

Sparse Recovery via Partial Regularization: Models, Theory and Algorithms *

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Abstract

In the context of sparse recovery, it is known that most of existing regularizers such as ℓ_1 suffer from some bias incurred by some leading entries (in magnitude) of the associated vector. To neutralize this bias, we propose a class of models with partial regularizers for recovering a sparse solution of a linear system. We show that every local minimizer of these models is sufficiently sparse or the magnitude of all its nonzero entries is above a uniform constant depending only on the data of the linear system. Moreover, for a class of partial regularizers, any global minimizer of these models is a sparsest solution to the linear system. We also establish some sufficient conditions for local or global recovery of the sparsest solution to the linear system, among which one of the conditions is weaker than the best known restricted isometry property (RIP) condition for sparse recovery by ℓ_1 . In addition, a first-order feasible augmented Lagrangian (FAL) method is proposed for solving these models, in which each subproblem is solved by a nonmonotone proximal gradient (NPG) method. Despite the complication of the partial regularizers, we show that each proximal subproblem in NPG can be solved as a certain number of one-dimensional optimization problems, which usually have a closed-form solution. We also show that any accumulation point of the sequence generated by FAL is a first-order stationary point of the models. Numerical results on compressed sensing and sparse logistic regression demonstrate that the proposed models substantially outperform the widely used ones in the literature in terms of solution quality.

Keywords: sparse recovery, partial regularization, null space property, restricted isometry property, proximal gradient method, augmented Lagrangian method, compressed sensing, sparse logistic regression

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1 Introduction

Over the last decade, seeking a sparse solution to a linear system has attracted a great deal of attention in science and engineering (e.g., see [33, 1, 34, 26, 38, 16]). It has found numerous applications in imaging processing, signal processing and statistics. Mathematically, it can be formulated as the ℓ_0 minimization problem

$$\min_{x \in \mathbb{R}^n} \{\|x\|_0 : \|Ax - b\| \leq \sigma\}, \quad (1)$$

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where $A \in \mathbb{R}^{m \times n}$ is a data matrix, $b \in \mathbb{R}^m$ is an observation vector, $\sigma \geq 0$ is the noisy level (i.e., $\sigma = 0$ for noiseless data and observation and $\sigma > 0$ for noisy data and/or observation), and $\|x\|_0$ denotes the cardinality of the vector x .

Given that $\|\cdot\|_0$ is an integer-valued, discontinuous and nonconvex function, it is generally hard to solve (1). One common approach in the literature (e.g., see [37, 12, 8]) is to solve the following ℓ_1 relaxation problem

$$\min_{x \in \mathbb{R}^n} \{\|x\|_1 : \|Ax - b\| \leq \sigma\}. \quad (2)$$

Under some condition on A such as restricted isometry property (RIP) and null space property (NSP), it has been shown in the literature that the solution of (1) can be stably recovered by that of (2) (e.g., see [8, 17, 18, 6, 7] and the references therein). It is well-known that $\|\cdot\|_1$ is generally a biased approximation to $\|\cdot\|_0$. To neutralize the bias in $\|x\|_1$ incurred by some leading entries (in magnitude) of x , one can replace $\|x\|_1$ by $\sum_{i=r+1}^n |x|_{[i]}$ in (2) and obtain a less biased relaxation for (1):

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{i=r+1}^n |x|_{[i]} : \|Ax - b\| \leq \sigma \right\}, \quad (3)$$

where $r \in \{0, 1, \dots, n-1\}$ and $|x|_{[i]}$ is the i th largest element in $\{|x_1|, |x_2|, \dots, |x_n|\}$. We now provide a simple example to illustrate this.

Example 1.1 *The general solution to $Ax = b$ is $x = (t, t, 1-t, 2-t, 3-t)^T$ for all $t \in \mathbb{R}$, where*

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

Thus $x^s = (0, 0, 1, 2, 3)$ is the sparsest solution to $Ax = b$. One can verify that $x^f = (1, 1, 0, 1, 2)^T$ is the unique solution of (2) with $\sigma = 0$, while $x^p = (0, 0, 1, 2, 3)^T$ is the unique solution of (3) with $\sigma = 0$ and $r = 2$ or 3. Clearly, the partial ℓ_1 model (3) recovers successfully the sparsest solution but the full ℓ_1 model (2) fails.

A more general relaxation of (1) considered in the literature is in the form of

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^n \phi(|x_i|) : \|Ax - b\| \leq \sigma \right\}, \quad (4)$$

where ϕ is some regularizer (e.g., see [21, 24, 19, 39, 10, 9, 11, 42, 41]). Some popular ϕ 's are listed as follows.

- (i) (ℓ_1 [37, 12, 8]): $\phi(t) = t \quad \forall t \geq 0$;
- (ii) (ℓ_q [21, 24]): $\phi(t) = t^q \quad \forall t \geq 0$;
- (iii) (Log [39, 9]): $\phi(t) = \log(t + \varepsilon) - \log(\varepsilon) \quad \forall t \geq 0$;
- (iv) (Capped- ℓ_1 [42]): $\phi(t) = \begin{cases} t & \text{if } 0 \leq t < \nu, \\ \nu & \text{if } t \geq \nu; \end{cases}$
- (v) (MCP [41]): $\phi(t) = \begin{cases} \lambda t - \frac{t^2}{2\alpha} & \text{if } 0 \leq t < \lambda\alpha, \\ \frac{\lambda^2\alpha}{2} & \text{if } t \geq \lambda\alpha; \end{cases}$

$$(vi) \text{ (SCAD [19]): } \phi(t) = \begin{cases} \lambda t & \text{if } 0 \leq t \leq \lambda, \\ \frac{-t^2 + 2\beta\lambda t - \lambda^2}{2(\beta-1)} & \text{if } \lambda < t < \lambda\beta, \\ \frac{(\beta+1)\lambda^2}{2} & \text{if } t \geq \lambda\beta, \end{cases}$$

where $0 < q < 1$, $\varepsilon > 0$, $\nu > 0$, $\lambda > 0$, $\alpha > 1$ and $\beta > 1$ are some parameters. One can observe that problem (2) is a special case of (4) with $\phi(t) = t$ for $t \geq 0$. Analogous to $\|\cdot\|_1$, $\sum_{i=1}^n \phi(|x_i|)$ is generally also a biased approximation to $\|\cdot\|_0$ in some extent. Similar as above, one can neutralize the bias in $\sum_{i=1}^n \phi(|x_i|)$ incurred by some leading entries (in magnitude) of x to obtain a less biased relaxation for (1):

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{i=r+1}^n \phi(|x|_{[i]}) : \|Ax - b\| \leq \sigma \right\} \quad (5)$$

for some $r \in \{0, 1, \dots, n-1\}$. In this paper we are interested in problem (5) for which ϕ satisfies the following assumption.

Assumption 1 ϕ is lower semi-continuous and increasing in $[0, \infty)$. Moreover, $\phi(0) = 0$.

Under Assumption 1, we study some theoretical properties of problem (5) and also develop an efficient algorithm for solving it. In particular, we show that problem (5) has at least an optimal solution. We also establish some sparsity inducing properties of (5) with $\sigma = 0$. It is shown that any local minimizer of (5) has at most r nonzero entries or the magnitude of all its nonzero entries is not too small, which is indeed above a uniform constant depending only on A and b . Moreover, we prove that for a class of ϕ , any global minimizer of (5) is a sparsest solution to $Ax = b$. In addition, we study some sufficient conditions for local or global recovery of a sparsest solution x^* of $Ax = b$ by model (5). It is shown that if a local null space property holds at x^* for A and ϕ , x^* is a *strictly local* minimizer of (5) with $\sigma = 0$. Specifically, for $\phi(t) = |t|^q$ with $q \in (0, 1]$, we develop some weaker RIP conditions for local recovery than the well-known optimal ones for (2). It is also shown that if a global null space property holds at x^* for A and ϕ , x^* is a *unique global* minimizer of (5) with $\sigma = 0$. Some RIP conditions are particularly derived for global recovery by (5) with $\sigma = 0$ and $\phi(t) = |t|^q$ with $q \in (0, 1]$. Furthermore, we study stable recoverability of (5) with $\phi(t) = |t|$ and $\sigma \geq 0$ under some RIP conditions. Additionally, we develop a first-order feasible augmented Lagrangian (FAL) method for solving problem (5) in which each subproblem is solved by a nonmonotone proximal gradient (NPG) method. Though the subproblems of NPG appear to be sophisticated due to $\sum_{i=r+1}^n \phi(|x|_{[i]})$, we are fortunate to show that they can be solved as $n - r$ number of one-dimensional optimization problems that have a closed-form solution for the widely used ϕ . We also show that any accumulation point of the sequence generated by FAL is a first-order stationary point of problem (5). Finally we conduct some numerical experiments to compare the performance of (5) with (4). Numerical results demonstrate that (5) substantially outperforms (4) in terms of solution quality.

The rest of this paper is organized as follows. In Section 2 we establish the existence of optimal solutions for problem (5). Some sparsity inducing properties for (5) are studied in Section 3. We establish some sufficient conditions for local or global recovery of a sparse solution by problem (5) and study its stable recoverability for in Section 4. We propose an efficient algorithm for solving (5) in Section 5. Numerical results are presented in Section 6. Finally we make some concluding remarks in Section 7.

1.1 Notation and terminology

In this paper the set of all nonnegative real numbers is denoted by \mathbb{R}_+ . For a real number t , let $t_+ = \max\{0, t\}$. Let \mathbb{R}_+^n denote the nonnegative orthant of \mathbb{R}^n . For any $x, y \in \mathbb{R}^n$, $x \geq y$ means $x - y \in \mathbb{R}_+^n$. $\{0, 1\}^n$ denotes the set of all n -dimensional vectors with binary entries. For any $x \in \mathbb{R}^n$, $\|x\|_0$ and $\|x\|$ denote the cardinality

(i.e., the number of nonzero entries) and the Euclidean norm of x , respectively, and $\|x\|_q = \sqrt[q]{\sum_{i=1}^n |x_i|^q}$ for any $q \in (0, 1]$. In addition, $x_{[i]}$ denotes the i th largest entry of x for $i = 1, \dots, n$ and $|x|$ stands for the n -dimensional vector whose i th entry is $|x_i|$. Given an integer $1 \leq k \leq n$, we denote by $x_{-\max(k)}$ the vector resulted from x by replacing the entries of x with k largest absolute value by zero, and set $x_{\max(k)} = x - x_{-\max(k)}$. A vector x is said to be k -sparse if $\|x\|_0 \leq k$. Given any $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A)$ stands for the rank of A , and the null space and range space of A are denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. Given an index set $J \subseteq \{1, \dots, n\}$, $|J|$ and J^c denote the size of J and the complement of J in $\{1, \dots, n\}$, respectively. For any $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{m \times n}$, x_J denotes the subvector formed by the entries of x indexed by J , and M_J denotes the submatrix formed by the columns of M indexed by J . For any closed set S , let $\text{dist}(x, S)$ denote the distance from x to S , and $\text{conv}(S)$ denote the convex hull of S .

We recall from [36, Definition 8.3] that for a proper lower semi-continuous function ϑ (not necessarily locally Lipschitz), the limiting and horizon subdifferentials are defined respectively as

$$\begin{aligned} \partial\vartheta(x) &:= \left\{ v \mid \exists x^k \xrightarrow{\vartheta} x, v^k \rightarrow v \text{ with } \liminf_{z \rightarrow x^k} \frac{\vartheta(z) - \vartheta(x^k) - \langle v^k, z - x^k \rangle}{\|z - x^k\|} \geq 0 \quad \forall k \right\}, \\ \partial^\infty \vartheta(x) &:= \left\{ v \mid \exists x^k \xrightarrow{\vartheta} x, \lambda_k v^k \rightarrow v, \lambda_k \downarrow 0 \text{ with } \liminf_{z \rightarrow x^k} \frac{\vartheta(z) - \vartheta(x^k) - \langle v^k, z - x^k \rangle}{\|z - x^k\|} \geq 0 \quad \forall k \right\}, \end{aligned}$$

where $\lambda_k \downarrow 0$ means $\lambda_k > 0$ and $\lambda_k \rightarrow 0$, and $x^k \xrightarrow{\vartheta} x$ means both $x^k \rightarrow x$ and $\vartheta(x^k) \rightarrow \vartheta(x)$. The following concept was introduced in [8] by Candès and Tao.

Definition 1 (restricted isometry constant) *Given $A \in \mathbb{R}^{m \times n}$ and an integer $1 \leq k \leq n$, the restricted isometry constant (RIC) of order k , denoted by δ_k^A , is the smallest number δ such that for all k -sparse vectors $x \in \mathbb{R}^n$,*

$$(1 - \delta)\|x\|^2 \leq \|Ax\|^2 \leq (1 + \delta)\|x\|^2.$$

We define δ_k^A as $\delta_{\lceil k \rceil}^A$ when $k > 0$ is not an integer.

2 Existence of optimal solutions

In this section we establish the existence of optimal solutions of problem (5). To this end, we first consider a more general class of problems in the form of

$$F^* = \inf_{x \in \mathbb{R}^n} \{F(x) := g(Ax - b) + \Psi(x)\}, \quad (6)$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and F^* , g and Ψ satisfy the following assumption.

Assumption 2 (a) $g : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\}$ is lower semi-continuous in \mathbb{R}^m and any level set of g is bounded.

(b) $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded below, lower semi-continuous, and moreover,

$$\Psi(y) \geq \Psi(z) \quad \text{if } |y| \geq |z|.$$

(c) F^* is finite.

We next show that under Assumption 2, the infimum of problem (6) is attainable at some point.

Theorem 2.1 *Under Assumption 2, the infimum F^* of problem (6) is achievable at some point, that is, (6) has at least an optimal solution.*

Proof. By the definition of F^* , there exists a sequence $\{x^k\}$ such that

$$\lim_{k \rightarrow \infty} F(x^k) = \lim_{k \rightarrow \infty} g(Ax^k - b) + \Psi(x^k) = F^*. \quad (7)$$

By Assumption 2(b) and 2(c), we know that $\{\Psi(x^k)\}$ is bounded below and F^* is finite. These together with (7) imply that $\{g(Ax^k - b)\}$ is bounded above. Using this and Assumption 2(a), we conclude that $\{Ax^k - b\}$ is bounded and so is $\{Ax^k\}$. Let $J = \{j : \{x_j^k\} \text{ is bounded}\}$. It then follows that $\{A_J x_J^k\}$ is bounded. This together with the boundedness of $\{Ax^k\}$ implies that $\{A_{J^c} x_{J^c}^k\}$ is bounded. Consider the linear system

$$A_{J^c} z = A_{J^c} x_{J^c}^k.$$

Clearly it has at least a solution $z = x_{J^c}^k$. By Hoffman Lemma [27], there exist z^k and some constant $\zeta > 0$ depending only on A_{J^c} such that

$$A_{J^c} z^k = A_{J^c} x_{J^c}^k, \quad \|z^k\| \leq \zeta \|A_{J^c} x_{J^c}^k\|. \quad (8)$$

Let $\tilde{x}^k = (x_J^k, z^k)$. Using (8), the boundedness of $\{A_{J^c} x_{J^c}^k\}$, and the definitions of J and \tilde{x}^k , one can observe that $A\tilde{x}^k = Ax^k$ and $\{\tilde{x}^k\}$ is bounded. Considering a convergent subsequence if necessary, assume for convenience that $\tilde{x}^k \rightarrow x^*$ for some x^* . By the definition of F and the lower semi-continuity of g and Ψ , we can see that F is lower semi-continuous and hence

$$\liminf_{k \rightarrow \infty} F(\tilde{x}^k) \geq F(x^*). \quad (9)$$

In addition, by the definition of J , we know that $\{x_{J^c}^k\}$ is unbounded. This along with the boundedness of $\{\tilde{x}_{J^c}^k\}$ implies $|x_{J^c}^k| \geq |\tilde{x}_{J^c}^k|$ for sufficiently large k . It follows from this and $\tilde{x}_J^k = x_J^k$ that $|x^k| \geq |\tilde{x}^k|$ for sufficiently large k . This and Assumption 2(b) implies that $\Psi(x^k) \geq \Psi(\tilde{x}^k)$ for sufficiently large k . By this relation, $A\tilde{x}^k = Ax^k$ and (6), we have that for sufficiently large k ,

$$F(x^k) = g(Ax^k - b) + \Psi(x^k) \geq g(A\tilde{x}^k - b) + \Psi(\tilde{x}^k) = F(\tilde{x}^k).$$

In view of this, (7) and (9), one has

$$F^* = \lim_{k \rightarrow \infty} F(x^k) \geq \liminf_{k \rightarrow \infty} F(\tilde{x}^k) \geq F(x^*).$$

This along with the definition of F^* implies $F(x^*) = F^*$ and hence x^* is an optimal solution of problem (6). ■

As a consequence of the above theorem, we next establish the existence of optimal solutions of problem (5).

