

New Douglas-Rachford Algorithmic Structures and Their Convergence Analyses

Yair Censor and Rafiq Mansour

Department of Mathematics

University of Haifa

Mt. Carmel, Haifa 3498838, Israel

December 24, 2014. Revised: June 23, 2015, September 2, 2015, October 20, 2015 and November 25, 2015.

Abstract

In this paper we study new algorithmic structures with Douglas-Rachford (DR) operators to solve convex feasibility problems. We propose to embed the basic two-set-DR algorithmic operator into the String-Averaging Projections (SAP) and into the Block-Iterative Projection (BIP) algorithmic structures, thereby creating new DR algorithmic schemes that include the recently proposed cyclic Douglas-Rachford algorithm and the averaged DR algorithm as special cases. We further propose and investigate a new multiple-set-DR algorithmic operator. Convergence of all these algorithmic schemes is studied by using properties of strongly quasi-nonexpansive operators and firmly nonexpansive operators.

1 Introduction

Contribution of this paper. We study new algorithmic structures for the Douglas-Rachford (DR) algorithm¹. Our starting points for the developments presented here are the two-set-DR original algorithm and the recent cyclic-DR algorithm of [14], designed to solve convex feasibility problems. They use the same basic algorithmic operator which reflects the current iterate consecutively into two sets and takes the midpoint between the current iterate and the end-point of the two consecutive reflections as the next iterate.

The convex feasibility problem (CFP) is to find an element $x \in C$ where C_i , $i \in I := \{1, 2, \dots, m\}$, form a finite family of closed convex sets in a Hilbert space \mathcal{H} , $C := \bigcap_{i \in I} C_i$ and $C \neq \emptyset$. There are many algorithms in the literature for solving CFPs, see, e.g., [6]. In particular, two algorithmic structures that encompass many specific feasibility-seeking algorithms are the String-Averaging Projections (SAP) method [21] and the Block-Iterative Projection (BIP) method [1].

In this paper we do two things: (i) create new algorithmic structures with the 2-set-DR algorithmic operator, and (ii) define and study an “ m -set-DR operator”.

First we employ the two-set-DR algorithmic operator and embed it in the SAP and BIP algorithmic structures. In doing so one obtains two new families of DR algorithms, of which the two-set-DR original algorithm and the recent cyclic-DR algorithm and the averaged DR algorithm are special cases. Convergence analyses of these new DR algorithms are provided.

In our String-Averaging Douglas-Rachford (SA-DR) scheme, we separate the index set I of the CFP into subsets, called “strings”, and proceed along each string by applying the basic two-set-DR operator (this can be done in parallel for all strings) sequentially along the string. Then a convex combination of the strings’ end-points is taken as the next iterate.

In our Block-Iterative Douglas-Rachford (BI-DR) scheme, the index set I of the CFP is again separated into subsets, now called “blocks”, dividing the family of convex closed sets into blocks of sets. The basic two-set-DR operator is applied in a specific way to each block and the algorithmic scheme proceeds sequentially over the blocks.

¹We use the terms “algorithm” and “algorithmic structures” for the iterative processes studied here although no termination criteria are present and only the asymptotic behavior of these processes is studied.

Finally, we propose and investigate a generalization of the 2-set-DR algorithmic operator itself. Instead of reflecting the current iterate consecutively into two sets and taking the midpoint between the current iterate and the end-point of the two consecutive reflections as the next iterate, we propose to allow the algorithmic operator to perform a finite number, say r (greater or equal 2), of consecutive reflections into r sets and only then take the midpoint between the current iterate and the end-point of the r consecutive reflections as the next iterate. We show how this “ m -set-DR operator” works algorithmically.

We study the convergence of all algorithmic schemes, under the assumption that C or its interior are nonempty by using properties of strongly quasi-nonexpansive operators and firmly nonexpansive operators. In particular, a corner stone of our results is the recognition that the 2-set-DR operator is not only firmly nonexpansive, thus nonexpansive, as stated in [14, Fact 2.2], but also strongly quasi-nonexpansive.

The framework. Since a reflection is a nonexpansive operator and the class of nonexpansive operators is closed under composition, the 2-set-DR operator is an averaged operator. Averaged operators form a very nice class since they are closed under compositions and convex combinations. Therefore, since all operators discussed in the paper are averaged one can get alternative proofs of our results from [26] or [7, Sections 5.2 and 5.3]. We chose, however, to work within the framework of various “quasi” operators recognizing the generality of this framework.

Current literature. Current literature witnesses a strong interest in DR algorithms in the framework of splitting methods for optimization, see, e.g., [27]. We are interested in the DR algorithm for the feasibility problem and in this direction there are several relevant publications that include also applications in various fields. An analysis of the behavior of the cyclic Douglas–Rachford algorithm when the target intersection set is possibly empty was undertaken in [16], consult this paper also for many additional relevant references. The work in [15] applies the DR algorithm to the problem of protein conformation. Recent positive experiences applying convex feasibility algorithms of Douglas-Rachford type to highly combinatorial and far from convex problems appear in [3]. Systematic investigation of the asymptotic behavior of averaged alternating reflections (AAR) which are also known as the 2-set-DR operators, in the general case when the sets do not necessarily intersect can be found in [8], and [9] presents a new averaged alternating reflections method which produces a strongly convergent sequence. General

recommendations for successful application of the DR algorithm to convex and nonconvex real matrix-completion problems are presented in [2].

For the convergence of the DR algorithm under various constraints, see, e.g., [30] which proves that any sequence generated by the DR algorithm converges weakly to a solution of an inclusion problem. In [11] the authors prove the two-set-DR algorithm's local convergence to a fixed point when the two sets are finite unions of convex sets, while [13] provides convergence results for a prototypical nonconvex two-sets scenario in which one of the sets is a Euclidean sphere. The results in [4] establish a region of convergence for the prototypical non-convex Douglas-Rachford iteration which finds a point in the intersection of a line and a circle. The work in [12] introduces regularity notions for averaged nonexpansive operators, and obtains linear and strong convergence results for quasi-cyclic, cyclic, and random iterations. New convergence results on the Borwein–Tam method (BTM) which is also known as the cyclic DR algorithm, and on the cyclically anchored Douglas–Rachford algorithm (CADRA) are also presented.

Relevant to our analyses are properties of operators under compositions and convex combinations. In, e.g., [5] one learns that the composition of projections onto closed convex sets in Hilbert space is asymptotically regular, and [10] proves that compositions and convex combinations of asymptotically regular firmly nonexpansive mappings are also asymptotically regular. In [26] a unified fixed point theoretic framework is proposed to investigate the asymptotic behavior of algorithms for finding solutions to monotone inclusion problems. The basic iterative scheme under consideration involves non-stationary compositions of perturbed averaged nonexpansive operators.

