A combined SDDP/benders decomposition approach with a risk-averse surface concept for reservoir operation in long term power generation planning

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Abstract

Power generation planning in hydrothermal systems is a complex optimization task, specially due to the high uncertainty in the inflows to hydro plants. Since it is impossible to traverse the huge scenario tree of the multistage problem, stochastic dual dynamic programming (SDDP) is the leading technique to solve it, originally from an expected-cost minimization perspective. However, there is a growing need to apply risk-averse/robust formulations to protect the system from critical hydrological scenarios. This is particularly important for predominantly hydro systems, because environmental issues prevent the construction of large reservoirs, thus reducing their water regulating capability. This paper proposes a two-level SDDP/Benders decomposition approach to include a new risk averse surface (SAR) concept for reservoir operation in power generation planning. The upper level problem is a SDDP solving strategy with expected-cost minimization criterion, where recourse functions for each time step are built through forward/backward passes. The second level consists in multi-period deterministic optimization subproblems for each node of the scenario tree, which are solved to ensure a desired level of protection from a given critical scenario several months ahead. We apply an inner iterative procedure for each SDDP stage/scenario, where feasibility cuts are included in the upper level subproblems to derive the SAR surface, which are multidimensional rule curves for reservoir operation. Such curves ensure that the policy provided by the SDDP algorithm yields storage levels in the reservoirs that are high enough to protect the system against such critical scenarios. Results are presented for the large-scale Brazilian system.

Keywords. OR in energy, power generation planning, stochastic dual dynamic programming, Benders decomposition, large-scale linear programming.

1 Introduction

Long term power generation planning under a minimum cost or a risk-averse perspective is a very complex optimization task. First of all, it is necessary to carefully model the high uncertainty in the water inflows to the reservoirs [1], [2], which are crucial to ensure power supply, specially in predominantly hydro systems, such as Brazil [3], Norway [4], Chile [5], Canada [6] and New Zealand [7]. Moreover, this is a multi-stage optimization problem, because availability of water in the reservoirs in the future depends on previous power generations levels. In such systems, there are reservoirs with weekly, monthly or even annual regulation capability, with historical records that may present long-duration dry periods. There is also a spatial dependency, for two main reasons: the operation of upstream plants affects the inflow of water to downstream plants in a same cascade, and transmission lines limit the amount of energy that can be transferred among different system areas in a given time step. In addition, there are nonlinear relations for thermal generation costs and the production function of hydroelectric plants [8]. Finally, it is a large-scale problem, specially for the Brazilian system, which comprises over 120 thermal units and 140 hydro plants spread in strongly linked river basins. As a result, the long-term hydrothermal planning problem [LTHTP] is a large-scale nonlinear stochastic program.

The main objective of the [LTHTP] problem is to obtain an optimal operation policy for the dispatch of hydro and thermal generation plants over time, so as to ensure that the demand requirements along a multi-year planning horizon are met, taking into account a trade-off between thermal generation costs and system security. Due to the complexity of the optimization problem that would be incurred when representing all nonlinear relations and the continuous distributions for the random variables, the problem is usually formulated as a stochastic linear program with a finite scenario tree. Since it is impossible to traverse the huge scenario tree of the multistage problem, the problem is solved by the well-know stochastic dual dynamic programming approach [9, 10], which is a sampling version of the multistage Benders decomposition approach [11]. There are several applications of SDDP for this type of problem, originally from an expected value minimization perspective [4, 5, 9], and later by introducing risk-aversion criteria [7, 12, 13, 14]. In particular, the operation planning in Brazil has been for more than ten years officially performed through a chain of optimization models [3, 15], in which SDDP is applied for the long-term planning and multi-stage Benders decomposition is applied for mid and short-term planning. Recently, there has been a switch from expected value minimization to a combined mean / conditional value at-risk (CVaR) approach [16]. Other examples of hierarchical approaches to address the power generation planning problem in practical hydrothermal systems can be found in [4, 17, 18, 19].

1.1 A review of rule curves and other risk-aversion approaches for power generation planning

Several approaches have been proposed in the literature to introduce a given level of protection against critical situations in power generation planning for different systems. Since there is a strong relation between the energy shortage risk and load shedding costs, an iterative procedure was proposed in [20] to calibrate load curtailment cost as a function of the desired reliability level for system supply, within a stochastic version of the traditional Bellman's dynamic programming approach (SDP) [21]. Also in the context of SDP, in [22] a new state variable was included in the recourse function of the multistage problem to account for the discrete number of load curtailments up to each stage t, and in [23] time-coupling constraints to assess the risk of supply from each stage t up to the end of the time horizon have been considered. Even though such procedures have shown to be very efficient under the SDP framework at that time, they could not be extended later to the SDDP approach due to the lack of convexity. In [24], a penalty was applied to values of revenue below some specified targets for the [LTHTP] problem in a market-based framework, where the problem had been solved by a hybrid SDP/SDDP approach.

Probabilistic constraints for minimum storage levels in the reservoirs have been considered in [25] for the energy management problem, solved by chance constrained programming and taking into account a continuous distribution for the random variables. Chance constraints are also formulated in [26] and turned into a hard constraint for a deterministic optimization problem, by assuming a Normal distribution for the inflows to the reservoirs. That work was later extended to include CVaR constraints in a rolling horizon strategy [27]. The CVaR risk measure in a profit maximization framework has been first applied in the SDDP framework in [28], for a two-stage problem, and extended to a multistage problem in [29], even though no implementation details have been provided. The application of CVaR criterion in the objective function for a multistage power generation problem solved by SDDP was presented for the first time in [7], following the nested CVaR risk measure approach proposed in [30]. Later on, it was applied in a number of papers [12, 13, 14, 16, 31], with some improvements.

Another alternative to address the security of supply that has been used for a long time in the literature is to derive the so-called rule curves for reservoir management [32], which are minimum acceptable storage levels for the reservoirs along time. In [33] such curve is defined to protect the system from the worst hydrological scenario of the historical record, and thermal generation should be enforced at its maximum value if system storage falls below its minimum required levels. A similar concept was later extended in [34] but taking into account the second worst historical scenario. A time-variant minimum storage zone has also been defined in [35]. Even though the idea of using minimum storage levels is attractive due to its intuitive meaning for the system operator, to the best of the authors knowledge its consideration under a stochastic framework is not well established in the literature.

1.2 Motivation, objective and relevance of the work

As can be seen from those recent applications to practical systems [7, 12, 16], there has been a growing need to apply risk-averse approaches to the [LTHTP] problem. This has become a critical issue for the operation of predominantly hydro systems, because environmental issues have prevented those systems from building large reservoirs. As a consequence, water regulation capacity has been decreasing over time due to a larger relative number of run-of-the-river plants. The objective of this paper is to propose a combined SDDP / Benders decomposition approach to obtain a risk-averse operation policy for the long term hydrothermal planning problem, composed of two levels. The upper level subproblem consists in the traditional SDDP solving strategy with an expected-cost minimization criterion, where recourse functions for each time step (called stages) are built based on forward and backward passes. The lower level consists in additional subproblems for each node of the scenario tree, where a multi-period deterministic optimization problem should be solved to ensure a desired level of protection of the system from a given critical scenario several months ahead. We apply an inner iterative procedure for each stage / scenario of the overall SDDP approach, where feasibility cuts that define the feasible region for the lower level subproblem are included in the upper level subproblems.

Besides providing a more risk-averse operation policy, one major advantage of this approach is that we obtain the so-called Risk-Averse Surfaces (SAR) for system operation, which are multidimensional rule curves for minimum storage levels in the reservoirs that can be also used for monitoring the system operation during critical scenarios. We obtain an operation policy which consists of a the set of recourse functions obtained by the SDDP approach that minimizes the expected value of thermal generation costs over the entire multistage scenario tree, and at the same time provides system protection from the occurrence of the specified critical scenario. As a consequence, the proposed methodology yields an adequate trade off between cost minimization and a secure operation policy.

In summary, the main contributions of this paper are: (i) from the methodological point of view, to propose a combined SDDP / two stage-Benders decomposition approach, which can be used not only for the type of problem considered in this work, but also to ensure satisfaction of constraints which are not explicitly defined in the upper level SDDP problem, but rather arise as a result of a lower level optimization problem; (ii) from the practical point of view, to propose a new risk-averse approach for long term power generation planning, which is more intuitive as compared to the CVaR approach of [16] in the point of view of system operation, since it ensures a safe system operation under the occurrence of given critical scenarios from which the decision makers require the system to be protected. Moreover, this so-called risk averse surface is an extension, to the multivariate case, of the concept of rule curves that has been traditionally considered in real hydrothermal generation systems.

The paper is organized as follows: in section 2 we review the long term hydrothermal planning problem [LTHTP] in its risk-neutral formulation and discuss one-dimensional and multidimensional rule curves that may be enforced for this problem. In section 3 we describe the risk averse surface (SAR) concept considered in this paper and in section 4 we detail the combined SDDP / two stage Benders decomposition approach that has been proposed to handle such constraints in the [LTHTP] problem. In section 5 we show numerical results for an application to the real large-scale Brazilian system and in section 6 we state the conclusions of the paper.

2 Long Term Hydrothermal Planning Problem([LTHTP])

The main objective of the [LTHTP] problem is to obtain an optimal monthly policy for operation of a hydrothermal system, for a planning horizon typically of several years. Due to the huge computational effort incurred in a detailed representation of both system constraints and the stochastic process behind the uncertainty on hydro inflows, the hydro generation system is represented by means of equivalent energy reservoirs [34, 36], one for each of the NS system areas in which the system is divided. We refer to [37] for details on the specific model of equivalent energy reservoirs that has been considered in this paper. The stochastic inflows are represented by a periodic autoregressive model [1] of order P, which takes into account both temporal and spatial correlations, but alternative models can be used [38, 39]. We make a finite discrete approximation of the random vector by employing a clustering technique called selective sampling approach, as described in [40]. Such technique is used to generate forward and backward scenarios of the SDDP approach described in section 2.1.

