(n+1)ij} + x_{(n+1)jk} - x_{(n+1)ik} \leq 1, \qquad l \in \mathcal{R}, i,j,k \in R_l,\ |\{i,j,k\}| = 3, \tag{14}$$

are implied for three departments i,j,k lying in the same row by (2), (4) and (8)–(10).

For proving this let $l \in \mathcal{R}, i,j,k \in R_l,\ |\{i,j,k\}| = 3$. First we show that $0 \leq x_{ji(n+1)} + x_{kj(n+1)} - x_{ki(n+1)}$ is implied. By (8)–(10) we know that

$$
\begin{aligned}
-x_{ki(n+1)} + x_{ji(n+1)} + x_{jik} &\geq 0,\\
-x_{ij(n+1)} + x_{kj(n+1)} + x_{ijk} &\geq 0,\\
-x_{jk(n+1)} + x_{ik(n+1)} + x_{ikj} &\geq 0.
\end{aligned}
$$

Summing up and together with $x_{ij(n+1)} + x_{ji(n+1)} = x_{ik(n+1)} + x_{ki(n+1)} = x_{jk(n+1)} + x_{kj(n+1)} = 1$, *which is implied by (2) and (4), we get*

$$- x_{ki(n+1)} + \underbrace{x_{ik(n+1)}}_{1-x_{ki(n+1)}} + x_{ji(n+1)} - \underbrace{x_{ij(n+1)}}_{1-x_{ji(n+1)}} + x_{kj(n+1)} - \underbrace{x_{jk(n+1)}}_{1-x_{kj(n+1)}} + \underbrace{x_{ijk} + x_{jik} + x_{ikj}}_{=1} \geq 0$$
$$\Leftrightarrow - 2x_{ki(n+1)} + 2x_{ji(n+1)} + 2x_{kj(n+1)} \geq 0 \;\Leftrightarrow\; x_{ji(n+1)} + x_{kj(n+1)} - x_{ki(n+1)} \geq 0.$$

Second, we prove that $x_{ji(n+1)} + x_{kj(n+1)} - x_{ki(n+1)} \leq 1$ *is implied by (2), (4) and (8)–(10). Multiplying each of the following inequalities by* -1

$$-x_{ji(n+1)} + x_{ki(n+1)} + x_{jik} \geq 0,$$
$$-x_{kj(n+1)} + x_{ij(n+1)} + x_{ijk} \geq 0,$$
$$-x_{ik(n+1)} + x_{jk(n+1)} + x_{ikj} \geq 0,$$

then summing up and using (2) we get

$$x_{ji(n+1)} - \underbrace{x_{ij(n+1)}}_{1-x_{ji(n+1)}} - x_{ki(n+1)} + \underbrace{x_{ik(n+1)}}_{1-x_{ki(n+1)}} + x_{kj(n+1)} - \underbrace{x_{jk(n+1)}}_{1-x_{kj(n+1)}} \leq 1$$
$$\Leftrightarrow 2x_{ji(n+1)} + 2x_{kj(n+1)} - 2x_{ki(n+1)} \leq 2 \;\Leftrightarrow\; x_{ji(n+1)} + x_{kj(n+1)} - x_{ki(n+1)} \leq 1.$$

Our model does not contain position variables for each of the departments explicitly as, e. g., in [11], but we can use the following result.

Remark 2. *In our model for the* (kPROP) *the position (along a horizontal line) of each department* $i \in [n]$, *i. e.,* p_i *in the problem description (1), is given by* $d_{i(n+1)}$. *Note that (11) and* $\ell_{n+1} = 0$ *imply that* $d_{i(n+1)}$ *equals half the length of department* $i \in [n]$ *plus the sum of the length of all departments between the left border of the layout and* i.

We can further improve the model with the clique constraints introduced in [6],

$$\sum_{\substack{i,j \in R_l \\ i<j}} \ell_i \ell_j d_{ij} = \tfrac{1}{6} \left(\Big(\sum_{i \in R_l} \ell_i \Big)^3 - \sum_{i \in R_l} \ell_i^3 \right), \qquad l \in \mathcal{R}, \tag{15}$$

$$\sum_{\substack{i,j \in S \\ i<j}} \ell_i \ell_j d_{ij} \geq \tfrac{1}{6} \left(\Big(\sum_{i \in S} \ell_i \Big)^3 - \sum_{i \in S} \ell_i^3 \right), \qquad l \in \mathcal{R}, S \subseteq R_l. \tag{16}$$

Depending on the measurement of the distances, the following inequalities are used for calculating inter-row distances. For distance-calculation type 1, we use

$$d_{ij} \geq d_{i(n+1)} - d_{j(n+1)}, \qquad l, o \in [m], l < o, i \in R_l, j \in R_o, \tag{17}$$
$$d_{ij} \geq d_{j(n+1)} - d_{i(n+1)}, \qquad l, o \in [m], l < o, i \in R_l, j \in R_o, \tag{18}$$
$$d_{ij} + d_{jk} \geq d_{ik}, \qquad i, j, k \in [n+1], |\{i, j, k\}| = 3, i < k, \tag{19}$$

for the calculation of the inter-row distances, where (17) and (18) model the absolute value of the difference of the center positions of $i, j \in [n], i < j$, and (19) are the classical triangle inequalities.

Considering type 2, the distances between departments $i, j \in [n], i \neq j, r_i \neq r_j$, lying in different rows are determined by summing up the distances of each of the two departments to the left border, i. e. to $n + 1$. We model this via

$$d_{ij} = d_{i(n+1)} + d_{j(n+1)}, \qquad l, o \in \mathcal{R}, l < o, i \in R_l, j \in R_o, \tag{20}$$
$$d_{ij} + d_{jk} \geq d_{ik}, \qquad l \in \mathcal{R}, i, j, k \in R_l, |\{i, j, k\}| = 3, i < k. \tag{21}$$

For distance type 3, we combine the formulas for type 1 and type 2. Indeed, inequalities (17)–(21) are slightly adapted, depending on whether two departments $i, j \in [n], i < j, r_i \neq r_j$, lie in rows that are neighboring or not:

$$
\begin{aligned}
&d_{ij} \geq d_{i(n+1)} - d_{j(n+1)}, && l \in \mathcal{R} \setminus \{m\}, i \in R_l, j \in R_{l+1}, && (22)\\
&d_{ij} \geq d_{j(n+1)} - d_{i(n+1)}, && l \in \mathcal{R} \setminus \{m\}, i \in R_l, j \in R_{l+1}, && (23)\\
&d_{ij} = d_{i(n+1)} + d_{j(n+1)}, && l, o \in \mathcal{R}, l \leq o-2, i \in R_l, j \in R_o, && (24)\\
&d_{ij} + d_{jk} \geq d_{ik}, && i, j, k \in [n], |\{i,j,k\}| = 3, i < k, && \\
& && \max\{|r(l_1) - r(l_2)| \colon l_1, l_2 \in \{i,j,k\}\} \leq 1. && (25)
\end{aligned}
$$

Note that we can only use the triangle inequalities (25) for departments lying in the same row or in two neighboring rows due to the different distance calculations involved.

It is also possible to reduce the number of the distance variables in our models by using equations (11) to replace distance variables for departments assigned to the same row by the sum of betweenness variables. For distance type 2 we do not need the distance variables at all as they can be eliminated with the help of (11) and (20).

Finally let us point out that all presented models for the different (kPROP) types are formulations.

Remark 3. *For all three distance-calculation types the models presented above are indeed formulations of the respective problem. Using mainly the betweenness model of [2] for each single row we know that the departments in each row correspond to a feasible ordering. The inner-row distances are calculated by (11) and the absolute value of the difference of the center positions of two departments lying in different rows is modeled via (17), (18), (22) and (23) (if the connectivities are positive the distance variable $d_{ij}, i, j \in [n], i \neq j$, equals the distance between departments i and j in all optimal solutions). All other inter-row distances are considered via (20) and (24).*

In Section 6 the models above are used in the enumerative scheme described in Algorithm 1 to obtain exact approaches for the (SF-DRFLP) and the (SF-MRFLP).

4 An MILP Model for the (MRFLP) with Fixed Row Assignment

In this section we propose a model for the (MRFLP) with fixed row assignment (FR-MRFLP). In comparison to the model presented in the last section spaces are allowed between departments lying next to each other in the same row and the rows do not need to have a common left border position. So we now have to ensure that neighboring departments do not overlap and that the distances between the departments are calculated correctly.

Our MILP model for the (FR-MRFLP) uses the same variables and many constraints of the model for (kPROP). In its basic form it is based on distance and betweenness variables, that correspond to ordering variables, see Remark 1. Additionally we set $M := \sum_{i=1}^{n} \ell_i$. Then our model reads

$$
\begin{aligned}
&&& \min \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij} && \\
&\mathrm{IP}_{(\mathrm{FR\text{-}MRFLP})} && \text{s.t. } (2)\text{–}(10), (12) \text{ and } (13), && \\
&&& d_{j(n+1)} - d_{i(n+1)} \geq M(x_{(n+1)ij} - 1) + \tfrac{\ell_i + \ell_j}{2}, && l \in \mathcal{R}, i, j \in R_l, i \neq j, \quad (26)\\
&&& d_{i(n+1)} \geq \tfrac{\ell_i}{2}, && i \in [n], \quad (27)\\
&&& d_{ij} \geq d_{i(n+1)} - d_{j(n+1)}, && i, j \in [n], i \neq j. \quad (28)
\end{aligned}
$$

By the considerations from the previous section we know that the betweenness constraints ensure a feasible ordering of the departments (that all lie between departments $n+1$ and $n+2$) in each of the rows. A minimal distance of $\frac{1}{2}(\ell_i + \ell_j)$ between departments $i, j \in R_l \cup \{n+1\}$ assigned to the same row $l \in \mathcal{R}$ is ensured via constraints (26) and (27). Inequalities (28) are special triangle inequalities that bound the distances between the departments. Additionally we can break some symmetry by setting $x_{ij(n+1)} = 0$, for some fixed pair $|\{i,j\}| = 2$ with $l \in \mathcal{R},\ i, j \in R_l$.

Theorem 4. *The model* `IP(FR-MRFLP)` *is a formulation for the* `(FR-MRFLP)`.