For details on BIP, SAP and other projection methods, see, e.g., [17] which studies the behavior of a class of BIP algorithms for solving convex feasibility problems. In [18] the simultaneous MART algorithm (SMART) and the expectation maximization method for likelihood maximization (EMML) are extended to block-iterative versions, called BI-SMART and BI-EMML, respectively, that converge to a solution in the feasible case. The work in [20] formulates a block-iterative algorithmic scheme for the solution of systems of linear inequalities and/or equations and analyzes its convergence. The excellent review [6] discusses projection algorithms for solving convex feasibility problems. More recently, [23] discusses the convergence of string-averaging projection schemes for inconsistent convex feasibility problems, [22] proposes a definition of sparseness of a family of operators and investigates a string-averaging algorithmic scheme that favorably handles the common fixed points

problem when the family of operators is sparse.

Potential computational advantages. We have no computational experience with the new DR algorithms proposed and studied here. Comparative computational performance can really be made only with exhaustive testing of the many possible specific variants of the new DR algorithms permitted by the general schemes and their various user-chosen parameters. The computational advantages of string-averaging and block-iterative algorithmic structures have been shown in the past for algorithms that use orthogonal projections² rather than DR operators. For example, the work on proton computed tomography (pCT) in [29] employs very efficiently a parallel code that uses a version of the string-averaging algorithm called component-averaged row projections (CARP), see [28], and a version of a block-iterative algorithm called diagonally-relaxed orthogonal projections (DRO), see [25].

It is plausible to hypothesize that since the string-averaging algorithmic structure and the cyclic DR algorithm of [14] have been demonstrated to be computationally useful separately then so might very well be their combination in the String-Averaging Douglas-Rachford (SA-DR). Admittedly, these practical questions should be resolved in future work, preferably within the context of a significant real-world application.

Structure of the paper. The paper is organized as follows: In Section 2, we give definitions and preliminaries. In Section 3, we introduce the new String-Averaging Douglas-Rachford (SA-DR) scheme and the new Block-Iterative Douglas-Rachford (BI-DR) scheme. For both the SA-DR and the BI-DR algorithms we prove strong convergence to a point in the intersection of the sets under the assumption that $\text{int} \bigcap_{i \in I} C_i \neq \emptyset$. In Section 4 we study our new “ m -set-DR operator” and show how it works.

2 Preliminaries

For the reader’s convenience we include in this section some properties of operators in Hilbert space that will be used to prove our results. We use the recent excellent book of Cegielski [19] as our desk-copy in which all the results of this section can be found [19, Chapter 2]. Let \mathcal{H} be a real Hilbert

²The term “orthogonal projection” is used here not only for the case when the sets are subspaces but in general. It means here “nearest point projection” operator or “metric projection” operator.

space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let $X \subseteq \mathcal{H}$ be a closed convex subset.

Definition 1. An operator $T : X \rightarrow \mathcal{H}$ is:

- i. *Fejér monotone* (FM) with respect to a nonempty subset $C \subseteq X$, if $\|T(x) - z\| \leq \|x - z\|$ for all $x \in X$ and $z \in C$.
- ii. *Strictly Fejér monotone* (sFM) with respect to a nonempty subset $C \subseteq X$, if $\|T(x) - z\| < \|x - z\|$ for all $x \notin C$ and $z \in C$.
- iii. *Nonexpansive* (NE), if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in X$.
- iv. *Firmly nonexpansive* (FNE), if $\langle T(x) - T(y), x - y \rangle \geq \|T(x) - T(y)\|^2$ for all $x, y \in X$.
- v. *Strongly nonexpansive* (SNE), if T is nonexpansive and for all sequences $\{x^k\}_{k=0}^\infty, \{y^k\}_{k=0}^\infty \subseteq X$ such that $(x^k - y^k)$ is bounded and $\|x^k - y^k\| - \|T(x^k) - T(y^k)\| \rightarrow 0$ it follows that $(x^k - y^k) - (T(x^k) - T(y^k)) \rightarrow 0$.

The following proposition is well-known, see, e.g., [19, Theorem 2.2.4] and [19, Corollary 2.2.20].

Proposition 2. *Every firmly nonexpansive operator is nonexpansive, and any convex combination of firmly nonexpansive operators is firmly nonexpansive.*

Definition 3. An operator $T : X \rightarrow \mathcal{H}$ having a fixed point is:

- i. *Quasi-nonexpansive* (QNE), if T is Fejér monotone with respect to the fixed points set $\text{Fix}T$, i.e., if $\|T(x) - z\| \leq \|x - z\|$ for all $x \in X$ and $z \in \text{Fix}T$.
- ii. *Strictly quasi-nonexpansive* (sQNE), if T is strictly Fejér monotone with respect to $\text{Fix}T$, i.e., if $\|T(x) - z\| < \|x - z\|$ for all $x \notin \text{Fix}T$ and $z \in \text{Fix}T$.
- iii. *C-strictly quasi-nonexpansive* (C-sQNE), where $C \neq \emptyset$ and $C \subseteq \text{Fix}T$, if T is quasi-nonexpansive and $\|T(x) - z\| < \|x - z\|$ for all $x \notin \text{Fix}T$ and $z \in C$.

- iv. α -strongly quasi-nonexpansive (α -SQNE), if $\|T(x) - z\|^2 \leq \|x - z\|^2 - \alpha\|T(x) - x\|^2$ for all $x \in X$ and $z \in \text{Fix}T$, where $\alpha \geq 0$. If $\alpha > 0$ then T is called *strongly quasi-nonexpansive* (SQNE).

The following implications follow directly from the definitions, see [19, page 47] and [19, Remark 2.1.44(iii)].

Proposition 4. *For an operator $T : X \rightarrow \mathcal{H}$ having a fixed point, the following statements hold:*

- i. *If T is sQNE then T is C -sQNE, where $C \subseteq \text{Fix}T$.*
- ii. *If T is $\text{Fix}T$ -sQNE then T is sQNE.*
- iii. *If T is strongly quasi-nonexpansive then it is strictly quasi-nonexpansive.*

From [19, Theorem 2.3.5], see also [19, Fig. 2.14], any composition of SNE operators is SNE, and any convex combination of SNE operators is SNE.