We consider a scenario tree with 60 monthly time steps and $K = 20$ possible realization per stage, including the first month, as shown in the left part of Fig. 1. Since it is impossible to traverse this huge scenario tree, the problem is solved by applying the SDDP approach [9] in

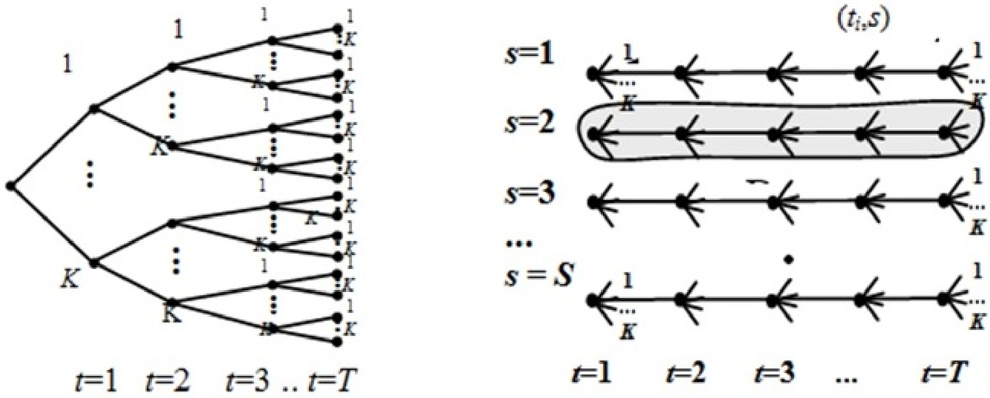

Figure 1: General scheme of the complete scenario tree of the [LTHTP] problem (left) and the scenarios traversed in the forward and backward scenarios of the SDDP solving strategy (right).

a more general context of temporal correlation among the values of the random variables [10]. The forward pass is performed by choosing S multistage scenarios, whereas in the backward pass all K one-stage scenarios are taken into account for each corresponding node traversed in the forward pass. A general sketch of this procedure is presented in the right hand side of Fig. 1, where each node is denoted by a pair (t, s). Alternative strategies to handle such multi-stage problem under the SDDP solving strategy can be seen for example in [41, 42, 43, 44].

In section 2.1 we present the basic formulation of the [LTHTP] problem and in section 2.2 we describe the traditional one-dimensional rule-curve that has been used in the literature in the past. In section 2.3 we present the new approach proposed in this paper which consists in using a multi-dimensional risk averse surface (SAR).

2.1 Problem Formulation

An abstract formulation of the [LTHTP] problem is presented in (1), based on [15]. The problem can be written in a nested formulation comprising all time steps and scenarios:

$$
\begin{aligned}
& Z = \mathop{\mathbb{E}}_{\xi_1|\xi_{[1],P}} \left[\varphi_1(x_{v_0}, \xi_1, \xi_{[1],P})\right] \\
& \text{where:} \\
& \varphi_1(x_{v_0}, \xi_1, \xi_{[1],P}) = \min_{x_1} \; {c_1}^\top x_1 + \left(\frac{1}{1+r}\right) \mathop{\mathbb{E}}_{\xi_2|\xi_{[2],P}} \left[\varphi_2(x_{v_1}, \xi_2, \xi_{[2],P})\right] \\
& \qquad\qquad s.t. \quad g_1(x_1) = b_1(x_{v_0}, \xi_1) \\
& \qquad\qquad\qquad\; x_1 \in X_1 \\
& \text{and, for all } t = 2, \ldots, T \text{ we have:} \\
& \varphi_t(x_{v_{t-1}}, \xi_t, \xi_{[t],P}) = \min_{x_t} \; {c_t}^\top x_t + \left(\frac{1}{1+r}\right) \mathop{\mathbb{E}}_{\xi_{t+1}|\xi_{[t+1],P}} \left[\varphi_{t+1}(x_{v_t}, \xi_{t+1}, \xi_{[t+1],P})\right] \\
& \qquad\qquad s.t. \quad g_t(x_t) = b_t(x_{v_{t-1}}, \xi_t) \\
& \qquad\qquad\qquad\; x_t \in X_t
\end{aligned}
\tag{1}
$$

where $\varphi_t(x_{v_{t-1}}, \xi_t, \xi_{[t],P})$ is the optimal value of the subproblem for the corresponding scenario of time step t in the nested formulation, and $\varphi_{T+1}(.) := 0$ for the subproblem related to the last time step T. The subproblem for each time step t has the following *state* variables:

$x_{v_{t-1}} \in \Re^{NS}$	:	subset of x_{t-1} related to the vector of energy storages in the reservoirs at the beginning of time step t (end of time step $t-1$). Values of initial storages x_{v_0} are known;
$\xi_t \in \Re^{NS}$	:	vector of inflows to the reservoirs at the corresponding scenario of time step t
$\xi_{[t],P} \in \Re^{NS \times P}$	:	recent history of the stochastic process up to previous time step $(t-P)$, defined as the set of vectors $\{\xi_{t-1}, \ldots, \xi_{t-P}\}$. We note that $\xi_{[t+1],P} = \xi_t \cup \xi_{[t],P-1}$.

We use the conditional notation $\{\xi_t | \xi_{t-1}, \ldots, \xi_{t-P}\}$ for the vector of hydro inflows ξ_t of each time step t because we take into account the serial correlation among these random variables by means of the aforementioned periodic autoregressive model [1], whose maximum order is P. Components $\{\xi_{1-p}\}_{p=1,\ldots,P}$ correspond to (known) values related to the p^{th} past months previous to first time step $t = 1$.

Additional parameters and variables are defined as follows:

$S \in \mathbb{Z}$	:	number of multistage scenarios in each forward pass of the SDDP algorithm, as shown in the right side of Fig. 1
$T \in \mathbb{Z}$	:	number of time steps;
$K \in \mathbb{Z}$	:	number of discrete scenarios for each time step, as shown in Fig. 1;
$\mathbb{T} \subset \mathbb{Z}$	:	set of indices for all time steps ($\{1, \ldots, T\}$);
$\mathbb{S} \subset \mathbb{Z}$	:	set of indices for all forward (multi-stage) scenarios in the SDDP algorithm ($\{1, \ldots, S\}$);
$\Omega \subset \mathbb{Z}$	:	set of indices of all (single step) scenarios of each time step ($\{1, \ldots, K\}$);
c_t	:	vector of unitary costs for the decision variables at time step t;
$r \in \Re$	:	discount rate applied to costs at future stages;
X_t	:	feasible set for the decision variables of time step t, defined by lower and upper bounds, which are time-dependent since some aspects of the problem (e.g., minimum water outflows for the reservoirs) may vary according to the season of the year;
$g_t(.)$	:	left hand side of the set of constraints of the problem;
$b_t(.)$	:	right hand side of the set of constraints of the problem, which may depend on state variables $x_{v_{t-1}}$ and random vectors $\{\xi_{t-1}, \ldots, \xi_{t-P}\}$ related to previous time steps;

The subproblem for each time step $t \in \mathbb{T}$, each scenario $s \in \mathbb{S}$, of the forward pass and each scenario $\omega \in \Omega$ in the backward pass of the SDDP algorithm is labeled hydrothermal dispatch subproblem $[\mathrm{HT}]_t^{s,\omega}$. Its formulation for iteration k of the SDDP procedure is given by (2).

$$
\begin{aligned}
\varphi_t(x^{s,\omega}_{v_{t-1}}, \xi^{s,\omega}_t, \xi^{s,\omega}_{[t],P}) = \min_{x^{s,\omega}_t} \; & {c_{gt}}^{\top} x^{s,\omega}_{gt_t} + {c_{df}}^{\top} x^{s,\omega}_{df_t} + \left(\frac{1}{1+r}\right) \varrho^{(k)}_{t+1}(x^{s,\omega}_{v_t}, \xi^{s,\omega}_{[t+1],P}) && (a) \\
s.t. \; & x^{s,\omega}_{gh_t} + M_{gt} x^{s,\omega}_{gt_t} + M_{int} x^{s,\omega}_{int_t} + x^{s,\omega}_{df_t} = d_t && (b) \\
& x^{s,\omega}_{v_t} + x^{s,\omega}_{gh_t} + x^{s,\omega}_{s_t} = x^{s,\omega}_{v_{t-1}} + {\xi_t}^{s,\omega} && (c) \\
& x^{s,\omega}_{gh_t} - \aleph(x^{s,\omega}_{v_t}) \leq 0 && (d) \\
& x^{s,\omega}_t \in X_t, && (e)
\end{aligned}
\qquad (2)
$$

The dimensions of the system are as follows:

NS : number of system areas, each one comprising one equivalent hydro plant with an energy reservoir and several thermal plants;

NI : number of interchanges (tie-lines) among system areas;

NT : number of thermal plants;

The decision vector $x_t^{s,\omega} \in \Re^{4NS+NT+NI}$ for node (t,s) and scenario ω is split in the following components:

$x_{v_t}^{s,\omega} \in \Re^{NS}$: vector of final storages in all reservoirs;

$x_{gh_t}^{s,\omega} \in \Re^{NS}$: vector of hydro generations in all reservoirs;

$x_{s_t}^{s,\omega} \in \Re^{NS}$: vector of spillages in all reservoirs;

$x_{gt_t}^{s,\omega} \in \Re^{NT}$: vector of generations in all thermal plants;

$x_{df_t}^{s,\omega} \in \Re^{NS}$: vector of energy shortages in all system areas;

$x_{int_t}^{s,\omega} \in \Re^{NI}$: vector of energy interchanges among different system areas;

and the vector of state variables is given by:

$x_{v_{t-1}}^{s,\omega} \in \Re^{NS}$: energy storage in the reservoirs at the beginning of node (t,s) and backward scenario ω;

$\xi_t^{s,\omega} \in \Re^{NS}$: vector of inflows to the reservoirs for node (t,s) and backward scenario ω;

$\xi_{[t],P}^{s,\omega} \in \Re^{NS \times P}$: recent history $\{\xi_{t-1}^{s,\omega}, \ldots, \xi_{t-P}^{s,\omega}\}$ of the stochastic process in the *forward scenario s* starting at time step $(t-P)$, which appear in the subproblem of time step t, because of the auto-regressive model. We also note that $\xi_{[t+1],P}^{s,\omega} = \xi_t^{s,\omega} \cup \xi_{[t],P-1}^{s,\omega}$, where the last $(P-1)$ components of this vector are the same for all backward scenarios ω of a given node (t,s).

Function $\varrho_{t+1}^{(k)}$, which is usually called "future cost function", is the piecewise linear approximation of the recourse function of the SDDP procedure obtained until iteration k of the SDDP algorithm described in section 4.1. Such function approximates the optimal expected cost φ_{t+1} at later stages, as shown in the nested formulation (1). Its arguments are the state variables for time step $(t+1)$, i.e., storages in the reservoirs at the end of time step t and energy inflows in the P previous periods to $(t+1)$. In particular, we set $\varrho_{T+1}^{(k)} := 0$ for the last time step T at all iterations k.