Theorem 2.2 *Suppose that Assumption 1 holds for ϕ and the feasible region of (5) is nonempty. Then problem (5) has at least an optimal solution.*

Proof. Let $\Omega = \{w \in \mathbb{R}^m : \|w\| \leq \sigma\}$ and ι_Ω be the indicator function of Ω , that is, $\iota_\Omega(w) = 0$ if $w \in \Omega$ and ∞ otherwise. One can observe that (5) is a special case of (6) with $g = \iota_\Omega$ and $\Psi(\cdot) = \sum_{i=r+1}^n \phi(|\cdot|_{[i]})$. Since ϕ satisfies Assumption 1, it is not hard to verify that Assumption 2 holds for such g and Ψ . The conclusion then follows from Theorem 2.1. ■

The following model has been widely used in the literature for sparse recovery:

$$\min_{x \in \mathbb{R}^n} \lambda \|Ax - b\|^2 + \Psi(x), \quad (10)$$

where $\lambda > 0$ and Ψ is a sparsity-inducing regularizer such as ℓ_0 [4, 5, 32, 30], ℓ_1 [37, 12, 8], ℓ_q [21, 24], Capped- ℓ_1 [42], Log [39, 9], MCP [41] and SCAD [19] that are presented in Section 1. It is not hard to observe that (10) is a special case of (6) with $g(\cdot) = \lambda \|\cdot\|_2^2$ for (10). The following result thus immediately follows from Theorem 2.1.

Corollary 2.3 *Problem (10) has at least an optimal solution if Ψ is one of the regularizers ℓ_0 , ℓ_1 , ℓ_q , Capped- ℓ_1 , Log, MCP and SCAD.*

3 Sparsity inducing property

In this section we study some sparsity inducing properties of model (5) with $\sigma = 0$, that is,

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{i=r+1}^n \phi(|x|_{[i]}) : Ax - b = 0 \right\}. \quad (11)$$

In particular, we show that any local minimizer of (11) is r -sparse or the magnitude of all its nonzero entries cannot be too small, which is indeed above a uniform constant depending only on A and b . We also show that under some assumption on ϕ , any global minimizer of (11) is a sparsest solution to the linear system $Ax = b$.

Before proceeding, we introduce some notations that will be used subsequently. Given any $x \in \mathbb{R}^n$ and $j \in \{1, \dots, n\}$, we define

$$\begin{aligned} \mathcal{I}_0(x) &= \{i : x_i = 0\}, & \mathcal{I}_0^c(x) &= \{i : x_i \neq 0\}, \\ \mathcal{I}_j^-(x) &= \{i : |x_i| = |x|_{[j]}\}, & \mathcal{I}_j^+(x) &= \{i : |x_i| > |x|_{[j]}\}, \\ \mathcal{I}_j^<(x) &= \{i : |x_i| < |x|_{[j]}\}, & \mathcal{I}_j^{<+}(x) &= \{i : 0 < |x_i| < |x|_{[j]}\}, \\ \mathcal{I}_j^{\leq}(x) &= \{i : |x_i| \leq |x|_{[j]}\}, & \mathcal{I}_j^{\leq+}(x) &= \{i : 0 < |x_i| \leq |x|_{[j]}\}. \end{aligned} \quad (12)$$

It is not hard to observe that

$$\begin{aligned} \mathcal{I}_j^+(x) \cup \mathcal{I}_j^-(x) \cup \mathcal{I}_j^<(x) &= \{1, \dots, n\}, & \mathcal{I}_0^c(x) &= \mathcal{I}_j^{\leq+}(x) \cup \mathcal{I}_j^+(x), \\ \mathcal{I}_j^{\leq}(x) &= \mathcal{I}_j^-(x) \cup \mathcal{I}_j^<(x), & \mathcal{I}_j^{\leq+}(x) &= \mathcal{I}_0(x) \cup \mathcal{I}_j^{\leq+}(x). \end{aligned}$$

The following lemma characterizes some nonzero components of a local minimizer of problem (11).

Lemma 3.1 *Assume $\phi''(t) < 0$ for all $t > 0$. Suppose x^* is a local minimizer of problem (11). Let $\mathcal{I}_{r+1}^> = \mathcal{I}_{r+1}^>(x^*)$, $\mathcal{I}_{r+1}^{\leq+} = \mathcal{I}_{r+1}^{\leq+}(x^*)$, \mathcal{J} a subset of $\mathcal{I}_{r+1}^>$ such that $\text{rank}(A_{\mathcal{J}}) = \text{rank}(A_{\mathcal{I}_{r+1}^>})$ and $A_{\mathcal{J}}$ has full column rank, and $B = [A_{\mathcal{I}_{r+1}^{\leq+}} \ A_{\mathcal{J}}]$, where $\mathcal{I}_{r+1}^>(x^*)$ and $\mathcal{I}_{r+1}^{\leq+}(x^*)$ are defined in (12). Then x^* is r -sparse or the following statements hold:*

(i) B has full column rank;

(ii) $x_{\mathcal{I}_{r+1}^{\leq+}}^* = [e_1 \cdots e_l]^T (B^T B)^{-1} B^T b$, where $l = |\mathcal{I}_{r+1}^{\leq+}|$ and e_i is the i th coordinate vector for all i .

Proof. If x^* is a r -sparse solution to $Ax = b$, the conclusion clearly holds. We now suppose that x^* is not a r -sparse solution to $Ax = b$. It implies that $\mathcal{I}_{r+1}^{\leq+} \neq \emptyset$ and $|x_i^*| > 0$ for all $i \in \mathcal{I}_{r+1}^{\leq+}(x^*)$. We next show that statements (i) and (ii) hold.

(i) One can observe from (12) that when x is sufficiently close to x^* ,

$$\mathcal{I}_{r+1}^<(x^*) \subseteq \mathcal{I}_{r+1}^<(x), \quad \mathcal{I}_{r+1}^>(x^*) \subseteq \mathcal{I}_{r+1}^>(x). \quad (13)$$

Let $t_1 = |\mathcal{I}_{r+1}^-(x^*)|$, $t_2 = n - r - |\mathcal{I}_{r+1}^<(x^*)|$ and

$$\mathcal{S} = \{\mathcal{K} : \mathcal{K} \subseteq \mathcal{I}_{r+1}^-(x^*), |\mathcal{K}| = t_2\}.$$

Observe that $t_1, t_2 > 0$ and $\mathcal{S} \neq \emptyset$. Using (13) and the definitions of t_1, t_2 and \mathcal{S} , we can observe that when x is sufficiently close to x^* ,

$$\sum_{i=r+1}^n \phi(|x|_{[i]}) = \sum_{i \in \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i|) + \min_{\mathcal{K} \in \mathcal{S}} \sum_{i \in \mathcal{K}} \phi(|x_i|) \leq \sum_{i \in \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i|) + \frac{t_2}{t_1} \sum_{i \in \mathcal{I}_{r+1}^{=}(x^*)} \phi(|x_i|). \quad (14)$$

Notice that

$$\sum_{i=r+1}^n \phi(|x^*|_{[i]}) = \sum_{i \in \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i^*|) + \frac{t_2}{t_1} \sum_{i \in \mathcal{I}_{r+1}^{=}(x^*)} \phi(|x_i^*|). \quad (15)$$

Since x^* is a local minimizer of problem (11), we have $\sum_{i=r+1}^n \phi(|x|_{[i]}) \geq \sum_{i=r+1}^n \phi(|x^*|_{[i]})$ when x is sufficiently close to x^* . By this relation, (14) and (15), one can see that when x is sufficiently close to x^* ,

$$\sum_{i \in \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i|) + \frac{t_2}{t_1} \sum_{i \in \mathcal{I}_{r+1}^{=}(x^*)} \phi(|x_i|) \geq \sum_{i \in \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i^*|) + \frac{t_2}{t_1} \sum_{i \in \mathcal{I}_{r+1}^{=}(x^*)} \phi(|x_i^*|).$$

Hence, x^* is a local minimizer of the problem

$$\min_x \left\{ \sum_{i \in \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i|) + \frac{t_2}{t_1} \sum_{i \in \mathcal{I}_{r+1}^{=}(x^*)} \phi(|x_i|) : Ax = b \right\}.$$

It then follows that $(x_{\mathcal{I}_{r+1}^{<+}}^*, x_{\mathcal{I}_{r+1}^{=}}^*, x_{\mathcal{I}_{r+1}^{>}}^*)$ is a local minimizer of the problem

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{I}_{r+1}^{<+}} \phi(|x_i|) + \frac{t_2}{t_1} \sum_{i \in \mathcal{I}_{r+1}^{=}} \phi(|x_i|) \\ & A_{\mathcal{I}_{r+1}^{<+}} x_{\mathcal{I}_{r+1}^{<+}} + A_{\mathcal{I}_{r+1}^{=}} x_{\mathcal{I}_{r+1}^{=}} + A_{\mathcal{I}_{r+1}^{>}} x_{\mathcal{I}_{r+1}^{>}} = b, \end{aligned} \quad (16)$$

where $\mathcal{I}_{r+1}^{<+} = \mathcal{I}_{r+1}^{<+}(x^*)$, $\mathcal{I}_{r+1}^{=} = \mathcal{I}_{r+1}^{=}(x^*)$ and $\mathcal{I}_{r+1}^{>} = \mathcal{I}_{r+1}^{>}(x^*)$. By the second-order necessary optimality condition of (16), one has

$$\sum_{i \in \mathcal{I}_{r+1}^{<+}} \phi''(|x_i^*|) h_i^2 + \frac{t_2}{t_1} \sum_{i \in \mathcal{I}_{r+1}^{=}} \phi''(|x_i^*|) h_i^2 \geq 0 \quad (17)$$

for all $h = (h_{\mathcal{I}_{r+1}^{<+}}, h_{\mathcal{I}_{r+1}^{=}}, h_{\mathcal{I}_{r+1}^{>}})$ satisfying

$$A_{\mathcal{I}_{r+1}^{<+}} h_{\mathcal{I}_{r+1}^{<+}} + A_{\mathcal{I}_{r+1}^{=}} h_{\mathcal{I}_{r+1}^{=}} + A_{\mathcal{I}_{r+1}^{>}} h_{\mathcal{I}_{r+1}^{>}} = 0. \quad (18)$$

Recall that $|x_i^*| > 0$ for all $i \in \mathcal{I}_{r+1}^{=}$. Hence, $\mathcal{I}_{r+1}^{<+} = \mathcal{I}_{r+1}^{<+} \cup \mathcal{I}_{r+1}^{=}$. This together with the assumption $\phi''(t) < 0$ for all $t > 0$ implies that $\phi''(|x_i^*|) < 0$ for all $i \in \mathcal{I}_{r+1}^{<+}$. Using this relation, $t_1, t_2 > 0$, $\mathcal{I}_{r+1}^{<+} = \mathcal{I}_{r+1}^{<+} \cup \mathcal{I}_{r+1}^{=}$, (17) and (18), we have

$$A_{\mathcal{I}_{r+1}^{<+}} h_{\mathcal{I}_{r+1}^{<+}} + A_{\mathcal{I}_{r+1}^{>}} h_{\mathcal{I}_{r+1}^{>}} = 0 \implies h_{\mathcal{I}_{r+1}^{<+}} = 0. \quad (19)$$

Recall that $\mathcal{J} \subseteq \mathcal{I}_{r+1}^{>}$ is such that $\text{rank}(A_{\mathcal{J}}) = \text{rank}(A_{\mathcal{I}_{r+1}^{>}})$ and $A_{\mathcal{J}}$ has full column rank. This along with (19) yields

$$A_{\mathcal{I}_{r+1}^{<+}} h_{\mathcal{I}_{r+1}^{<+}} + A_{\mathcal{J}} h_{\mathcal{J}} = 0 \implies h_{\mathcal{J} \cup \mathcal{I}_{r+1}^{<+}} = 0.$$

It implies that $B = [A_{\mathcal{I}_{r+1}^{<+}} \ A_{\mathcal{J}}]$ has full column rank.

(ii) Notice that $Ax^* = b$. It then follows that

$$A_{\mathcal{I}_{r+1}^{<+}} x_{\mathcal{I}_{r+1}^{<+}}^* + A_{\mathcal{I}_{r+1}^{>}} x_{\mathcal{I}_{r+1}^{>}}^* = b. \quad (20)$$

Since $\mathcal{J} \subseteq \mathcal{I}_{r+1}^>$ and $\text{rank}(A_{\mathcal{J}}) = \text{rank}(A_{\mathcal{I}_{r+1}^>})$, there exists some $\tilde{x}^* \mathfrak{R}^{|\mathcal{J}|}$ such that $A_{\mathcal{J}} \tilde{x}^* = A_{\mathcal{I}_{r+1}^>} x_{\mathcal{I}_{r+1}^>}^*$. This together with (20) yields

$$A_{\mathcal{I}_{r+1}^{\leq+}} x_{\mathcal{I}_{r+1}^{\leq+}}^* + A_{\mathcal{J}} \tilde{x}^* = b.$$

In view of this and the fact that $B = [A_{\mathcal{I}_{r+1}^{\leq+}} \ A_{\mathcal{J}}]$ has full column rank, one has

$$\begin{pmatrix} x_{\mathcal{I}_{r+1}^{\leq+}}^* \\ \tilde{x}^* \end{pmatrix} = (B^T B)^{-1} B^T b,$$

which immediately implies that statement (ii) holds. \blacksquare

As a consequence of the above lemma, we next show that any local minimizer of (11) is r -sparse or the magnitude of all its nonzero entries cannot be too small, which is indeed above a uniform constant depending only on A and b .

Theorem 3.2 *Assume $\phi''(t) < 0$ for all $t > 0$. Let*

$$\begin{aligned} \mathcal{B} &:= \{ \mathcal{I} \subseteq \{1, \dots, n\} : A_{\mathcal{I}} \text{ has full column rank and } (A_{\mathcal{I}})^T b \neq 0 \}, \\ \delta_{\mathcal{I}} &:= \min \left\{ \left| [(A_{\mathcal{I}})^T A_{\mathcal{I}}]^{-1} (A_{\mathcal{I}})^T b \right|_i : \begin{array}{l} |[(A_{\mathcal{I}})^T A_{\mathcal{I}}]^{-1} (A_{\mathcal{I}})^T b|_i > 0, \\ 1 \leq i \leq |\mathcal{I}| \end{array} \right\} \quad \forall \mathcal{I} \in \mathcal{B}, \\ \delta &:= \min \{ \delta_{\mathcal{I}} : \mathcal{I} \in \mathcal{B} \}. \end{aligned} \tag{21}$$

Let x^* be an arbitrary local minimizer of problem (11) and

$$K = \min \{ \|x\|_0 : Ax = b \}. \tag{22}$$

There hold:

- (i) If $0 \leq r < K$, $|x_i^*| \geq \delta$ for all $i \in \mathcal{I}_0^c(x^*)$;
- (i) If $K \leq r \leq n-1$, x^* is r -sparse or $|x_i^*| \geq \delta$ for all $i \in \mathcal{I}_0^c(x^*)$.

Proof. Notice $\mathcal{I}_0^c(x^*) = \mathcal{I}_{r+1}^{\leq+}(x^*) \cup \mathcal{I}_{r+1}^>(x^*)$. In view of this, Lemma 3.1 and the definition of δ , one can observe that if x^* is non- r -sparse, then $\mathcal{I}_{r+1}^{\leq+}(x^*) \neq \emptyset$ and

$$|x_i^*| \geq \min \{ |x_j^*| : j \in \mathcal{I}_{r+1}^{\leq+}(x^*) \} \geq \delta \quad \forall i \in \mathcal{I}_0^c(x^*).$$

In addition, by the definition of K , one can see that if $0 \leq r < K$, x^* must be non- r -sparse. The conclusion of this theorem then immediately follows from these observations and Lemma 3.1. \blacksquare

Remark: The quantity δ given in (21) is independent to r , but only depends on A and b . In addition, the above theorem implies that the components of a non- r -sparse local minimizer of (11) with magnitude below δ must be zero. Therefore, any local minimizer of (11) is either r -sparse or the magnitude of its nonzero entries is above δ .

In the context of sparse recovery, the regularizers often associate with some parameter that controls the proximity to the ℓ_0 regularizer. For example, the well-known ℓ_q regularizer is one of them. We next consider a

special class of model (11) with $\phi(\cdot) = \psi(\cdot, q)$ for some parameter q , namely,

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \sum_{i=r+1}^n \psi(|x|_{[i]}, q) \\ \text{s.t.} \quad & Ax = b, \end{aligned} \tag{23}$$

where $\psi : \mathbb{R}_+ \times (0, \alpha) \rightarrow \mathbb{R}_+$ for some $\alpha \in (0, \infty]$ satisfies the following assumption.