Theorem 5. [19, Corollary 2.1.42] *Let $U_i : X \rightarrow \mathcal{H}$, $i \in I$, be quasi-nonexpansive with $C := \bigcap_{i \in I} \text{Fix}U_i \neq \emptyset$ and let: $U := U_m U_{m-1} \cdots U_1$. If $\text{int } C \neq \emptyset$, then $\text{Fix}U = \bigcap_{i \in I} \text{Fix}U_i$ and U is $\text{int}C$ -strictly quasi-nonexpansive.*

Denoting by $\Delta_m := \{u \in \mathbb{R}^m \mid u \geq 0, \sum_{i=1}^m u_i = 1\}$ the standard simplex, and by $\text{ri } \Delta_m$ its relative interior, a function $w : X \rightarrow \Delta_m$, with $w(x) = (w_1(x), w_2(x), \dots, w_m(x))$ is called a *weight function*. According to [19, Definition 2.1.25 and text on Page 50] a weight function $w : X \rightarrow \text{ri } \Delta_m$ is *appropriate* with respect to any family of operators $\{U_i\}_{i \in I}$ if: (i) $w \in \text{ri } \Delta_m$ is a vector of constant weights, or if (ii) $w_i(x) > 0$ for all $x \notin \text{Fix}U_i$ and all $i \in I$. Throughout the paper, we use only the first option of constant weights.

We have also the following.

Theorem 6. [19, Theorem 2.1.26] *Let the operators $U_i : X \rightarrow X$, $i \in I$, with $\bigcap_{i \in I} \text{Fix}U_i \neq \emptyset$, be C -strictly quasi-nonexpansive, where $C \subseteq \bigcap_{i \in I} \text{Fix}U_i$, $C \neq \emptyset$. If U has one of the following forms:*

- i. $U := \sum_{i \in I} w_i U_i$ and the weight function $w : X \rightarrow \Delta_m$ is appropriate,
- ii. $U := U_m U_{m-1} \cdots U_1$,

then

$$\text{Fix}U = \bigcap_{i \in I} \text{Fix}U_i, \quad (2.1)$$

and U is C -strictly quasi-nonexpansive.

Theorem 7. [19, Theorem 2.1.14] Let $U_i : X \rightarrow \mathcal{H}$, $i \in I := \{1, 2, \dots, m\}$, be nonexpansive operators with a common fixed point and let $U := \sum_{i \in I} w_i U_i$ with the weight function $w(x) = (w_1(x), w_2(x), \dots, w_m(x)) \in \text{ri } \Delta_m$. Then

$$\text{Fix}U = \bigcap_{i \in I} \text{Fix}U_i. \quad (2.2)$$

Definition 8. An operator $U : X \rightarrow X$ is *asymptotically regular* if for all $x \in X$,

$$\lim_{k \rightarrow \infty} \|U^{k+1}(x) - U^k(x)\| = 0. \quad (2.3)$$

By combining the results of [19, Theorem 3.4.3], [19, Corollary 3.4.6] for $\lambda = 1$, and [19, Theorem 3.4.9], see also [19, Fig. 3.2], we can state the following.

Theorem 9. Let $U : X \rightarrow X$ be an operator with a fixed point. If U is strongly quasi-nonexpansive or strongly nonexpansive, then U is asymptotically regular.

Definition 10. An operator $T : X \rightarrow \mathcal{H}$ is *demi-closed* at 0 if for any sequence $x^k \rightharpoonup y \in X$ with $T(x^k) \rightarrow 0$ we have $T(y) = 0$.

If we replace the weak convergence $x^k \rightharpoonup y$ by the strong one in Definition 10, then we obtain the definition of the closedness of T at 0. If \mathcal{H} is finite-dimensional, then the notions of a demi-closed operator and a closed operator coincide, see, e.g., [19, Page 107].

Denoting by Id the identity operator we have the following theorem.

Theorem 11. [19, Theorem 3.5.2] Let $X \subseteq \mathcal{H}$ be nonempty closed convex subset of a finite-dimensional Hilbert space \mathcal{H} and let $U : X \rightarrow X$ be an operator with a fixed point and such that $U - \text{Id}$ is closed at 0. If U is quasi-nonexpansive and asymptotically regular, then, for arbitrary $x \in X$, the sequence $\{U^k(x)\}_{k=0}^{\infty}$ converges to a point $z \in \text{Fix}U$.

Finally, here is the well-known theorem due to Opial.

Theorem 12. [19, Theorem 3.5.1] *Let $X \subseteq \mathcal{H}$ be a nonempty closed convex subset of a Hilbert space \mathcal{H} and let $U : X \rightarrow X$ be a nonexpansive and asymptotically regular operator with a fixed point. Then, for any $x \in X$, the sequence $\{U^k(x)\}_{k=0}^{\infty}$ converges weakly to a point $z \in \text{Fix}U$.*

3 String-Averaging and Block-Iterative Douglas-Rachford

In this section we describe our new String-Averaging Douglas-Rachford (SADR) algorithmic scheme and our new Block-Iterative Douglas-Rachford (BIDR) algorithmic scheme and prove their convergence. Let C_1, C_2, \dots, C_m , be nonempty closed convex subsets of \mathcal{H} , defining a convex feasibility problem (CFP) of finding an element in their intersection. We call $I := \{1, 2, \dots, m\}$ the index set of the convex feasibility problem.

3.1 The algorithms

Definition 13. Let $T : X \rightarrow \mathcal{H}$ and $\lambda \in [0, 2]$. The operator $T_\lambda : X \rightarrow \mathcal{H}$ defined by $T_\lambda := (1 - \lambda)\text{Id} + \lambda T$ is called a λ -relaxation or, in short, *relaxation* of the operator T . If $\lambda = 2$, then T_λ is called the *reflection* of T .

Note that if $\lambda = 1$ then $T_\lambda = T_1 = T$. The 2-set-Douglas-Rachford operator is defined as follows, see, e.g., [14, Equation (2)].

Definition 14. Let $A, B \subseteq \mathcal{H}$ be closed convex subsets and let P_A and P_B be the orthogonal projections onto A and B , respectively. The operators $R_A := 2P_A - \text{Id}$ and $R_B := 2P_B - \text{Id}$ are the reflection operators into A and B , respectively. The operator $T_{B,A} : \mathcal{H} \rightarrow \mathcal{H}$, defined by,

$$T_{B,A} := \frac{1}{2}(\text{Id} + R_A R_B) \tag{3.1}$$

is the “2-set-Douglas-Rachford” (2-set-DR) operator .

This operator was termed earlier “averaged alternating reflection” (AAR), see [19, Subsection 4.3.5] for details and references. For $t = 1, 2, \dots, M$, let the “string” I_t be an ordered finite nonempty subset of I of the CFP, of the form

$$I_t = (i_1^t, i_2^t, \dots, i_{\gamma(t)}^t), \quad (3.2)$$

where the “length” of the string I_t , denoted by $\gamma(t)$, is the number of elements in I_t . Denoting T_{C_i, C_j} by $T_{i,j}$, the String-Averaging Douglas-Rachford-(SA-DR) algorithmic scheme is as follows.