Additional parameters of the problem are as follows:

$c_{gt} \in \Re^{NT}$: vector of unitary generation costs in all thermal plants;

$c_{df} \in \Re^{NS}$: vector of unitary energy shortage costs in all system areas;

$d_t \in \Re^{NS}$: vector of power demand in all system areas at time step t;

$M_{gt} \in \{0,1\}^{NS \times NT}$: incidence matrix to define to which area each thermal generator belongs;

$M_{int} \in \{-1,0,1\}^{NS \times NI}$: incidence matrix to define existence and directions of interchanges among system areas;

$\aleph(.) \in \Re^{NS} \to \Re^{NS}$: maximum generation for the equivalent power a plant as a function of reservoir levels, due to water head effects. Each component i is a concave piecewise linear function of the level of reservoir i;

The objective function $(2a)$ is the sum of generation costs for thermal plants, energy shortage (deficit) costs and the approximated value for future cost given by the approximated function $\varrho_{t+1}^{(k)}$ obtained so far. The main constraints of subproblem (2) correspond to system-wide energy balance equations ($2b$), water conservation equations ($2c$) in the reservoirs and the maximum generation ($2d$) of reservoirs as a function of storage. A detailed formulation of all constraints of the problem is provided in [3, 37, 45].

2.2 Traditional risk-averse Curves (CAR) in the [LTHTP] problem

The traditional way of including a certain level of system protection in the [LTHTP] problem (1) is by the risk averse curve (CAR), which is a rule curve formally defined as follows:

Definition 1. ***Risk averse curve (CAR)***. *Consists of individual minimum storage levels for the reservoirs – either in water or in energy – specified in some time periods along the planning horizon [32, 33, 34, 35]. They are included as inequality constraints, with slack variables to allow the complete recourse property* [1] *for the [LTHTP] stochastic program (1), and are formulated in each subproblem (2) for scenario* (s, ω) *of time step* t *as follows:*

$$x_{v_t}^{s,\omega} + \delta_{CAR_t}^{s,\omega} \geq \underline{x_{v_t}}. \tag{3}$$

where:

$\underline{x_{v_t}} \in \Re^{NS}$	:	vector of minimum energy storages in all reservoirs for time step t;
$\delta_{CAR_t}^{s,\omega} \in \Re^{NS}$	:	vector of violations of minimum storage constraints in all reservoirs, for scenario (s, ω) of time step t;

The use of such constraints was the official procedure used in the Brazilian system until Sept. 2013, when it was replaced by the CVaR criterion described in [16]. There are two main drawbacks of using such individual rules curves:

- the CAR methodology assumes univariate values for the storage levels. However, the minimum secure level of energy in a given equivalent reservoir should depend on the storage level (and thus the operation) in the other reservoirs (see Fig. 3 later);
- the interconnection among reservoirs is not taken into account (or is defined a priori) when computing the CAR values. However, such energy interchanges among system areas are subject to the current system state and hydrological conditions along the year;

As a consequence, the traditional rule curves may be too conservative, specially when the reservoirs are in different hydrological conditions in a given time of the year.

2.3 Proposed Risk Averse [LTHTP] problem with risk-averse surfaces (SAR)

The strategy presented in this paper aims to overcome the issues mentioned in the previous section for the use of traditional rule curves. We propose the use of the so-called risk aversion surface (SAR), which would turn the original risk-aversion curve into a multivariate surface that comprises the storage levels in all equivalent reservoirs of the system. This idea was first mentioned in [46] but no detailed procedure was presented on how to formulate the underlying risk-averse subproblem and how to compute the cuts. In addition, time and space dependency between the water inflows to the reservoirs - which are crucial to assess the system performance, specially in dry periods - had not been discussed.

[1] the complete recourse property in stochastic programming states that the subproblem related to stage t of a decomposed problem remains feasible, regardless of the values of the decision variables x_τ in previous stages $\tau \leq t$.

However, in the multivariate case it would be very difficult to obtain the complete risk-aversion surface a priori (as done in the CAR methodology) for two reasons: (i) its much higher dimension as compared to constraints (3); (ii) the nonlinear variation of power output with the water head, which should be taken into account when trading-off the storages in different reservoirs, as described in [15]. As a result, the surface may become smooth, which would make it impossible to obtain an exact a priori representation with linear cuts. In this paper, since this approximation is done in an iterative way - in the neighborhood of the system states reached in each scenario - it is possible to obtain a good approximation with a reasonable number of cuts.

In summary, we neither apply a risk-measure for the operation cost nor specify fixed lower level curves for the equivalent energy reservoirs. Rather, we enforce a more accurate level of security by including a lower level deterministic multiperiod problem as a subproblem for each node (t, s), which represents the system operation under a critical scenario several months ahead. In such case, even though we still work with the expected value formulation in the objective function of the overall [LTHTP] problem (1), we favor operation policies that are able to protect the system against critical scenarios. Such scenarios can be taken from the historical record or synthetically generated by a periodical autoregressive model, as described in section 3.2.

The concept of risk averse surface (SAR) and the corresponding subproblem that is solved to obtain its piecewise linear approximation are described in the next section.

3 Risk-Averse Surface

Risk aversion is enforced in our model by defining minimum conditions for the system state at each time step, in order to ensure that system demand is met in the subsequent months of this stage in the event of occurrence of a given (pessimistic) dry scenario. First we define the concept of Risk Averse Surface (SAR), originally sketched in [46] and further improved in [47, 45] and then formulate the underlying optimization problem that induces the existence of such surface. Finally, we describe how such risk-averse criterion is embedded in the overall SDDP solving strategy applied to our [LTHTP] problem (1).

Definition 2. ***Risk Averse Surface (SAR)****, for each time step t, is a multidimensional curve that defines required conditions for the system state - namely, energy storage levels $x_{v_t}^{s,\omega}$ in the reservoirs at the end of time step t for each scenario (s, ω) and the vector $\xi_{[t+1],P}^{s,\omega}$ of past energy inflows to time step $(t + 1)$, so that it is possible to perform a deterministic operation of the system under a critical inflow scenario ranging from $(t+1)$ until $(t+T_{SAR}^t)$, where T_{SAR}^t is the number of months until the end of the critical scenario, meeting the two following requirements:*

- *no energy shortage is allowed along this scenario, i.e., the system should be able to fully satisfy the power demand during this period with its available hydro and thermal resources;*
- *a minimum energy storage target is met at the end of time step $(t + T_{SAR}^t)$ for each equivalent reservoir, so that the system has not fully exhausted its hydro resources at the end of such critical period.*

The motivation for such deterministic procedure is the assumption that once the system state falls below (or in the boundary) of the risk averse surface, the independent system operator would switch from a non-anticipativity operation based on expected value of costs under a set of possible inflow scenarios to a worst-case operation procedure, where all necessary thermal generation resources are used in order to protect the system from the occurrence of critical inflows. Indeed, this was just the situation recently observed in the Brazilian system operation during the years of 2014 and 2015 [48].

Conceptually, we consider risk control at every stage, i.e., the combination of final storages in the reservoirs should be constrained at each time step. However, because of the heavy time-linking nature of the hydrothermal coordination problem, such secure set of system states should be set by assessing the system operation several time steps in the future. More specifically, the two conditions above can be mathematically formulated as a deterministic hydrothermal dispatch problem along the chosen scenario, which is called "SAR subproblem", as described next.

3.1 Risk Averse Surface subproblem

The SAR subproblem is a multistage deterministic hydrothermal scheduling problem, whose aim is to satisfy both the required power demand at all time steps and the end storage targets with the available resources: initial storages in the reservoirs, the water inflows along the critical scenario and the thermal plants generation capacity. Each so called $[\text{SAR}]_t$ subproblem is related to a given time step t in the outer [LTHTP] problem (1), and is formulated as in (4) below:

$$
\begin{array}{llrl}
\beta_t(x_{v_t}^{s,\omega}, \xi_t^{s,\omega}, \xi_{[t],P-1}^{s,\omega}) = \min\limits_{\tilde{x}} \ \mathbb{1}^\top_{T^t_{SAR}NS}\tilde{x}_{df} & & & (a) \\
s.t. \ \ \tilde{x}_{gh_\tau} + M_{gt}\tilde{x}_{gt_\tau} + M_{int}\tilde{x}_{int_\tau} + \tilde{x}_{df_\tau} = d_\tau, & \tau = t+1,\dots,t+T^t_{SAR} & & (b) \\
\tilde{x}_{v_\tau} + \tilde{x}_{gh_\tau} + \tilde{x}_{s_\tau} = x_{v_t}^{s,\omega} + \tilde{\xi}_\tau |\{\xi_t^{s,\omega}, \xi_{[t],P-1}^{s,\omega}\}, & \tau = t+1 & & (c1) \\
\tilde{x}_{v_\tau} - \tilde{x}_{v_{\tau-1}} + \tilde{x}_{gh_\tau} + \tilde{x}_{s_\tau} = \tilde{\xi}_\tau |\{\xi_t^{s,\omega}, \xi_{[t],P-1}^{s,\omega}\}, & \tau = t+2,\dots,t+T^t_{SAR} & & (c2) \\
\tilde{x}_{gh_\tau} - \aleph(\tilde{x}_{v_\tau}) \leq 0, & \tau = t+1,\dots,t+T^t_{SAR} & & (d) \\
\tilde{x}_\tau \in X_\tau, & \tau = t+1,\dots,t+T^t_{SAR} & & (e) \\
\tilde{x}_{v_\tau} \geq \underline{x_{v_{SAR}}}, & \tau = t+T^t_{SAR} & & (f)
\end{array} \qquad (4)
$$

The length of the horizon of the $[\text{SAR}]_t$ subproblem for time step t is T^t_{SAR}. The decision variables $\tilde{x} := \{\tilde{x}_{v_\tau}, \tilde{x}_{gh_\tau}, \tilde{x}_{s_\tau}, \tilde{x}_{gt_\tau}, \tilde{x}_{df_\tau}, \tilde{x}_{int_\tau}\}_{\tau=(t+1),\dots,(t+T^t_{SAR})} \in \Re^{(4NS+NT+NI)\times(T^t_{SAR})}$ are of the same type as in subproblem (2), thus they are split in the following components, for each time step τ:

$\tilde{x}_{v_\tau} \in \Re^{NS}$	:	vector of final storages in the reservoirs;
$\tilde{x}_{gh_\tau} \in \Re^{NS}$	:	vector of hydro generations in all reservoirs;
$\tilde{x}_{s_\tau} \in \Re^{NS}$	:	vector of spillages in all reservoirs;
$\tilde{x}_{df_\tau} \in \Re^{NS}$	:	vector of energy shortages in all system areas;
$\tilde{x}_{gt_\tau} \in \Re^{NT}$	:	vector of generations in all thermal plants;
$\tilde{x}_{int_\tau} \in \Re^{NI}$	:	vector of energy interchanges among different system areas.