Proof. It is well-known that the 3-cycle inequalities on the ordering variables (14), implied by (2), (4) and (8)–(10) according to Remark 1, together with integrality conditions (12) suffice to describe feasible orderings, see, e. g., [15, 16, 30, 32]. The inequalities (26) connect the ordering variables $(x_{(n+1)ij}, i, j \in [n], i \neq j)$ with continuous position variables $(d_{i(n+1)}, i \in [n])$ and ensure that all position variables are feasible. Indeed, let a feasible ordering according to the $x_{(n+1)ij}$-variables be given and assume, w. l. o. g., that departments $1, \ldots, h$ lie in row 1 in ascending order. Then the distances $d_{ij}, i, j \in [h], i < j$, fulfill $d_{ij} \geq d_{j(n+1)} - d_{i(n+1)} \geq \frac{\ell_i + \ell_j}{2} + \sum_{k=i+1}^{j-1} \ell_k$ by summing up $d_{j(n+1)} - d_{(j-1)(n+1)} \geq \frac{\ell_j + \ell_{j-1}}{2}, d_{(j-1)(n+1)} - d_{(j-2)(n+1)} \geq \frac{\ell_{j-1} + \ell_{j-2}}{2}, \ldots, d_{(i+1)(n+1)} - d_{i(n+1)} \geq \frac{\ell_{i+1} + \ell_i}{2}$ (see (26)) and (28). (Note, the value of the distance variables d_{ij} might be larger then the actual distance because we modeled the absolute value of the distances between the departments, but assuming positive connectivities the distances have the correct value in all optimal solutions.) □

Replacing the integrality constraints $x_{ijk} \in \{0,1\}$ by bounds on the betweenness variables $0 \leq x_{ijk} \leq 1$ gives a basic LP relaxation for the `(FR-MRFLP)`. To improve the tightness of this relaxation, we can add constraints (16) and (19) introduced above. Equations (11) are not valid anymore, but the position of each department $i \in [n]$, represented by the distance $d_{i(n+1)}$, can be bounded from below by

$$d_{i(n+1)} \geq \sum_{k \in R_l \setminus \{i\}} \ell_k x_{ik(n+1)} + \tfrac{1}{2}\ell_i, \qquad l \in \mathcal{R}, i \in R_l, \tag{29}$$

$$d_{ij} \geq \sum_{k \in R_l \setminus \{i,j\}} \ell_k x_{ikj} + \tfrac{\ell_i + \ell_j}{2}, \qquad l \in \mathcal{R}, i, j \in R_l, i < j. \tag{30}$$

Remark 5. *The* `(MRFLP)` *model that we obtain by combining the* `(FR-MRFLP)` *model above and Algorithm 1 can easily be extended to cover further aspects, which might be of practical relevance depending on the application. The main reason for this is that in each step of Algorithm 1 the assignment of the departments to the rows is fixed.*

- *Smith et al. [29] considered not only the weighted sum of the distances but also the size of the smallest rectangle that contains all departments in the objective function. Given not only the width but also the height of each department this size can easily be calculated or bounded by additional constraints. On the one hand the maximal height in each row is predetermined by the row assignment and on the other hand the length of each row (including the spaces between the departments) equals the maximal distance of the rightmost point of a department in this row to department $n+1$, i. e. to the left border of the layout.*

- *In our models the inter-row distances are neglected. However, they can easily be handled by our enumeration scheme because they lead to fix costs in the restricted models. In particular, the size of the aisle can be incorporated into the model.*

- *The model of Chung and Tanchoco [11] (see also Zhang and Murray [33]) contained minimum clearance conditions, i. e., a minimal distance between two departments if they lie next to each other in the same row. These clearance conditions can easily be included in* *our model if they fulfill some kind of triangle inequality: the minimal distance between two departments $i, j \in [n], i \neq j$, is not larger than the sum of the minimal distance between i and a third department $k \in [n] \setminus \{i, j\}$, the minimal distance between j and k and the length ℓ_k. Assuming these triangle inequalities, the clearance conditions can be ensured by slightly adapting (26).*

5 Combinatorial Properties for Speeding Up our Enumeration

In the following subsection we aim to further improve the computational performance of Algorithm 1 for the (SF-MRFLP) by excluding row assignments that are too unbalanced. Later on in Section 5.2 we also speed up Algorithm 1 for the (MRFLP) by reducing the big-M value M in (26).

5.1 Excluding Unbalanced Assignments for the (SF-DRFLP) and the (SF-MRFLP).

In the following we show how to speed up Algorithm 1 for the (SF-DRFLP) and the (SF-MRFLP) by excluding some row assignments. We denote the sum of the lengths of the departments in row $i \in \mathcal{R}$ by $L_i := \sum_{j \in R_i} \ell_j$. Obviously we can restrict to all row assignments with $L_i \geq L_j \; i, j \in \mathcal{R}, \; i < j$, in Algorithm 1. But further row assignments can be neglected.

We start with the double-row case and distance-calculation type 1. Our aim is to determine the smallest number $g \in \mathbb{R}_+$ such that there always exists an optimal solution of the (SF-DRFLP), independent of the objective function, where $g \geq L_1 - L_2$ holds for the corresponding row assignment. Note that in general we have $g \geq \ell_{\max,1} = \max_{i \in R_1} \ell_i$, see Figure 3 for a small instance where this value for g is attained.

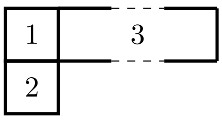


Figure 3: We consider an instance with $\ell_1 = \ell_2 = 1, \; \ell_3 = k \geq 2, \; w_{12} = 4, \; w_{13} = w_{23} = 1$. In the optimal space-free double-row layout g equals $k = \max_{i \in R_1} \ell_i$.

In the following lemma we show that $\ell_{\max,1}$ is an upper bound to g.

Lemma 6. *For distance-calculation type 1 there always exists an optimal solution (r^*, p^*) of the (SF-DRFLP) with row assignment r^*, where $\ell_{\max,1} \geq L_1 - L_2$ is fulfilled for r^*.*

Proof. We prove this by contradiction. Let us assume that an optimal space-free double-row layout L^* and a corresponding row assignment r^* are given such that the row lengths fulfill $L_1^* - L_2^* > \ell_{\max,1}$. Furthermore assume that there does not exist an optimal solution L that additionally fulfills $L_1 - L_2 \leq \ell_{\max,1}$. Our aim is now to reorder the departments such that afterwards the length are somehow balanced and that the objective value is not increased, a contradiction.

We assume, w. l. o. g., that departments $1, \ldots, t$ lie in row 1 and departments $t+1, \ldots, n$ lie in row 2 in the optimal layout L^* and that these departments are numbered consecutively from left to right in row 1 and from right to left in row 2. Now we define $B_1 := \{i \in R_1 \colon \sum_{j=1}^{i} \ell_j \leq L_2^*\}$ with $|B_1| = s \leq t$. B_1 contains departments $1, \ldots, s$ and the total length of these departments does not exceed L_2^*. We refer to Figure 4 for an illustration of the departments in B_1.

By our assumption $L_1^* - L_2^* > \ell_{\max,1}$ we know that the set $R_1 \setminus B_1$ contains at least two departments. We suggest a new layout $\hat{L}$ with row assignment $\hat{r}$ that has objective value less or equal to L^* fulfilling the desired property $\hat{L}_1 - \hat{L}_2 \leq \ell_{\max,1}$.

For constructing $\hat{L}$ we first determine $u \in \{s+1, \ldots, t\}$ minimal such that $\sum_{i=1}^{u-1} \ell_i \geq \sum_{i=u+1}^{n} \ell_i$. This implies $\sum_{i=u}^{n} \ell_i > \sum_{i=1}^{u-2} \ell_i$ and $\sum_{i=1}^{\hat{u}-1} \ell_i \geq \sum_{i=\hat{u}+1}^{n} \ell_i$ for all $t \geq \hat{u} \geq u$. Now in the layout

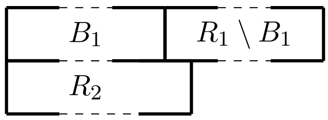


Figure 4: Illustration of the partition of the departments in row 1.

$\hat{L}$ all departments in B_1, R_2 and $B_2 := \{s+1, \dots, u-1\}$ remain at their old position and departments $B_3 := \{u, \dots, t\}$ are assigned to row 2 in reversed order compared to their order in row 1, i. e., t is assigned right to $t+1$, $t-1$ then right to t and so on. In Figure 5 we depict L^* and $\hat{L}$. Note, it might happen that now $\hat{L}_2 > \hat{L}_1$, then we have to change the role of row 1 and row 2 later on. But by the choice of u we know $|\hat{L}_1 - \hat{L}_2| \leq \ell_{\max,1}$ by

$$\hat{L}_1 - \hat{L}_2 = \sum_{i=1}^{u-1} \ell_i - \sum_{i=u}^{n} \ell_i = \ell_{u-1} + \underbrace{\sum_{i=1}^{u-2} \ell_i - \sum_{i=u}^{n} \ell_i}_{<0} < \ell_{u-1} \leq \ell_{\max,1},$$

$$\hat{L}_2 - \hat{L}_1 = \sum_{i=u}^{n} \ell_i - \sum_{i=1}^{u-1} \ell_i = \ell_u + \underbrace{\sum_{i=u+1}^{n} \ell_i - \sum_{i=1}^{u-1} \ell_i}_{\leq 0} \leq \ell_u \leq \ell_{\max,1}.$$

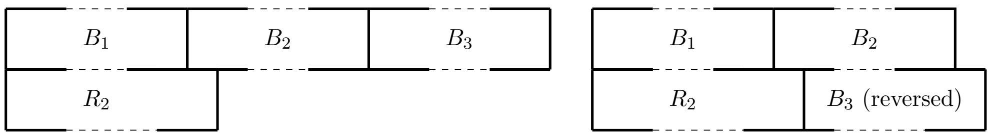


Figure 5: Comparison of the layouts L^* on the left and $\hat{L}$ on the right.

Clearly the distances between pairs of departments from $B_1 \cup B_2 \cup R_2$ are the same in L^* and $\hat{L}$. The same is true for pairs of departments from B_3.

- The distances between any department from B_3 and any department from $B_1 \cup R_2$ decrease: On the one hand the centers of the departments from B_3 are still right from the centers of the departments in $B_1 \cup R_2$ due to the definition of B_1. On the other hand the centers of the departments from B_3 are shifted to the left in $\hat{L}$ in comparison to L^*. Indeed, it suffices to consider the position of u. The position of u is $p_u^* = \sum_{i=1}^{u-1} \ell_i + \frac{\ell_u}{2}$ in L^* and $\hat{p}_u = \sum_{i=u+1}^{n} \ell_i + \frac{\ell_u}{2}$ in $\hat{L}$. By the choice of u we have $\hat{p}_u \leq p_u^*$.
- It remains to consider the distances between arbitrary departments $v \in B_2$ and $w \in B_3$. The positions are $p_v^* = \sum_{i=1}^{v-1} \ell_i + \frac{\ell_v}{2}$, $p_w^* = \sum_{i=1}^{w-1} \ell_i + \frac{\ell_w}{2}$ in L^* with distance $d_{vw}^* = \sum_{i=v+1}^{w-1} \ell_i + \frac{\ell_w}{2} + \frac{\ell_v}{2}$ and $\hat{p}_v = p_v^*$, $\hat{p}_w^* = \sum_{i=w+1}^{n} \ell_i + \frac{\ell_w}{2}$ in $\hat{L}$. If $\hat{p}_w \geq \hat{p}_v = p_v$ then the new distance $\hat{d}_{vw}$ between v and w fulfills $\hat{d}_{vw} \leq d_{vw}$ because w has been moved to the left. If, otherwise, $\hat{p}_w < \hat{p}_v$, then by $v < u \leq w$ and the choice of u we have

$$\hat{d}_{vw} - d_{vw}^* = \sum_{i=w+1}^{n} \ell_i + \frac{\ell_w}{2} - \sum_{i=1}^{v-1} \ell_i - \frac{\ell_v}{2} - \sum_{i=v+1}^{w-1} \ell_i - \frac{\ell_w}{2} - \frac{\ell_v}{2} = \sum_{i=w+1}^{n} \ell_i - \sum_{i=1}^{w-1} \ell_i \leq 0.$$

In summary, $\hat{L}$ is also an optimal layout, but for the corresponding row assignment $\hat{r}$ the inequality $|\hat{L}_1 - \hat{L}_2| \leq \ell_{\max,1}$ holds. This proves the statement. □

In the equidistant case, i. e. for the Space-Free Double-Row Equidistant Facility Layout Problem `(SF-DREFLP)`, the result from Lemma 6 further simplifies.

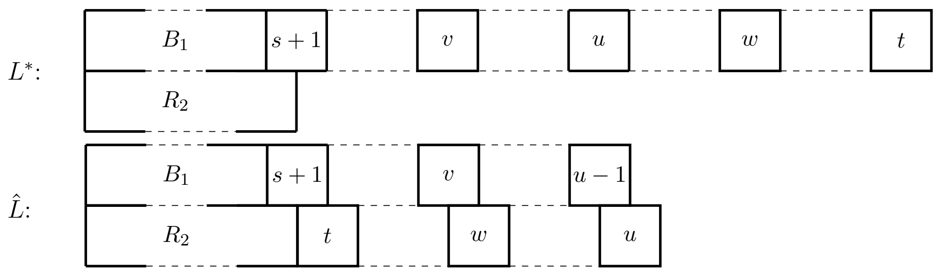


Figure 6: Illustration of the reduction of the distances between department $v \in B_2$ and department $w \in B_3$. Note that the argument also works for the case $v = u - 1$ and/or $w = u$.