Assumption 3 (a) $\psi(\cdot, q)$ is lower semi-continuous and increasing in \mathbb{R}_+ for any $q \in (0, \alpha)$;

(b) $\frac{\partial^2 \psi}{\partial t^2}(t, q) < 0$ for all $t > 0$ and $q \in (0, \alpha)$;

(c) $\psi(0, q) = 0$ and $\lim_{q \rightarrow 0^+} \psi(t, q) = 1$ for all $t > 0$.

One can observe from Theorem 2.1 that under Assumption 3(a), problem (23) has at least an optimal solution. In addition, under Assumption 3(b), Theorem 3.2 also holds for problem (23). We next show that when q is sufficiently small, any global minimizer of (23) is a sparsest solution to the system $Ax = b$. This indicates that problem (23) is capable of inducing a sparse solution.

Theorem 3.3 Suppose that Assumption 3 holds for ψ . Let K be defined in (22). Then for any $0 \leq r \leq \min\{K, n-1\}$ and sufficiently small $q > 0$, any global minimizer x^* of problem (23) is also that of problem (22), that is, x^* is a sparsest solution to the system $Ax = b$.

Proof. Let $r \in [0, \min\{K, n-1\}]$ be arbitrarily chosen, \tilde{x}^* and x^* a global minimizer of (22) and (23), respectively, and let δ be given in (21). In view of Assumption 3(c), $\|\tilde{x}^*\|_0 = K$ and $r \leq K$, one can observe that

$$\lim_{q \rightarrow 0^+} \frac{\sum_{i=r+1}^n \psi(|\tilde{x}^*|_{[i]}, q)}{\psi(\delta, q)} = K - r.$$

Hence, there exists some $0 < \epsilon < \alpha$ such that $\psi(\delta, q) > 0$ and

$$\frac{\sum_{i=r+1}^n \psi(|\tilde{x}^*|_{[i]}, q)}{\psi(\delta, q)} \leq K - r + \frac{1}{2} \quad \forall q \in (0, \epsilon). \tag{24}$$

By Theorem 3.2, we know that x^* is a r -sparse solution to $Ax = b$ or $|x_i^*| \geq \delta$ for all $i \in \mathcal{I}_0^c(x^*)$. We next show that x^* is an optimal solution of (22) if $q \in (0, \epsilon)$ by considering two cases.

Case 1): x^* is a r -sparse solution to $Ax = b$. This together with the assumption $r \leq K$ implies that x^* is a K -sparse solution to $Ax = b$. It then follows from (22) that the conclusion holds.

Case 2): $|x_i^*| \geq \delta$ for all $i \in \mathcal{I}_0^c(x^*)$. This together with the monotonicity of $\psi(\cdot, q)$ implies that $\psi(|x_i^*|, q) \geq \psi(\delta, q)$. Since $A\tilde{x}^* = b$ and x^* is a global minimizer of problem (23), one has

$$\sum_{i=r+1}^n \psi(|x^*|_{[i]}, q) \leq \sum_{i=r+1}^n \psi(|\tilde{x}^*|_{[i]}, q).$$

Using these two relations and $\psi(0, q) = 0$, we obtain that

$$(n - r - |\mathcal{I}_0(x^*)|) \psi(\delta, q) \leq \sum_{i=r+1}^n \psi(|x^*|_{[i]}, q) \leq \sum_{i=r+1}^n \psi(|\tilde{x}^*|_{[i]}, q) \quad \forall q \in (0, \epsilon),$$

which together with (24) and $\psi(\delta, q) > 0$ implies $n - r - |\mathcal{I}_0(x^*)| \leq K - r + 1/2$ and hence $|\mathcal{I}_0(x^*)| \geq n - K - 1/2$. Since $|\mathcal{I}_0(x^*)|$ is integer, it then follows that $|\mathcal{I}_0(x^*)| \geq n - K$ and hence $|\mathcal{I}_0^c(x^*)| \leq K$. This along with (22) implies that x^* is an optimal solution of (22) if $q \in (0, \epsilon)$. \blacksquare

4 Sparse recovery

In this section we first establish some sufficient conditions for local or global recovery of a sparsest solution to $Ax = b$ by model (5) with $\sigma = 0$, namely, (11). Then we study the stable recoverability of model (5) with $\phi(\cdot) = |\cdot|$.

4.1 Local sparse recovery by model (11)

In this subsection we study some sufficient conditions for local recovery of a sparsest solution x^* to the system $Ax = b$ by model (11). In particular, we show that if a local null space property holds, x^* is a strictly local minimizer of (11). We also establish a local recovery result for (11) under some RIP conditions and the assumption $\phi(\cdot) = |\cdot|^q$ for $q \in (0, 1]$. To this end, we first introduce the definition of local null space property.

Definition 2 (local null space property) *The local null space property $\text{LNSP}(r, \phi)$ holds for the system $Ax = b$ at a solution x^* with $\|x^*\|_0 \geq r$ if there exists some $\epsilon > 0$ such that*

$$\sum_{i \in \mathcal{I}_0(x^*)} \phi(|h_i|) - \max_{J \in \mathcal{J}} \sum_{i \in J \cup \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|h_i|) > 0 \quad (25)$$

for all $h \in \mathcal{N}(A)$ with $0 < \|h\| < \epsilon$, where

$$\mathcal{J} = \{J \subseteq \mathcal{I}_{r+1}^=(x^*) : |J| = \|x^*\|_0 - r - |\mathcal{I}_{r+1}^{<+}(x^*)| \}. \quad (26)$$

We next show that for a class of ϕ , a solution x^* of $Ax = b$ is a strictly local minimizer of (11) if the local null space property holds at x^* .

Theorem 4.1 *Suppose that Assumption 1 holds for ϕ and additionally*

$$\phi(s) \geq \phi(t) - \phi(|s - t|) \quad \forall s, t \geq 0. \quad (27)$$

Let $x^* \in \mathbb{R}^n$ be such that $Ax^* = b$. Assume that $\text{LNSP}(r, \phi)$ holds at x^* for some $0 \leq r \leq \|x^*\|_0$. Then x^* is a strictly local minimizer of problem (11) with such ϕ and r .

Proof. One can observe that $\mathcal{I}_{r+1}^{<}(x^*) \subseteq \mathcal{I}_{r+1}^{<}(x^* + h)$ and $\mathcal{I}_{r+1}^{>}(x^*) \subseteq \mathcal{I}_{r+1}^{>}(x^* + h)$ for sufficiently small h . Hence, for every sufficiently small h , there exists some $J \subseteq \mathcal{I}_{r+1}^=(x^*)$ (dependent on h) with $|J| = \|x^*\|_0 - r - |\mathcal{I}_{r+1}^{<+}(x^*)|$ such that

$$\sum_{i=r+1}^n \phi(|x^* + h|_{[i]}) = \sum_{i \in J \cup \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i^* + h_i|) + \sum_{i \in \mathcal{I}_0(x^*)} \phi(|h_i|). \quad (28)$$

By $|x_i^* + h_i| \geq ||x_i^*| - |h_i||$, the monotonicity of ϕ and (27), one has

$$\phi(|x_i^* + h_i|) \geq \phi(|x_i^*| - |h_i|) \geq \phi(|x_i^*|) - \phi(|h_i|) \quad \forall i. \quad (29)$$

Using this relation and (28), we obtain that

$$\sum_{i=r+1}^n \phi(|x^* + h|_{[i]}) \geq \sum_{i \in J \cup \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i^*|) - \sum_{i \in J \cup \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|h_i|) + \sum_{i \in \mathcal{I}_0(x^*)} \phi(|h_i|). \quad (30)$$

Since $\phi(0) = 0$, one has $\phi(x_i^*) = 0$ for all $i \in \mathcal{I}_0(x^*)$. By this and the definition of J , we have

$$\sum_{i \in J \cup \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i^*|) = \sum_{i \in J \cup \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|x_i^*|) + \sum_{i \in \mathcal{I}_0(x^*)} \phi(|x_i^*|) = \sum_{i=r+1}^n \phi(|x^*|_{[i]}).$$

This relation and (30) yield

$$\sum_{i=r+1}^n \phi(|x^* + h|_{[i]}) \geq \sum_{i=r+1}^n \phi(|x^*|_{[i]}) - \sum_{i \in J \cup \mathcal{I}_{r+1}^{<+}(x^*)} \phi(|h_i|) + \sum_{i \in \mathcal{I}_0(x^*)} \phi(|h_i|).$$

By this and (25), we see that for all sufficiently small $h \in \mathcal{N}(A) \setminus \{0\}$, there holds

$$\sum_{i=r+1}^n \phi(|x^* + h|_{[i]}) > \sum_{i=r+1}^n \phi(|x^*|_{[i]}).$$

Hence, x^* is a strictly local minimizer of problem (11). ■

Remark: It is not hard to see that $\phi(t) = |t|^q$ for some $q \in (0, 1]$ satisfies the assumption stated in Theorem 4.1.

As an immediate consequence, we obtain that under some suitable assumption, a sparsest solution x^* to the system $Ax = b$ is a strictly local minimizer of (11). One implication of this result is that an optimization method applied to (11) likely converges to x^* if its initial point is near x^* .

Corollary 4.2 (local sparse recovery) *Suppose that Assumption 1 and (27) hold for ϕ , and x^* is a sparsest solution to the system $Ax = b$. Assume that the local null space property $\text{LNSP}(r, \phi)$ holds at x^* for some $0 \leq r \leq K$, where K is defined in (22). Then x^* is a strictly local minimizer of problem (11) with such ϕ and r .*

The above local sparse recovery is established based on the local null space property $\text{LNSP}(r, \phi)$. We next show that the local sparse recovery holds under some RIP conditions when $\phi(t) = |t|^q$ for some $q \in (0, 1]$. Before proceeding, we state two useful lemma as follows. The proof of the first lemma is similar to that of [6, Theorem 3.1]. The second lemma can be proven similarly as [7, Theorem 1] and [35, Theorems 1 and 2]. Due to the paper length limitation, we omit the proof of them.

Lemma 4.3 *Let $k \in \{1, 2, \dots, n-1\}$ and $q \in (0, 1]$ be given. Suppose $\delta_k^A < 1/3$. Then there holds*

$$\|h_{-\max(k)}\|_q^q > \|h_{\max(k)}\|_q^q \quad \forall h \in \mathcal{N}(A) \setminus \{0\}.$$

Lemma 4.4 *Let $k \in \{1, 2, \dots, n-1\}$ and $q \in (0, 1]$ be given. Suppose $\delta_{\gamma k}^A < \frac{1}{\sqrt{(\gamma-1)^{1-2/q}+1}}$ for some $\gamma > 1$. Then there holds*

$$\|h_{-\max(k)}\|_q^q > \|h_{\max(k)}\|_q^q \quad \forall h \in \mathcal{N}(A) \setminus \{0\}.$$

We are now ready to establish a local sparse recovery result for (11) with $\phi(t) = |t|^q$ for $q \in (0, 1]$ under some suitable RIP conditions.

Theorem 4.5 *Let $q \in (0, 1]$ be given and K be defined in (22). Assume that $\delta_{K-\lfloor r/2 \rfloor}^A < \frac{1}{3}$ or $\delta_{\gamma(K-\lfloor r/2 \rfloor)}^A < \frac{1}{\sqrt{(\gamma-1)^{1-2/q}+1}}$ for some $\gamma > 1$. Suppose that x^* is a sparsest solution to the system $Ax = b$. Then x^* is a strictly local minimizer of problem (11) with $0 \leq r \leq K$ and $\phi(t) = |t|^q$.*

Proof. Since $\delta_{K-\lfloor r/2 \rfloor}^A < 1/3$ or $\delta_{\gamma(K-\lfloor r/2 \rfloor)}^A < \frac{1}{\sqrt{(\gamma-1)^{1-2/q}+1}}$ for some $\gamma > 1$, it follows from Lemmas 4.3 and 4.4 with $k = K - \lfloor r/2 \rfloor$ that

$$\|h_{-\max(K-\lfloor r/2 \rfloor)}\|_q^q > \|h_{\max(K-\lfloor r/2 \rfloor)}\|_q^q \quad \forall h \in \mathcal{N}(A) \setminus \{0\}. \quad (31)$$

In addition, we observe that

$$\begin{aligned} \|h_{-\max(K-\lfloor r/2 \rfloor)}\|_q^q &= \|h_{-\max(K)}\|_q^q + \sum_{i=K-\lfloor r/2 \rfloor+1}^K |h|_{[i]}^q, \\ \|h_{\max(K-\lfloor r/2 \rfloor)}\|_q^q &= \|h_{\max(K-r)}\|_q^q + \sum_{i=K-r+1}^{K-\lfloor r/2 \rfloor} |h|_{[i]}^q. \end{aligned}$$

By these relations, (31) and the fact $\sum_{i=K-r+1}^{K-\lfloor r/2 \rfloor} |h|_{[i]}^q \geq \sum_{i=K-\lfloor r/2 \rfloor+1}^K |h|_{[i]}^q$, we have that for all $h \in \mathcal{N}(A) \setminus \{0\}$,

$$\begin{aligned} \|h_{-\max(K)}\|_q^q - \|h_{\max(K-r)}\|_q^q &= \|h_{-\max(K-\lfloor r/2 \rfloor)}\|_q^q - \|h_{\max(K-\lfloor r/2 \rfloor)}\|_q^q \\ &\quad + \sum_{i=K-r+1}^{K-\lfloor r/2 \rfloor} |h|_{[i]}^q - \sum_{i=K-\lfloor r/2 \rfloor+1}^K |h|_{[i]}^q > 0. \end{aligned} \quad (32)$$

Let \mathcal{J} be defined in (26). Then for any $J \in \mathcal{J}$, one has $J \subseteq \mathcal{I}_{r+1}^-(x^*)$ and $|J| = \|x^*\|_0 - r - |\mathcal{I}_{r+1}^{<+}(x^*)|$. Using these and $\|x^*\|_0 = K$, we obtain that for all $h \in \mathfrak{R}^n$,

$$\begin{aligned} \sum_{i \in J \cup \mathcal{I}_{r+1}^{<+}(x^*)} |h_i|^q &\leq \|h_{\max(\|x^*\|_0-r)}\|_q^q = \|h_{\max(K-r)}\|_q^q, \\ \sum_{i \in \mathcal{I}_0(x^*)} |h_i|^q &\geq \|h_{-\max(\|x^*\|_0)}\|_q^q = \|h_{-\max(K)}\|_q^q. \end{aligned}$$

It then follows from these inequalities and (32) that for all $h \in \mathcal{N}(A) \setminus \{0\}$,

$$\sum_{i \in \mathcal{I}_0(x^*)} |h_i|^q - \max_{J \in \mathcal{J}} \sum_{i \in J \cup \mathcal{I}_{r+1}^{<+}(x^*)} |h_i|^q \geq \|h_{-\max(K)}\|_q^q - \|h_{\max(K-r)}\|_q^q > 0.$$

This implies that $\text{LNSP}(r, \phi)$ holds at x^* for $\phi(t) = |t|^q$. In addition, it is not hard to observe that such ϕ satisfies the assumptions stated in Theorem 4.1. The conclusion of this theorem then follows from Corollary 4.2. \blacksquare

Remark: Cai and Zhang [6, 7] recently established sparse recovery by model (11) with $r = 0$ and $\phi(\cdot) = |\cdot|$ under the condition $\delta_k^A < 1/3$ or $\delta_{\gamma k}^A < \frac{1}{\sqrt{(\gamma-1)^{-1}+1}}$. In addition, Song and Xia [35] established sparse recovery by model (11) with $r = 0$ and $\phi(\cdot) = |\cdot|^q$ for $q \in (0, 1]$ under the condition $\delta_{\gamma k}^A < \frac{1}{\sqrt{(\gamma-1)^{1-2/q}+1}}$ for some $\gamma > 1$. As seen from Theorem 4.5, our RIC bounds for local sparse recovery are weaker than their bounds when $0 < r \leq K$. Clearly, there exist some problems satisfying our RIP bounds but violating their bounds.

4.2 Global sparse recovery by (11)

In this subsection we study some sufficient conditions for global recovery of a sparsest solution x^* to the system $Ax = b$ by model (11). In particular, we show that if a global null space property holds, x^* is a global minimizer of (11). We also establish a global recovery result for (11) under some RIP conditions and the assumption $\phi(\cdot) = |\cdot|^q$ for $q \in (0, 1]$. To proceed, we first introduce the definition of global null space property.