In principle, the decision vector $\tilde{x}$ should have additional sub/superscripts to indicate the node (t, s) and the specific backward scenario ω to which the $[\text{SAR}]_t$ subproblem belongs, but we suppress such indices in order to alleviate notation.

The vector of hydro inflows for all equivalent reservoirs in each time step τ of the [SAR] subproblem is a function of past inflows because of the same Par-p autoregressive model [1] considered previously. In this sense, the critical scenario $\{\tilde{\xi}_\tau\}_{t=(t+1),\dots,(t+T^t_{SAR})}$ may depend on the inflows from the P time periods previous to $t+1$, which are divided in two parts: the inflow $\xi_t^{s,\omega}$ in backward scenario ω of node (t, s) and the inflows from time steps $(t+1-P)$ to $(t-1)$

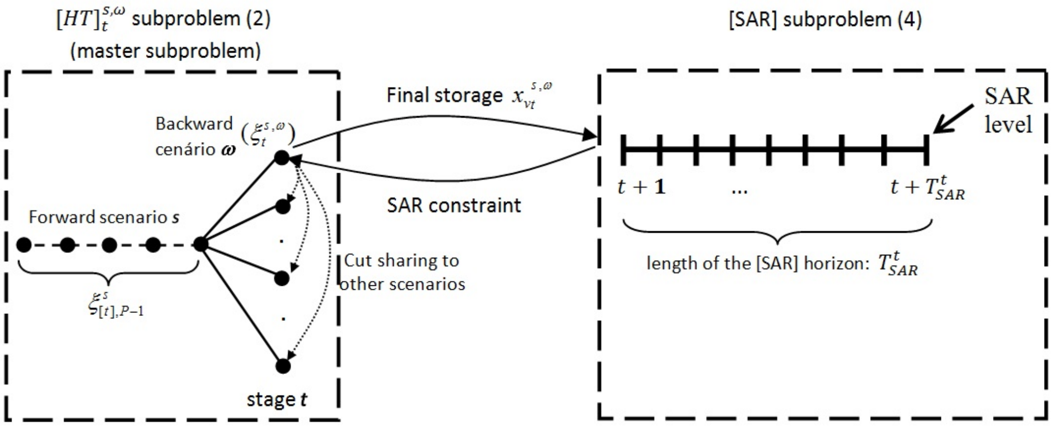

Figure 2: Two-state Benders decomposition to solve each $[\mathrm{HT}]_t^{s,\omega}$ subproblem (2), in order to include feasibility cuts related to the $[\mathrm{SAR}]_t$ subproblem (4).

in the forward scenario s of the SDDP procedure, which consists in the matrix $\xi_{[t],P-1}^{s,\omega}$. In this sense, we define the set of *state* variables for this problem as $u_t := \{x_{v_t}^{s,\omega}, \xi_t^{s,\omega}, \xi_{[t],P-1}^{s,\omega}\}$. The interplay between each $[\mathrm{HT}]_t^{s,\omega}$ subproblem (2) and its corresponding $[\mathrm{SAR}]_t$ subproblem (4) is sketched in Fig. 2.

The objective function β_t in ($4a$) is minimization of the sum of energy shortage in all system areas and all time steps, which ideally should be equal to zero. The term $\mathbb{1}_n$ indicates a vector of ones with dimension n. Our aim is to check whether the vector of storages $x_{v_t}^{s,\omega}$ for subproblem (2) is sufficient to yield β_t^*=0 for subproblem (4) i.e., the system is able to operate from $(t+1)$ to $(t+T_{SAR}^t)$ under the critical scenario $\tilde{\xi}$ without facing energy deficits and meeting end storage requirements $\underline{x_{v_{SAR}}}$ defined by ($4f$). The constraints and related data of this problem for each time step are similar to the corresponding sets of constraints in subproblem (2), namely, power balances ($4b$) for each area, energy balances ($4c$) and maximum generation ($4d$) for all equivalent reservoirs. Finally, the feasible domain X_τ in ($4e$) is the same of the corresponding time step $t = \tau$ in the upper level subproblem (2).

In order to illustrate the SAR concept, we consider a very simple example of two areas interconnected by a transmission line, with the data shown in Table 1, where $T_{SAR}^t = 2$. In this example area 2 has better structural and state conditions, since it has more thermal generation capacity and its expected inflows are higher as compared to area 1. The corresponding risk averse surface for the storages in the reservoirs of this system at the beginning of time step $\tau = 1$ is shown in Fig. 3.

If both areas were completely isolated (i.e., $\overline{x_{int}}_{1\to 2} = 0$), the minimum storages to ensure that no deficit occurs would be 65 and 30 for areas 1 and 2, respectively, which are the corresponding values of $\sum_{\tau=1}^{2}(d_\tau - \tilde{\xi}_\tau - \overline{x_{gt}})$, indicated by point P in Fig. 3. However, the system can benefit from energy interchange among areas, so that each system area is allowed to decrease its initial storage value as compared to point P, as long as the other area offsets this decrease by having more storage. This leads to the diagonal portion of the risk averse surface. However, there are minimum initial storage levels for each system area which are equal to 45 and 10, respectively, due to maximum interchange limit between the areas.

Table 1: Data for the example of the $[\text{SAR}]_t$ subproblem.

area	1		2	
time step	$\tau = 1$	$\tau = 2$	$\tau = 1$	$\tau = 2$
Demand (d_τ)	50	50	50	50
Hydro Inflows ($\tilde{\xi}$)	10	5	20	10
Max. thermal generation ($\overline{x_{gt}}$)	10	10	20	20
Max. Interchange ($\overline{x_{int}}$)	10	10	10	10

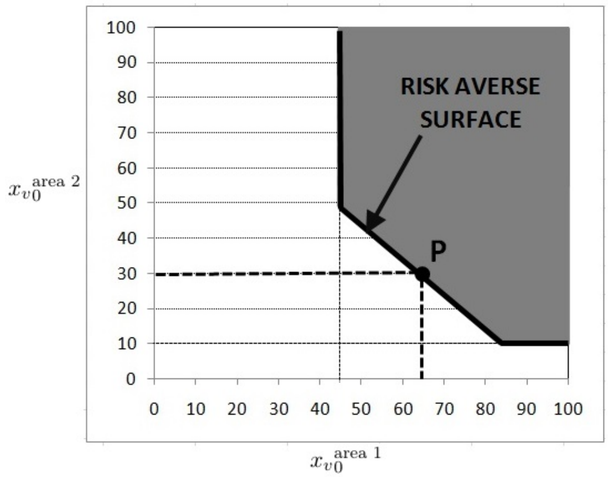

Figure 3: Risk Averse Surface for the [SAR] subproblem (4) with the data shown in Table 1.

3.2 Master subproblem with SAR constraints

The subproblem (2) for node (t,s) and backward scenario ω with the explicit inclusion of constraints related to the approximated recourse function $\varrho_{t+1}^{(k)}$ at the k^{th} SDDP iteration and also feasibility cuts induced by subproblem (4) is now denoted $[\text{HT}_{\text{SAR}}]_t^{s,\omega}$ and becomes as follows:

$$
\begin{aligned}
& Z_t^{s,\omega}(x_{v_{t-1}}^{s,\omega}, \xi_t^{s,\omega}, \xi_{[t],P}^{s,\omega}) = \\
& \min_{x_t^{s,\omega}} \; c_{gt}{}^\top x_{gt_t}^{s,\omega} + c_{df}{}^\top x_{df_t}^{s,\omega} + p_{SAR}\delta_{SAR_t}^{s,\omega} + \left(\frac{1}{1+r}\right)\alpha_t^{s,\omega} && (a) \\
& s.t. \; x_{gh_t}^{s,\omega} + M_{gt}x_{gt_t}^{s,\omega} + M_{int}x_{int_t}^{s,\omega} + x_{df_t}^{s,\omega} = d_t && (b) \\
& \quad x_{v_t}^{s,\omega} + x_{gh_t}^{s,\omega} + x_{s_t}^{s,\omega} = x_{v_{t-1}}^{s,\omega} + \xi_t{}^{s,\omega} && (c) \\
& \quad x_{gh_t}^{s,\omega} - \aleph(x_{v_t}^{s,\omega}) \le 0 && (d) \quad (5) \\
& \quad x_t^{s,\omega} \in X_t, && (e) \\
& \quad \pi_{v}{}_{SAR_t}^{(j)}{}^\top x_{v_t}^{s,\omega} + \delta_{SAR_t}^{s,\omega} \ge \pi_0{}_{SAR_t}^{(j)} - \sum_{p=1}^{P}[\pi_{\xi,p}{}_{SAR_t}^{(j)}{}^\top \xi_{t+1-p}^{s,\omega}], \quad j = 1, \ldots NCUT_{SAR}^t && (f) \\
& \quad \pi_{v}{}_{FCF_t}^{(j)}{}^\top x_{v_t}^{s,\omega} + \alpha_t^{s,\omega} \ge \pi_0{}_{FCF_t}^{(j)} - \sum_{p=1}^{P}[\pi_{\xi,p}{}_{FCF_t}^{(j)}{}^\top \xi_{t+1-p}^{s,\omega}], \quad j = 1, \ldots NCUT_{FCF}^t && (g)
\end{aligned}
$$

where $Z_t^{s,\omega}$ is the value of the objective function $(5a)$ and $\alpha_t^{s,\omega} \in \Re$ is the future cost given by the approximated model obtained so far for the recourse function of time step t, comprised by $NCUT_{FCF}^t$ cuts indicated in $(5g)$, where each cut j is defined by the following terms:

$\pi 0_{FCF_t}^{(j)} \in \Re$	:	independent term of the j^{th} cut of the FCF;
$\pi v_{FCF_t}^{(j)} \in \Re^{NS}$	:	vector of coefficients related to storage in the reservoirs, in the j^{th} cut of the FCF;
$\pi_{\xi,p}{}_{FCF_t}^{(j)} \in \Re^{NS}$	:	vector of coefficients related to the p^{th} past energy inflows previous to time step $(t+1)$, in the j^{th} cut of the FCF.

Constraints $(5b)$-$(5e)$ are exactly the same as $(2b)$-$(2e)$, and expression $(5f)$ corresponds to the set of $NCUT_{SAR}^t$ SAR constraints obtained so far for time step t, each one defined by the following terms:

$\pi 0_{SAR_t}^{(j)} \in \Re$	:	independent term of the j^{th} SAR constraint;
$\pi v_{SAR_t}^{(j)} \in \Re^{NS}$	:	vector of coefficients related to storage in the reservoirs in the j^{th} SAR constraint;
$\pi_{\xi,p}{}_{SAR_t}^{(j)} \in \Re^{NS}$	:	vector of coefficients related to the p^{th} past energy inflows previous to time step $(t+1)$, in the j^{th} SAR constraint.