Corollary 7. *For distance-calculation type 1 there always exists an optimal* (SF-DREFLP) *solution, where*

- *half of the departments are assigned to each of the two rows if n is even and*
- *one row contains $\frac{n+1}{2}$ departments and the other row contains $\frac{n-1}{2}$ departments if n is odd.*

For solving the (SF-DREFLP) we can also use an approach presented in [9]. In order to handle the spaces between departments the authors introduced dummy departments that were added to a (DREFLP) instance. With the help of these dummy departments the (DREFLP) reduced to a (SF-DRFLP). Hence for the (SF-DREFLP) with even n we can directly apply the approach from [9] because then both rows have the same length and no spaces occur. For the (SF-DREFLP) with odd n, there is a space of length 1 at the end of row 2 and thus exactly one dummy department is needed. The condition that the dummy department has to be the last department on row 2 can easily be included in the different models from [9]. For instance one can add a constraint guaranteeing that the dummy department does not lie between other departments.

We can also extend the results above to the multi-row case. For this, note that the proof above does not depend on the number of rows m. Hence we can formulate the following corollary.

Corollary 8. *For distance-calculation type 1 $\ell_{\max,1}$ is the smallest g such that there always exists an optimal solution to the* (SF-MRFLP) *with $m \geq 2$, where $L_i \geq L_j, i, j \in \mathcal{R}, i < j$, and $g \geq L_1 - L_2$ holds for the corresponding row assignment r^*.*

Remark 9. *Note that Corollary 8 does not hold for distance-calculation type 2 with $m \geq 2$ and distance-calculation type 3 with $m \geq 3$: For these cases it might be preferable to arrange two departments next to each other in the same row instead of putting them below of each other in two different (for type 3 non-neighboring) rows because then the distance of the departments equals the sum of the distances of both departments to the left border of the layout.*

In the following two examples we want to show that in general the difference between the row lengths of arbitrary rows cannot be bounded. Let us first consider an instance with $m \geq 3$ odd, $\tilde{m} := \frac{m+1}{2}$, and some $k \in \mathbb{N}, k \geq 2$, that was originally introduced in [9]. In slight abuse of notation, in order to improve the readability, we denote each department by a tuple (i, j) with $i \in [\tilde{m}], j \in [k]$. Then each optimal solution of an equidistant instance with $n = \tilde{m} \cdot k$ departments and

$$w_{(i,j),(i,j+1)} = 1, \qquad i \in [\tilde{m}], j \in [k-1], \tag{31a}$$

$$w_{(i,j),(i+1,j)} = n, \qquad i \in [\tilde{m}-1], j \in [k], \tag{31b}$$

does not use the rows $L_i, i \in \{\tilde{m}+1, \dots, m\}$. For an illustration of an optimal layout we refer to Figure 7.

Secondly, we demonstrate that the differences of the row lengths $L_i - L_{i+1}$, $i \in \{2, \dots, m-1\}$, can be arbitrarily large in optimal multi-row layouts. To do so we consider the following instance

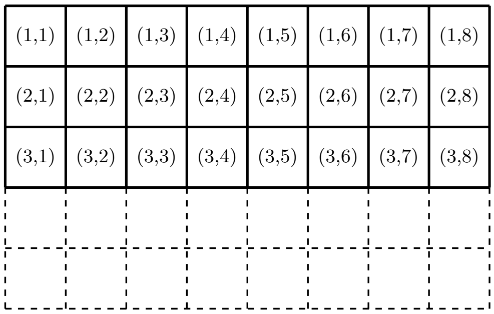


Figure 7: Illustration of an optimal space-free multi-row layout for the equidistant instance given in (31) with $m = 5$ and $k = 8$. The row lengths fulfill $L_1 = \ldots = L_{\tilde{m}} = k$ and $L_{\tilde{m}+1} = \ldots = L_m = 0$ with $\tilde{m} := \frac{m+1}{2} = 3$.

for fixed $k \in \mathbb{N}$, where we assume, w. l. o. g., that k is even. As above each department is denoted by a tuple (i, j), here with $i \in [m-1]$, and $j \in [k(m-2)]$ if $i = 1$ and $j \in [k(m-i)]$ otherwise. We collect all tuples of the departments in the set D. The department lengths and connectivities are chosen as follows:

$$\ell_{i,j} = 1 + 2^{j-1}\varepsilon, \qquad (i,j) \in D,\ \varepsilon \ll 1, \tag{32a}$$

$$w_{(i,j),(i+1,j)} = 1, \qquad (i,j),(i+1,j) \in D. \tag{32b}$$

In all optimal space-free layouts $L_i - L_{i+1} > k,\ \ i \in \{2, \ldots, m-2\}$, holds for arbitrary k. For an illustration of the optimal layout structure attaining objective value 0 we refer to Figure 8.

Indeed, the structure of all optimal solutions is the same by the following considerations:

- First note that 0 is indeed the objective value of our proposed layout as the centers of pairs of departments with connectivities greater than zero have the same x-coordinate.
- Departments with two arbitrary but different column indices cannot be moved to one column without either violating the space-free property or forcing the centers of two departments with connectivity 1 to be arranged on different x-coordinates, which results in non-optimal layouts.
- We can change the order of the departments within a so called block $B_i,\ i \in \{0, 1, \ldots, m-3\}$, of k neighboring department, as long as we change the order of the departments in all blocks with the same index i in the same way. These blocks are visualized in Figure 8. If we want to avoid this freedom, we can introduce further connectivities with a small enough value ε for pairs of departments in neighboring columns.
- Additionally, we can interchange rows 1 and 2 as long as we do not change the order of the departments in each of these rows.

Note that this example also shows that Corollary 8 is tight in the following sense: We cannot bound the difference of row lengths in space-free layouts in any reasonable way except for the two longest rows.

5.2 Reducing the Big-M value M in the Formulation of the `(FR-DRFLP)`.

In order to improve our enumeration scheme for the `(DRFLP)` we will show that the big-M-value M in $\texttt{IP}_{\texttt{(FR-DRFLP)}}$ and more generally in any `(DRFLP)` model can be chosen smaller than $\sum_{i=1}^{n} \ell_i$, which is usually used in the literature, see, e. g., [4, 11].

We again start with the double-row case and distance-calculation type 1. Let a feasible `(DRFLP)` layout be given and let us assume, w. l. o. g., that the leftmost department i of the considered

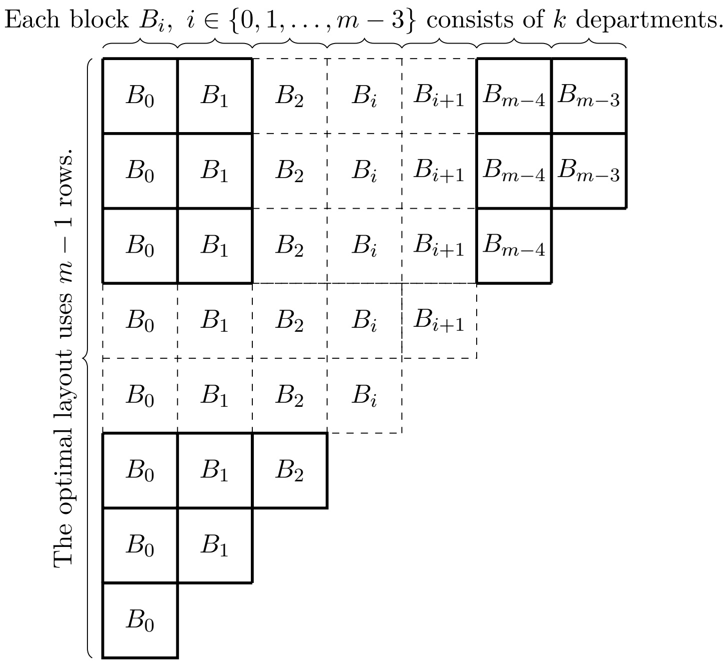


Figure 8: Illustration of the unique optimal space-free multi-row layout for the instance given in (32). The blocks B_i, $i \in \{0, 1, \dots, m-3\}$, consist of k departments with lengths $1 + 2^{ik}\varepsilon, \dots, 1 + 2^{((i+1)k-1)}\varepsilon$. Row m is empty.

layout starts at position 0 with its center at $p_i = \frac{\ell_i}{2}$ and that the rightmost department j of this layout finishes at t with its center at $p_j = t - \frac{\ell_j}{2}$. Clearly there exists an optimal layout that does not have space on both rows at any position p with $0 \le p \le t$. Now we want to give a bound on t such that there always exists an optimal double-row layout just using the interval $[0, t]$. A straightforward bound for t is $\sum_{i=1}^{n} \ell_i$. In the following lemma we deduce a tighter bound for t.

Lemma 10. *Given an (DRFLP) instance fulfilling, w. l. o. g., $\ell_i \le \ell_{i+1},\ i \in [n-1]$. Then for distance-calculation type 1 there always exists an optimal double-row layout on the interval $[0, t]$ with*

$$t \le \sum_{i=\lfloor \frac{n+1}{3} \rfloor + 1}^{n} \ell_i. \tag{33}$$

Proof. First we are going to suggest the basic operations of shifting and switching a block of departments that can be applied to an optimal layout to reduce t without deteriorating its objective value. In the following we concentrate on optimal layouts that cannot be further contracted through these operations. For these layouts we prove the suggested upper bound on t that is also tight, see the example depicted in Figure 12.

The first simple operation for reducing t is shifting a block of departments. The rightmost point of department d is at coordinate x. Now we consider the length s_r of the space right of x on the same row and the length s_l of the space left of x on the other row. Additionally we determine the distance of the closest center left or right of x on the other row and denote it by c_n. Then we can shift all departments in $[x, t]$ by $\min\{s_r, s_l, c_n\}$ to the left without deteriorating the objective value of the layout because all pairwise distances are reduced or remain the same. We refer to Figure 9 for an example.

Another operation to possibly decrease the layout length without deteriorating its objective value is switching the rows for a set of departments and afterwards shifting a block of departments. Switching all departments in the interval $[x_1, x_2],\ x_1 < x_2$, without switching any other department is possible if there do not exist departments that start (finish) inside the interval

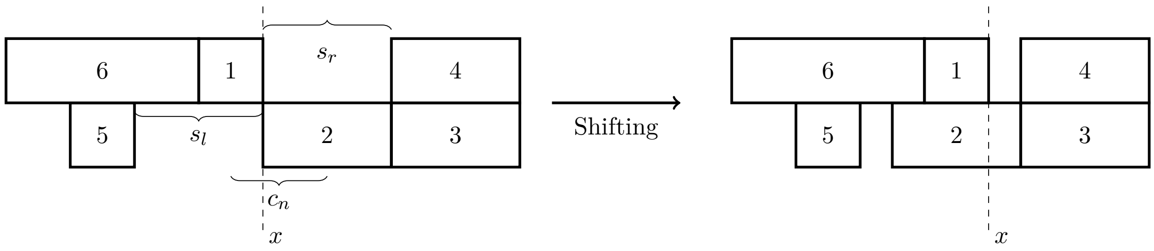


Figure 9: Shifting a block of departments. In the layout above we shift all departments in the interval $[x, t]$ by $c_n = \min\{s_r, s_l, c_n\}$ to the left.

$[x_1, x_2]$ and finish (start) outside the interval. See Figure 10 for an illustration of the switching operation. If switching is possible, it clearly does not change the objective value.