Definition 3 (global null space property) *The global null space property $\text{GNSP}(r, \phi)$ holds for the system $Ax = b$ at a solution x^* with $\|x^*\|_0 \geq r$ if*

$$\sum_{i \in J_0} \phi(|h_i|) - \sum_{i \in J_1} \phi(|h_i|) > 0 \quad \forall h \in \mathcal{N}(A) \setminus \{0\} \quad (33)$$

holds for every $(J_0, J_1) \in \tilde{\mathcal{J}}$, where

$$\tilde{\mathcal{J}} = \{(J_0, J_1) : J_0 \subseteq \mathcal{I}_0(x^*), J_1 \subseteq \mathcal{I}_0^c(x^*), |J_0| + |J_1| = n - r\}. \quad (34)$$

We next show that for a class of ϕ , a solution x^* of $Ax = b$ is a unique global minimizer of (11) if the global null space property holds at x^* .

Theorem 4.6 *Suppose that Assumption 1 and (27) hold for ϕ . Let $x^* \in \mathbb{R}^n$ be such that $Ax^* = b$. Assume that $\text{GNSP}(r, \phi)$ holds at x^* for some $0 \leq r \leq \|x^*\|_0$. Then x^* is a unique global minimizer of problem (11) with such ϕ and r .*

Proof. One can observe that for every $h \in \mathbb{R}^n$, there exist some $J_0 \subseteq \mathcal{I}_0(x^*)$ and $J_1 \subseteq \mathcal{I}_0^c(x^*)$ (dependent on h) with $|J_0| + |J_1| = n - r$ such that

$$\sum_{i=r+1}^n \phi(|x^* + h|_{[i]}) = \sum_{i \in J_1} \phi(|x_i^* + h_i|) + \sum_{i \in J_0} \phi(|h_i|).$$

We also notice that (29) holds here due to (27) and the monotonicity of ϕ . It then follows from the above equality and (29) that

$$\sum_{i=r+1}^n \phi(|x^* + h|_{[i]}) \geq \sum_{i \in J_1} \phi(|x_i^*|) - \sum_{i \in J_1} \phi(|h_i|) + \sum_{i \in J_0} \phi(|h_i|). \quad (35)$$

Observe that $\phi(|x^*|_{[i]}) = 0$ for all $i > \|x^*\|_0$ due to $\phi(0) = 0$. Since $J_0 \subseteq \mathcal{I}_0(x^*)$, one has $|J_0| \leq n - \|x^*\|_0$, which together with $|J_0| + |J_1| = n - r$ implies $|J_1| \geq \|x^*\|_0 - r$. Using this, $J_1 \subseteq \mathcal{I}_0^c(x^*)$ and $\phi(|x^*|_{[i]}) = 0$ for all $i > \|x^*\|_0$, one can observe that

$$\sum_{i \in J_1} \phi(|x_i^*|) \geq \sum_{i=r+1}^{\|x^*\|_0} \phi(|x^*|_{[i]}) = \sum_{i=r+1}^n \phi(|x^*|_{[i]}). \quad (36)$$

In addition, we see that $(J_0, J_1) \in \tilde{\mathcal{J}}$, where $\tilde{\mathcal{J}}$ is given in (34). It thus follows from the global null space property that (33) holds for the above (J_0, J_1) . In view of (33), (35) and (36), we obtain that

$$\sum_{i=r+1}^n \phi(|x^* + h|_{[i]}) > \sum_{i=r+1}^n \phi(|x^*|_{[i]}) \quad \forall h \in \mathcal{N}(A) \setminus \{0\},$$

and hence x^* is a unique global minimizer of problem (11). ■

As an immediate consequence, we obtain that under some suitable assumption, a sparsest solution x^* to the system $Ax = b$ is a unique global minimizer of (11).

Corollary 4.7 (global sparse recovery) *Suppose that Assumption 1 and (27) hold for ϕ , and x^* is a sparsest solution to the system $Ax = b$. Assume that the global null space property $\text{GNSP}(r, \phi)$ holds at x^* for some $0 \leq r \leq K$, where K is defined in (22). Then x^* is a unique global minimizer of problem (11) with such ϕ and r .*

The above global sparse recovery is established based on the global null space property $\text{GNSP}(r, \phi)$. We next show that the global sparse recovery holds under some RIP conditions when $\phi(t) = |t|^q$ for some $q \in (0, 1]$.

Theorem 4.8 *Let $q \in (0, 1]$ be given and K be defined in (22). Assume that $\delta_{K+\lfloor r/2 \rfloor}^A < 1/3$ or $\delta_{\gamma(K+\lfloor r/2 \rfloor)}^A < \frac{1}{\sqrt{(\gamma-1)^{1-2/q}+1}}$ for some $\gamma > 1$. Suppose that x^* is a sparsest solution to the system $Ax = b$. Then x^* is a unique global minimizer of problem (11) with $0 \leq r \leq K$ and $\phi(t) = |t|^q$.*

Proof. Since $\delta_{K+\lfloor r/2 \rfloor}^A < 1/3$ or $\delta_{\gamma(K+\lfloor r/2 \rfloor)}^A < \frac{1}{\sqrt{(\gamma-1)^{1-2/q}+1}}$ for some $\gamma > 1$, it follows from Lemmas 4.3 and 4.4 with $k = K + \lfloor r/2 \rfloor$ that

$$\|h_{-\max(K+\lfloor r/2 \rfloor)}\|_q^q > \|h_{\max(K+\lfloor r/2 \rfloor)}\|_q^q \quad \forall h \in \mathcal{N}(A) \setminus \{0\}. \quad (37)$$

Observe that

$$\begin{aligned} \|h_{-\max(K+\lfloor r/2 \rfloor)}\|_q^q &= \|h_{-\max(K+r)}\|_q^q + \sum_{i=K+\lfloor r/2 \rfloor+1}^{K+r} |h|_{[i]}^q, \\ \|h_{\max(K+\lfloor r/2 \rfloor)}\|_q^q &= \|h_{\max(K)}\|_q^q + \sum_{i=K+1}^{K+\lfloor r/2 \rfloor} |h|_{[i]}^q. \end{aligned}$$

Using these relations, (37) and the fact $\sum_{i=K+1}^{K+\lfloor r/2 \rfloor} |h|_{[i]}^q \geq \sum_{i=K+\lfloor r/2 \rfloor+1}^{K+r} |h|_{[i]}^q$, we see that for all $h \in \mathcal{N}(A) \setminus \{0\}$,

$$\begin{aligned} \|h_{-\max(K+r)}\|_q^q - \|h_{\max(K)}\|_q^q &= \|h_{-\max(K+\lfloor r/2 \rfloor)}\|_q^q - \|h_{\max(K+\lfloor r/2 \rfloor)}\|_q^q \\ &\quad + \sum_{i=K+1}^{K+\lfloor r/2 \rfloor} |h|_{[i]}^q - \sum_{i=K+\lfloor r/2 \rfloor+1}^{K+r} |h|_{[i]}^q > 0. \end{aligned} \quad (38)$$

Let $\tilde{\mathcal{J}}$ be defined in (34). For any $(J_0, J_1) \in \tilde{\mathcal{J}}$, one has $J_0 \subseteq \mathcal{I}_0(x^*)$ and $J_1 \subseteq \mathcal{I}_0^c(x^*)$ with $|J_0| + |J_1| = n - r$. Since $J_1 \subseteq \mathcal{I}_0^c(x^*)$, one has $|J_1| \leq \|x^*\|_0$, which together with $|J_0| + |J_1| = n - r$ implies $|J_0| \geq n - \|x^*\|_0 - r$. Using these and $\|x^*\|_0 = K$, we obtain that for all $h \in \mathfrak{R}^n$,

$$\begin{aligned} \sum_{i \in J_0} |h_i|^q &\geq \|h_{-\max(\|x^*\|_0+r)}\|_q^q = \|h_{-\max(K+r)}\|_q^q, \\ \sum_{i \in J_1} |h_i|^q &\leq \|h_{\max(\|x^*\|_0)}\|_q^q = \|h_{\max(K)}\|_q^q. \end{aligned}$$

It then follows from these relations and (38) that for all $h \in \mathcal{N}(A) \setminus \{0\}$ and $(J_0, J_1) \in \tilde{\mathcal{J}}$,

$$\sum_{i \in J_0} |h_i|^q - \sum_{i \in J_1} |h_i|^q \geq \|h_{-\max(K+r)}\|_q^q - \|h_{\max(K)}\|_q^q > 0.$$

This implies that $\text{GNSP}(r, \phi)$ holds at x^* for $\phi(t) = |t|^q$. In addition, such ϕ satisfies the assumptions stated in Theorem 4.6. The conclusion of this theorem then follows from Corollary 4.7. \blacksquare

4.3 Stable recovery by (5)

In this subsection we study the stable recoverability of model (5) for a sparsest solution \tilde{x}^* to the noisy system $Ax = b + \xi$ with $\|\xi\| \leq \sigma$. In particular, we derive an error bound between \tilde{x}^* and a global minimizer of (5) with $\phi(\cdot) = |\cdot|$. To proceed, we establish a technical lemma as follows.

Lemma 4.9 Suppose that Assumption 1 and (27) hold for ϕ . Assume that x^* is an optimal solution of (5). Let $x \in \mathbb{R}^n$ be a feasible solution of problem (5) and $h = x - x^*$. Then there holds:

$$\sum_{i=k+\lceil r/2 \rceil+1}^n \phi(|h|_{[i]}) \leq \sum_{i=1}^{k+\lceil r/2 \rceil} \phi(|h|_{[i]}) + 2 \sum_{i=k+1}^n \phi(|x|_{[i]}) \quad (39)$$

for all $k \geq 1$, $r \geq 0$ and $k+r \leq n-1$.

Proof. Since x and x^* are a feasible solution and an optimal solution of (5), respectively, one has

$$\sum_{i=r+1}^n \phi(|x|_{[i]}) \geq \sum_{i=r+1}^n \phi(|x^*|_{[i]}). \quad (40)$$

Let J be the set of indices corresponding to the k largest entries of $|x|$. Clearly $|J| = k$. By the definitions of J and h , one can see that

$$\sum_{i=r+1}^n \phi(|x^*|_{[i]}) = \sum_{i=r+1}^n \phi(|x - h|_{[i]}) = \sum_{i \in J_1} \phi(|x_i - h_i|) + \sum_{i \in J_2} \phi(|x_i - h_i|), \quad (41)$$

for some $J_1 \subseteq J$, $J_2 \subseteq J^c$ with $|J_1| + |J_2| = n - r$. Due to (27), $\phi(0) = 0$ and the monotonicity of ϕ , one has

$$\phi(|x_i - h_i|) \geq \phi(|x_i| - |h_i|) \geq \max\{\phi(|x_i|) - \phi(|h_i|), \phi(|h_i|) - \phi(|x_i|)\} \quad \forall i.$$

Using this and (41), we have

$$\begin{aligned} \sum_{i=r+1}^n \phi(|x^*|_{[i]}) &\geq \sum_{i \in J_1} \phi(|x_i|) - \sum_{i \in J_1} \phi(|h_i|) + \sum_{i \in J_2} \phi(|h_i|) - \sum_{i \in J_2} \phi(|x_i|) \\ &= \sum_{i \in J_1 \cup J_2} \phi(|x_i|) + \sum_{i \in J_2} \phi(|h_i|) - \sum_{i \in J_1} \phi(|h_i|) - 2 \sum_{i \in J_2} \phi(|x_i|). \end{aligned} \quad (42)$$

Since $J_1 \subseteq J$, $|J_1| + |J_2| = n - r$ and $|J| = k$, one has $|J_1| \leq k$ and $|J_2| \geq n - k - r$. These together with the monotonicity of ϕ imply that

$$\sum_{i \in J_1} \phi(|h_i|) \leq \sum_{i=1}^k \phi(|h|_{[i]}), \quad \sum_{i \in J_2} \phi(|h_i|) \geq \sum_{i=k+r+1}^n \phi(|h|_{[i]}). \quad (43)$$

In addition, by $|J_1| + |J_2| = n - r$, $J_2 \subseteq J^c$, the definition of J , and the monotonicity of ϕ , one has

$$\sum_{i \in J_1 \cup J_2} \phi(|x_i|) \geq \sum_{i=r+1}^n \phi(|x|_{[i]}), \quad \sum_{i \in J_2} \phi(|x_i|) \leq \sum_{i \in J^c} \phi(|x_i|) = \sum_{i=k+1}^n \phi(|x|_{[i]}).$$

These relations, (42) and (43) yield

$$\sum_{i=r+1}^n \phi(|x^*|_{[i]}) \geq \sum_{i=r+1}^n \phi(|x|_{[i]}) + \sum_{i=k+r+1}^n \phi(|h|_{[i]}) - \sum_{i=1}^k \phi(|h|_{[i]}) - 2 \sum_{i=k+1}^n \phi(|x|_{[i]}),$$

which together with (40) implies

$$\sum_{i=k+r+1}^n \phi(|h|_{[i]}) \leq \sum_{i=1}^k \phi(|h|_{[i]}) + 2 \sum_{i=k+1}^n \phi(|x|_{[i]}).$$

Using this relation and the fact $\sum_{i=k+\lceil r/2 \rceil+1}^{k+r} \phi(|h|_{[i]}) \leq \sum_{i=k+1}^{k+\lceil r/2 \rceil} \phi(|h|_{[i]})$, we obtain that

$$\begin{aligned}
& \sum_{i=k+\lceil r/2 \rceil+1}^n \phi(|h|_{[i]}) - \sum_{i=1}^{k+\lceil r/2 \rceil} \phi(|h|_{[i]}) \\
&= \sum_{i=k+r+1}^n \phi(|h|_{[i]}) + \sum_{i=k+\lceil r/2 \rceil+1}^{k+r} \phi(|h|_{[i]}) - \sum_{i=1}^k \phi(|h|_{[i]}) - \sum_{i=k+1}^{k+\lceil r/2 \rceil} \phi(|h|_{[i]}) \\
&\leq \sum_{i=k+r+1}^n \phi(|h|_{[i]}) - \sum_{i=1}^k \phi(|h|_{[i]}) \leq 2 \sum_{i=k+1}^n \phi(|x|_{[i]}).
\end{aligned}$$

This implies that (39) holds as desired. \blacksquare

We are now ready to establish error bounds on the sparse recovery by (5) with $\phi(\cdot) = |\cdot|$ under some RIP conditions. It shall be mentioned that these results can be generalized to the case where $\phi(t) = |t|^q$ with $q \in (0, 1)$ by using (39) and the similar techniques used in the proof of [6, Theorem 3.3] and [35, Theorem 2].

Theorem 4.10 *Let $k \geq 2$ and $r \geq 0$ be given such that $k + r \leq n - 1$. Suppose that x^* is an optimal solution of problem (5) with such r , $\phi(t) = |t|$ and some $\sigma \geq 0$, and $\tilde{x}^* \in \mathbb{R}^n$ satisfies $\|A\tilde{x}^* - b\| \leq \sigma$. The following statements hold:*

(i) *If $\delta = \delta_{k+\lceil r/2 \rceil}^A < \frac{1}{3}$, then*

$$\|\tilde{x}^* - x^*\| \leq \frac{2\sqrt{2(1+\delta)}\sigma}{1-3\delta} + \frac{2\sqrt{2}(2\delta + \sqrt{(1-3\delta)\delta})}{1-3\delta} \frac{\|\tilde{x}_{-\max(k)}^*\|_1}{\sqrt{k + \lceil r/2 \rceil}};$$

(ii) *If $\delta = \delta_{\gamma(k+\lceil r/2 \rceil)}^A < \sqrt{(\gamma-1)/\gamma}$ for some $\gamma > 1$, then*

$$\|\tilde{x}^* - x^*\| \leq \frac{2\sqrt{2(1+\delta)}\sigma}{1 - \sqrt{\gamma/(\gamma-1)}\delta} + \left(\frac{\sqrt{2}\delta + \sqrt{\gamma(\sqrt{(\gamma-1)/\gamma} - \delta)\delta}}{\gamma(\sqrt{(\gamma-1)/\gamma} - \delta)} + 1 \right) \frac{2\|\tilde{x}_{-\max(k)}^*\|_1}{\sqrt{k + \lceil r/2 \rceil}}.$$

Proof. Let $h = \tilde{x}^* - x^*$. In view of (39) with $\phi(t) = |t|$ and $x = \tilde{x}^*$, we obtain that

$$\|h_{-\max(k+\lceil r/2 \rceil)}\|_1 \leq \|h_{\max(k+\lceil r/2 \rceil)}\|_1 + 2\|\tilde{x}_{-\max(k)}^*\|_1.$$

The conclusion of this theorem then follows from this relation and [6, Theorem 3.3] and [7, Theorem 2.1]. \blacksquare

Remark: If \tilde{x}^* is a k -sparse vector satisfying $\|A\tilde{x}^* - b\| \leq \sigma$, the second term in the above error bounds vanishes due to $\tilde{x}_{-\max(k)}^* = 0$.