Finally, $\delta_{SAR_t}^{s,\omega} \in \Re$ is the slack variable of the SAR constraint, to ensure relatively complete recourse in the SDDP solving strategy, i.e., subproblem $[\mathrm{HT_{SAR}}]_t^{s,\omega}$ remains feasible regardless of the values of the decision variables in the subproblems of previous time steps. We use the same slack variable for all SAR cuts, since they can be viewed as a unique piecewise linear constraint that expresses the requirements for system security. The unitary cost p_{SAR} applied to this variable is a high penalty parameter for violation of these constraints.

4 Combined SDDP / Two-stage Benders decomposition approach

In our real case presented in section 5, the SAR surface comprises several system areas, which increases the complexity in determining all its hyperplanes. In this sense, it is very difficult to obtain an a priori model for the SAR surface and an iterative process is proposed in this paper in order to define its linear approximations as the [LTHTP] problem is solved by the SDDP approach. As a consequence, the proposed approach consists in two iterative procedures: the upper level SDDP algorithm, which comprises a set of forward and backward passes along the time horizon of the [LTHTP] problem (1), and a lower level two-stage Benders decomposition to solve the $[\mathrm{HT_{SAR}}]_t^{s,\omega}$ subproblem (4) for each node (t,s) and backward scenario ω shown in Fig. 1.

4.1 Upper level SDDP solving strategy

As described in [9, 3] and illustrated in Fig.1, the SDDP solving strategy consists in a time decomposition of the problem, combined with a sampling strategy to select a set of S multistage scenarios to be traversed in the forward pass of the algorithm, in order to select "smart" states in which to approximate the recourse function for each stage. In a backward pass all K possible scenarios for the corresponding time step t are solved for the current value of the state variables, and Benders cut to approximate the recourse function of each stage are built. The SDDP algorithm can be summarized as follows [9]:

Algorithm 1. *SDDP solving procedure*
<u>Step 0</u>. *Initialization. Set* $NCUT_{FCF_t} = 0, \forall t$; *lower bound* $\underline{Z} := 0$; *upper bound* $\overline{Z} := \infty$; *SDDP iteration counter* $iter_{SDDP} = 1$.
<u>Step 1</u>. *Forward pass. For* $t = 1, \ldots, T$, *solve subproblem (5) for each forward scenario* s, *which is related to a given realization* ω *for the random variables. Record the vector of final storages* $x_{v_t}^{s,\omega}$ *for all nodes* (t, s), *which will be the initial system state for scenario* s *and time step* $(t+1)$ *in the backward pass.*
<u>Step 2</u>. *Update upper bound. Set* $\overline{Z} := \min\{\overline{Z}, Z^{(iter)}\}$, *where* $Z^{(iter)}$ *is the average present cost, i.e., excluding future cost* $\alpha_t^{s,\omega}$ *of subproblems (5) related to all* S *scenarios and* T *time steps.*
<u>Step 3</u>. *Stopping criteria. Check the following three conditions:*

(i) the lower bound $\underline{Z}$ *and upper bound* $\overline{Z}$ *for the optimal solution of problem (1) satisfy the statistical criterion defined in [9];*

(ii) the value of $\underline{Z}$ *stabilizes along* N *iterations;*

(iii) A maximum number $niter_{SDDP}$ *of SDDP iterations is reached.*

If at least one of these conditions is met, STOP: the set of $NCUT_{FCF_t}$ *cuts for all time steps* $t = 1, \ldots, T$ *is the output policy for system operation along the planning horizon. Otherwise, proceed to step 4.*
<u>Step 4</u>. *Backward pass. For* $t = T, \ldots, 2$ *and each scenario* $s \in \mathbb{S}$:

- solve subproblem (2) for all backward scenarios $\omega \in \Omega$ *of time step* t;

- based on optimal value $Z_t^{s,\omega*}$ *of (5) and optimal Lagrange multipliers* $\lambda_t^{s,\omega*}$ *related to constraints (5c),(5f),(5g), build a new Benders cut of type (5g) for time step* $(t-1)$;

- update $NCUT_{FCF_{t-1}} := NCUT_{FCF_{t-1}} + 1$.

<u>Step 5.</u> *Update lower bound. Set* $\underline{Z} = \max\{\underline{Z}, Z_1\}$, *where* Z_1 *is the average cost of subproblems (5) related to all* S *scenarios of time step 1.*
<u>Step 6</u>. *Recursive loop. Set* $iter_{SDDP} := iter_{SDDP} + 1$. *Go to Step 1.*

4.2 Lower level iterative approach to solve $[\mathbf{HT_{SAR}}]_t^{s,\omega}$ subproblems

As discussed previously, it would be very difficult to identify a priori the set of SAR constraints $(5f)$ that would ensure that the solution $x_{v_t}^{s,\omega}$ of subproblem (5) yields a zero value for subproblem (4). In order to avoid this extra computational effort, we propose in this paper a dynamic identification of these constraints, where only the necessary cuts are added to (5). In other words, the SAR surface will be shaped "on demand", i.e., depending on the values of the state variables $\{x_{v_{t-1}}^{s,\omega}, \xi_t^{s,\omega}, \xi_{[t],P}^{s,\omega}\}$ reached during the course of the SDDP algorithm. However, this requires an extra two-stage Benders decomposition approach to solve subproblem (5), where an iterative procedure is performed to include SAR constraints, as described in Algorithm 2 below.

Algorithm 2. *Iterative procedure to solve* $[HT_{SAR}]_t^{s,\omega}$ *subproblem (5) related to node* (t, s) *and backward scenario* ω, *for a given SDDP iteration* $iter_{SDDP}$.
<u>Step 0</u>. *Initialization. If* $iter_{SDDP} = 1$, $s = 1$ *and* $\omega = 1$, *set* $NCUT_{SAR}^t = 0$. *Set SAR iteration counter* $iter_{SAR} = 1$.
<u>Step 1</u>. $[HT_{SAR}]_t^{s,\omega}$ *solving procedure. Solve subproblem (5) for the current number* $NCUT_{SAR}^t$ *of SAR constraints. Obtain final storages* $x_{v_t}^{s,\omega}$ *in all reservoirs.*
<u>Step 2</u>. $[SAR]_t$ *solving procedure. Solve* $[SAR]_t$ *subproblem (4) with input parameters* $x_{v_t}^{s,\omega}$ *and taking as vector of past inflows the composition of* $\xi_t^{s,\omega}$ *for time step* t *and* $\xi_{[t],P-1}^{s,\omega}$ *for previous time steps, which are related to the corresponding multistage forward scenario* s.
<u>Step 3</u>. *Stopping criteria. If* $\beta_t^* = 0$ *or* $iter_{SAR}$ *reaches the maximum number of iterations*

$niter_{SAR}$, STOP. Go to the next node/backward scenario of the SDDP procedure. Otherwise, proceed to step 4.
Step 4. SAR cut construction. Based on the optimal value β^* of subproblem (4) and Lagrange multipliers ${\lambda_{SAR_t}}^*$ related to energy conservation constraints (4c), build a new SAR cut of type (5f), as described later in section 4.3. Update $NCUT^t_{SAR} := NCUT^t_{SAR} + 1$.
Step 5. Recursive loop. Set $iter_{SAR} := iter_{SAR} + 1$. Go to Step 1.

Such iterative scheme has been illustrated in Fig. 2. We note that the SAR constraints can be shared for all scenarios of time step t, regardless of the inflow scenario $\xi_t^{s,\omega}$ and the history $\xi_{[t],P-1}^{s,\omega}$ related to the corresponding forward scenario s, as illustrated in that figure. This is due to the fact that the impact of these variables is already taken into account in the formulation of the SAR constraint, therefore its right hand side will be properly adjusted according to the values of such state variables achieved in other forward scenarios and/or further SDDP iterations. For details on cut sharing we refer to [41].

We note that the scheme in Fig. 2 is performed to solve each subproblem for stage t, forward simulation s and backward scenario ω of Fig. 1. By merging the SDDP iterative procedure of Algorithm 1 with the iterative procedure of Algorithm 2 to solve each SDDP subproblem (5), we obtain the scheme shown in Fig. 4 for the solving procedure of the overall [LTHTP] problem (1).

4.3 Construction of SAR cuts

The risk averse constraints $(5f)$ are built based on the optimal value β_t^* of the $[\mathrm{SAR}]_t$ subproblem (4) and sensitivities $\frac{\partial \beta}{\partial u_t}(\hat{u}_t)$ of such objective function related to its state variables u_t, evaluated at their current values $\hat{u}_t$ defined as:

$$\hat{u}_t := \{\hat{x}_{v_t}^{s,\omega}, \hat{\xi}_t^{s,\omega}, \hat{\xi}_{[t],P-1}^{s,\omega}\}, \tag{6}$$

where $\hat{x}_{v_t}^{s,\omega}$, $\hat{\xi}_t^{s,\omega}$ and $\hat{\xi}_{[t],P-1}^{s,\omega}$ are the values used in (4), coming from the solution of the corresponding subproblem (5). Since the optimal value of subproblem (4) is non-negative, the requirement of zero deficit in subproblem can be expressed as an inequality of type $\beta_t < 0$. A Taylor expansion for the value of the objective function β_t around the solution in this subproblem yields the condition:

$$\beta_t^* + \frac{\partial \beta_t}{\partial u_t}(\hat{u}_t)^\top (u_t - \hat{u}_t) \leq 0 \tag{7}$$

which leads to a new constraint j of type $(5f)$ with the following terms:

$$\begin{aligned}
{\pi_0}_{SAR_t}^{(j)} &:= \beta_t^* - \frac{\partial \beta_t}{\partial u_t}(\hat{u}_t)^\top \hat{u}_t, && (a)\\
{\pi_v}_{SAR_t}^{(j)} &:= -\frac{\partial \beta_t}{\partial u_{v_t}}(\hat{x}_{v_t}^{s,\omega}), && (b)\\
{\pi_{\xi,p}}_{SAR_t}^{(j)} &:= -\frac{\partial \beta_t}{\partial u_{\xi_p}}(\hat{\xi}_{t+1-p}^{s,\omega}), \quad p = 1, \ldots, P, && (c)
\end{aligned} \tag{8}$$

where u_{v_t} and u_{ξ_p} are the terms related to x_{v_t} and ξ_{t+1-p} in the state vector u_t.