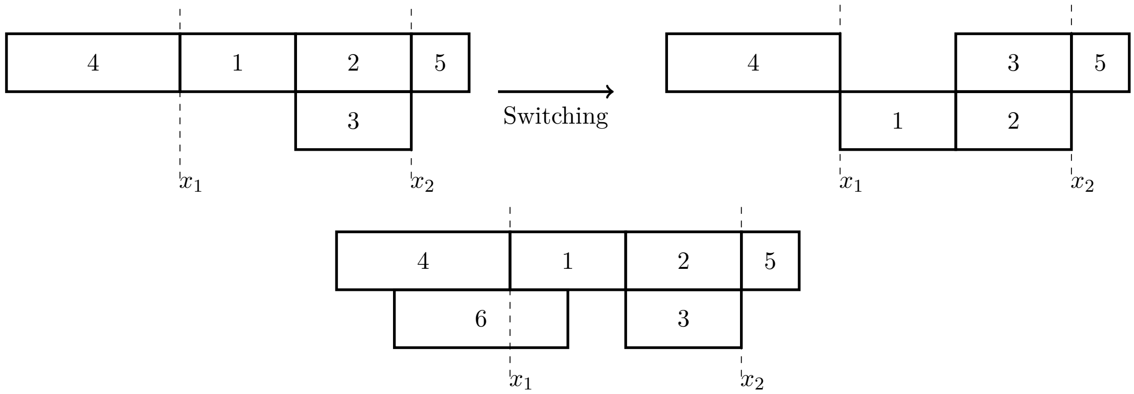


Figure 10: Illustration of the possibility of switching. In the upper layout switching is possible on the interval $[x_1, x_2]$, in the lower layout switching is not possible.

Now suppose department d starts at coordinate x and on the other row there is space of length s_l left and space of length s_r right of x. Additionally c_n gives the distance of the closest centers left and right of x. Then we can switch all departments in $[x, t]$ and shift them $\min\{s_l, s_r, c_n\}$ to the left without increasing the objective value of the layout. See Figure 11 for an illustration. Hence in the following we focus on layouts that do not allow for shifting a block of departments even after a possible switching.

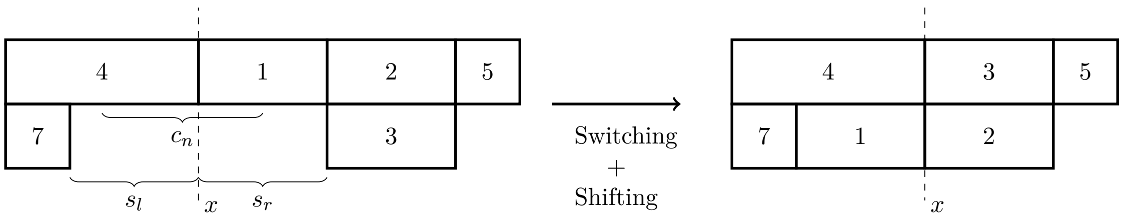


Figure 11: Illustration of switching plus shifting. The objective value of the layout does not increase but value t is reduced by ℓ_1.

Next we are going to bound t. We will prove that there exists a sequence of at most $n - \left\lfloor \frac{n+1}{3} \right\rfloor$ departments that covers the interval $[0, t]$. To determine such a sequence we condense our optimal layout in one row. The leftmost department belongs to the sequence. If there are two such departments, then we choose the longer one. Whenever we reach the coordinate x of the rightmost point of our current department, we add another department that starts at or covers x. If there are two such departments we choose the one with the larger coordinate of its rightmost point. If the two departments have the same rightmost point, then we choose the department with the smaller coordinate of its leftmost point. If also the smaller coordinates are the same, then we choose the one with the smaller index. Clearly the departments of the constructed sequence cover the interval $[0, t]$. The sequences for the layouts in Figures 9 and 11 are $(6, 2, 3)$ and $(4, 2, 5)$ respectively.

Finally let us show that at least $\left\lfloor \frac{n+1}{3} \right\rfloor$ departments of the layout do not belong to this sequence. Therefore we distinguish two cases:

Case 1: We have two candidates for the i^{th} department of the sequence, i. e. two departments are covering coordinate x. Then the $(i-1)^{\text{th}}$ and i^{th} department of the sequence cover another department not contained in the sequence: the department covering x that we do not choose does not end after the i^{th} department (due to our selection rule) and does not start before the $(i-1)^{\text{th}}$ department (due to our choice in the previous selection step).

Case 2: There is only one candidate department covering coordinate x that defines the rightmost point of the $(i-1)^{\text{th}}$ department of the sequence. Then on one of the two rows there is space right of x. There cannot be space left of x because then we would be able to perform a block shifting (maybe switching is needed beforehand) at x. But if there is no space left of x on the other row, then the $(i-1)^{\text{th}}$ department covers the other department ending at x due to our selection rule.

In summary each pair of departments in the sequence totally covers one department not contained in the sequence. This property directly yields our upper bound on t.

□

The following example shows that the bound from Lemma 10 is tight with respect to the shifting and switching operations suggested. We are given an instance with n fulfilling $\text{mod}(n, 3) = 1$ and

$$\ell_i = \begin{cases} 1 + \varepsilon, & i \in [n], \text{mod}(i, 3) = 2, \\ 1, & i \in [n], \text{mod}(i, 3) \neq 2, \end{cases}$$
$$w_{ij} = 1, \; i \in [n], \text{mod}(i, 3) = 2, j \in \{i-1, i+1, i+2\}.$$

In the optimal double-row layout, depicted in Figure 12, the longer row of the optimal layout has length $n - \left\lfloor \frac{n+1}{3} \right\rfloor = n - \frac{n-1}{3}$ and this is optimal because the length cannot be reduced by the shifting and switching operations suggested above. The bound on the maximal length t according to (33) is $n - \frac{n-1}{3} + \frac{n-1}{3} \cdot \varepsilon$, so the bound is tight.

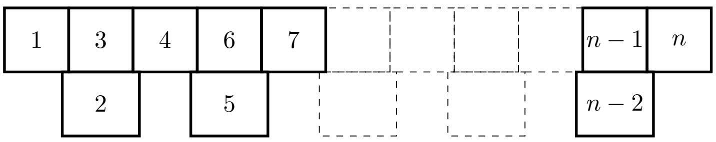


Figure 12: We are given a (DRFLP) instance with lengths $\ell_i = 1 + \varepsilon$ for $i \in [n]$, $\text{mod}(i, 3) = 2$, and $\ell_i = 1$ for $i \in [n]$, $\text{mod}(i, 3) \neq 2$, and connectivities $w_{ij} = 1$, $i \in [n]$, $\text{mod}(i, 3) = 2$, $j \in \{i-1, i+1, i+2\}$. For all optimal double-row layouts the bound on t from Lemma 10 is tight with respect to the shifting and switching operations suggested.

Remark 11. *The upper bound t on the maximal horizontal length of the double-row layout cannot only be used for improving our MILP model by setting M to a smaller value, but it can also be applied to reduce the number of row assignments that have to be considered in our enumeration scheme as we can neglect all assignments that are unbalanced, i. e., where the sum of the lengths of the departments in one of the rows exceeds t.*

Remark 12. *Note that Lemma 10 cannot be straightforwardly generalized to the multi-row case using $m \geq 3$ rows and hence we leave the deduction of possible even smaller values of t for larger m for future research. Furthermore Lemma 10 does not hold for distance-calculation type 2 because it might again be preferable to arrange two departments next to each other in the same row instead of putting them below of each other in the two different rows and hence the switching operations suggested do not work. If we considered inter-row distances (e. g. modelling the size of the aisle) in our enumeration scheme then none of the combinatorial results determined in this section would apply anymore.*

Finally let us indicate that the length of each row might still be arbitrarily close to $\sum_{i \in [n]} \ell_i$ for specially structured instances. For $m = 2$ we consider 5 departments with lengths $\ell_1 = \ell_2 = \ell_3 = \ell_4 = \varepsilon,\ \ell_5 = 1$ and connectivities $w_{12} = w_{34} = 1,\ w_{15} = w_{35} = \varepsilon$. The structure of an optimal layout for this instance is illustrated in Figure 13. Note that both rows have the same length $L = 1 + 2\varepsilon$ and it holds

$$\lim_{\varepsilon \to 0} \frac{L}{\sum_{i \in [n]} \ell_i} = \lim_{\varepsilon \to 0} \frac{1 + 2\varepsilon}{1 + 4\varepsilon} = 1.$$

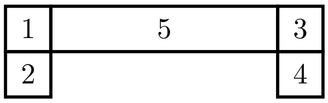


Figure 13: We consider a (DRFLP) instance with $\varepsilon \ll 1$, lengths $\ell_1 = \ell_2 = \ell_3 = \ell_4 = \varepsilon,\ \ell_5 = 1$, connectivities $w_{12} = w_{34} = 1,\ w_{15} = w_{35} = \varepsilon$, and depict an optimal layout with row length $1 + 2\varepsilon$ for both rows.

This construction can easily be generalized to an arbitrary number of rows: For each additional row we add two departments of length ε and connectivity 1 with two other departments i and j of length ε that are not connected, i. e. $w_{ij} = 0$.

6 Computational Experiments

In this section we present the computational results for several `(PROP)`, `(kPROP)`, `(SF-DRFLP)` and `(DRFLP)` instances. We used the larger instances from [5] for the `(PROP)` and the instances from [20] for `(kPROP)`. Furthermore, we tested all instances used by Amaral in [3] for the `(SF-DRFLP)` and [4] for the `(DRFLP)`, respectively. Apart from these instances, we created new ones according to the construction schemes in [4, 5, 20]. All instances can be downloaded from [22]. All experiments were performed on a `QUAD-Core INTEL-Core-I7-4770` (4×3400 MHz) with 32 GB RAM in single processor mode. We used CPLEX 12.6.2 [26] as an IP solver. For all different problems we tested two versions. In `Full` we included all inequalities at once, and in `Cuts` we only used some of the constraints at once and separated the respective variant of the triangle inequalities (see (19) and (21)) and inequalities (7)–(10). For the `(PROP)` we also considered the effect of adding the clique constraints (15) and (16), leading to `Full-C` and `Cuts-C`. Some preliminary tests, not contained in this paper, indicated that CPLEX should be forced to use all detected violated inequalities until the end. All running times are given in seconds and we used a time limit of six hours for the `(PROP)` and the `(kPROP)` instances as well as of twelve hours for the `(SF-DRFLP)` and `(DRFLP)` instances.

Considering the `(PROP)` instances in Table 1, we see that in comparison to [5] the running times could be reduced significantly. This can be seen especially for instances where only few departments are contained in the first row. These instances were extremely hard when treated with the model of Amaral. The average running time for instances with $n = 23$ and only 4 departments in row 1 was 344,869 seconds in [5] on an `Intel Core i3-M330` (2.13 GHz) with 4

GB RAM using CPLEX 12.4. With the fastest variant we solved all these instances in less then 20 minutes. But also for more balanced row assignments our approach is much better, allowing us to solve all instances with $n = 25$ in at most six hours with the best variant. Comparing the results of `Full` and `Cuts` one can see that using separation often pays off if n is rather large, but for smaller n and balanced row assignments `Full` is often better than `Cuts`. One reason for this effect is that the number of the constraints highly depends on the assignment. For each row $i \in \mathcal{R}$ the number of constraints (2) grows cubic in the size $|R_i|$ and with a power of 4 in $|R_i|$ for constraints (7)–(10). So the number of constraints is much smaller for balanced row assignments than for unbalanced ones. Using additionally the clique constraints is advantageous for some instances, but in general there is no clear winner comparing the four different solution variants. Here we also want to note that in our tests the solution time was partially highly influenced by the time and the quality of the (upper bound) solutions found by CPLEX during the solution process. The optimal values for all instances can be found in Table 2.