Algorithm 1. The String-Averaging Douglas-Rachford Scheme.

Initialization: $x^0 \in \mathcal{H}$ is arbitrary.

Iterative Step: Given the current iterate x^k ,

(i) Calculate, for all $t = 1, 2, \dots, M$,

$$T_t(x^k) := T_{i_{\gamma(t)}^t, i_1^t} T_{i_{\gamma(t)-1}^t, i_{\gamma(t)}^t} \dots T_{i_2^t, i_3^t} T_{i_1^t, i_2^t}(x^k), \quad (3.3)$$

(ii) Calculate the convex combination of the strings’ end-points by

$$x^{k+1} = \sum_{t=1}^M w_t T_t(x^k), \quad (3.4)$$

with $w_t > 0$, for all $t = 1, 2, \dots, M$, and $\sum_{t=1}^M w_t = 1$.

Note that the work on the strings in part (i) of the Iterative Step in the SA-DR scheme of Algorithm 1 can be performed in parallel on all strings.

In order to present the Block-Iterative Douglas-Rachford scheme, we again look at nonempty subsets

$$I_t = (i_1^t, i_2^t, \dots, i_{\gamma(t)}^t), \quad (3.5)$$

of the index set I of the CFP, which are now called “blocks”.

Algorithm 2. The Block-Iterative Douglas-Rachford Scheme.

Initialization: $x^0 \in \mathcal{H}$ is arbitrary.

Iterative Step: Given the current iterate x^k , pick a block index $t = t(k)$ according to a cyclic rule $t(k) = k \bmod M + 1$.

(i) Calculate intermediate points obtained by applying the 2-set-Douglas-Rachford operator to pairs of sets in the t -th block as follows. For all $\ell = 1, 2, \dots, \gamma(t) - 1$ define

$$z_\ell = T_{i_\ell^t, i_{\ell+1}^t}(x^k), \quad (3.6)$$

and for $\ell = \gamma(t)$ let

$$z_{\gamma(t)} = T_{i_{\gamma(t)}^t, i_1^t}(x^k). \quad (3.7)$$

(ii) Calculate the convex combination of the intermediate points

$$x^{k+1} = \sum_{\ell=1}^{\gamma(t)} w_{\ell}^t z_{\ell}, \quad (3.8)$$

with $w_{\ell}^t > 0$, for all $\ell = 1, 2, \dots, \gamma(t)$, and $\sum_{\ell=1}^{\gamma(t)} w_{\ell}^t = 1$.

Note that the work on the pairs of sets in a block in part (i) of the Iterative Step in the BI-DR scheme of Algorithm 2 can be performed in parallel on all pairs.

3.2 Convergence proofs

The following lemma characterizes the fixed points of the 2-set Douglas-Rachford operator.

Lemma 15. [8, Corollary 3.9] *Let $A, B \subseteq \mathcal{H}$ be closed and convex with nonempty intersection. Then*

$$P_A \text{Fix} T_{A,B} = A \cap B. \quad (3.9)$$

If the right-hand side of (3.9) has nonempty interior additional information is available from [19, Corollary 4.3.17(ii)].

Lemma 16. [19, Corollary 4.3.17(ii)] *Let $A, B \subseteq \mathcal{H}$ be closed and convex sets and let $T_{A,B}$ be their 2-set-DR operator. If $\text{int}(A \cap B) \neq \emptyset$ then $A \cap B = \text{Fix} T_{A,B}$.*

A corner stone of our subsequent results is the recognition that the 2-set-DR operator is not only firmly nonexpansive, as stated in [14, Fact 2.2], but also strongly quasi-nonexpansive.

Lemma 17. *Every 2-set-DR operator $T_{A,B}$ is:*

- i. Strongly quasi-nonexpansive (SQNE),*
- ii. Strongly nonexpansive (SNE).*

Proof. (i) Follows from [19, Corollary 2.2.9].

(ii) By [19, Theorem 2.3.4] with $\lambda = 1$ and $T = T_1$ we obtain that T is SNE. \square

Our convergence theorems for the above two algorithmic schemes can now be stated and proved.

Theorem 18. (SA-DR). *Let $C_1, C_2, \dots, C_m \subseteq \mathcal{H}$ be closed and convex sets that define a convex feasibility problem (CFP). If*

$$\text{int} \bigcap_{i \in I} C_i \neq \emptyset \quad (3.10)$$

then for any $x^0 \in \mathcal{H}$, any sequence $\{x^k\}_{k=0}^\infty$, generated by the SA-DR Algorithm 1, with strings such that $I = I_1 \cup I_2 \cup \dots \cup I_M$, converges strongly to a point $x^ \in \bigcap_{i \in I} C_i$.*

Proof. From Lemma 17 we have that the 2-set-DR operators $T_{i_\ell^t, i_{\ell+1}^t}$ for all $\ell = 1, 2, \dots, \gamma(t) - 1$ and $T_{i_{\gamma(t)}^t, i_1^t}$ are strongly quasi-nonexpansive. Denoting the intersection of all fixed points sets of all 2-set-DR operators within the t -th string by

$$\Gamma_t := \left(\bigcap_{\ell=1}^{\gamma(t)-1} \text{Fix} T_{i_\ell^t, i_{\ell+1}^t} \right) \cap \text{Fix} T_{i_{\gamma(t)}^t, i_1^t}, \quad (3.11)$$

we first show that $\text{int} \Gamma_t$ is nonempty. Each point in the nonempty intersection $\bigcap_{i \in I} C_i$ is a fixed point for any 2-set-DR operator with respect to any pair of sets from the family of sets in the CFP, meaning that $\bigcap_{i \in I} C_i \neq \emptyset$ is included in the fixed points set of any of the operators that appear in the right-hand side of (3.11). So we have

$$\Gamma_t \supseteq \bigcap_{i=1}^{\gamma(t)} C_i \neq \emptyset, \quad (3.12)$$

and, by (3.10),

$$\text{int} \Gamma_t \neq \emptyset. \quad (3.13)$$

Identifying the 2-set-DR operators in a string with the individual operators in Theorem 5, and since any SQNE operator is also QNE, the conditions of Theorem 5 are met and we conclude that any string operator T_t in (3.3) is $\text{int} \Gamma_t$ -strictly quasi-nonexpansive, and

$$\text{Fix} T_t = \Gamma_t, \quad (3.14)$$

for all $t = 1, 2, \dots, M$. Looking at the intersection of the fixed point sets of all string operators