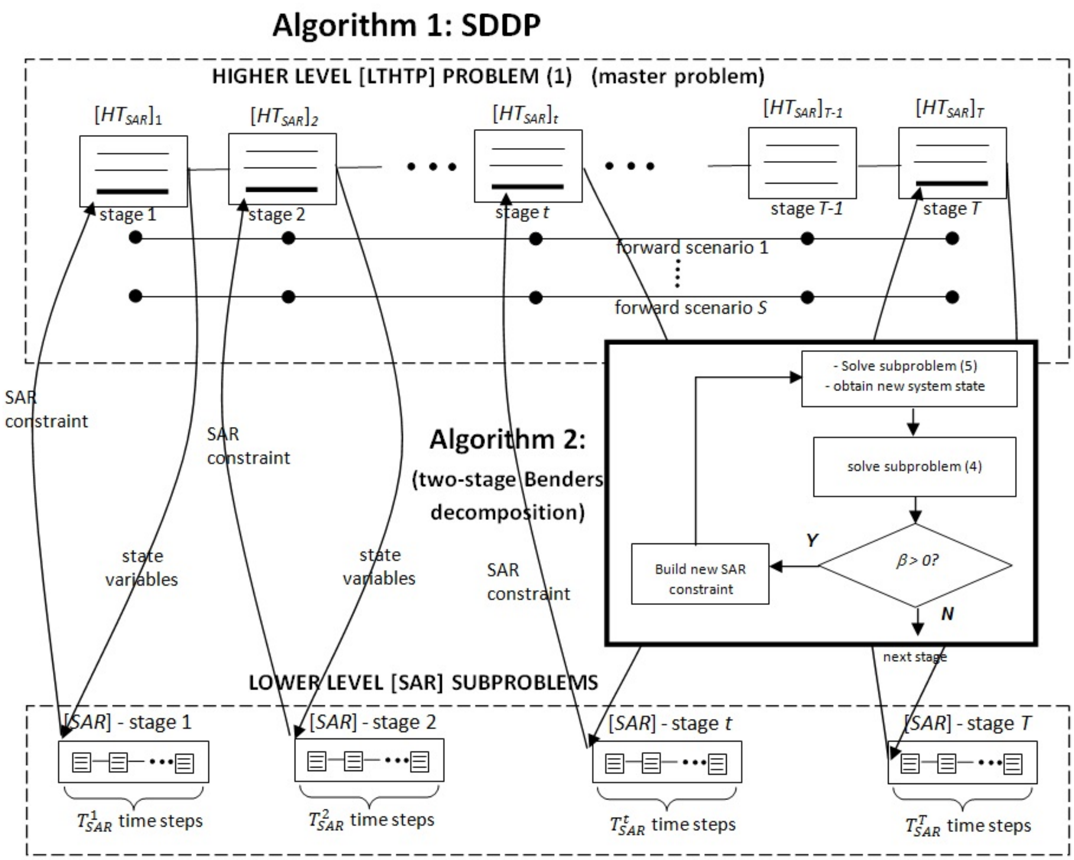

Figure 4: Overall SDDP/two-stage Benders decomposition to solve the stochastic multistage [LTHTP] problem (1) with subproblems (4) that yield SAR constraints (5g) for the $[\mathrm{HT}_{\mathrm{SAR}}]_t^{s,\omega}$ subproblem (5)

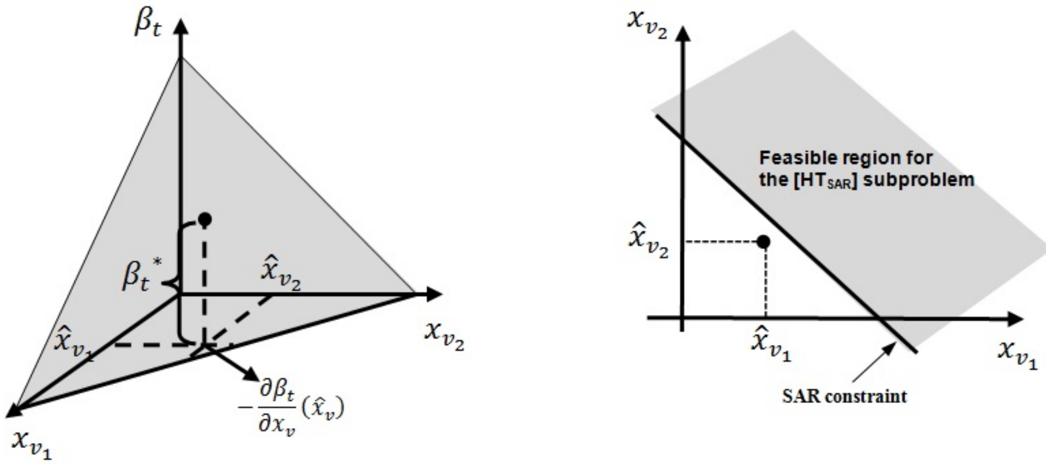

Figure 5: Graphic interpretation of the SAR constraint: as a sensitivity analysis of the optimal solution of the [SAR] subproblem (left) and as a feasibility cut included in the upper level subproblem (5) (right).

4.4 Properties of SAR constraints

Taking into account the particular case where there is no serial correlation among the energy inflows, it can be seen from the formulation in $(5f)$ that the SAR constraints impose the condition that a linear combination of energy storages should meet some minimum target value, as shown in Fig. 3. Any cut that is parallel to one axis corresponds to the case where the storage related to that axis does not contribute to meeting the requirement due to maximum interchange limits. In Fig. 5 we present two alternative graphic interpretations of the SAR constraint: the first one (in the left hand side) shows the hyperplane that indicates how the objective function of the $[\mathrm{SAR}]_t$ subproblem varies with the value of its state variables. The second interpretation, shown in the right hand side of this figure, is the cut that is actually included in the upper level $[\mathrm{HT_{SAR}}]_t^{s,\omega}$ subproblem (5), which is the interception of the hyperplane in the left side with the hyperplane that represents the condition $\beta_t = 0$.

The computation and physical interpretation of coefficients $\pi_{\xi, pSAR_t}$ related to past inflows is more involving. Due to application of the coefficientes of the autoregressive model in hydro balance equations $(4c)$ of the $[\mathrm{SAR}]_t$ subproblem, the energy inflows in each previous period $(t+1-p)$, for $p = 1, \ldots, P$, may directly impact the right hand side of all such constraints from time step $(t+1)$ to $(t+1-p+P)$. By defining $\phi_{t_i}^p$ as the coefficient of order p for the autoregressive model of reservoir i at time step t, we can write each component ξ_{ti} of the vector ξ_t of hydro inflows at time step t as follows:

$$\xi_{ti} = \sum_{p=1}^{P} \phi_{t\,i}^{p} \xi_{(t-p)_i}. \tag{9}$$

As a result, all terms in the right hand side of equations $(4c)$ for reservoir i from time steps $(t{+}1)$ to $(t{+}min\{T_{SAR}^t, P\})$ may depend *directly* on past inflows $\{\hat{\xi}_{t_i}^{s,\omega}, \ldots, \hat{\xi}_{(t+1-P)_i}^{s,\omega}\}$, which are state variables of the [SAR] subproblem. Table 2 summarizes such terms for the more general case where $P \leq T_{SAR}^t$.

Table 2: Terms related to past inflows that may directly impact the right hand side of equations (4*c*) of the $[\text{SAR}]_t$ subproblem, for each time step τ and reservoir i.

Previous time steps to $t+1$	Time steps τ within the horizon of the [SAR] subproblem				
	$t+1$	$t+2$	...	$t+P-1$	$t+P$
$(t+1)-1$	$\phi^{1}_{(t+1)_i}\hat{\xi}^{s,\omega}_{t_i}$	$\phi^{2}_{(t+2)_i}\hat{\xi}^{s,\omega}_{t_i}$	...	$\phi^{P-1}_{(t+P-1)_i}\hat{\xi}^{s,\omega}_{t_i}$	$\phi^{P}_{(t+P)_i}\hat{\xi}^{s,\omega}_{t_i}$
$(t+1)-2$	$\phi^{2}_{(t+1)_i}\hat{\xi}^{s,\omega}_{(t-1)_i}$	$\phi^{3}_{(t+2)_i}\hat{\xi}^{s,\omega}_{(t-1)_i}$	...	$\phi^{P}_{(t+P-1)_i}\hat{\xi}^{s,\omega}_{(t-1)_i}$	–
...	...	...	...	–	–
$(t+1)-(P-1)$	$\phi^{P-1}_{(t+1)_i}\hat{\xi}^{s,\omega}_{(t+2-P)_i}$	$\phi^{P}_{(t+2)_i}\hat{\xi}^{s,\omega}_{(t+2-P)_i}$	–	–	–
$(t+1)-P$	$\phi^{P}_{(t+1)_i}\hat{\xi}^{s,\omega}_{(t+1-P)_i}$	–	–	–	–

For each time step $\tau = (t+1), \dots, (t+T^t_{SAR})$ within the time horizon of the $[\text{SAR}]_t$ subproblem, the earliest time step that may contribute to the energy inflows for that time step is $(\tau - P)$. In the case where $P < (\tau - t)$, there are no contributions from previous time steps.

We note that Table 2 indicates only the *direct* contributions of past inflows $\{\xi^{s,\omega}_{t+1-p}\}_{,p=1,\dots,P}$ to the inflows of the $[\text{SAR}]_t$ subproblem. However, each term $\tilde{\xi}_\tau$ also depends on previous inflows $\{\tilde{\xi}_{\tau-j,}\}_{j=1,\dots,(\tau-t-1)}$ and each of such terms $\tilde{\xi}_{\tau-j}$ may also depend on past inflows to $(t+1)$, either directly (as in Table 2) or indirectly (in the same way as $\tilde{\xi}_\tau$ above). All such recursive relations must be taken into account when applying the chain rule for computing the derivatives of the SAR constraint regarding the vector of past inflows to time step $(t+1)$, leading to the expression below:

$$\frac{\partial \beta_t}{\partial \xi_{(t+1-p)_i}}(\hat{\xi}^{s,\omega}_{(t+1-p)_i}) = \sum_{t'=1}^{T^t_{SAR}} \Phi_{(t',p)_i}, \quad p = 1, \dots, P. \tag{10}$$

Each term $\Phi_{(t',p)}$ corresponds to the contributions of previous inflow $\xi^{s,\omega}_{(t+1-p)_i}$ to the inflow $\tilde{\xi}_{\tau_i}$of each time period $\tau = (t+t')$ within the time horizon of the $[\text{SAR}]_t$ subproblem, which are given by the recursive formula:

$$\Phi_{(t',p)_i} = \phi^{t'+p-1}_{(t+t')_i} + \sum_{j=1}^{min\{t'-1,P\}} (\phi^{j}_{(t+t')_i})\Phi_{(t'-j,p)_i}, \quad t' = 1, \dots, T^t_{SAR}, \tag{11}$$

where the first term corresponds to the direct contribution and only applies if $(t'+p-1) \leq P$. The summation in the second term comprises all composite relations of autoregressive components that link time steps $(t-p)$ and $(t+t')$.