5 Numerical algorithms

In this section we study numerical algorithms for solving models (5) and (11). In particular, we propose a feasible augmented Lagrangian (FAL) method for solving them, which solves a sequence of partially regularized unconstrained optimization problems in the form of

$$\min_{x \in \mathbb{R}^n} \left\{ F(x) := f(x) + \lambda \sum_{i=r+1}^n \phi(|x|_{[i]}) \right\}. \quad (44)$$

5.1 Partially regularized unconstrained optimization

In this subsection we propose a nonmonotone proximal gradient (NPG) method for solving problem (44), which arises as a subproblem in our subsequent FAL method, and establish its convergence. We also show that its associated proximal subproblems generally can be solved efficiently. Before proceeding, we make the following assumption on problem (44) throughout this subsection.

Assumption 4 (i) f is continuously differentiable in $\mathcal{U}(x^0; \Delta)$ for some $x^0 \in \mathbb{R}^n$ and $\Delta > 0$, and moreover, there exists some $L_f > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L_f \|x - y\|, \quad \forall x, y \in \mathcal{U}(x^0; \Delta),$$

where

$$\begin{aligned} \mathcal{U}(x^0, \Delta) &:= \{x : \|x - z\| \leq \Delta \text{ for some } z \in \mathcal{S}(x^0)\}, \\ \mathcal{S}(x^0) &:= \{x \in \mathbb{R}^n : F(x) \leq F(x^0)\}. \end{aligned}$$

(ii) F is bounded below and uniformly continuous in $\mathcal{S}(x^0)$.

(iii) The quantities A and B defined below are finite:

$$A := \sup_{x \in \mathcal{S}(x^0)} \|\nabla f(x)\|, \quad B := \sup_{x \in \mathcal{S}(x^0)} \sum_{i=r+1}^n \phi(|x|_{[i]}).$$

Following a general framework proposed by Wright et al. [40], we now present an NPG method for (44) as follows. The detailed explanation of this method can be found in [40].

Algorithm 1: A nonmonotone proximal gradient (NPG) method for (44)

Choose $0 < L_{\min} < L_{\max}$, $\tau > 1$, $c > 0$ and integer $N \geq 0$ arbitrarily. Set $k = 0$.

1) Choose $L_k^0 \in [L_{\min}, L_{\max}]$ arbitrarily. Set $L_k = L_k^0$.

1a) Solve the proximal subproblem

$$x^{k+1} \in \underset{x \in \mathbb{R}^n}{\text{Argmin}} \left\{ f(x^k) + \nabla f(x^k)^T (x - x^k) + \frac{L_k}{2} \|x - x^k\|^2 + \lambda \sum_{i=r+1}^n \phi(|x|_{[i]}) \right\}. \quad (45)$$

1b) If

$$F(x^{k+1}) \leq \max_{[k-N]^+ \leq i \leq k} F(x^i) - \frac{c}{2} \|x^{k+1} - x^k\|_2^2 \quad (46)$$

is satisfied, then go to Step 2).

1c) Set $L_k \leftarrow \tau L_k$ and go to (1a).

2) Set $\bar{L}_k = L_k$, $k \leftarrow k + 1$ and go to Step 1).

end

Remark:

- (i) When $N = 0$, $\{F(x^k)\}$ is decreasing. Otherwise, this sequence may increase at some iterations. Thus the above method is generally a nonmonotone method when $N > 0$.
- (ii) A practical choice of L_k^0 is by the following formula proposed by Barzilai and Borwein [2] (see also [3]):

$$L_k^0 = \max \left\{ L_{\min}, \min \left\{ L_{\max}, \frac{\langle s^k, y^k \rangle}{\|s^k\|^2} \right\} \right\}, \quad (47)$$

where $s^k = x^k - x^{k-1}$, $y^k = \nabla f(x^k) - \nabla f(x^{k-1})$.

- (iii) The main computational burden of Algorithm 1 lies in solving the proximal subproblem (45). As shown at the end of this subsection, (45) can be solved as $n - r$ number of one-dimensional problems in the form of

$$\nu(t) = \min_{u \in \mathfrak{R}} \left\{ \frac{1}{2}(u - t)^2 + \phi(|u|) \right\} \quad (48)$$

for some $t \in \mathfrak{R}$. Clearly the latter problem can be efficiently solved for some commonly used ϕ such as ℓ_0 , ℓ_1 , ℓ_q , Capped- ℓ_1 , Log, MCP and SCAD.

The following theorem states the number of inner iterations is uniformly bounded at each outer iteration of Algorithm 1 and thus this algorithm is well defined. Its proof follows from that of [13, Proposition A.1] with $C = \sup_{x \in S(x^0)} \sum_{i=r+1}^n \phi(|x|_{[i]}) \geq 0$.

Theorem 5.1 *Let $\{x^k\}$ and \bar{L}_k be generated in Algorithm 1, and let*

$$\bar{L} := \max\{L_{\max}, \tau L, \tau(L_f + c)\}, \quad \underline{L} := \frac{2(A\Delta + B)}{\Delta^2},$$

where A , B , L_f and Δ are given in Assumption 4. Under Assumption 4, the following statements hold:

- (i) For each $k \geq 0$, the inner termination criterion (46) is satisfied after at most

$$\left\lceil \frac{\log \bar{L} - \log L_{\min}}{\log \gamma} + 1 \right\rceil$$

inner iterations;

- (ii) $F(x^k) \leq F(x^0)$ and $\bar{L}_k \leq \bar{L}$ for all $k \geq 0$.

Before studying the global convergence of the NPG method for solving problem (44), we introduce two definitions as follows, which can be found in [36]. In addition, for convenience of presentation, let $\Phi(x) := \sum_{i=r+1}^n \phi(|x|_{[i]})$.

Definition 4 (first-order stationary point) $x^* \in \mathfrak{R}^n$ is a first-order stationary point of (44) if

$$0 \in \nabla f(x^*) + \lambda \partial \Phi(x^*).$$

Definition 5 (first-order ϵ -stationary point) Given any $\epsilon \geq 0$, $x^* \in \mathfrak{R}^n$ is a first-order ϵ -stationary point of (44) if

$$\text{dist}(0, \nabla f(x^*) + \lambda \partial \Phi(x^*)) \leq \epsilon.$$

We are now ready to establish the global convergence of the NPG method. In particular, we show that any accumulation point of $\{x^k\}$ is a first-order stationary point of problem (44).

Theorem 5.2 *Let the sequence $\{x^k\}$ be generated by Algorithm 1. There holds:*

- (i) $\|x^{k+1} - x^k\| \rightarrow 0$ as $k \rightarrow \infty$;
- (ii) Any accumulation point of $\{x^k\}$ is a first-order stationary point of (44);
- (iii) For any $\epsilon > 0$, x^k is a first-order ϵ -stationary point of problem (44) when k is sufficiently large.

Proof. (i) The proof of statement (i) is similar to that of [40, Lemma 4].

(ii) It follows from Theorem 5.1 that $\{\bar{L}_k\}$ is bounded. By the first-order optimality condition of (45), we have

$$0 \in \nabla f(x^k) + \bar{L}_k(x^{k+1} - x^k) + \lambda \partial \Phi(x^{k+1}), \quad (49)$$

where $\Phi(x) = \sum_{i=r+1}^n \phi(|x|_{[i]})$. Let x^* be an accumulation point of $\{x^k\}$. Then there exists a subsequence \mathcal{K} such that $\{x^k\}_{\mathcal{K}} \rightarrow x^*$, which together with $\|x^{k+1} - x^k\| \rightarrow 0$ implies that $\{x^{k+1}\}_{\mathcal{K}} \rightarrow x^*$. Using this, the boundedness of $\{\bar{L}_k\}$, the continuity of ∇f and the outer semi-continuity of $\partial \Phi(\cdot)$, and taking limits on both sides of (49) as $k \in \mathcal{K} \rightarrow \infty$, we obtain

$$0 \in \nabla f(x^*) + \lambda \partial \Phi(x^*).$$

Hence, x^* is a first-order stationary point of (44).

(iii) It follows from (49) with k replaced by $k-1$ that

$$\nabla f(x^k) - \nabla f(x^{k-1}) + \bar{L}_{k-1}(x^{k-1} - x^k) \in \nabla f(x^k) + \lambda \partial \Phi(x^k),$$

which yields

$$\text{dist}(0, \nabla f(x^k) + \lambda \partial \Phi(x^k)) \leq \|\nabla f(x^{k-1}) - \nabla f(x^k) + \bar{L}_k(x^k - x^{k-1})\|. \quad (50)$$

In addition, one can observe from Theorem 5.1 that $\{x^k\} \subseteq \mathcal{S}(x^0) = \{x : F(x) \leq F(x^0)\}$. By Assumption 4, we know that ∇f is uniformly continuous in $\mathcal{S}(x^0)$. It follows from these and Theorem 5.2 (i) that

$$\|\nabla f(x^{k-1}) - \nabla f(x^k) + \bar{L}_k(x^k - x^{k-1})\| \leq \epsilon \quad (51)$$

holds when k is sufficiently large. By (50) and (51), we see that for sufficiently large k , there holds

$$\text{dist}(0, \nabla f(x^k) + \lambda \partial \Phi(x^k)) \leq \epsilon, \quad (52)$$

and hence x^k is a first-order ϵ -stationary point of problem (44). ■

Remark: Recall that $\Phi(x) = \sum_{i=r+1}^n \phi(|x|_{[i]})$. Due to the complication of this function, it is generally hard to evaluate $\partial \Phi$ and hence verify (52) directly. Nevertheless, one can see from the above proof that (51) implies (52). Therefore, (51) can be used as a verifiable termination criterion for Algorithm 1 to generate a first-order ϵ -stationary point of (44).

In the remainder of this subsection, we discuss how to efficiently solve the proximal subproblem (45) of Algorithm 1. In particular, we show that (45) can be solved as $n - r$ number of one-dimensional problems in the form of (48). To proceed, we first study some properties of the function $\nu(t)$ that is defined in (48). The following lemma shows that ν is locally Lipschitz everywhere, and it also characterizes the Clarke subdifferential of ν .

Lemma 5.3 *Let ν be defined in (48). Suppose that ϕ satisfies Assumption 1. Then the following properties hold:*

- (i) ν is well-defined in \mathbb{R} ;

(ii) ν is locally Lipschitz in \mathfrak{R} ;

(iii) $\partial^c \nu(t) = \text{conv}(\{t - u : u \in \mathcal{M}(t)\})$, where $\partial^c \nu(t)$ denotes the Clarke subdifferential of ν at t , and $\mathcal{M}(t)$ is the set of the optimal solutions of (48) at t , that is,

$$\mathcal{M}(t) = \left\{ u \in \mathfrak{R} : \frac{1}{2}(u - t)^2 + \phi(|u|) = \nu(t) \right\}. \quad (53)$$

Proof. (i) The well-definedness of ν follows from Theorem 2.1.

(ii) Let $t_1 < t_2$ be arbitrarily chosen. For any $t \in [t_1, t_2]$ and $u^* \in \mathcal{M}(t)$, it follows from (48) and $\phi(0) = 0$ that

$$\frac{1}{2}(u^* - t)^2 + \phi(|u^*|) = \nu(t) \leq \frac{1}{2}(0 - t)^2 + \phi(0) = \frac{1}{2}t^2,$$

with together with $\phi(|u^*|) \geq 0$ implies $|u^* - t| \leq |t|$. By this and $t \in [t_1, t_2]$, one has

$$-2 \max\{|t_1|, |t_2|\} \leq t - |t| \leq u^* \leq t + |t| \leq 2 \max\{|t_1|, |t_2|\}. \quad (54)$$

Hence, we obtain that for any $t \in [t_1, t_2]$,

$$|u^*| \leq 2 \max\{|t_1|, |t_2|\} \quad \forall u^* \in \mathcal{M}(t). \quad (55)$$

For any $\tilde{t}_1, \tilde{t}_2 \in [t_1, t_2]$, let $u_1^* \in \mathcal{M}(\tilde{t}_1)$ and $u_2^* \in \mathcal{M}(\tilde{t}_2)$. In view of (48) and (55), we can see that

$$\begin{aligned} \nu(\tilde{t}_1) &\leq \frac{1}{2}(u_2^* - \tilde{t}_1)^2 + \phi(|u_2^*|) = \frac{1}{2}(u_2^* - \tilde{t}_2)^2 + \phi(|u_2^*|) + (\tilde{t}_2 - \tilde{t}_1) \left(u_2^* - \frac{\tilde{t}_1 + \tilde{t}_2}{2} \right) \\ &\leq \nu(\tilde{t}_2) + |\tilde{t}_1 - \tilde{t}_2| \left(|u_2^*| + \frac{1}{2}|\tilde{t}_1| + \frac{1}{2}|\tilde{t}_2| \right) \leq \nu(\tilde{t}_2) + 3 \max\{|t_1|, |t_2|\} |\tilde{t}_1 - \tilde{t}_2|. \end{aligned}$$

Analogously, one can show that

$$\nu(\tilde{t}_2) \leq \nu(\tilde{t}_1) + 3 \max\{|t_1|, |t_2|\} |\tilde{t}_1 - \tilde{t}_2|.$$

It then follows that

$$|\nu(\tilde{t}_1) - \nu(\tilde{t}_2)| \leq 3 \max\{|t_1|, |t_2|\} |\tilde{t}_1 - \tilde{t}_2| \quad \forall \tilde{t}_1, \tilde{t}_2 \in [t_1, t_2],$$

and hence statement (ii) holds.

(iii) Since ν is locally Lipschitz everywhere, it is differentiable almost everywhere, and moreover, $\partial^c \nu(t) = \text{conv}(S(t))$ for all $t \in \mathfrak{R}$, where

$$S(t) = \{\xi : \exists t_k \rightarrow t \text{ such that } \nu'(t_k) \rightarrow \xi\} \quad \forall t \in \mathfrak{R},$$

and $\nu'(t_k)$ denotes the standard derivative of ν at t_k .

Given any $t \in \mathfrak{R}$, $u^* \in \mathcal{M}(t)$ and $h \in \mathfrak{R}$, we have

$$\begin{aligned} (t - u^*)h &= \lim_{\delta \downarrow 0} \frac{1}{\delta} \left[\frac{1}{2} \|t + \delta h - u^*\|^2 - \frac{1}{2} \|t - u^*\|^2 \right] \\ &= \lim_{\delta \downarrow 0} \frac{1}{\delta} \left[\left(\frac{1}{2} \|t + \delta h - u^*\|^2 + \phi(|u^*|) \right) - \left(\frac{1}{2} \|t - u^*\|^2 + \phi(|u^*|) \right) \right] \\ &\geq \liminf_{\delta \downarrow 0} \frac{\nu(t + \delta h) - \nu(t)}{\delta}, \end{aligned}$$

where the last inequality follows from $u^* \in \mathcal{M}(t)$ and the definition of ν . Using this inequality and [14, Corollary 1.10], we obtain that $t - u^* \in \partial^c \nu(t)$. Due to the arbitrariness of $u^* \in \mathcal{M}(t)$ and the convexity of $\partial^c \nu(t)$, one further has

$$\text{conv}(\{t - u : u \in \mathcal{M}(t)\}) \subseteq \partial^c \nu(t). \quad (56)$$

It remains to show $\partial^c \nu(t) \subseteq \text{conv}(\{t - u : u \in \mathcal{M}(t)\})$. Indeed, let $\xi \in S(t)$ be arbitrarily chosen. Then there exists some sequence $\{t_k\}$ such that $t_k \rightarrow t$ and $\nu'(t_k) \rightarrow \xi$. Choose $u_k \in \mathcal{M}(t_k)$ arbitrarily and let \underline{t}, \bar{t} be such that $\{t_k\} \subseteq [\underline{t}, \bar{t}]$. From (54), we know that $|u_k| \leq 2 \max\{|\underline{t}|, |\bar{t}|\}$ and hence $\{u_k\}$ is bounded. Considering a convergent subsequence if necessary, assume without loss of generality that $u_k \rightarrow u^*$ for some $u^* \in \mathfrak{R}$. It follows from $u_k \in \mathcal{M}(t_k)$ and the definition of $\mathcal{M}(\cdot)$ that

$$\frac{1}{2}(u - t_k)^2 + \phi(|u|) \geq \frac{1}{2}(u_k - t_k)^2 + \phi(|u_k|) \quad \forall u \in \mathfrak{R}.$$

Taking \liminf on both sides of this relation as $k \rightarrow \infty$ and using the lower semi-continuity of ϕ , we have

$$\frac{1}{2}(u - t)^2 + \phi(|u|) \geq \liminf_{k \rightarrow \infty} \left(\frac{1}{2}(u_k - t_k)^2 + \phi(|u_k|) \right) \geq \frac{1}{2}(u^* - t)^2 + \phi(|u^*|) \quad \forall u \in \mathfrak{R}.$$

This yields $u^* \in \mathcal{M}(t)$. In addition, since ν is differentiable at t_k , one has $\partial^c \nu(t_k) = \{\nu'(t_k)\}$, which together with $u_k \in \mathcal{M}(t_k)$ and (56) implies $\nu'(t_k) = t_k - u_k$ for all k . In view of this, $t_k \rightarrow t$, $u_k \rightarrow u^*$ and $\nu'(t_k) \rightarrow \xi$, one has

$$\xi = \lim_{k \rightarrow \infty} \nu'(t_k) = \lim_{k \rightarrow \infty} (t_k - u_k) = t - u^*.$$

It follows from this, $u^* \in \mathcal{M}(t)$ and the arbitrariness of $\xi \in S(t)$ that $S(t) \subseteq \{t - u : u \in \mathcal{M}(t)\}$, which along with the fact $\partial^c \nu(t) = \text{conv}(S(t))$ implies that $\partial^c \nu(t) \subseteq \text{conv}(\{t - u : u \in \mathcal{M}(t)\})$. The conclusion of statement (iii) immediately follows from this relation and (56). \blacksquare

We next show that ν is increasing with respect to $|t|$.