$$\Gamma := \bigcap_{t=1}^M \text{Fix}T_t, \quad (3.15)$$

we conclude again, re-applying the previous considerations, that

$$\Gamma \supseteq \bigcap_{i \in I} C_i \neq \emptyset, \quad (3.16)$$

and that for any subset Θ of $\text{int } \Gamma$, the operators T_t are Θ -strictly quasi-nonexpansive, since they are $\text{int}\Gamma_t$ -strictly quasi-nonexpansive. So, by Theorem 6(i),

$$\text{Fix} \left(\sum_{t=1}^M w_t T_t \right) = \Gamma, \quad (3.17)$$

and $\sum_{t=1}^M w_t T_t$ is $\text{int}\Gamma$ -strictly quasi-nonexpansive, which in turn implies that it is a quasi-nonexpansive operator. By Lemma 17(ii) the 2-set-DR operators $T_{i_\ell^t, i_{\ell+1}^t}$ for all $\ell = 1, 2, \dots, \gamma(t) - 1$ and $T_{i_{\gamma(t)}^t, i_1^t}$ are strongly nonexpansive, therefore, their composition and the convex combination of their compositions $\sum_{t=1}^M w_t T_t$ are strongly nonexpansive, thus, nonexpansive, see Definition 1(v). Using the demi-closedness principle embodied in [19, Lemma 3.2.5] and [19, Definition 3.2.6] for the operator $\sum_{t=1}^M w_t T_t$, the operator $(\sum_{t=1}^M w_t T_t - \text{Id})$ is demi-closed at 0. By Theorem 9, $\sum_{t=1}^M w_t T_t$ is asymptotically regular. Finally, by Theorem 11, $\{x^k\}_{k=0}^\infty$, generated by the SA-DR Algorithm 1, converges weakly to a point

$$x^* \in \text{Fix} \left(\sum_{t=1}^M w_t T_t \right) = \bigcap_{t=1}^M \text{Fix}T_t = \bigcap_{t=1}^M \Gamma_t = \Gamma. \quad (3.18)$$

Using the assumption of (3.10) that $\text{int} \bigcap_{i \in I} C_i \neq \emptyset$ we apply [19, Corollary 4.3.17(ii)] to all pairs of sets in each string Γ_t and obtain

$$\Gamma = \bigcap_{i \in I} C_i. \quad (3.19)$$

Therefore, $x^* \in \bigcap_{i \in I} C_i$.

Since $\sum_{t=1}^M w_t T_t$ is a quasi-nonexpansive operator, it is Fejér monotone. Therefore, $\{x^k\}_{k=0}^\infty$, generated by the SA-DR Algorithm 1 is Fejér monotone sequence. By [7, Proposition 5.10], the convergence of $\{x^k\}_{k=0}^\infty$ to x^* is strong. \square

Note that the operator T_t is nonexpansive, but without the assumption $\text{int} \bigcap_{i \in I} C_i \neq \emptyset$ it need not be Θ -strictly quasi-nonexpansive which would have prevented us from getting (3.17) from Theorem 6(i), which is a corner stone in the proof of Theorem 17.

For the Block-Iterative Douglas-Rachford algorithm we prove the following convergence result.

Theorem 19. (BI-DR). *Let $C_1, C_2, \dots, C_m \subseteq \mathcal{H}$ be closed and convex sets with a nonempty intersection. For any $x^0 \in \mathcal{H}$, the sequence $\{y^k\}_{k=0}^\infty$ of iterates of the BI-DR Algorithm 2 with $I = I_1 \cup I_2 \cup \dots \cup I_M$, after full sweeps through all blocks, converges*

(i) *weakly to a point y^* such that $P_{C_{i_\ell}^t}(y^*) \in \bigcap_{\ell=1}^{\gamma(t)} C_{i_\ell}^t$ for $\ell = 1, 2, \dots, \gamma(t)$ and $t = 1, 2, \dots, M$, and*

(ii) *strongly to a point y^* such that $y^* \in \bigcap_{i=1}^m C_i$ if the additional assumption $\text{int} \bigcap_{i \in I} C_i \neq \emptyset$ holds.*

Proof. (i) Define $Q_t := \sum_{\ell=1}^{\gamma(t)-1} w_\ell^t T_{i_\ell^t, i_{\ell+1}^t} + w_{\gamma(t)}^t T_{i_{\gamma(t)}^t, i_1^t}$ and let us look at the sequence

$$y^{k+1} = Q_M Q_{M-1} \cdots Q_2 Q_1 (y^k) := \left(\prod_{t=1}^M Q_t \right) (y^k), \quad (3.20)$$

wherein the order of multiplication of operators is as indicated. By [19, Corollary 4.3.17(iv)], the operators on the right-hand sides of (3.6) and (3.7) are firmly nonexpansive, for all $\ell = 1, 2, \dots, \gamma(t)$, so, by Proposition 2, they are nonexpansive. Each point in the nonempty intersection $\bigcap_{i \in I} C_i$, is a fixed point for each 2-set-DR operator, thus, the operators on the right-hand sides of (3.6) and (3.7) have a common fixed point. Then by Theorem 7, for $w_\ell^t > 0$, for all $\ell = 1, 2, \dots, \gamma(t)$, and $\sum_{\ell=1}^{\gamma(t)} w_\ell^t = 1$,

$$\text{Fix} Q_t = \left(\bigcap_{\ell=1}^{\gamma(t)-1} \text{Fix} T_{i_\ell^t, i_{\ell+1}^t} \right) \cap \text{Fix} T_{i_{\gamma(t)}^t, i_1^t} \supseteq \bigcap_{i=1}^m C_i \neq \emptyset. \quad (3.21)$$

By Proposition 2 the operator Q_t is firmly nonexpansive, thus, nonexpansive. Furthermore, by (3.21) it has a fixed point and, so, by Theorem 9 it is asymptotically regular. By (3.21)

$$\bigcap_{t=1}^M (\text{Fix} Q_t) \supseteq \bigcap_{i=1}^m C_i \neq \emptyset, \quad (3.22)$$

so, by [14, Lemma 2.3]

$$\text{Fix} \left(\prod_{t=1}^M Q_t \right) = \bigcap_{t=1}^M (\text{Fix} Q_t). \quad (3.23)$$

By [10, Theorem 4.6] the operator $\prod_{t=1}^M Q_t$ is asymptotically regular. Since the composition of firmly nonexpansive operators is always nonexpansive, by Theorem 12, $\{y^k\}_{k=0}^\infty$, generated by (3.20) and the BI-DR Algorithm 2, converges weakly to a point y^* such that:

$$y^* \in \text{Fix} \left(\prod_{t=1}^M Q_t \right). \quad (3.24)$$

By (3.24), (3.23) and (3.21) we have:

$$y^* \in \bigcap_{t=1}^M \left(\left(\bigcap_{\ell=1}^{\gamma(t)-1} \text{Fix} T_{i_\ell^t, i_{\ell+1}^t} \right) \cap \text{Fix} T_{i_{\gamma(t)}^t, i_1^t} \right). \quad (3.25)$$