4.5 Practical implementation aspects

In this section we describe several issues that emerged during the practical implementation of the proposed SDDP / two stage Benders decomposition and the procedures that have been performed in order to handle such aspects.

4.5.1 Parallelization and cut sharing

Parallelization is an important feature in SDDP solving strategies [49, 50]. In our proposed approach, each SAR constraint $(5f)$ of a given time step can be shared with any subproblem related to that stage. However, we use a parallel processing scheme in the backward pass of the SDDP approach to solve the subproblems related to each system state generated in the previous forward pass. Therefore, in order not to lose efficiency of this parallel scheme, we only share the SAR constraints built at iteration k of the SDDP algorithm in the next SDDP iteration. This may result in replication of SAR constraints by different processors, but this is less harmful than the cost of exchanging constraints among processors during the course of the inner two-state Benders decomposition approach, due to some extra CPU time that would be spent for data communication and synchronization.

4.5.2 Elimination of redundant constraints

Since SAR constraints are built independently in the parallel SDDP approach, as described above, it may happen that the same (or a very similar) SAR constraint is built by different processors for the same time step. In this sense, after we gather all SAR constraints that are built for a given stage t at the end of a forward or backward pass, we run a checking procedure that eliminates redundant constraints $(5f)$ in the $[\mathrm{HT_{SAR}}]_t^{s,\omega}$ subproblems.

4.5.3 Penalty for violation of SAR constraints

The inclusion of a SAR constraint in a given upper level $[\mathrm{HT_{SAR}}]_t^{s,\omega}$ subproblem (5) may turn it to be infeasible, probably because of a poor initial state $x_{vt-1}^{s,\omega}$ that was given by subproblem $(t-1)$ in the forward pass of the SDDP procedure. For this reason, it was necessary to include the slack variable $\delta_{SAR_t}^{s,\omega}$ in the set of constraints for each time step, as shown in $(5a)$,$(5f)$. The penalty parameter p_{SAR} applied to such variable had to be tuned in order not to cause a huge increase in system costs as well as system marginal costs. This may occur due to the fact that there are some very severe multistage scenarios, where it is impossible to reach the energy storage levels required by the lower level $[\mathrm{SAR}]_t$ subproblems (4). In section 5.4 of the numerical results we provide a discussion on the calibration of the value of this penalty.

5 Numerical results

We present numerical results to a real case for the large-scale Brazilian hydrothermal system, composed of 149 hydro plants and 127 thermal plants, whose total power capacity are roughly 92GW and 24GW, respectively. The Brazilian National Interconnected System (SIN) is divided into four major system areas, labeled as Southeast (SE), South (S), Northeast (NE) and North (N) subsystems. Hydro plants within each area are represented as a single equivalent reservoir. Fig. 6 shows a sketch of the Brazilian system.

The time horizon is 10 years, split into $T = 120$ monthly decision stages. We consider a scenario tree of $K = 20$ inflows per stage (including the first one), which leads to a decision tree comprising 20^{120} leaves at the last stage. As discussed previously, we applied SDDP approach to solve the problem, with the combined two-stage Benders decomposition approach described in section 4 in order to handle the risk averse surface constraints. The size of each $[\mathrm{HT_{SAR}}]_t^{s,\omega}$ subproblem (5) may vary according to time dependent constraints and different number of SAR

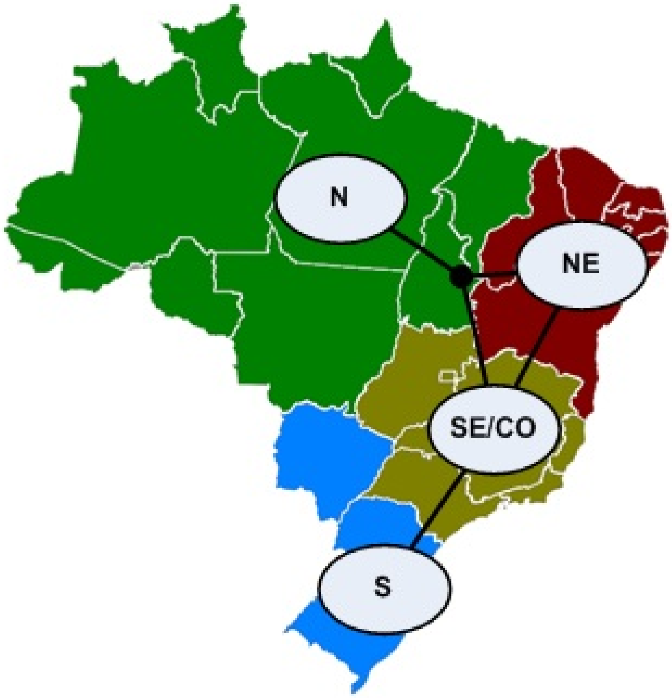

Figure 6: Sketch of the Brazilian system considered in the application of the proposed approach.

constraints that are added, but in average it has 570 decision variables, 80 system constraints ($5b$)-($5e$), 120 SAR constraints ($5f$) and 200 FCF constraints ($5g$) per SDDP iteration. The largest $[\text{SAR}]_t$ subproblem (4) has 7 time steps (from May no November) and contains 280 constraints and 590 variables.

5.1 SDDP convergence

First we present convergence results of the SDDP approach in three situations:

- **RN**: the risk-neutral case, which corresponds to problem (1)-(2) with no rule-curve or SAR constraints;
- **CAR**: Problem (1)-(2) with individual rule-curve constraints (3), which have been traditionally used for reservoir operation and was the official risk criterion in Brazil until late 2013;
- **SAR**: Problem (1) with subproblems (5), where the multivariate risk-averse surface constraints proposed in this paper are built, through the two-stage iterative procedure involving subproblems (5) and (4), for each node (t, s) and backward scenario ω.

Fig. 7 shows the evolution of lower and upper bounds for convergence of the SDDP approach, along with the confidence interval for the upper bound, which is used for the statistical stopping criteria defined in [9]. We note that each case takes a different number of iterations to converge. It can be seen that both CAR and SAR risk-averse approaches are more expensive than the

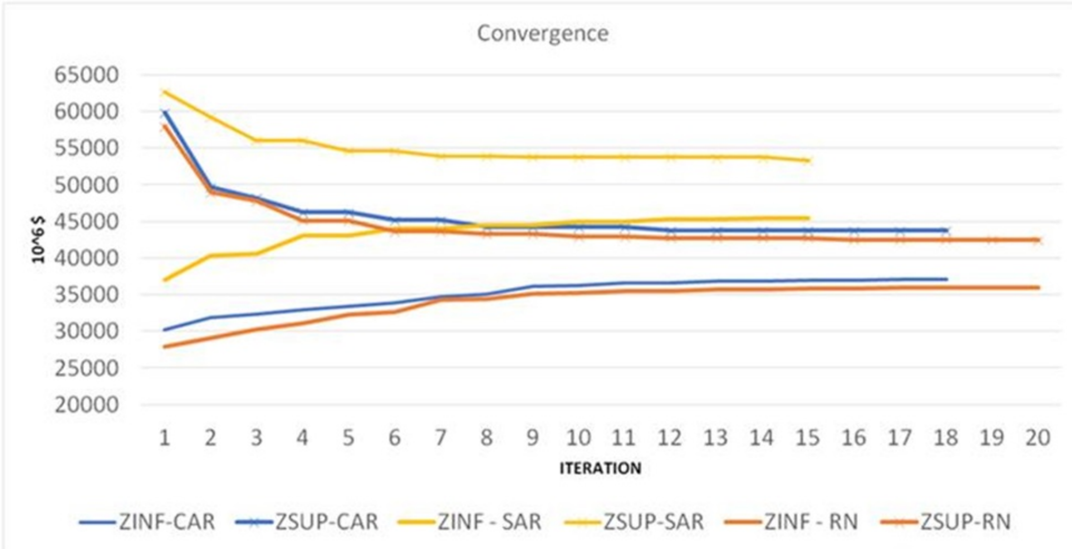

Figure 7: Overall convergence of the SDDP approach for the risk neutral (RN), rule-curve (CAR) and the risk-averse surface constraints [SAR].

risk-neutral case. Moreover, the proposed SAR approach leads to a much higher operation cost, which is somehow expected because of the requirement of providing a secure operation for all $[\text{SAR}]_t$ subproblems.

5.2 CPU times

The CPU time to solve the problem is heavily dependent on the number of SDDP iterations until convergence. In order to show comparative results, we present in Table 3 both the CPU times until convergence and the CPU times with a fixed number of 15 SDDP iterations, for the RN, CAR and SAR cases. The parallelization scheme comprised 48 processors.

Table 3: CPU times until convergence and for the same number of SDDP iterations for the RN, CAR and SAR cases.

	RN	CAR	SAR
CPU time until the 15^{th} SDDP iteration (min)	50.2	48.0	75.4
CPU time until convergence (min)	71.1	78.2	75.4

The CPU time of the proposed SAR approach is significantly higher than the other two strategies if the number of SDDP iterations is the same. However, since the SAR surface tends to restrict the storage values of the reservoirs during the course of the SDDP algorithm (because slack variables $\delta^{s,\omega}_{SAR_t}$ in (5) are penalized in the objective function), the SAR methodology might lead to a lower number of iterations until one of the stopping criteria in Algorithm 1 is reached (specially, the stability in the lower bound), because of its smaller feasible region.