The results for the (kPROP) are presented in Tables 3 to 5. They show that the type of the distance calculation has a large effect on the running time. It seems that instances with distance-calculation type 2 are rather easy. Using `Cuts` all instances could be solved to optimality in less than one second. This is even much faster than the running times in [20] where one only gets lower bounds. Using type 3, the running times are a bit higher, but we also only need at most three minutes. Similarly as for (PROP), there is no clear winner between `Full` and `Cuts` for distance-calculation type 1. All instances could by solved to optimality with `Full`, while `Cuts` failed on seven instances. `Full` is always faster than `Cuts` for instances with three rows. `Cuts` is often better for four and five rows for the instances with $n \leq 23$ and several times slower for instances with $n \in \{24, 25\}$.

Iterating over all possible row assignments and neglecting all unbalanced assignments, we can solve the (SF-DRFLP) and the (DRFLP). Although the number of row assignments grows exponentially and so lots of NP-hard problems have to be solved, we could solve instances with up to 16 departments in reasonable time, see Tables 6 and 7. Note, the previously largest instance of the (SF-DRFLP) contained 13 departments and the largest (DRFLP) instances contained 12 departments. To allow the reader a direct comparison to the approach of Amaral we also tested his models and included the running times as well as the gaps in percent, calculated by $(\frac{\text{optimal}}{\text{lower bound}} - 1) \cdot 100$, for instances not solved within the time limit of twelve hours in Tables 6 and 7. With our new approach all instances with $n \leq 15$ could be solved in less than two hours, only the instances with $n = 16$ were costlier, but could be solved within the time limit. This was not possible for five (SF-DRFLP) instances and for six (DRFLP) instances using the approaches of Amaral [3, 4]. But note that his approaches are faster than our new ones for small instances. Comparing `Full` and `Cuts` there is no clear winner, although separation often helps if n is increased.

In order to show the impact of the investigations in Section 5 we also tested our enumeration scheme for the (SF-DRFLP) without using Lemma 6, i. e., we considered all assignments with $L_1 \geq L_2$, neglecting the ones with $L_1 = L_2$ and $r_1 = 2$. In our newly presented variant we additionally restrict to assignments with $\ell_{\max,1} \geq L_1 - L_2$. For the (DRFLP) we tested $\mathtt{IP}_{\mathtt{(FR\text{-}DRFLP)}}$ with the big-M-value $M = \sum_{i=1}^{n} \ell_i$ and without excluding row assignments where the sum of the lengths of the departments in one of the rows exceeds t defined in (33). All results are included in Table 6 for the (SF-DRFLP) and Table 7 for the (DRFLP). These tables show that reducing the number of assignments that have to be considered allows to reduce the running times of the (SF-DRFLP) significantly. For the (DRFLP) the combined effect of the improved big-M-value and the exclusion of unbalanced row assignments is much smaller for our model, here only small improvements are possible. One reason for this behavior might be that inequalities (29) and (30) bound the distances between the departments rather well from below. Additionally we can only exclude few row assignments based on the maximal layout length t. The effect is stronger for the approach of Amaral. In most cases the running times or the gaps can be improved significantly using the better big-M-value and the exclusion of unbalanced row assignments.

7 Conclusion and Future Work

In this paper we presented a new formulation for the (kPROP). Combining this formulation and a slightly modified model allowing spaces with an enumeration scheme over all relevant row assignments we were able to solve (SF-DRFLP) and (DRFLP) instances with up to 16 departments for the first time. To further speed-up the enumeration scheme we proved with the help of combinatorial arguments that very unbalanced row assignments do not have to be considered in the enumeration in the both cases, i. e., with and without allowing space between neighboring departments.

It remains for future work to improve this approach. One direction could be the study of the corresponding polyhedra deriving stronger relaxations for the (kPROP) and the (FR-MRFLP). Furthermore, assigning most but not all of the departments to the rows, a large amount of the total objective value is predetermined. It would be nice if one could detect situations that cannot lead to optimal solutions beforehand, e. g., by calculating some combinatorial bounds.

In our enumeration scheme we did not use, the partially known, upper bounds on the optimal objective value, determined by some heuristic. We only compared to the currently best solution. So it would also be worth to study if one can find some criteria which row assignments should be considered first. At the moment we use the same order of the row assignments for all instances.

Our approach is mainly based on the combination of a fast solution of (kPROP) and (FR-MRFLP) instances and an enumeration scheme. In general, one could try to combine the assignment of the departments to the rows and the inter-row distance calculation via betweenness variables and further variables, coupled with appropriate constraints, in one model. For this note that all MILP models for the (DRFLP) in the literature do not use betweenness variables, although these were successfully employed for the (SRFLP).

As a next step it would also be interesting to consider facility layout problems with more complex path structures, if, e. g., there are two paths in the shape of a T or an X.

name	source	Full				Cuts				Full-C				Cuts-C			
		$i=2$	$i=3$	$i=4$	$i=5$	$i=2$	$i=3$	$i=4$	$i=5$	$i=2$	$i=3$	$i=4$	$i=5$	$i=2$	$i=3$	$i=4$	$i=5$
P16a	[5]	16.69	24.50	10.22	17.41	14.99	21.33	14.75	29.80	10.16	27.84	18.31	20.02	11.41	21.96	13.99	18.95
P16b	[5]	10.39	17.33	20.43	5.79	13.31	14.69	12.41	8.11	9.98	20.89	19.16	8.10	10.30	16.84	15.30	6.41
P20a	[5]	560.07	339.49	736.42	156.60	810.33	303.93	223.16	84.72	665.12	418.95	486.15	154.01	930.15	247.38	220.67	125.67
P20b	[5]	146.74	377.18	214.11	190.21	138.64	207.05	277.98	187.86	180.46	222.61	213.77	232.20	140.28	293.94	231.71	145.37
P21a	[5]	416.29	431.36	150.10	371.08	539.52	481.73	257.60	334.30	411.16	341.37	218.42	372.96	712.43	243.08	240.82	304.66
P21b	[5]	285.66	543.21	228.78	139.61	193.44	318.61	102.34	76.91	323.66	375.47	216.65	173.70	148.05	250.82	160.03	79.84
P21c	[5]	199.49	836.53	417.31	436.67	152.44	1435.74	144.04	364.41	268.64	922.35	246.32	646.33	392.72	1353.01	200.77	433.11
P21d	[5]	545.44	242.81	542.90	287.13	345.66	272.51	238.40	122.90	398.25	233.26	341.18	451.84	352.04	265.10	220.51	110.65
P21e	[5]	2466.78	636.03	243.24	71.30	931.46	326.87	98.92	58.84	1358.18	454.48	218.51	197.46	1385.26	478.23	92.58	52.43
P22a	[5]	574.54	1425.96	313.25	836.17	394.96	1211.30	377.33	564.46	806.99	615.27	482.95	1033.82	564.77	777.05	577.98	827.25
P22b	[5]	647.62	1334.10	956.02	178.62	878.32	786.60	251.57	146.11	769.92	1325.68	461.98	231.76	558.73	655.10	243.04	157.05
P22c	[5]	835.26	975.46	359.15	607.63	4496.06	880.35	392.44	212.22	1737.41	1663.77	656.70	807.90	767.87	706.93	287.95	199.39
P22d	[5]	801.84	830.01	249.89	196.68	1261.22	805.13	95.27	72.59	803.56	863.17	328.92	332.76	636.09	786.36	112.95	69.05
P22e	[5]	1033.20	3039.47	938.50	407.99	1808.94	1347.98	420.30	176.77	634.38	1084.93	591.70	802.95	923.14	923.37	417.69	186.82
P23a	[5]	1182.54	928.22	1320.54	1148.85	2077.39	2919.68	509.13	924.31	1448.90	2069.22	1691.74	1154.24	1531.78	1582.90	658.29	646.76
P23b	[5]	509.26	2584.86	592.71	471.48	470.24	536.95	391.99	161.01	710.65	1577.20	534.49	1677.29	444.18	757.42	581.18	126.92
P23c	[5]	498.26	538.40	377.01	726.30	199.88	413.00	156.98	311.12	199.70	736.87	434.80	2260.05	642.97	479.85	159.23	286.23
P23d	[5]	1509.04	996.46	1183.74	1528.57	1034.83	1308.82	713.72	391.90	1759.00	2353.65	1220.26	1109.01	1649.75	672.09	634.76	891.88
P23e	[5]	807.83	798.59	330.48	103.26	729.59	304.25	191.71	95.54	1431.42	948.25	286.76	1934.19	593.02	462.79	154.40	98.03
P24a	new	4060.84	2542.48	2056.15	2164.15	10488.58	3121.71	1358.03	1148.58	3825.91	12310.51	1485.62	12225.13	5350.22	2116.35	1722.79	1587.88
P24b	new	1683.30	3259.37	2185.47	1070.26	1408.49	2747.25	2204.36	347.82	1568.79	13563.39	5653.16	8781.06	1100.97	3078.87	1778.20	321.02
P24c	new	1934.47	5234.74	3576.10	6201.10	6912.44	9847.60	1271.51	788.75	2657.01	5826.02	4610.27	4469.53	8108.55	4218.73	1576.76	777.56
P24d	new	1087.15	3618.04	2251.63	1161.50	826.05	1385.67	588.68	812.78	1190.72	1933.80	2099.18	4070.07	490.48	938.66	588.06	552.79
P24e	new	1345.51	2464.95	1988.84	280.02	1499.19	4661.78	2016.66	79.53	1194.11	12191.93	2556.88	2321.69	919.62	5685.33	1523.20	79.70
P25a	new	4337.45	8691.46	2860.70	3168.47	9067.30	5187.36	2529.52	924.81	3035.14	9695.73	9078.21	5670.31	TL	6795.39	1991.49	1326.79
P25b	new	6272.78	8708.71	5997.29	7862.11	5097.33	21396.53	4174.19	2303.28	7011.16	TL	14617.46	11475.14	4491.80	7258.51	4104.73	1617.29
P25c	new	3595.96	9422.10	5819.61	2279.24	8227.48	5566.18	2176.12	715.07	4991.11	13181.00	5044.94	4722.25	5476.65	5330.85	1716.65	1331.58
P25d	new	3357.97	4610.75	1111.77	2189.01	2880.91	2110.27	737.46	487.19	4447.51	8904.91	1879.68	8153.88	3722.41	2726.21	1237.28	505.66
P25e	new	3700.78	3241.67	14184.93	4822.80	1853.67	2813.87	2953.62	1906.79	2633.25	6524.08	14596.36	7196.28	4820.50	3990.80	2114.11	1588.69
AV25_1	[8]	TL	13870.64	6051.03	1836.68	20904.15	7878.95	2053.88	643.81	TL	TL	6376.36	2460.24	12947.90	8824.26	1744.82	371.70
AV25_2	[8]	15765.34	3327.52	4088.58	1269.99	5555.20	1492.89	1366.40	1438.46	18680.46	2170.09	3033.33	2261.11	12133.07	1377.56	1288.24	697.73
AV25_3	[8]	8196.42	2562.48	2183.90	1690.54	TL	912.56	989.46	641.33	4459.43	4857.43	7709.38	2780.82	TL	1025.32	1124.97	658.12
AV25_4	[8]	TL	2203.12	3931.55	1701.42	6818.28	2012.71	1299.00	1750.69	3271.57	2556.46	4113.56	1729.10	4592.42	2248.63	2702.15	1081.45
AV25_5	[8]	4891.63	2379.67	3222.64	1879.48	5444.14	996.72	1469.08	1050.25	13817.28	4610.18	3510.85	1011.39	7933.29	1683.79	1196.85	1217.65

Table 1: Running times in seconds for (PROP) instances with $\lfloor \frac{n}{i} \rfloor$ departments in row 1 (time limit TL of six hours). In Full-C and Cuts-C we also add the clique constraints (15) and (16) to our basic model (2)–(13).