By Lemma 15, $P_{C_{i_\ell^t}} \text{Fix} T_{i_\ell^t, i_{\ell+1}^t} = C_{i_\ell^t} \cap C_{i_{\ell+1}^t}$ for $\ell = 1, 2, \dots, \gamma(t) - 1$, and $P_{C_{i_{\gamma(t)}^t}} \text{Fix} T_{i_{\gamma(t)}^t, i_1^t} = C_{i_{\gamma(t)}^t} \cap C_{i_1^t}$. So, $P_{C_{i_\ell^t}}(y^*) \in C_{i_{\ell+1}^t}$ for $\ell = 1, 2, \dots, \gamma(t) - 1$, and $P_{C_{i_{\gamma(t)}^t}}(y^*) \in C_{i_1^t}$. By the characterization of projections, e.g., [19, Theorem 1.2.4] we prove the following:

$$\begin{aligned} 0 &\geq 2 \sum_{\ell=1}^{\gamma(t)-1} \left\langle y^* - P_{C_{i_{\ell+1}^t}}(y^*), P_{C_{i_\ell^t}}(y^*) - P_{C_{i_{\ell+1}^t}}(y^*) \right\rangle \\ &\quad + 2 \left\langle y^* - P_{C_{i_1^t}}(y^*), P_{C_{i_{\gamma(t)}^t}}(y^*) - P_{C_{i_1^t}}(y^*) \right\rangle \\ &= \sum_{\ell=1}^{\gamma(t)-1} \left\| y^* - P_{C_{i_{\ell+1}^t}}(y^*) \right\|^2 + \left\| y^* - P_{C_{i_1^t}}(y^*) \right\|^2 - \sum_{\ell=1}^{\gamma(t)-1} \left\| y^* - P_{C_{i_\ell^t}}(y^*) \right\|^2 \\ &\quad - \left\| y^* - P_{C_{i_{\gamma(t)}^t}}(y^*) \right\|^2 + \sum_{\ell=1}^{\gamma(t)-1} \left\| P_{C_{i_{\ell+1}^t}}(y^*) - P_{C_{i_\ell^t}}(y^*) \right\|^2 + \left\| P_{C_{i_1^t}}(y^*) - P_{C_{i_{\gamma(t)}^t}}(y^*) \right\|^2 \\ &= \sum_{\ell=1}^{\gamma(t)-1} \left\| P_{C_{i_{\ell+1}^t}}(y^*) - P_{C_{i_\ell^t}}(y^*) \right\|^2 + \left\| P_{C_{i_1^t}}(y^*) - P_{C_{i_{\gamma(t)}^t}}(y^*) \right\|^2. \quad (3.26) \end{aligned}$$

Since the right-hand side of (3.26) is nonnegative it must be equal to zero. So, we have $P_{C_{i_{\ell+1}^t}}(y^*) = P_{C_{i_\ell^t}}(y^*)$ for $\ell = 1, 2, \dots, \gamma(t) - 1$, and $P_{C_{i_1^t}}(y^*) = P_{C_{i_{\gamma(t)}^t}}(y^*)$. Therefore, $P_{C_{i_\ell^t}}(y^*) \in \bigcap_{\ell=1}^{\gamma(t)} C_{i_\ell^t}$ for $t = 1, 2, \dots, M$.
(ii) Continuing from (3.25), using the additional assumption $\text{int} \bigcap_{i \in I} C_i \neq \emptyset$ and making repeated use of Lemma 16 yields,

$$y^* \in \bigcap_{t=1}^M \left(\bigcap_{\ell=1}^{\gamma(t)} C_{i_\ell^t} \right) = \bigcap_{i=1}^m C_i. \quad (3.27)$$

On the other hand, $\bigcap_{i=1}^m C_i = \text{Fix} \left(\prod_{t=1}^M Q_t \right)$ by applying the right-hand side expression of (3.25) and Lemma 16. Since $\prod_{t=1}^M Q_t$ is a nonexpansive operator with a fixed point, it is quasi-nonexpansive [19, Lemma 2.1.20], and, therefore, it is Fejér monotone with respect to $\text{Fix} \left(\prod_{t=1}^M Q_t \right)$, i.e., with respect to $\bigcap_{i=1}^m C_i$. Thus, $\{y^k\}_{k=0}^\infty$, generated by Algorithm 2 is a Fejér monotone sequence with respect to $\bigcap_{i=1}^m C_i$. By [7, Proposition 5.10], the convergence of $\{y^k\}_{k=0}^\infty$ to y^* is strong. \square

3.3 Special cases

For the String-Averaging Douglas-Rachford algorithm: (i) if all the sets are included in one string then we obtain the Cyclic Douglas-Rachford (CDR) algorithm of [14, Section 3], (ii) if there are exactly two sets in each string then we get an algorithm that can legitimately be called the Simultaneous Douglas-Rachford (SDR) algorithm. This SDR algorithm includes as a special case, when the weights are all equal, the Averaged Douglas-Rachford algorithm of [14, Theorem 3.3]

For the Block-Iterative Douglas-Rachford algorithm: (i) if all the sets are included in one block then we get the same Simultaneous Douglas-Rachford (SDR) algorithm as above, including again the Averaged Douglas-Rachford algorithm of [14, Theorem 3.3] as a special case. (ii) if there are exactly two sets in each block, so that each two consecutive blocks have a common set, then we get again the Cyclic Douglas-Rachford (CDR) algorithm.

Another case worth mentioning occurs if the initial point is included in the first set of each string or each block. Then all the reflections become orthogonal projections and our algorithms coincide with the Strings-Averaging

Projection (SAP) method [21] and with the Block-Iterative Projection (BIP) method [1], respectively. See also [14, Corollary 3.1].

4 A generalized m -set-Douglas-Rachford operator and algorithm

In this section we propose a generalization of the 2-set-DR original operator of Definition 14 that is applicable to m sets and formulate an algorithmic structure to employ it. Instead of reflecting consecutively into two sets and taking the midpoint between the original point and the end-point of the two consecutive reflections as the outcome of the 2-set-DR operator, we propose to allow a finite number, say r (greater or equal 2), of consecutive reflections into r sets and then taking the midpoint between the original point and the end-point of the r consecutive reflections as the outcome of the newly defined operator. We name this as the “generalized r -set-DR operator”, and show how it works algorithmically. Before presenting those we need the following preliminary results.