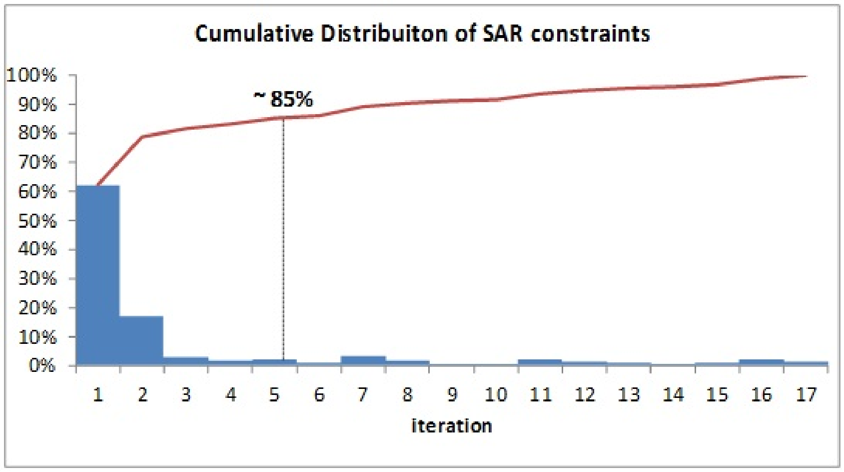

Figure 8: Percentage of the total number of cuts $(5f)$ that were built in each iteration of the SDDP procedure.

5.3 Construction of SAR constraints

Now we assess the efficiency of the proposed two-stage Benders decomposition approach to build the necessary SAR constraints. Fig. 8 shows in blue and red, respectively, the probability function and the cumulative distribution of the total number of SAR constraints that are added to the upper level subproblems for all time steps in each iteration of the outer SDDP procedure. Since most of the necessary constraints are built up to the 5th iteration, in our further experiments we cease to check feasibility of the $[\text{SAR}]_t$ subproblems from the sixth iteration on in order to avoid unnecessary increase in CPU time performance.

5.4 Sensitivity analysis on the model parameters

We present next a sensitivity analysis of the two main outputs of the model - thermal generation costs and deficit costs - as a function of the two model parameters that have to be set by the system operator in order to apply the proposed approach: the minimum storage level required for each equivalent reservoir at the end of the deterministic $[\text{SAR}]_t$ subproblems and the penalty p_{SAR} for violation of the SAR constraints in the upper level SDDP subproblem. For the first parameter we considered three situations: 30% or 40% storage level for both SE and NE subsystems and a third variant with 47% and 35% for SE and NE subsystems, respectively, which were the "ideal" values desired by the Brazilian ISO. For the penalty parameter p_{SAR} we chose values of \$50, \$100, \$150 and \$940/MWh. The highest value is close to the unitary deficit cost, which is the "intuitive" value that in principle would be adopted by the ISO.

The points in Fig. 9 show the average values of thermal generation costs and deficit costs for each alternative, for 2000 synthetic hydrological scenarios simulated with the operation policy (set of recourse functions for each time step) obtained at the end of the SDDD convergence, performed with the corresponding parameters. Squares, diamonds, circles and triangles indicate the different values of penalty costs, while the colors distinguish different end target levels. We note that a decrease in deficit costs is directly related to a lower value for the expected energy not supplied (EENS), which is intended to be obtained with the SAR approach, at the expense

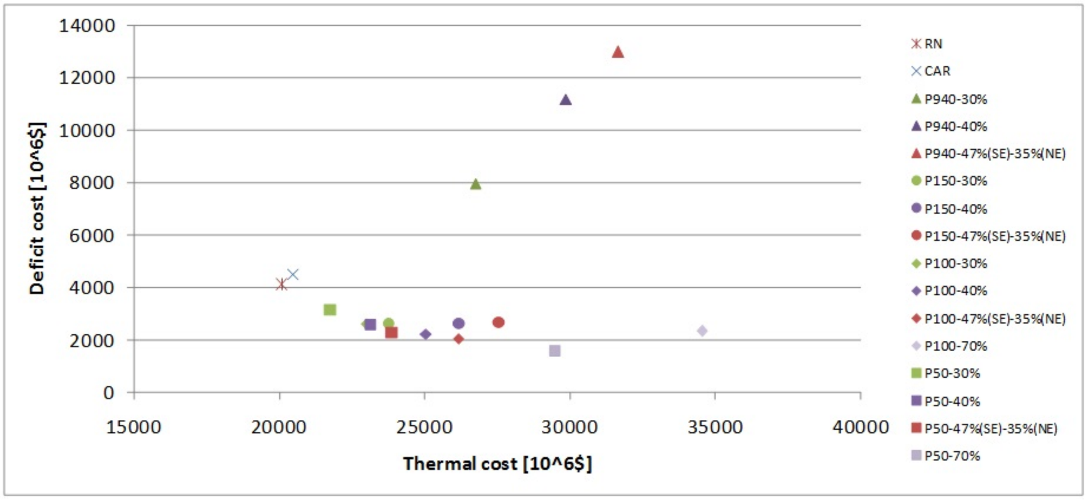

Figure 9: Sensitivity analysis of system operation costs and expected value of energy not supplied (EENS) with the values of the two main parameters of the proposed SAR approach.

of higher operation costs.

The values of deficit and total operation costs should decrease and increase, respectively, as the penalty value increases. However, we note that when the penalty value is as high as the value of the first step of the official piecewise linear deficit cost curve used in Brazil ($940/MWh), both thermal generation and deficit costs increase. This is due to the fact that the incremental cost for the energy deficit is modeled as a staircase function, with higher values as the relative amount of deficit is increased. In this sense, the optimization model curtails load in lower percentage levels in several time steps in order to decrease (or avoid) energy deficits in higher percentage levels. This is what can been called "energy deficit for optimization purposes". The advantage of the proposed risk-averse surface approach as compared to the use of rule-curves (CAR) for penalty values up to $150/MWh is clear: we both decrease system operation costs and energy deficits as compared to the methodology that had been used in Brazil. The reason for this is that the SAR surface can take better advantages of the trade-off between storage values in different energy reservoirs, while the CAR methodology sets individual minimum storage levels regardless of the (unknown) energy interchanges values among system areas.

5.5 System Operation

Finally, we assess the behavior of some key variables for the system operation – system marginal costs, energy storage levels, thermal generation and spillage and – for some combinations of SAR parameters, as compared to the results obtained by the traditional rule-curve approach (CAR). The evolution along time for the average values of these variables over 2000 simulated scenarios are shown in Fig. 10. We can see an increase in system marginal costs and thermal generation levels as the SAR parameters become stricter. As a consequence, the storage levels in the reservoirs reach higher levels, which leads to a more secure operation, at the expense of a higher risk of spillage, as shown in one of the charts.

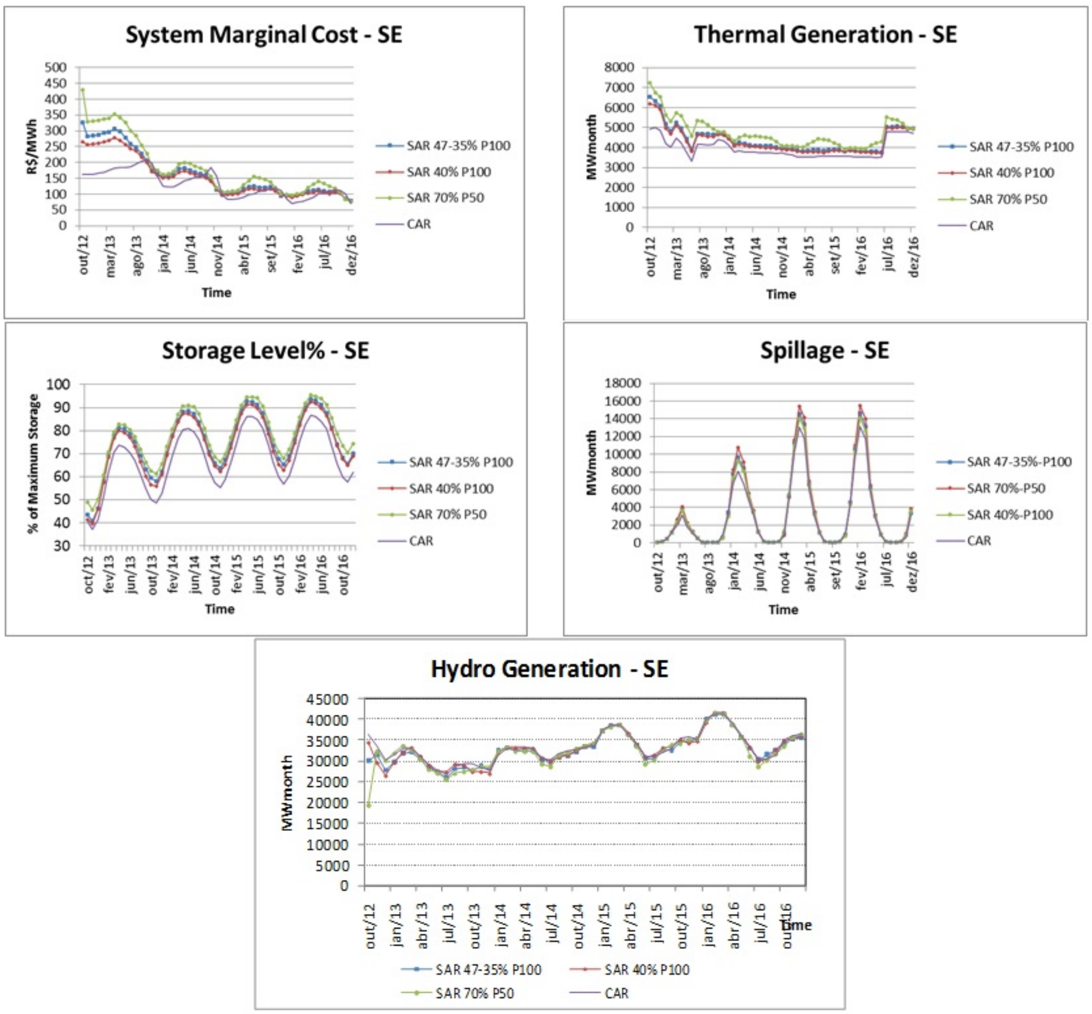

Figure 10: Output of key variables for the operation of the larger system area of the Brazilian system: marginal costs (\$/MWh), thermal generation levels (MWmonth), storage levels (% of maximum storage), spilled energy (MWmonth) and hydro generation levels (MWmonth).

6 Conclusions

This paper proposes a combined stochastic dual dynamic programming (SDDP)/two-stage Benders decomposition approach to solve the risk-averse long term hydrothermal planning problem, taking into account the so-called risk-averse surface (SAR) constraints, which is an extension, to the multivariate case, of traditional rule-curve constraints that have been used in the literature of reservoir management. The proposed methodology can be applied either to (static) critical historical scenarios or to (dynamic) critical scenarios generated by synthetic autoregressive periodical hydrological models. Results for the real large-scale Brazilian system show the efficiency of the proposed approach, which yields a more secure and economic operation as compared to the use of a rule-curve, which was the approach that had been officially used in Brazil.

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