name	source	$i=2$	$i=3$	$i=4$	$i=5$
P16a	[5]	7630.0	9813.0	11409.0	12279.0
P16b	[5]	6239.5	9091.5	9636.5	11256.5
P20a	[5]	12609.5	15874.5	18185.5	21215.5
P20b	[5]	12936.0	19167.0	22801.0	23902.0
P21a	[5]	7006.5	9141.5	11765.5	12382.5
P21b	[5]	11705.0	13887.0	18564.0	20825.0
P21c	[5]	11434.0	12758.0	16888.0	19481.0
P21d	[5]	12289.0	14988.0	19471.0	20685.0
P21e	[5]	13112.5	15711.5	19865.5	22423.5
P22a	[5]	8874.0	12238.0	15385.0	16114.0
P22b	[5]	15714.0	19183.0	23534.0	25044.0
P22c	[5]	14693.0	19963.0	24221.0	25545.0
P22d	[5]	16355.0	19981.0	25180.0	26796.0
P22e	[5]	14815.5	20112.5	24515.5	27161.5
P23a	[5]	10242.0	14294.0	17812.0	18619.0
P23b	[5]	15802.5	21116.5	26004.5	29892.5
P23c	[5]	15542.0	21511.0	26040.0	27553.0
P23d	[5]	17174.0	23522.0	27922.0	30694.0
P23e	[5]	16481.5	20798.5	27574.5	29810.5
P24a	new	11778.0	14730.0	18757.0	21729.0
P24b	new	17629.5	21015.5	25311.5	29881.5
P24c	new	17378.5	18630.5	23909.5	29041.5
P24d	new	19630.0	21786.0	26949.0	32132.0
P24e	new	21400.0	26998.0	33235.0	38800.0
P25a	new	12889.5	16411.5	20865.5	22990.5
P25b	new	17459.0	23635.0	27162.0	29496.0
P25c	new	23148.5	33228.5	40086.5	41264.5
P25d	new	22421.0	29450.0	35283.0	38373.0
P25e	new	21048.5	24664.5	32078.5	34018.5
AV25_1	[8]	2349.0	3077.0	3705.0	4039.0
AV25_2	[8]	19138.5	23826.5	26229.5	30193.5
AV25_3	[8]	12549.0	18714.0	23081.0	23167.0
AV25_4	[8]	24922.5	30647.5	33584.5	38689.5
AV25_5	[8]	8011.0	10126.0	11289.0	12951.0

Table 2: Optimal values of (PROP) instances with $\lfloor \frac{n}{i} \rfloor$ departments in row 1.

name	m	i	Full 1	Full 2	Full 3	Cuts 1	Cuts 2	Cuts 3	opt 1	opt 2	opt 3
P16a	3	3	4.42	0.02	0.41	6.81	0.00	1.42	5234.0	13936.0	7910.0
P16a	3	4	3.27	0.02	0.54	8.88	0.00	1.59	6639.0	13541.0	9615.0
P16a	3	5	2.76	0.07	0.89	4.42	0.01	0.93	9746.0	14843.0	12845.0
P16a	4	4	1.73	0.01	0.10	1.16	0.00	0.18	3896.0	10883.0	7191.0
P16a	4	5	2.63	0.02	0.17	1.73	0.00	0.35	6021.0	12370.0	9630.0
P16a	5	5	1.15	0.00	0.02	0.55	0.00	0.02	3181.0	9412.0	7073.0
P16b	3	3	2.02	0.01	0.26	3.04	0.00	0.92	4289.5	11375.5	6330.5
P16b	3	4	2.33	0.02	0.55	4.01	0.00	1.21	5228.5	11123.5	7583.5
P16b	3	5	2.76	0.07	0.74	3.87	0.01	0.77	9123.5	12644.5	10981.5
P16b	4	4	1.44	0.00	0.08	0.82	0.00	0.20	3401.5	9028.5	5993.5
P16b	4	5	1.97	0.01	0.20	1.37	0.00	0.40	4717.5	9834.5	7299.5
P16b	5	5	0.71	0.00	0.03	0.50	0.00	0.08	2786.5	7688.5	5934.5
P20a	3	3	90.74	0.03	3.71	951.35	0.00	4.98	8587.5	21440.5	13317.5
P20a	3	4	65.61	0.07	6.17	463.84	0.01	9.83	11223.5	22952.5	16161.5
P20a	3	5	91.29	0.44	14.06	522.71	0.02	16.44	14171.5	23096.5	18129.5
P20a	4	4	28.47	0.01	0.76	20.88	0.00	1.50	6697.5	17663.5	12064.5
P20a	4	5	24.88	0.02	0.41	17.85	0.00	0.65	7653.5	17694.5	13524.5
P20a	5	5	7.11	0.01	0.07	3.99	0.00	0.17	5210.5	14317.5	10532.5
P20b	3	3	47.70	0.03	3.45	207.55	0.01	4.21	9396.0	22750.0	14073.0
P20b	3	4	18.54	0.06	7.65	192.32	0.01	9.03	11441.0	23777.0	15892.0
P20b	3	5	45.54	0.32	12.38	90.97	0.03	14.56	15505.0	25114.0	19615.0
P20b	4	4	26.67	0.01	0.32	9.13	0.00	0.71	7092.0	18972.0	12160.0
P20b	4	5	10.42	0.03	0.37	6.81	0.00	0.98	8645.0	19081.0	15005.0
P20b	5	5	10.90	0.01	0.07	2.44	0.00	0.12	5408.0	15055.0	11038.0
P21a	3	3	42.64	0.02	4.82	198.20	0.01	8.87	4867.5	12526.5	7462.5
P21a	3	4	54.81	0.17	13.54	205.99	0.01	15.51	6344.5	13135.5	8405.5
P21a	3	5	188.13	0.65	19.55	1279.80	0.03	29.21	8549.5	13969.5	10482.5
P21a	4	4	16.63	0.02	0.36	12.88	0.00	0.92	3921.5	10431.5	7729.5
P21a	4	5	17.52	0.04	0.52	15.50	0.01	0.89	4732.5	10405.5	8181.5
P21a	5	5	14.07	0.01	0.06	4.58	0.00	0.11	3032.5	8563.5	6076.5
P21b	3	3	51.11	0.03	4.49	199.80	0.01	4.94	8139.0	21966.0	12388.0
P21b	3	4	86.56	0.16	5.92	290.93	0.01	6.91	9809.0	22333.0	13575.0
P21b	3	5	353.44	0.60	18.67	436.49	0.03	22.33	12928.0	22681.0	16204.0
P21b	4	4	29.54	0.02	0.29	13.53	0.01	0.67	6282.0	17987.0	11948.0
P21b	4	5	21.77	0.04	0.13	13.82	0.00	0.08	7931.0	17942.0	13682.0
P21b	5	5	5.50	0.01	0.08	2.46	0.00	0.14	5030.0	14957.0	10483.0
P21c	3	3	29.20	0.03	7.28	90.24	0.01	13.45	7846.0	20665.0	12427.0
P21c	3	4	134.38	0.18	37.24	1538.44	0.01	29.71	8961.0	21437.0	13227.0
P21c	3	5	478.57	0.86	112.13	2868.55	0.06	62.27	11470.0	22351.0	15882.0
P21c	4	4	21.06	0.02	0.48	12.82	0.00	1.02	6804.0	17283.0	12625.0
P21c	4	5	28.58	0.04	0.64	17.77	0.01	0.97	7141.0	17053.0	13211.0
P21c	5	5	8.17	0.01	0.07	4.43	0.00	0.06	5323.0	14221.0	10377.0
P21d	3	3	39.89	0.03	4.50	261.03	0.00	7.28	8260.0	21814.0	12886.0
P21d	3	4	52.83	0.17	19.48	452.65	0.01	13.99	10683.0	22677.0	14756.0
P21d	3	5	236.98	0.73	40.83	731.73	0.04	59.80	15094.0	24338.0	19221.0
P21d	4	4	34.41	0.02	0.25	19.18	0.01	0.62	6940.0	18225.0	12828.0
P21d	4	5	15.33	0.04	0.46	10.06	0.01	0.45	8888.0	18817.0	14877.0
P21d	5	5	7.74	0.01	0.07	4.53	0.00	0.10	5268.0	15208.0	11024.0
P21e	3	3	132.96	0.03	3.57	1372.73	0.00	9.59	9156.5	23443.5	13624.5
P21e	3	4	62.98	0.18	16.62	321.66	0.02	17.82	11574.5	25059.5	15808.5
P21e	3	5	275.28	0.81	112.25	1737.27	0.14	70.48	15436.5	26552.5	19670.5
P21e	4	4	63.80	0.02	0.31	42.07	0.00	0.86	7351.5	19820.5	13372.5
P21e	4	5	13.58	0.04	0.67	12.44	0.01	1.53	8970.5	20592.5	15645.5
P21e	5	5	14.94	0.01	0.10	6.94	0.00	0.12	5541.5	16655.5	11441.5

Table 3: Running times in seconds and optimal values "opt" for (kPROP) instances from [20] with $n \in \{16, 20, 21\}$ with $\lfloor \frac{n}{i} \rfloor$ departments in each of the first $m-1$ rows. We apply our basic model (2)–(13) for both variants Full and Cuts for the three distance types $\{1, 2, 3\}$.