Proposition 20. *[7, Corollary 4.10] Let C be a nonempty closed convex subset of \mathcal{H} . Then $\text{Id} - P_C$ is firmly nonexpansive and $2P_C - \text{Id}$ is nonexpansive.*

By [19, Lemma 2.1.12] and [19, Fig. 2.14] any composition of nonexpansive (NE) operators is NE and any convex combination of NE operators is NE.

Theorem 21. *[19, Theorem 2.2.10(i)-(iii)] Let $T : X \rightarrow \mathcal{H}$. Then the following conditions are equivalent:*

- i. T is firmly nonexpansive.*
- ii. T_λ is nonexpansive for any $\lambda \in [0, 2]$.*
- iii. T has the form $T = \frac{1}{2}(S + \text{Id})$, where $S : X \rightarrow \mathcal{H}$ is a nonexpansive operator.*

Our generalized Douglas-Rachford operator is presented in the following definition.

Definition 22. Let $C_1, C_2, \dots, C_m \subseteq \mathcal{H}$ be nonempty closed convex sets. For $r = 2, 3, \dots, m$ ($m \geq 2$) define the composite reflection operator $V_{C_1, C_2, \dots, C_r} : \mathcal{H} \rightarrow \mathcal{H}$ by

$$V_{C_1, C_2, \dots, C_r} := R_{C_r} R_{C_{r-1}} \cdots R_{C_1}. \quad (4.1)$$

The generalized r -set-DR operator $T_{C_1, C_2, \dots, C_r} : \mathcal{H} \rightarrow \mathcal{H}$ is defined by

$$T_{C_1, C_2, \dots, C_r} := \frac{1}{2} (\text{Id} + V_{C_1, C_2, \dots, C_r}). \quad (4.2)$$

For $r = 2$ the generalized r -set-DR operator coincides with the 2-set-DR operator. For $r = 3$ the generalized r -set-DR operator coincides with the 3-set-DR iteration defined in [3, Eq. (2)].

We will make use of the following corollary which extends [19, Corollary 4.3.17(ii)].

Corollary 23. Let $C_1, C_2, \dots, C_m \subseteq \mathcal{H}$ be nonempty closed convex sets with a nonempty intersection, and let $V_{C_1, C_2, \dots, C_m} : \mathcal{H} \rightarrow \mathcal{H}$ and T_{C_1, C_2, \dots, C_m} be as in Definition 22. If $\text{int} \bigcap_{i=1}^m C_i \neq \emptyset$ then

$$\bigcap_{i=1}^m C_i = \text{Fix} T_{C_1, C_2, \dots, C_m}. \quad (4.3)$$

Proof. It is clear that $\bigcap_{i=1}^m \text{int} C_i = \bigcap_{i=1}^m \text{int} \text{Fix} R_{C_i} \subseteq \bigcap_{i=1}^m \text{Fix} R_{C_i}$. By [19, Proposition 2.1.41] R_{C_i} , for $i = 1, 2, \dots, m$, are C -strictly quasi-nonexpansive. Theorem 6(ii) and the fact that $\text{Fix} R_{C_i} = C_i$, for $i = 1, 2, \dots, m$, yield

$$\text{Fix} T_{C_1, C_2, \dots, C_m} = \text{Fix} V_{C_1, C_2, \dots, C_m} = \bigcap_{i=1}^m \text{Fix} R_{C_i} = \bigcap_{i=1}^m C_i. \quad (4.4)$$

□

Next we present the algorithm that uses generalized r -set-DR operators and prove its convergence.

Algorithm 3.

Initialization: $x^0 \in \mathcal{H}$ is arbitrary.

Iterative Step: Given the current iterate x^k , calculate, for all $r = 2, 3, \dots, m$,

$$x^{k+1} = \sum_{r=2}^m w_r T_{C_1, C_2, \dots, C_r}(x^k) \quad (4.5)$$

with $w_r > 0$, for all $r = 2, 3, \dots, m$, and $\sum_{r=2}^m w_r = 1$.

Theorem 24. *Let $C_1, C_2, \dots, C_m \subseteq \mathcal{H}$ be nonempty closed convex sets with a nonempty intersection. if*

$$\text{int} \bigcap_{i \in I} C_i \neq \emptyset \quad (4.6)$$

then for any $x^0 \in \mathcal{H}$, any sequence $\{x^k\}_{k=0}^\infty$, generated by Algorithm 3, converges strongly to a point $x^* \in \bigcap_{i \in I} C_i$.

Proof. By Proposition 20, the reflection R_{C_i} for all $i = 1, 2, \dots, m$, is non-expansive operator. By the facts noted after Proposition 20, the operator V_{C_1, C_2, \dots, C_r} defined in (4.1) is nonexpansive. By Theorem 21(i) and (iii), the operator T_{C_1, C_2, \dots, C_r} defined in (4.2) is firmly nonexpansive, and by Proposition 2 it is nonexpansive. Since $\bigcap_{i=1}^m C_i \neq \emptyset$, any point in the intersection is a fixed point of the reflection R_{C_i} for all $i = 1, 2, \dots, m$, and such point is also a fixed point of any operator T_{C_1, C_2, \dots, C_r} for all $r = 2, 3, \dots, m$. By Theorem 7 for $w_r > 0$, for all $r = 2, 3, \dots, m$, and $\sum_{r=2}^m w_r = 1$,

$$\text{Fix} \left(\sum_{r=2}^m w_r T_{C_1, C_2, \dots, C_r} \right) = \bigcap_{r=2}^m \text{Fix} T_{C_1, C_2, \dots, C_r} \supseteq \bigcap_{i=1}^m C_i \neq \emptyset. \quad (4.7)$$

Using again Proposition 2, the operator $\sum_{r=2}^m w_r T_{C_1, C_2, \dots, C_r}$ is firmly non-expansive, thus, nonexpansive, and has a fixed point according to (4.7). So, by Theorem 9, it is asymptotically regular, therefore, by Theorem 12, $\{x^k\}_{k=0}^\infty$, generated by Algorithm 3, converges weakly to a point $x^* \in \mathcal{H}$ for which

$$x^* \in \text{Fix} \left(\sum_{r=2}^m w_r T_{C_1, C_2, \dots, C_r} \right). \quad (4.8)$$

By (4.7) and Corollary 23 we have

$$\text{Fix} \left(\sum_{r=2}^m w_r T_{C_1, C_2, \dots, C_r} \right) = \bigcap_{r=2}^m \text{Fix} T_{C_1, C_2, \dots, C_r} = \bigcap_{r=2}^m \left(\bigcap_{i=1}^r C_i \right) = \bigcap_{d=1}^m C_d, \quad (4.9)$$

therefore, $x^* \in \bigcap_{i \in I} C_i$. To prove that the convergence is strong recall that $\sum_{r=2}^m w_r T_{C_1, C_2, \dots, C_r}$ is firmly nonexpansive, thus nonexpansive, according to [19, Theorem 2.2.4], with a fixed point. Therefore, it is quasi-nonexpansive [19, Lemma 2.1.20], and thus is Fejér monotone. Hence, $\{x^k\}_{k=0}^\infty$, generated by the Algorithm 3, is a Fejér monotone sequence. By [7, Proposition 5.10], the convergence of $\{x^k\}_{k=0}^\infty$ to x^* is strong. \square

If we replace the reflections by projections, we get the convergence to a point in the intersection of the sets, because the algorithm then becomes a special case of string-averaging projections (SAP), see, [24].