Lemma 5.4 *Let ν be defined in (48). Suppose that ϕ satisfies Assumption 1. Then $\nu(t_1) \geq \nu(t_2)$ for any t_1, t_2 such that $|t_1| \geq |t_2|$.*

Proof. Observe that $\nu(-t) = \nu(t)$ for all $t \in \mathfrak{R}$. To prove this lemma, it thus suffices to show that $\nu(t)$ is increasing on $[0, \infty)$. We first claim that $\mathcal{M}(t) \subseteq (-\infty, t]$ for all $t \geq 0$, where $\mathcal{M}(t)$ is defined in (53). Suppose for contradiction that there exists some $t \geq 0$ and $u^* \in \mathcal{M}(t)$ such that $u^* > t$. Let $u = (u^* + t)/2$. Clearly $0 \leq t < u < u^*$. By this and the monotonicity of ϕ on $[0, \infty)$, one can see that

$$\frac{1}{2}(u - t)^2 + \phi(|u|) < \frac{1}{2}(u^* - t)^2 + \phi(|u^*|),$$

which contradicts $u^* \in \mathcal{M}(t)$. Hence, $\mathcal{M}(t) \subseteq (-\infty, t]$ for all $t \geq 0$. This together with Lemma 5.3 (iii) implies that $\partial^c \nu(t) \in [0, \infty)$ for all $t \geq 0$. Further, by the mean value theorem [15, Theorem 2.3.7], one has

$$\nu(t_1) - \nu(t_2) \in \partial^c \nu(\tilde{t})(t_1 - t_2) \quad \forall t_1, t_2 \geq 0,$$

where \tilde{t} is some point between t_1 and t_2 . Hence, $\nu(t)$ is increasing in $[0, \infty)$ and the conclusion of this lemma holds. \blacksquare

We are now ready to discuss how to solve efficiently the subproblem (45) of Algorithm 1. Clearly, (45) is equivalent to

$$\min_{x \in \mathfrak{R}^n} \left\{ \frac{1}{2} \left\| x - \left(x^k - \frac{1}{L_k} \nabla f(x^k) \right) \right\|_2^2 + \frac{\lambda}{L_k} \sum_{i=r+1}^n \phi(|x|_{[i]}) \right\},$$

which is a special case of a more general problem

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|x - a\|_2^2 + \tilde{\lambda} \sum_{i=r+1}^n \phi(|x|_{[i]}) \right\}, \quad (57)$$

for some $a \in \mathbb{R}^n$ and $\tilde{\lambda} > 0$. In what follows, we show that problem (57) can be solved as $n - r$ number of one-dimensional problems in the form of (48).

Theorem 5.5 *Suppose that ϕ satisfies Assumption 1. Let \mathcal{I}^* be the index set corresponding to the $n - r$ smallest entries of $|a|$ and $x^* \in \mathbb{R}^n$ be defined as follows:*

$$x_i^* \in \begin{cases} \underset{u \in \mathbb{R}}{\text{Argmin}} \left\{ \frac{1}{2} (u - a_i)^2 + \tilde{\lambda} \phi(|u|) \right\} & \text{if } i \in \mathcal{I}^*, \\ \{a_i\} & \text{otherwise} \end{cases} \quad i = 1, \dots, n.$$

Then x^* is an optimal solution of problem (57).

Proof. Let ν be defined in (48). In view of the monotonicity of ϕ and the definition of ν , one can observe that

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - a\|^2 + \tilde{\lambda} \sum_{i=r+1}^n \phi(|x|_{[i]}) &= \min_{x \in \mathbb{R}^n} \left\{ \min_{\substack{s \in \{0,1\}^n \\ \|s\|_0 = n-r}} \frac{1}{2} \|x - a\|^2 + \tilde{\lambda} \sum_{i=1}^n s_i \phi(|x_i|) \right\} \\ &= \min_{\substack{s \in \{0,1\}^n \\ \|s\|_0 = n-r}} \left\{ \min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - a\|^2 + \tilde{\lambda} \sum_{i=1}^n s_i \phi(|x_i|) \right\} = \min_{\substack{I \subseteq \{1, \dots, n\} \\ |I| = n-r}} \sum_{i \in I} \nu(a_i) = \sum_{i \in \mathcal{I}^*} \nu(a_i), \end{aligned}$$

where the last equality follows from Lemma 5.4 and the definition of \mathcal{I}^* . In addition, by the definition of x^* , one can verify that

$$\frac{1}{2} \|x^* - a\|^2 + \tilde{\lambda} \sum_{i=r+1}^n \phi(|x^*|_{[i]}) = \sum_{i \in \mathcal{I}^*} \nu(a_i),$$

and hence the conclusion holds. ■

5.2 Sparse recovery for a noiseless linear system

In this subsection we propose a feasible augmented Lagrangian (FAL) method for solving the partially regularized sparse recovery model for a noiseless linear system, namely, problem (11). For convenience of presentation, let $\Phi(x) := \sum_{i=r+1}^n \phi(|x|_{[i]})$ throughout this subsection.

For any given $\varrho > 0$ and $\mu \in \mathbb{R}^m$, the augmented Lagrangian (AL) function for problem (11) is defined as

$$\mathcal{L}(x; \mu, \varrho) := w(x; \mu, \varrho) + \Phi(x), \quad (58)$$

where

$$w(x; \mu, \varrho) := \mu^T (Ax - b) + \frac{\varrho}{2} \|Ax - b\|^2. \quad (59)$$

Similar to the classical AL method, our FAL method solves a sequence of AL subproblems in the form of

$$\min_x \mathcal{L}(x; \mu, \varrho) \quad (60)$$

while updating the Lagrangian multiplier μ and the penalty parameter ϱ . It is easy to observe that problem (60) is a special case of (44) and satisfies Assumption 4. Therefore, it can be suitably solved by Algorithm 1.

It is well known that the classical AL method may converge to an infeasible point. To remedy this pathological behavior, we adopt the same strategies proposed by Lu and Zhang [31] so that the values of the AL function along the solution sequence generated by the FAL method are bounded from above, and the magnitude of penalty parameters outgrows that of Lagrangian multipliers.

Throughout this section, let x^{feas} be an arbitrary feasible point of problem (11). We are now ready to present the FAL method for solving (11).

Algorithm 2: A feasible augmented Lagrangian method for (11)

Choose $\mu^0 \in \mathbb{R}^m$, $x^0 \in \mathbb{R}^n$, $\varrho_0 > 0$, $\gamma \in (1, \infty)$, $\theta > 0$, $\eta \in (0, 1)$, a decreasing positive sequence $\{\epsilon_k\}$ with $\lim_{k \rightarrow \infty} \epsilon_k = 0$, and a constant $\Upsilon \geq \max\{\Phi(x^{\text{feas}}), \mathcal{L}(x^0, \mu^0, \varrho_0)\}$. Set $k = 0$.

- 1) Apply Algorithm 1 to find an approximate stationary point x^{k+1} for problem (60) with $\mu = \mu^k$ and $\varrho = \varrho_k$ such that

$$\text{dist}(0, \nabla w(x^{k+1}; \mu^k, \varrho_k) + \partial \Phi(x^{k+1})) \leq \epsilon_k, \quad \mathcal{L}(x^{k+1}, \mu^k, \varrho_k) \leq \Upsilon. \quad (61)$$

- 2) Set $\mu^{k+1} = \mu^k + \varrho_k(Ax^{k+1} - b)$.

- 3) If $\|Ax^{k+1} - b\| \leq \eta\|Ax^k - b\|$, then set $\varrho_{k+1} = \varrho_k$. Otherwise, set

$$\varrho_{k+1} = \max\{\gamma\varrho_k, \|\mu^{k+1}\|^{1+\theta}\}.$$

- 4) Set $k \leftarrow k + 1$ and go to Step 1).

end

We now discuss how to find by Algorithm 1 an approximate stationary point x^{k+1} of problem (60) with $\mu = \mu^k$ and $\varrho = \varrho_k$ such that (61) holds. Given $k \geq 0$, let x_{init}^k denote the initial point of Algorithm 1 when applied to solve problem (60) with $\mu = \mu^k$ and $\varrho = \varrho_k$, which is specified as follows:

$$x_{\text{init}}^k = \begin{cases} x^{\text{feas}} & \text{if } \mathcal{L}(x^k, \mu^k, \varrho_k) > \Upsilon, \\ x^k & \text{otherwise,} \end{cases}$$

where x^k is the approximate stationary point of problem (60) with $\mu = \mu^{k-1}$ and $\varrho = \varrho_{k-1}$ obtained at the $(k-1)$ th outer iteration of Algorithm 2. Notice that $Ax^{\text{feas}} = b$. It follows from (58) and the definition of Υ that

$$\mathcal{L}(x^{\text{feas}}; \mu^k, \varrho_k) = \Phi(x^{\text{feas}}) \leq \Upsilon.$$

This inequality and the above choice of x_{init}^k imply $\mathcal{L}(x_{\text{init}}^k, \mu^k, \varrho_k) \leq \Upsilon$. In addition, by Theorem 5.1, we know that all subsequent objective values of Algorithm 1 are bounded above the initial objective value. It follows that

$$\mathcal{L}(x^{k+1}; \mu^k, \varrho_k) \leq \mathcal{L}(x_{\text{init}}^k, \mu^k, \varrho_k) \leq \Upsilon,$$

and hence the second relation of (61) is satisfied at x^{k+1} . Finally, as seen from Theorem 5.2 (iii), Algorithm 1 is capable of finding an approximate solution x^{k+1} satisfying the first relation of (61).

We next establish a global convergence result for the above FAL method.

Theorem 5.6 *Let $\{x^k\}$ be the sequence generated by Algorithm 2. Suppose that x^* is an accumulation point of $\{x^k\}$. Then the following statements hold:*

- (i) x^* is a feasible point of problem (11);

(ii) Suppose additionally that $\partial^\infty \Phi(x^*) \cap \mathcal{R}(A^T) = \{0\}$. Then x^* is a first-order stationary point of problem (11), that is, $Ax^* = b$ and $0 \in A^T \mu^* + \partial \Phi(x^*)$ for some $\mu^* \in \mathbb{R}^m$.

Proof. Suppose that x^* is an accumulation point of $\{x^k\}$. Considering a convergent subsequence if necessary, assume without loss of generality that $x^k \rightarrow x^*$ as $k \rightarrow \infty$.

(i) We show that x^* is a feasible point of problem (11), that is, $Ax^* = b$, by considering two separate cases as follows.

Case (a): $\{\varrho_k\}$ is bounded. This together with Step 3) of Algorithm 2 implies that $\|Ax^{k+1} - b\| \leq \eta \|Ax^k - b\|$ holds for all sufficiently large k . It then follows from this and $x^k \rightarrow x^*$ that $Ax^* = b$.

Case (b): $\{\varrho_k\}$ is unbounded. Considering a subsequence if necessary, assume for convenience that $\{\varrho_k\}$ strictly increases and $\varrho_k \rightarrow \infty$ as $k \rightarrow \infty$. This along with Step 3) of Algorithm 2 implies that $\varrho_{k+1} = \max\{\gamma \varrho_k, \|\mu^{k+1}\|^{1+\theta}\}$ holds for all k . It then follows from this and $\varrho_k \rightarrow \infty$ that

$$0 \leq \varrho_k^{-1} \|\mu^k\| \leq \varrho_k^{-\theta/(1+\theta)} \rightarrow 0. \quad (62)$$

In view of (58) and the second relation in (61), one has

$$(\mu^k)^T (Ax^{k+1} - b) + \frac{\varrho_k}{2} \|Ax^{k+1} - b\|^2 + \Phi(x^{k+1}) \leq \Upsilon.$$

It then follows that

$$\|Ax^{k+1} - b\|^2 \leq 2\varrho_k^{-1} (\Upsilon - \Phi(x^{k+1})) + 2\varrho_k^{-1} \|\mu^k\| \|Ax^{k+1} - b\|.$$

Taking limits on both sides of this relation as $k \rightarrow \infty$, and using (62), $x^k \rightarrow x^*$, $\varrho_k \rightarrow \infty$ and the lower boundedness of Φ , we obtain $Ax^* = b$ again.

(ii) We show that x^* is a first-order stationary point of problem (11). It follows from (59), the first relation in (61) and Step 2) of Algorithm 2 that there exist some $\xi^k \in \mathbb{R}^n$ and $v^k \in \partial \Phi(x^{k+1})$ such that $\|\xi^k\| \leq \epsilon_k$ and

$$\xi^k = A^T \mu^{k+1} + v^k. \quad (63)$$

Since $\|\xi^k\| \leq \epsilon_k$ and $\epsilon_k \rightarrow 0$, we have $\|\xi^k\| \rightarrow 0$. Claim that $\{v^k\}$ is bounded. Suppose for contradiction that $\{v^k\}$ is unbounded. Considering a subsequence if necessary, assume for convenience that $\|v^k\| \rightarrow \infty$ and $v^k/\|v^k\| \rightarrow v$ for some $v \in \mathbb{R}^n$. Clearly $v \neq 0$. By $x^k \rightarrow x^*$, $v^k \in \partial \Phi(x^{k+1})$, $\|v^k\| \rightarrow \infty$ and $v^k/\|v^k\| \rightarrow v$, one can see that $v \in \partial^\infty \Phi(x^*)$. Dividing both sides of (63) by $\|v^k\|$ and taking limits as $k \rightarrow \infty$, we obtain $0 \neq v \in \partial^\infty \Phi(x^*) \cap \mathcal{R}(A^T)$, which contradicts the assumption $\partial^\infty \Phi(x^*) \cap \mathcal{R}(A^T) = \{0\}$. Thus $\{v^k\}$ is bounded. Since (63) is a linear system with respect to μ^{k+1} , it follows from Hoffman lemma [27] that there exist $\bar{\mu}^{k+1} \in \mathbb{R}^m$ and a constant $\theta > 0$ (independent of k) such that

$$A^T \bar{\mu}^{k+1} = \xi^k - v^k, \quad \|\bar{\mu}^{k+1}\| \leq \theta \|\xi^k - v^k\|. \quad (64)$$

Using the second relation of (64) and the boundedness of $\{v^k\}$ and $\{\xi^k\}$, one can see that $\{\bar{\mu}^{k+1}\}$ is bounded. Considering a subsequence if necessary, assume for convenience that $\bar{\mu}^{k+1} \rightarrow \mu^*$ and $v^k \rightarrow v^*$ for some $\mu^* \in \mathbb{R}^m$ and $v^* \in \mathbb{R}^n$. By $x^k \rightarrow x^*$, $v^k \in \partial \Phi(x^{k+1})$ and the outer semi-continuity of $\partial \Phi$, one has $v^* \in \partial \Phi(x^*)$. Using $\xi^k \rightarrow 0$, $\bar{\mu}^{k+1} \rightarrow \mu^*$, $v^k \rightarrow v^* \in \partial \Phi(x^*)$ and taking limits on both sides of the first relation of (64) as $k \rightarrow \infty$, we obtain $0 \in A^T \mu^* + \partial \Phi(x^*)$. This along with $Ax^* = b$ implies that x^* is a first-order stationary point of (11). ■

5.3 Sparse recovery for a noisy linear system

In this subsection we propose a FAL method for solving the partially regularized sparse recovery model for a noisy linear system, namely, problem (5). For convenience of presentation, let $\Phi(x) := \sum_{i=r+1}^n \phi(|x|_{[i]})$ throughout this subsection.