name	m	i	Full 1	Full 2	Full 3	Cuts 1	Cuts 2	Cuts 3	opt 1	opt 2	opt 3
P22a	3	3	138.72	0.04	7.53	1747.61	0.01	16.24	6213.0	15797.0	9829.0
P22a	3	4	179.49	0.41	33.66	988.06	0.02	22.19	8928.0	16823.0	11308.0
P22a	3	5	479.21	1.24	54.91	3553.97	0.09	73.22	11634.0	18002.0	14082.0
P22a	4	4	42.57	0.02	0.61	33.92	0.01	1.85	5476.0	13453.0	10477.0
P22a	4	5	42.78	0.07	1.42	21.61	0.02	1.11	6870.0	13629.0	11136.0
P22a	5	5	18.16	0.02	0.04	4.88	0.00	0.05	4286.0	10994.0	8438.0
P22b	3	3	238.58	0.04	13.06	726.17	0.02	41.64	10398.0	26847.0	16698.0
P22b	3	4	113.98	0.44	40.35	4519.16	0.03	50.63	13030.0	28196.0	18219.0
P22b	3	5	365.96	1.25	45.48	2639.33	0.06	27.22	18470.0	30211.0	23371.0
P22b	4	4	50.81	0.03	0.40	58.06	0.00	0.48	8168.0	22115.0	15805.0
P22b	4	5	43.92	0.09	1.52	46.93	0.01	1.98	10509.0	23566.0	18285.0
P22b	5	5	22.08	0.02	0.09	7.74	0.00	0.10	6615.0	19201.0	14224.0
P22c	3	3	160.98	0.03	14.19	3445.79	0.00	16.52	10194.0	25805.0	16217.0
P22c	3	4	169.94	0.43	36.33	712.92	0.02	28.64	15139.0	27997.0	19293.0
P22c	3	5	339.82	1.46	11.38	1431.32	0.10	10.73	20267.0	29877.0	24351.0
P22c	4	4	53.45	0.02	0.35	32.00	0.00	0.84	8564.0	21739.0	15642.0
P22c	4	5	33.83	0.07	1.43	27.85	0.01	0.64	12139.0	23237.0	18810.0
P22c	5	5	26.04	0.01	0.05	11.41	0.00	0.03	6673.0	18255.0	13325.0
P22d	3	3	93.78	0.04	8.41	406.31	0.00	9.46	11053.0	28052.0	17142.0
P22d	3	4	244.92	0.36	35.28	1565.81	0.03	20.82	13503.0	29154.0	18148.0
P22d	3	5	300.09	1.12	32.84	1734.83	0.06	5.77	18950.0	30516.0	23411.0
P22d	4	4	53.23	0.02	0.57	53.26	0.01	1.25	8308.0	23241.0	15939.0
P22d	4	5	44.18	0.07	0.93	33.67	0.01	0.19	9990.0	23305.0	17778.0
P22d	5	5	38.72	0.01	0.12	17.80	0.00	0.07	6624.0	19337.0	14165.0
P22e	3	3	126.63	0.04	5.63	2417.27	0.02	11.59	10336.5	25655.5	16349.5
P22e	3	4	85.02	0.50	27.15	230.22	0.08	36.48	17355.5	30707.5	22528.5
P22e	3	5	258.95	1.71	66.28	473.69	0.37	69.81	21414.5	31821.5	25897.5
P22e	4	4	34.17	0.02	0.26	16.80	0.00	0.60	8801.5	22152.5	15680.5
P22e	4	5	45.46	0.08	1.33	21.64	0.01	3.06	12024.5	23554.5	19220.5
P22e	5	5	11.95	0.01	0.06	5.18	0.00	0.06	7004.5	18244.5	13972.5
P23a	3	3	213.48	0.07	15.40	2842.13	0.02	17.60	7321.0	18212.0	11377.0
P23a	3	4	244.08	0.71	127.17	4823.24	0.06	75.23	10654.0	19247.0	13364.0
P23a	3	5	636.84	2.64	174.93	2562.23	0.49	109.39	13675.0	20533.0	16281.0
P23a	4	4	60.44	0.03	1.77	38.78	0.01	4.17	6583.0	15593.0	12294.0
P23a	4	5	75.16	0.22	4.54	61.14	0.07	4.32	8310.0	15823.0	13054.0
P23a	5	5	32.00	0.02	0.14	13.60	0.00	0.14	5293.0	12773.0	10104.0
P23b	3	3	119.50	0.05	12.05	1490.11	0.01	13.14	11493.5	27227.5	18063.5
P23b	3	4	81.81	0.69	59.26	790.86	0.04	36.08	16696.5	29712.5	21965.5
P23b	3	5	292.07	2.47	159.66	1297.92	0.18	85.93	21930.5	31951.5	26267.5
P23b	4	4	27.63	0.03	0.40	10.04	0.00	0.73	9645.5	23236.5	17241.5
P23b	4	5	46.46	0.18	2.29	31.44	0.02	1.43	14308.5	24984.5	21016.5
P23b	5	5	36.85	0.02	0.11	13.94	0.00	0.14	8213.5	19185.5	15057.5
P23c	3	3	281.27	0.05	7.36	1073.28	0.01	12.45	11293.0	26913.0	17061.0
P23c	3	4	135.25	0.72	38.34	590.85	0.04	39.86	17701.0	29983.0	22247.0
P23c	3	5	333.62	2.24	155.01	1469.41	0.13	77.36	22115.0	31484.0	26304.0
P23c	4	4	32.20	0.03	0.50	25.77	0.00	0.70	9533.0	22710.0	16584.0
P23c	4	5	54.28	0.16	3.32	36.46	0.02	3.70	13675.0	24735.0	20354.0
P23c	5	5	17.17	0.02	0.04	13.63	0.01	0.08	8001.0	19280.0	14882.0
P23d	3	3	351.30	0.05	25.76	13668.14	0.01	19.11	11908.0	29764.0	18976.0
P23d	3	4	157.91	0.84	106.36	1364.31	0.06	60.45	19822.0	33930.0	25325.0
P23d	3	5	749.31	2.68	220.85	1322.22	0.23	127.01	25008.0	35905.0	29756.0
P23d	4	4	175.84	0.02	1.10	71.46	0.01	1.31	10152.0	25210.0	18232.0
P23d	4	5	115.46	0.21	3.18	85.98	0.02	2.92	14529.0	27499.0	22196.0
P23d	5	5	20.75	0.02	0.14	13.85	0.00	0.18	8267.0	21500.0	16596.0
P23e	3	3	162.63	0.04	21.82	4516.00	0.01	18.43	11199.5	29222.5	17644.5
P23e	3	4	320.47	0.56	76.18	2453.74	0.03	34.74	16011.5	30390.5	20315.5
P23e	3	5	750.58	1.98	148.95	1657.76	0.13	75.31	20218.5	32014.5	23977.5
P23e	4	4	145.79	0.03	1.05	59.21	0.01	1.87	9299.5	24129.5	17153.5
P23e	4	5	45.60	0.21	1.80	26.86	0.02	1.49	12694.5	25861.5	20616.5
P23e	5	5	44.43	0.02	0.05	23.58	0.01	0.06	7778.5	20840.5	15339.5

Table 4: Running times in seconds and optimal values "opt" for (kPROP) instances from [20] with $n \in \{22, 23\}$ with $\lfloor \frac{n}{i} \rfloor$ departments in each of the first $m-1$ rows for the three distance types. We apply our basic model (2)–(13) for both variants Full and Cuts for the three distance types $\{1, 2, 3\}$.

name	m	i	`Full 1`	`Full 2`	`Full 3`	`Cuts 1`	`Cuts 2`	`Cuts 3`	opt 1	opt 2	opt 3
P24a	3	3	166.02	0.04	13.34	3592.49	0.01	16.18	7786.0	20415.0	12529.0
P24a	3	4	230.04	0.46	77.75	2056.57	0.03	57.32	10206.0	21008.0	14529.0
P24a	3	5	966.70	3.54	368.70	5206.19	0.72	159.78	16196.0	23426.0	19037.0
P24a	4	4	172.81	0.02	0.88	162.83	0.01	1.66	6188.0	16634.0	11974.0
P24a	4	5	207.66	0.45	6.84	225.68	0.03	3.98	10515.0	18875.0	15578.0
P24a	5	5	84.93	0.03	0.15	21.19	0.01	0.13	6828.0	15278.0	12441.0
P24b	3	3	229.25	0.05	15.32	2624.55	0.01	12.32	12181.5	30661.5	19127.5
P24b	3	4	531.99	0.42	39.21	4417.06	0.03	53.20	15535.5	32002.5	22680.5
P24b	3	5	1350.67	3.94	199.87	16499.98	0.65	135.34	23361.5	35162.5	28885.5
P24b	4	4	739.70	0.02	1.69	894.95	0.00	2.79	9775.5	24864.5	18587.5
P24b	4	5	84.25	0.39	6.11	129.93	0.03	4.05	15551.5	28839.5	24020.5
P24b	5	5	69.87	0.02	0.38	54.87	0.00	0.54	10142.5	22914.5	18711.5
P24c	3	3	511.20	0.04	49.21	6652.70	0.01	36.18	11724.5	29736.5	18531.5
P24c	3	4	587.42	0.44	53.99	TL	0.04	36.33	12823.5	29341.5	19445.5
P24c	3	5	1144.53	4.25	112.74	TL	0.50	130.71	20319.5	32661.5	25373.5
P24c	4	4	130.96	0.02	3.11	72.94	0.01	3.97	9014.5	23968.5	17645.5
P24c	4	5	107.84	0.39	4.45	58.98	0.04	2.37	12304.5	25963.5	20728.5
P24c	5	5	73.90	0.03	0.39	30.37	0.01	0.96	8335.5	21249.5	17221.5
P24d	3	3	131.16	0.04	8.15	1076.54	0.01	9.42	13153.0	34340.0	20817.0
P24d	3	4	184.94	0.37	30.81	3726.89	0.03	21.21	16013.0	34898.0	24223.0
P24d	3	5	1063.32	2.85	148.76	15317.29	0.26	50.64	23580.0	37270.0	29879.0
P24d	4	4	66.74	0.02	1.75	90.68	0.01	2.86	10643.0	28053.0	20682.0
P24d	4	5	90.00	0.38	2.77	82.94	0.03	5.28	15915.0	30837.0	25182.0
P24d	5	5	36.30	0.03	0.26	21.31	0.00	0.40	10210.0	25017.0	20133.0
P24e	3	3	1732.46	0.05	13.06	TL	0.01	14.17	14847.0	37580.0	22442.0
P24e	3	4	180.62	0.34	28.49	5634.37	0.02	18.52	18347.0	37504.0	26053.0
P24e	3	5	887.69	3.64	347.53	5469.19	0.66	147.56	30005.0	44191.0	35715.0
P24e	4	4	197.11	0.02	1.21	444.91	0.00	2.02	11586.0	29778.0	21738.0
P24e	4	5	75.17	0.33	4.49	96.22	0.02	2.41	18411.0	34610.0	28030.0
P24e	5	5	71.69	0.03	0.14	25.27	0.00	0.05	11108.0	27583.0	21450.0
P25a	3	3	603.24	0.07	15.23	8323.14	0.02	22.30	8780.5	22323.5	13974.5
P25a	3	4	420.82	0.81	154.96	5620.59	0.06	91.31	11537.5	23033.5	16080.5
P25a	3	5	1219.63	2.15	163.96	6005.02	0.09	123.32	14441.5	24268.5	17641.5
P25a	4	4	306.61	0.03	1.34	346.63	0.01	2.88	6796.5	18182.5	13266.5
P25a	4	5	186.59	0.12	12.25	448.98	0.02	11.84	9548.5	20384.5	16460.5
P25a	5	5	176.88	0.02	0.38	192.88	0.00	0.20	5768.5	15860.5	11994.5
P25b	3	3	4424.82	0.08	17.68	TL	0.02	15.24	12129.0	30894.0	18268.0
P25b	3	4	1179.09	0.78	106.84	20284.50	0.12	60.78	16150.0	31616.0	22842.0
P25b	3	5	767.46	2.51	246.82	6691.79	0.35	146.58	20351.0	33785.0	25577.0
P25b	4	4	404.04	0.03	1.23	664.19	0.01	2.11	9249.0	24873.0	18005.0
P25b	4	5	1262.72	0.08	3.53	562.24	0.02	4.10	11535.0	26776.0	19962.0
P25b	5	5	648.06	0.02	0.24	436.94	0.01	0.87	7729.0	21576.0	15786.0
P25c	3	3	1150.17	0.07	38.65	TL	0.02	31.29	16183.5	41087.5	25683.5
P25c	3	4	487.02	0.86	239.47	4153.91	0.11	84.33	21955.5	42947.5	29963.5
P25c	3	5	575.21	2.69	200.58	2878.46	0.30	137.00	28287.5	45689.5	34432.5
P25c	4	4	282.22	0.03	2.65	796.48	0.01	4.51	12655.5	32883.5	24086.5
P25c	4	5	687.32	0.07	2.53	191.77	0.02	4.69	17132.5	35907.5	27800.5
P25c	5	5	460.03	0.02	0.22	178.27	0.00	0.22	9801.5	28173.5	20661.5
P25d	3	3	3553.53	0.06	29.21	TL	0.01	26.25	15379.0	39071.0	23844.0
P25d	3	4	510.21	0.69	69.76	TL	0.03	57.12	19769.0	39277.0	27603.0
P25d	3	5	2672.71	2.35	176.94	7546.81	0.26	136.24	24073.0	42118.0	30638.0
P25d	4	4	1599.85	0.03	1.64	787.12	0.01	2.93	12059.0	31212.0	22699.0
P25d	4	5	1137.70	0.07	3.15	791.77	0.01	2.86	14899.0	33490.0	25399.0
P25d	5	5	233.12	0.02	0.21	290.53	0.00	0.06	9761.0	27381.0	19716.0
P25e	3	3	626.11	0.07	24.74	1903.20	0.01	25.71	14361.5	36781.5	22521.5
P25e	3	4	936.13	0.73	133.28	706.61	0.10	65.64	16990.5	37058.5	24521.5
P25e	3	5	2683.72	2.20	200.21	815.58	0.19	179.66	20420.5	38502.5	26683.5
P25e	4	4	182.13	0.03	2.72	481.51	0.00	3.78	11278.5	30009.5	21618.5
P25e	4	5	399.39	0.08	3.38	173.03	0.02	5.43	13310.5	31756.5	23752.5
P25e	5	5	140.15	0.02	0.32	103.87	0.00	0.92	9329.5	26121.5	19098.5

Table 5: Running times in seconds and optimal values "opt" for new `(kPROP)` instances (constructed according to the rules in [20]) with $n \in \{24, 25\}$ with $\lfloor \frac{n}{i} \rfloor$ departments in each of the first $m-1$ rows for the three distance types (time limit TL of six hours). We apply our basic model (2)–(13) for both variants `Full` and `Cuts` for the three distance types $\{1, 2, 3\}$.

name	source	standard		Lemma 6		Amaral [3]	
		`Full`	`Cuts`	`Full`	`Cuts`	time (gap)	optimal
HA5	[23]	0.04	0.02	0.02	0.01	0.02	52.5
HA6	[23]	0.17	0.16	0.09	0.13	0.06	190.5
HA7	[23]	0.39	0.36	0.25	0.29	0.08	166.0
HA8	[23]	1.65	1.51	1.00	1.06	0.57	205.0
HA9	[23]	6.13	6.76	3.79	4.95	4.34	492.5
HA10	[23]	18.04	19.86	11.00	14.57	22.85	838.0
HA11	[23]	30.97	22.88	16.06	16.59	33.89	796.0
HA12	[23]	173.73	164.76	115.03	133.52	331.32	1028.0
HA13	[23]	335.08	227.52	175.57	164.40	832.24	1530.5
HA14	[23]	1868.29	1418.52	1131.58	1043.26	13315.28	1841.0
HA15	[23]	7907.46	7010.24	5449.31	5685.67	TL (2.50)	2643.5
s9	[3, 4]	4.16	3.35	1.47	1.97	9.03	1181.5
s9h	[3, 4]	18.43	28.28	10.99	18.64	55.96	2294.5
s10	[3, 4]	11.78	8.63	4.37	4.39	25.23	1374.5
s11	[3, 4]	48.04	42.65	22.25	24.36	152.59	3439.5
Am12a	[3, 4]	177.61	146.16	110.23	116.41	356.88	1529.0
Am12b	[3, 4]	116.68	84.20	56.44	56.79	421.01	1609.5
Am13a	[3]	446.76	341.49	292.51	282.80	2033.46	2467.5
Am13b	[3]	402.71	267.21	239.40	203.14	1580.31	2870.0
Am14_1	new	2150.47	1645.00	1482.12	1429.62	10058.04	2756.5
Am15_1	[1]	3892.73	1693.54	2077.85	1241.81	TL (5.39)	3195.0
HK15	[19]	3291.68	1132.41	1576.67	764.08	TL (8.73)	16640.0
P16_a	[5]	41468.10	37276.28	29152.76	29800.27	TL (53.48)	7370.0
P16_b	[5]	15619.68	8688.04	8636.34	6466.20	$*$ (45.08)	5884.5

Table 6: Running times in seconds and optimal values "optimal" for the `(SF-DRFLP)` obtained by applying our MILP models for both variants `Full` and `Cuts` for each relevant row assignment in the standard variant ($L_1 \geq L_2$) and according to Lemma 6. Additionally, the table shows the running times and the gaps in percent after a time limit TL of twelve hours in brackets using the model of Amaral [3]. The symbol "$*$" indicates that the computer ran out of memory.

		$M = \sum_{i=1}^{n} \ell_i$			$M = \sum_{i=\lfloor \frac{n+1}{3} \rfloor + 1}^{n} \ell_i$			
name	source	`Full`	`Cuts`	Amaral	`Full`	`Cuts`	Amaral	optimal
HA5	[23]	0.04	0.03	0.01	0.03	0.02	0.01	52.5
HA6	[23]	0.14	0.15	0.04	0.12	0.13	0.05	190.5
HA7	[23]	0.33	0.46	0.08	0.31	0.44	0.08	159.0
HA8	[23]	1.17	1.30	1.55	1.01	1.15	0.86	189.5
HA9	[23]	4.37	5.94	6.53	3.85	5.66	3.98	486.5
HA10	[23]	15.93	22.04	30.44	15.12	23.13	24.71	821.0
HA11	[23]	29.59	29.10	35.74	27.08	28.15	35.06	773.5
HA12	[23]	164.31	214.54	961.93	157.55	211.65	785.04	1021.0
HA13	[23]	342.74	367.42	2585.63	320.37	358.94	2251.47	1520.5
HA14	[23]	1816.84	1819.97	38358.91	1749.95	1798.38	41622.08	1833.5
HA15	[23]	6711.04	6436.54	TL (18.76)	6568.02	6357.56	TL (14.38)	2624.5
s9	[3, 4]	3.93	4.82	7.79	3.62	4.85	5.15	1179.0
s9h	[3, 4]	18.52	42.64	142.82	15.01	35.70	139.31	2293.0
s10	[3, 4]	9.92	11.41	36.66	9.17	11.09	29.74	1351.0
s11	[3, 4]	38.16	46.26	317.60	36.16	45.92	374.51	3424.5
Am12a	[3, 4]	122.78	110.48	615.05	114.91	110.67	516.66	1493
Am12b	[3, 4]	107.69	106.62	670.45	100.82	107.72	684.36	1606.5
Am13a	[3]	442.96	472.60	8417.00	427.59	462.51	4991.75	2456.5
Am13b	[3]	378.97	412.76	4089.15	360.27	404.04	3439.71	2864.0
Am14_1	new	1828.83	1493.73	TL (3.53)	1765.89	1493.21	TL (4.24)	2738.5
Am15_1	[1]	4285.01	2899.47	$*$ (34.24)	4174.01	2818.79	TL (15.89)	3195.0
HK15	[19]	3423.49	2147.26	TL (5.04)	3268.04	2145.75	TL (7.34)	16570.0
P16_a	[5]	37370.53	37417.40	TL (214.80)	36765.17	36572.42	TL (206.00)	7365.5
P16_b	[5]	16732.72	11398.87	TL (189.16)	15846.97	11218.22	TL (159.55)	5870.5

Table 7: Running times in seconds and optimal values "optimal" for the (`DRFLP`) obtained by applying our MILP models for both variants `Full` and `Cuts` for each relevant row assignment with big-M-value as given. Additionally, the table shows the running times and the gaps in percent after a time limit TL of twelve hours in brackets using the model of Amaral [4]. The symbol "$*$" indicates that the computer ran out of memory.

References

[1] A. R. S. Amaral. On the exact solution of a facility layout problem. *European Journal of Operational Research*, 173(2):508–518, 2006.

[2] A. R. S. Amaral. A new lower bound for the single row facility layout problem. *Discrete Applied Mathematics*, 157(1):183–190, 2009.

[3] A. R. S. Amaral. The corridor allocation problem. *Computers & Operations Research*, 39 (12):3325–3330, 2012.

[4] A. R. S. Amaral. Optimal solutions for the double row layout problem. *Optimization Letters*, 7(2):407–413, 2013.

[5] A. R. S. Amaral. A parallel ordering problem in facilities layout. *Computers & Operations Research*, 40(12):2930–2939, 2013.

[6] A. R. S. Amaral and A. N. Letchford. A polyhedral approach to the single row facility layout problem. *Mathematical Programming*, 141(1-2):453–477, 2013.

[7] M. F. Anjos and F. Liers. Global approaches for facility layout and VLSI floorplanning. In M. F. Anjos and J. B. Lasserre, editors, *Handbook on Semidefinite, Conic and Polynomial Optimization*, volume 166 of *International Series in Operations Research & Management Science*, pages 849–877. Springer US, 2012.

[8] M. F. Anjos and A. Vannelli. Computing Globally Optimal Solutions for Single-Row Layout Problems Using Semidefinite Programming and Cutting Planes. *INFORMS Journal On Computing*, 20(4):611–617, 2008.

[9] M. F. Anjos, A. Fischer, and P. Hungerländer. Solution approaches for equidistant double- and multi-row facility layout problems. Cahiers du GERAD G-2015-06, GERAD, Montreal, QC, Canada, 2015.

[10] M. Brusco and S. Stahl. Using quadratic assignment methods to generate initial permutations for least-squares unidimensional scaling of symmetric proximity matrices. *Journal of Classification*, 17(2):197–223, 2000.

[11] J. Chung and J. Tanchoco. The double row layout problem. *International Journal of Production Research*, 48(3):709–727, 2010.

[12] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman & Co., New York, NY, USA, 1979.

[13] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1(3):237–267, 1976.

[14] A. Geoffrion and G. Graves. Scheduling parallel production lines with changeover costs: practical applications of a quadratic assignment/LP approach. *Operations Research*, 24: 595–610, 1976.

[15] M. Grötschel, M. Jünger, and G. Reinelt. A cutting plane algorithm for the linear ordering problem. *Operations Reasearch*, 32(6):1195–1220, 1984.

[16] M. Grötschel, M. Jünger, and G. Reinelt. Facets of the linear ordering polytope. *Mathematical Programming*, 33:43–60, 1985.

[17] M. M. D. Hassan. Machine layout problem in modern manufacturing facilities. *International Journal of Production Research*, 32(11):2559–2584, 1994.

[18] S. S. Heragu and A. Kusiak. Machine Layout Problem in Flexible Manufacturing Systems. *Operations Research*, 36(2):258–268, 1988.

[19] S. S. Heragu and A. Kusiak. Efficient models for the facility layout problem. *European Journal of Operational Research*, 53(1):1–13, 1991.

[20] P. Hungerländer. A semidefinite optimization approach to the parallel row ordering problem. Technical report, Alpen-Adria Universität Klagenfurt, Mathematics, Optimization Group, TR-ARUK-M-O-14-05, 2014.

[21] P. Hungerländer. Single-row equidistant facility layout as a special case of single-row facility layout. *International Journal of Production Research*, 52(5):1257–1268, 2014.

[22] P. Hungerländer. DRFLP benchmark problems, 2015. URL `http://philipphungerlaender.jimdo.com/benchmark-libraries/`.

[23] P. Hungerländer and M. F. Anjos. A semidefinite optimization approach to space-free multi-row facility layout. Cahiers du GERAD G-2012-03, GERAD, Montreal, QC, Canada, 2012.

[24] P. Hungerländer and M. F. Anjos. A semidefinite optimization-based approach for global optimization of multi-row facility layout. *European Journal of Operational Research*, 245 (1):46–61, 2015. ISSN 0377-2217. doi: http://dx.doi.org/10.1016/j.ejor.2015.02.049. URL `http://www.sciencedirect.com/science/article/pii/S0377221715001691`.

[25] P. Hungerländer and F. Rendl. A computational study and survey of methods for the single-row facility layout problem. *Computational Optimization and Applications*, 55(1): 1–20, 2013.

[26] IBM. *IBM ILOG CPLEX V12.6.2 User's Manual for CPLEX*, 2015.

[27] B. Keller and U. Buscher. Single row layout models. *European Journal of Operational Research*, 245(3):629–644, 2015.

[28] G. Laporte and H. Mercure. Balancing hydraulic turbine runners: A quadratic assignment problem. *European Journal of Operational Research*, 35(3):378–381, 1988.

[29] A. E. Smith, C. C. Murray, and X. Zuo. An extended double row layout problem. In *Proceedings of the 12th International Material Handling Research Colloquium*, pages 554–569.

[30] A. W. Tucker. On directed graphs and integer programs. Technical report, IBM Mathematical Research Project, 1960.

[31] B. Wess and T. Zeitlhofer. On the phase coupling problem between data memory layout generation and address pointer assignment. In H. Schepers, editor, *Software and Compilers for Embedded Systems*, volume 3199 of *Lecture Notes in Computer Science*, pages 152–166. Springer Berlin Heidelberg, 2004.

[32] D. H. Younger. Minimum feedback arc sets for a directed graph. *IEEE Transactions on Circuit Theory*, 10(2):238–245, 1963.

[33] Z. Zhang and C. C. Murray. A corrected formulation for the double row layout problem. *International Journal of Production Research*, 50(15):4220–4223, 2012.