For any given $\varrho > 0$ and $\mu \geq 0$, the AL function for problem (5) is defined as

$$\tilde{\mathcal{L}}(x; \mu, \varrho) := \tilde{w}(x; \mu, \varrho) + \Phi(x), \quad (65)$$

where

$$\tilde{w}(x; \mu, \varrho) := \frac{1}{2\varrho} ([\mu + \varrho(\|Ax - b\|^2 - \sigma^2)]_+^2 - \mu^2). \quad (66)$$

Similar to the classical AL method, our FAL method solves a sequence of AL subproblems in the form of

$$\min_x \tilde{\mathcal{L}}(x; \mu, \varrho). \quad (67)$$

By a similar argument as in [13], one can verify that for any given $\mu \geq 0$ and $\rho \geq 0$, problem (67) is a special case of (44) and satisfies Assumption 4. Therefore, the NPG method presented in Algorithm 1 can be suitably applied to solve (67).

For a similar reason as mentioned in Subsection 5.2, we adopt the same strategies proposed in [31] so that the feasibility of any accumulation point of the solution sequence generated by our method is guaranteed. Throughout this section, let x^{feas} be an arbitrary feasible point of problem (5). We are now ready to present the FAL method for solving (5).

Algorithm 3: A feasible augmented Lagrangian method for (5)

Choose $\mu_0 \geq 0$, $x^0 \in \mathbb{R}^n$, $\varrho_0 > 0$, $\gamma \in (1, \infty)$, $\theta > 0$, $\eta \in (0, 1)$, a decreasing positive sequence $\{\epsilon_k\}$ with $\lim_{k \rightarrow \infty} \epsilon_k = 0$, and a constant $\Upsilon \geq \max\{\Phi(x^{\text{feas}}), \tilde{\mathcal{L}}(x^0, \mu_0, \varrho_0)\}$. Set $k = 0$.

- 1) Apply Algorithm 1 to find an approximate stationary point x^{k+1} for problem (67) with $\mu = \mu_k$ and $\varrho = \varrho_k$ such that

$$\text{dist}(0, \nabla \tilde{w}(x^{k+1}; \mu_k, \varrho_k) + \partial \Phi(x^{k+1})) \leq \epsilon_k, \quad \tilde{\mathcal{L}}(x^{k+1}, \mu_k, \varrho_k) \leq \Upsilon. \quad (68)$$

- 2) Set $\mu_{k+1} = [\mu_k + \varrho_k(\|Ax^{k+1} - b\|^2 - \sigma^2)]_+$.

- 3) If $[\|Ax^{k+1} - b\| - \sigma]_+ \leq \eta[\|Ax^k - b\| - \sigma]_+$, then set $\varrho_{k+1} = \varrho_k$. Otherwise, set

$$\varrho_{k+1} = \max\{\gamma \varrho_k, \mu_{k+1}^{1+\theta}\}.$$

- 4) Set $k \leftarrow k + 1$ and go to Step 1).

end

By a similar argument as in Subsection 5.2, one can observe that x^{k+1} satisfying (68) can be found by Algorithm 1. We next establish a global convergence result for the above FAL method.

Theorem 5.7 *Let $\{x^k\}$ be a sequence generated by Algorithm 3. Suppose that x^* is an accumulation point of $\{x^k\}$. Then the following statements hold:*

- (i) x^* is a feasible point of problem (5);
- (ii) Suppose additionally that $\partial^\infty \Phi(x^*) \cap \mathcal{C} = \{0\}$ and $A^T(Ax^* - b) \neq 0$ if $\|Ax^* - b\| = \sigma$, where

$$\mathcal{C} := \{tA^T(b - Ax^*) : t \geq 0\}.$$

Then x^ is a first-order stationary point of problem (5), that is, there exists $\mu^* \geq 0$ such that*

$$\begin{aligned} 0 &\in 2\mu^* A^T(Ax^* - b) + \partial \Phi(x^*), \\ \|Ax^* - b\| &\leq \sigma, \quad \mu^*(\|Ax^* - b\| - \sigma) = 0. \end{aligned} \quad (69)$$

Proof. Suppose that x^* is an accumulation point of $\{x^k\}$. Considering a convergent subsequence if necessary, assume for convenience that $x^k \rightarrow x^*$ as $k \rightarrow \infty$.

(i) We show that x^* is a feasible point of problem (11), that is, $\|Ax^* - b\| \leq \sigma$, by considering two separate cases as follows.

Case (a): $\{\varrho_k\}$ is bounded. This together with Step 3) of Algorithm 3 implies that

$$[\|Ax^{k+1} - b\| - \sigma]_+ \leq \eta[\|Ax^k - b\| - \sigma]_+$$

holds for all sufficiently large k . It then follows from this and $x^k \rightarrow x^*$ that $[\|Ax^* - b\| - \sigma]_+ = 0$ and hence $\|Ax^* - b\| \leq \sigma$.

Case (b): $\{\varrho_k\}$ is unbounded. Considering a subsequence if necessary, assume for convenience that $\{\varrho_k\}$ strictly increases and $\varrho_k \rightarrow \infty$ as $k \rightarrow \infty$. This along with Step 3) of Algorithm 2 implies that $\varrho_{k+1} = \max\{\gamma\varrho_k, \|\mu^{k+1}\|^{1+\theta}\}$ holds for all k . It then follows from this and $\rho_k \rightarrow \infty$ that

$$0 \leq \varrho_k^{-1}\mu_k \leq \varrho_k^{-\theta/(1+\theta)} \rightarrow 0. \quad (70)$$

In view of (65) and the second relation in (68), one has

$$\frac{1}{2\varrho_k} \left([\mu_k + \varrho_k (\|Ax^{k+1} - b\|^2 - \sigma^2)]_+^2 - \mu_k^2 \right) + \Phi(x^{k+1}) \leq \Upsilon.$$

It follows that

$$(\varrho_k^{-1}\mu_k + \|Ax^{k+1} - b\|^2 - \sigma^2)_+^2 \leq 2\varrho_k^{-1}[\Upsilon - \Phi(x^{k+1})] + \varrho_k^{-2}\mu_k^2.$$

Taking limits on both sides of this relation as $k \rightarrow \infty$, and using (70), $x^k \rightarrow x^*$, $\varrho_k \rightarrow \infty$ and the lower boundedness of Φ , we obtain $(\|Ax^* - b\|^2 - \sigma^2)_+ = 0$, which clearly yields $\|Ax^* - b\| \leq \sigma$.

(ii) We show that x^* is a first-order stationary point of problem (5). It follows from (66), the first relation in (68) and Step 2) of Algorithm 3 that there exist some $\xi^k \in \mathbb{R}^n$ and $v^k \in \partial\Phi(x^{k+1})$ such that $\|\xi^k\| \leq \epsilon_k$ and

$$\xi^k = 2\mu_{k+1}A^T(Ax^{k+1} - b) + v^k. \quad (71)$$

since $\|\xi^k\| \leq \epsilon_k$ and $\epsilon_k \rightarrow 0$, we have $\|\xi^k\| \rightarrow 0$. We next consider two separate cases to complete the rest of the proof.

Case (a): $\|Ax^* - b\| < \sigma$. This together with $x^k \rightarrow x^*$ implies that $\|Ax^k - b\| \leq \hat{\sigma}$ for some $\hat{\sigma} < \sigma$ and all sufficiently large k . It then follows from this, (70) and Step 2) of Algorithm 3 that for all sufficiently large k ,

$$\mu_{k+1} = [\mu_k + \varrho_k(\|Ax^{k+1} - b\|^2 - \sigma^2)]_+ = \varrho_k[\varrho_k^{-1}\mu_k + \|Ax^{k+1} - b\|^2 - \sigma^2]_+ = 0.$$

This together with (71) and $\xi^k \rightarrow 0$ implies $v^k \rightarrow 0$. It follows from this, $v^k \in \partial\Phi(x^{k+1})$, $x^k \rightarrow x^*$ and the semi-continuity of $\partial\Phi$ that $0 \in \partial\Phi(x^*)$. Hence, (69) holds with $\mu^* = 0$.

Case (b): $\|Ax^* - b\| = \sigma$. By this and the assumption, we know $A^T(Ax^* - b) \neq 0$. Claim that $\{\mu_{k+1}\}$ is bounded. Suppose for contradiction that $\{\mu_{k+1}\}$ is unbounded. Assume without loss of generality that $\mu_{k+1} \rightarrow \infty$. Recall that $x^k \rightarrow x^*$ and $\xi^k \rightarrow 0$. Dividing both sides of (71) by μ_{k+1} and taking limits as $k \rightarrow \infty$, we obtain

$$\lim_{k \rightarrow \infty} v^k / \mu_{k+1} = -2A^T(Ax^* - b).$$

This together with $v^k \in \partial\Phi(x^k)$, $x^k \rightarrow x^*$ and $\mu_{k+1} \rightarrow \infty$ implies $-2A^T(Ax^* - b) \in \Phi^\infty(x^*)$ and hence $-2A^T(Ax^* - b) \in \Phi^\infty(x^*) \cap \mathcal{C}$, which contradicts the assumption $\Phi^\infty(x^*) \cap \mathcal{C} = \{0\}$ due to $A^T(Ax^* - b) \neq 0$. Therefore, $\{\mu_{k+1}\}$ is bounded. This together with (71) and the boundedness of $\{x^k\}$ and $\{\xi^k\}$ implies that $\{v^k\}$ is

bounded. Considering a convergent subsequence if necessary, assume for convenience that $(\mu_{k+1}, v^k) \rightarrow (\mu^*, v^*)$. Using this, $x^k \rightarrow x^*$, $\xi^k \rightarrow 0$, and taking limits on both sides of (71) as $k \rightarrow \infty$, we obtain that

$$2\mu^* A^T(Ax^* - b) + v^* = 0.$$

Notice $v^* \in \partial\Phi(x^*)$. Hence, the first relation of (69) holds with such μ^* . In addition, the other two relations of (69) hold due to the assumption $\|Ax^* - b\| = \sigma$. This completes the proof. \blacksquare

6 Numerical results

In this section we conduct numerical experiments to test the performance of the partially regularized models studied in this paper. Due to the paper length limitation, we only present the computational results for the partial ℓ_1 regularized model, that is, $\phi(\cdot) = |\cdot|$. We also compare the performance of this model with some existing models. The codes for them are written in Matlab. All computations are performed by Matlab R2015b running on a Dell desktop with a 3.40-GHz Intel Core i7-3770 processor and 16 GB of RAM.

6.1 Sparse logistic regression

Sparse logistic regression has numerous applications in machine learning, computer vision, data mining, bioinformatics and signal processing (e.g., see [28, 32]). Given m samples $\{a^1, \dots, a^m\}$ with n features, and m binary outcomes b_1, \dots, b_m , the goal of sparse logistic regression is to seek a sparse vector to minimize the *average logistic loss* function that is defined as

$$l_{\text{avg}}(w) := \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-b_i w^T a^i)) \quad w \in \mathbb{R}^n.$$

This problem can be formulated as

$$\min_w \{l_{\text{avg}}(w) : \|w\|_0 \leq r\}, \quad (72)$$

where $r \in [1, n]$ is some integer for controlling the sparsity of the solution. Given that problem (72) is generally hard to solve, one popular approach for finding an approximate solution of (72) is by solving the full ℓ_1 regularized problem:

$$\min_w l_{\text{avg}}(w) + \lambda \sum_{i=1}^n |w|_i \quad (73)$$

for some $\lambda > 0$ (e.g., see [28]). Since the full ℓ_1 regularization is typically biased, we are interested in finding an approximate solution of (72) by solving the partial ℓ_1 regularized model:

$$\min_w l_{\text{avg}}(w) + \hat{\lambda} \sum_{i=r+1}^n |w|_{[i]} \quad (74)$$

for some $\hat{\lambda} > 0$. Our aim below is to compare the performance of models (73) and (74). In particular, we solve them with suitable λ and $\hat{\lambda}$ to find a solution with the same cardinality and then compare their respective average logistic loss.

We apply Algorithm 1 (the NPG method) presented in Subsection 5.1 to solve problems (73) and (74). In our implementation of Algorithm 1, we set $L_{\min} = 10^{-8}$, $L_{\max} = 10^8$, $\tau = 2$, $c = 10^{-4}$ and $N = 5$. In addition, we choose the initial point to be zero and set L_k^0 according to (47). We use (51) as the termination criterion for Algorithm 1 and set the associated accuracy parameter ϵ to be 10^{-5} .

In the first experiment, we compare the solution quality of models (73) and (74) on the random data of six different sizes. For each size, we generate 10 instances in which the samples $\{a^1, \dots, a^m\}$ and their outcomes b_1, \dots, b_m are generated in the same manner as described in [28]. In detail, for each instance we choose an equal number of positive and negative samples, that is, $m_+ = m_- = m/2$, where m_+ (resp., m_-) is the number of samples with outcome 1 (resp., -1). The features of positive (resp., negative) samples are independent and identically distributed, drawn from a normal distribution $N(\mu, 1)$, where μ is in turn drawn from a uniform distribution on $[0, 1]$ (resp., $[-1, 0]$). For each instance, we first apply Algorithm 1 to solve (73) with four different values of λ , which are $0.5\lambda_{\max}$, $0.25\lambda_{\max}$, $0.1\lambda_{\max}$ and $0.01\lambda_{\max}$, where $\lambda_{\max} = \|\sum_{i=1}^m b_i a^i\|_{\infty}/(2m)$. It is not hard to verify that $w = 0$ is an optimal solution of (73) for all $\lambda \geq \lambda_{\max}$ and thus λ_{\max} is the upper bound on the useful range of λ . For each λ , let K be the cardinality of the approximate solution of (73) found by Algorithm 1. Then we apply Algorithm 1 to solve (74) respectively with $r = \lceil 0.1K \rceil, \lceil 0.2K \rceil, \dots, \lceil 0.9K \rceil, K$ and some $\hat{\lambda}$ (depending on r) so that the resulting approximate solution has cardinality no greater than K . The average l_{avg} over 10 instances of each size for different λ and r is presented in Figure 1. The result of model (73) corresponds to the part of this figure with $r = 0$. We can observe that model (74) with the aforementioned positive r substantially outperforms model (73) in terms of the solution quality since it generally achieves lower average logistic loss while the sparsity of their solutions is similar. In addition, the average logistic loss of for model (74) becomes smaller as r gets closer to K , which indicates more alleviation on the bias of the solution.

In the second experiment, we compare the solution quality of models (73) and (74) on three small- or medium-sized benchmark data sets which are from the UCI machine learning bench market repository [29] and other sources [25]. The first data set is the colon tumor gene expression data [25] with more features than samples; the second one is the ionosphere data [29] with fewer features than samples; and the third one is the musk data [29] where we take as many samples as features. We discard the samples with missing data, and standardize each data set so that the sample mean is zero and the sample variance is one. We repeat the above experiment for these data sets and present the computational results in Figure 2. One can see that the solution of model (74) achieves lower average logistic loss than that of model (73) while they have same cardinality. In addition, the average logistic loss of model (74) decreases as r increases.

6.2 Sparse recovery for a linear system

In this subsection we consider finding a sparse solution to a linear system, which can be formulated as

$$\min_x \{ \|x\|_0 : \|Ax - b\|^2 \leq \sigma^2 \}, \quad (75)$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $\sigma \geq 0$ is the noise level (that is, $\sigma = 0$ for the noiseless case and $\sigma > 0$ for the noisy case). Given that problem (75) is typically hard to solve, one common approach in the literature for finding an approximate solution of (75) is by solving the model:

$$\min_x \left\{ \sum_{i=1}^n \phi(|x|_i) : \|Ax - b\|^2 \leq \sigma^2 \right\} \quad (76)$$

for some regularizer $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Our aim below is to compare the performance of model (76) with the partial ℓ_1 regularized model

$$\min_x \left\{ \sum_{i=r+1}^n |x|_{[i]} : \|Ax - b\|^2 \leq \sigma^2 \right\} \quad (77)$$

for some integer $r > 0$. In our comparison, the particular ϕ for (76) is chosen to be the regularizers $\ell_1, \ell_q, \text{Log}, \text{Capped-}\ell_1, \text{MCP}$ and SCAD , which are presented in Section 1. The associated parameters for them are chosen as $q = 0.5, \varepsilon = 10^{-3}, \nu = 10^{-2}, \lambda = 1, \alpha = 2.7$ and $\beta = 3.7$ that are commonly used in the literature.

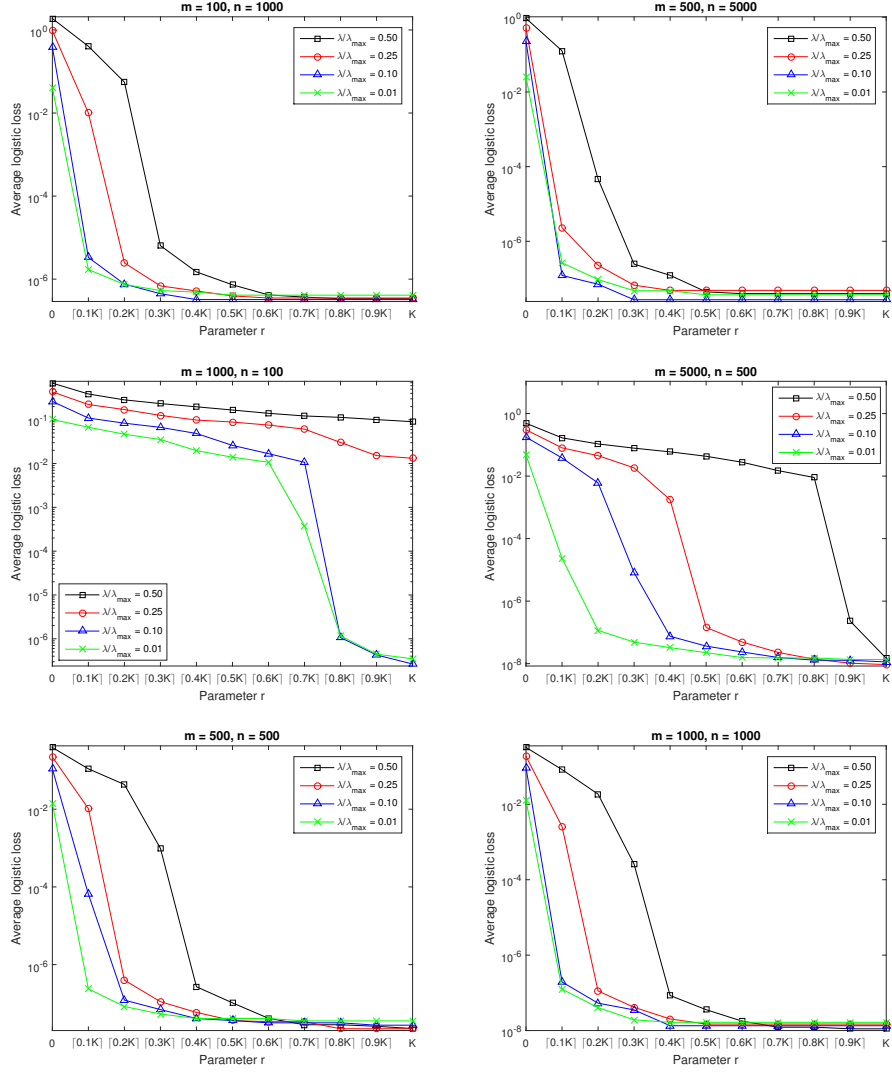


Figure 1: Computational results on six random problems

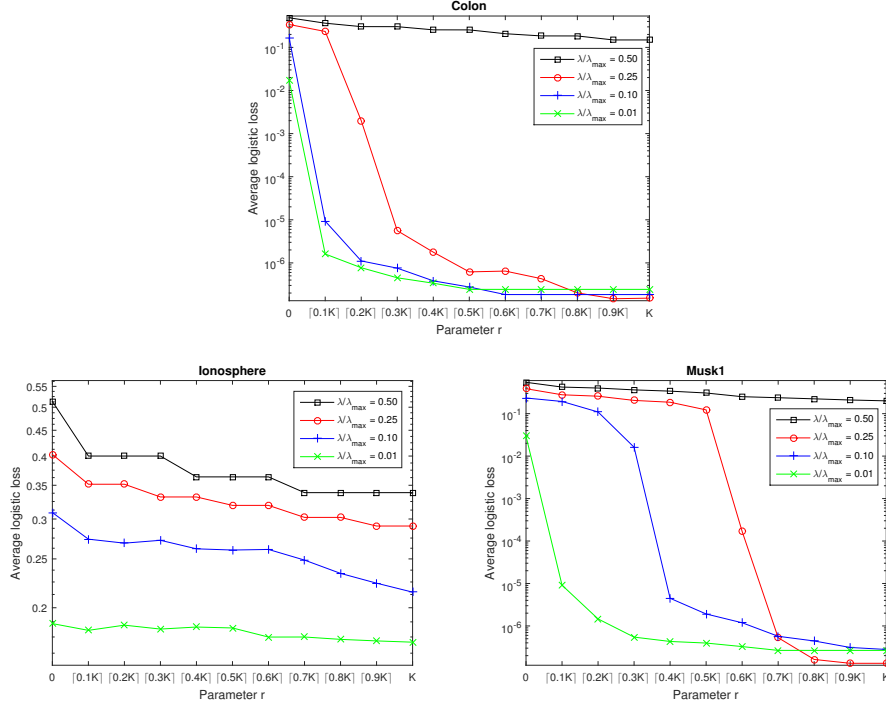
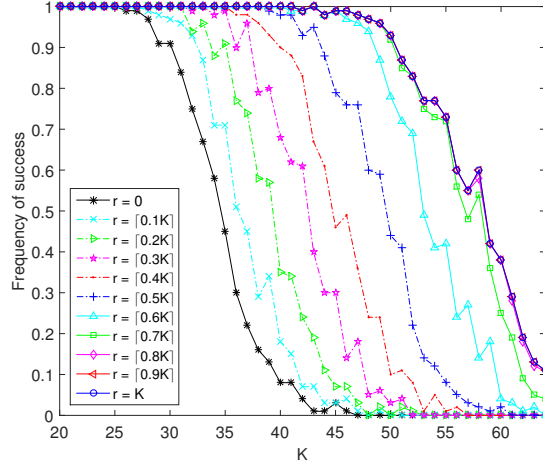


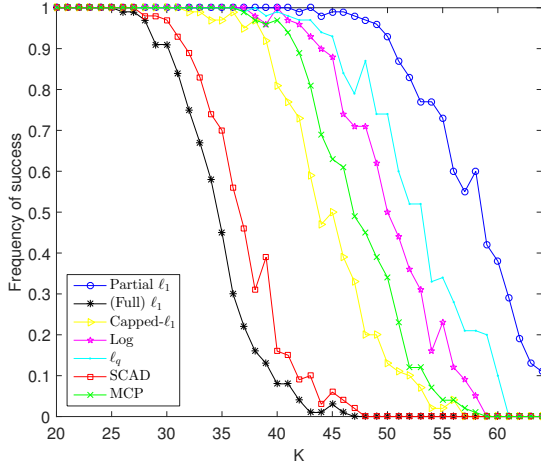
Figure 2: Computational results on three real data sets

We apply FAL methods (Algorithms 2 and 3) proposed in Subsections 5.2 and 5.3 to solve both (76) and (77) with $\sigma = 0$ and $\sigma > 0$, respectively. We now address the initialization and termination criterion for Algorithms 2 and 3. In particular, we set $x^{\text{feas}} = A \setminus b$, $x^0 = 0$, $\mu^0 = 0$, $\mu_0 = 0$, $\rho_0 = 1$, $\gamma = 5$, $\eta = 0.25$, $\theta = 10^{-2}$, $\epsilon_0 = 1$ and $\epsilon_k = \max\{0.1\epsilon_{k-1}, 10^{-5}\}$ for $k \geq 1$. In addition, we choose the termination criterion $\|Ax^k - b\|_\infty \leq 10^{-5}$ and $\epsilon_k \leq 10^{-4}$ for Algorithm 2, and $|\|Ax^k - b\|_\infty - \sigma|_+ \leq 10^{-5}$ and $\epsilon_k \leq 10^{-4}$ for Algorithm 3. The augmented Lagrangian subproblems arising in FAL are solved by Algorithm 1, whose parameter setting and termination criterion are the same as those specified in Subsection 6.1.

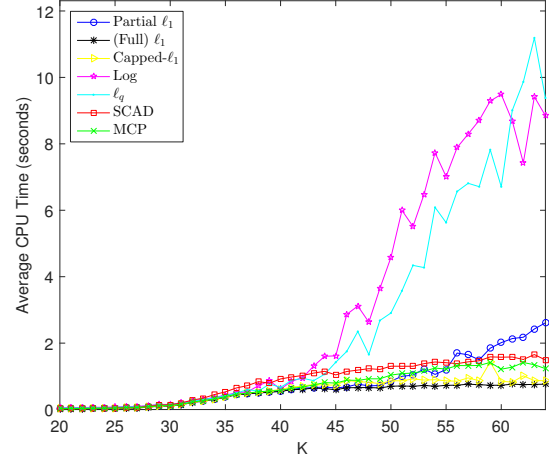
We next conduct numerical experiments to compare the performance of models (76) and (77) for the aforementioned ϕ . In the first experiment, we compare them on some noiseless random data that is generated in the same manner as described in [20]. We set $m = 128$ and $n = 512$. For each $K \in \{1, 2, \dots, 64\}$, we first randomly generate 100 pairs of K -sparse vector $x^* \in \mathbb{R}^n$ and $m \times n$ data matrix A . In particular, we randomly choose K numbers from $\{1, 2, \dots, 512\}$ as the support for x^* and then generate the nonzero entries of x^* and matrix A according to the standard Gaussian distribution. Finally we orthonormalize the rows of A and set $b = Ax^*$. We apply Algorithm 2 to solve models (76) and (77) with $r = \lceil 0.1K \rceil, \lceil 0.2K \rceil, \dots, \lceil 0.9K \rceil, K$, $\sigma = 0$ and the aforementioned ϕ . It was observed in our experiment that all models can exactly recover x^* for $K \in \{1, 2, \dots, 19\}$. Therefore, we will not report the results for these K . To evaluate the solution quality for the rest of K , we adopt a criterion as described in [9]. We say that a sparse vector x is *successfully recovered* by an estimate \hat{x} if $\|x - \hat{x}\| < 10^{-3}$. The computational results for the instances generated above are plotted in Figure 3. In detail, the average frequency of success over all instances for model (77) with $r = \lceil 0.1K \rceil, \lceil 0.2K \rceil, \dots, \lceil 0.9K \rceil, K$ against $K \in \{20, 21, \dots, 64\}$ is presented in the first subfigure. The average frequency of success of model (77) with $r = K$ and model (76) with the aforementioned ϕ against $K \in \{20, 21, \dots, 64\}$ is plotted in the second subfigure, where the partial ℓ_1 represents model (77) with $r = K$ and the others represent model (76) with various ϕ , respectively. Analogously, the accumulated CPU time over all instances for each model against



(a) Frequency of success for model (77)



(b) Frequency of success for models (76) and (77)

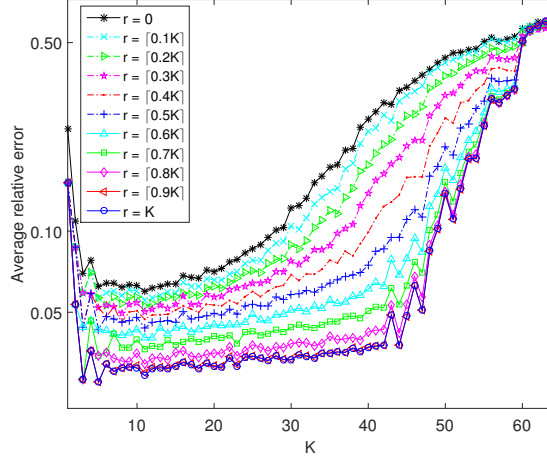


(c) Average CPU time for models (76) and (77)

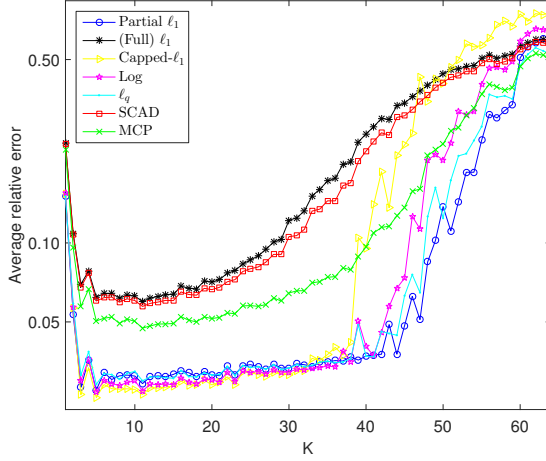
Figure 3: Computational results on random noiseless data

$K \in \{20, 21, \dots, 64\}$ is displayed in the last subfigure. One can observe from the first subfigure that for each K , the average frequency of success of model (77) becomes higher as r increases. In addition, from the last two subfigures, we can see that for each $K \in \{20, 21, \dots, 64\}$, model (77) with $r = K$ generally has higher average frequency of success than and comparable average CPU time to model (76) with various ϕ .

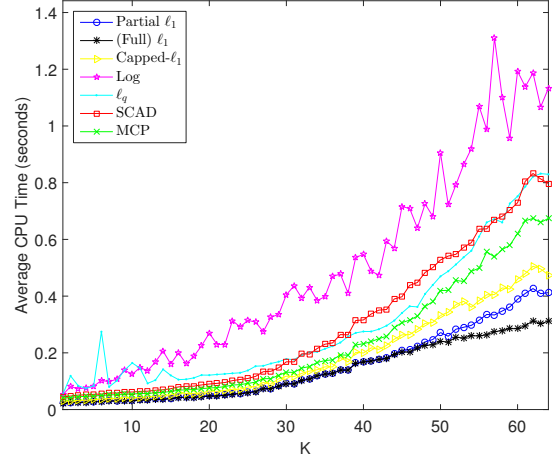
In the second experiment, we compare the performance of models (76) and (77) with the aforementioned ϕ on some noisy random data. For each $K \in \{1, 2, \dots, 64\}$, we generate 100 pairs of K -sparse vector $x^* \in \mathbb{R}^n$ and $m \times n$ data matrix A in the same manner as above. For each pair of x^* and A , we set $b = Ax^* + \xi$ and $\sigma = \|\xi\|$, where ξ is drawn from a normal distribution with mean 0 and variance 10^{-4} . For each such instance, we apply Algorithm 3 to solve the models (76) and (77) and compute their corresponding relative error according to $\text{rel.err} = \|\hat{x} - x\|/\|x\|$, which evaluates how well a sparse vector x is recovered by an estimate \hat{x} as described in [9]. The computational results of this experiment are presented in Figure 4. We plot the average relative error



(a) Average relative error for model (77)



(b) Average relative error for models (76) and (77)



(c) Average CPU time for models (76) and (77)

Figure 4: Computational results on random noisy data

over all instances for model (77) with $r = \lceil 0.1K \rceil, \lceil 0.2K \rceil, \dots, \lceil 0.9K \rceil$, K against K in the first subfigure. The average relative error of model (77) with $r = K$ and (76) with the aforementioned ϕ against K is displayed in the second subfigure, where the partial ℓ_1 represents model (77) with $r = K$ and the others represent model (76) with various ϕ , respectively. Similarly, the accumulated CPU time over all instances for each model against K is presented in the last subfigure. One can see from the first subfigure that for each K , the average relative error of model (77) becomes lower as r increases. From the second subfigure, we can observe that when $0 < K < 38$, the average relative error of model (77) with $r = K$ is comparable to that of model (76) with ϕ chosen as ℓ_q , Log, and Capped- ℓ_1 , while it is lower than the one with ϕ chosen as ℓ_1 , SCAD and MCP. Nevertheless, when $38 \leq K \leq 64$, model (77) with $r = K$ has lower average relative error than the other models. In addition, as seen from the last subfigure, model (77) with $r = K$ generally has comparable average CPU time to model (76) with various ϕ .

7 Concluding remarks

In this paper we proposed a class of models with partial regularization for recovering a sparse solution of a linear system. We studied some theoretical properties for these models including sparsity inducing, local or global recovery and also stable recovery. We also developed an efficient first-order feasible augmented Lagrangian method for solving them whose subproblems are solved by a nonmonotone proximal gradient method. The global convergence of these methods were also established. Numerical results on compressed sensing and sparse logistic regression demonstrate that our proposed models substantially outperform the widely used ones in the literature in terms of solution quality.

It shall be mentioned that the partially regularized model has recently been proposed in [22, 23] for finding a low-rank matrix lying in a certain convex subset of positive semidefinite cone. In particular, the function $\sum_{i=r+1}^n \lambda_i(X)$ was used as a partial regularizer there, where $0 < r < n$ and $\lambda_i(X)$ is the i th largest eigenvalue of an $n \times n$ positive semidefinite matrix X . Clearly, this regularizer can be generalized to a class of partial regularizers in the form of $\sum_{i=r+1}^l \phi(\sigma_i(X))$, where $\sigma_i(X)$ is the i th largest singular value of a $m \times n$ matrix X , $l = \min\{m, n\}$ and ϕ satisfies Assumption 1. Most results of this paper can be generalized to the following model for finding a low-rank matrix:

$$\min_{X \in \mathbb{R}^{m \times n}} \left\{ \sum_{i=r+1}^l \phi(\sigma_i(X)) : \|\mathcal{A}(X) - b\| \leq \sigma \right\},$$

where $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ is a linear operator, $b \in \mathbb{R}^p$ and $\sigma \geq 0$. This is beyond this paper and left as a future research.

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