Acknowledgments. We thank Andrzej Cegielski for having read carefully our work and for his insightful comments. We thank Rafał Zalas for a discussion on an earlier draft of this paper. We greatly appreciate the insightful and constructive comments of two anonymous referees and the Associate Editor which helped us improve the paper. This work was supported by Research Grant No. 2013003 of the United States-Israel Binational Science Foundation (BSF).

References

- [1] R. Aharoni and Y. Censor, *Block-iterative projection methods for parallel computation of solutions to convex feasibility problems*, Linear Algebra and its Applications **120** (1989), 165–175.
- [2] F. J. A. Artacho, J. M. Borwein, and M. K. Tam, *Douglas-Rachford feasibility methods for matrix completion problems*, The ANZIAM Journal **55** (2014), 299–326.
- [3] ———, *Recent results on Douglas-Rachford methods for combinatorial optimization problems*, Journal of Optimization Theory and Applications **163** (2014), 1–30.
- [4] F. J. Aragón Artacho and J. M. Borwein, *Global convergence of a non-convex Douglas-Rachford iteration*, Journal of Global Optimization **57** (2013), 753–769.
- [5] H. H. Bauschke, *The composition of projections onto closed convex sets in Hilbert space is asymptotically regular*, Proceedings of the American Mathematical Society **131** (2003), 141–146.
- [6] H. H. Bauschke and J. M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Review **38** (1996), 367–426.
- [7] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, NY, USA, 2011.

- [8] H. H. Bauschke, P. L. Combettes, and D. R. Luke, *Finding best approximation pairs relative to two closed convex sets in Hilbert spaces*, Journal of Approximation Theory **127** (2004), 178–192.
- [9] ———, *A strongly convergent reflection method for finding the projection onto the intersection of two closed convex sets in a Hilbert space*, Journal of Approximation Theory **141** (2006), 63–69.
- [10] H. H. Bauschke, V. Martín-Márquez, S. M. Moffat, and X. Wang, *Compositions and convex combinations of asymptotically regular firmly non-expansive mappings are also asymptotically regular*, Fixed Point Theory and Applications **2012:53** (2012), 1–11.
- [11] H. H. Bauschke and D. Noll, *On the local convergence of the Douglas-Rachford algorithm*, Archiv der Mathematik **102** (2014), 589–600.
- [12] H. H. Bauschke, D. Noll, and H. M. Phan, *Linear and strong convergence of algorithms involving averaged nonexpansive operators*, Journal of Mathematical Analysis and Applications **421** (2015), 1–20.
- [13] J. M. Borwein and B. Sims, *The Douglas-Rachford algorithm in the absence of convexity*, Fixed-Point Algorithms for Inverse Problems in Science and Engineering (H. H. Bauschke, R. S. Burachik, P. L. Combettes, V. Elser, D. R. Luke, and H. Wolkowicz, eds.), Springer, 2011, pp. 93–109.
- [14] J. M. Borwein and M. K. Tam, *A cyclic Douglas-Rachford iteration scheme*, Journal of Optimization Theory and Applications **160** (2014), 1–29.
- [15] ———, *Reflection methods for inverse problems with application to protein conformation determination*, arXiv preprint arXiv:1408.4213 (2014).
- [16] ———, *The cyclic Douglas-Rachford method for inconsistent feasibility problems*, Nonlinear Convex Analysis **16** (2015), 537–584.
- [17] D. Butnariu and Y. Censor, *On the behavior of a block-iterative projection method for solving convex feasibility problems*, International Journal of Computer Mathematics **34** (1990), 79–94.

- [18] C. L. Byrne, *Block-iterative methods for image reconstruction from projections*, IEEE Transactions on Image Processing **5** (1995), 792–794.
- [19] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Lecture Notes in Mathematics 2057, Springer-Verlag, Berlin, Heidelberg, Germany, 2012.
- [20] Y. Censor and T. Elfving, *Block-iterative algorithms with diagonally scaled oblique projections for the linear feasibility problem*, SIAM Journal on Matrix Analysis and Applications **24** (2002), 40–58.
- [21] Y. Censor, T. Elfving, and G. T. Herman, *Averaging strings of sequential iterations for convex feasibility problems*, Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (D. Butnariu, Y. Censor, and S. Reich, eds.), Elsevier Science Publishers, Amsterdam, The Netherlands, 2001, pp. 101–114.
- [22] Y. Censor and A. Segal, *On the string averaging method for sparse common fixed-point problems*, International Transactions in Operational Research **16** (2009), 481–494.
- [23] Y. Censor and E. Tom, *Convergence of string-averaging projection schemes for inconsistent convex feasibility problems*, Optimization Methods and Software **18** (2003), 543–554.
- [24] Y. Censor and A. J. Zaslavski, *Convergence and perturbation resilience of dynamic string-averaging projection methods*, Computational Optimization and Applications **54** (2013), 65–76.
- [25] Y. Censor, T. Elfving, G. T. Herman, and T. Nikazad, *On diagonally relaxed orthogonal projection methods*, SIAM Journal on Scientific Computing **30** (2008), 473–504.
- [26] P. L. Combettes, *Solving monotone inclusions via compositions of non-expansive averaged operators*, Optimization **53** (2004), 475–504.
- [27] P. L. Combettes and J. C. Pesquet, *Proximal splitting methods in signal processing*, Fixed-Point Algorithms for Inverse Problems in Science and Engineering (H. H. Bauschke, R. S. Burachik, P. L. Combettes, V. Elser, D. R. Luke, and H. Wolkowicz, eds.), Springer, 2011, pp. 185–212.

- [28] D. Gordon and R. Gordon, *Component-averaged row projections: A robust, block-parallel scheme for sparse linear systems*, SIAM Journal on Scientific Computing **27** (2005), 1092–1117.
- [29] N. T. Karonis, K. L. Duffin, C. E. Ordoñez, B. Erdely, T. D. Uram, E. C. Olson, G. Coutrakon, and M. E. Papka, *Distributed and hardware accelerated computing for clinical medical imaging using proton computed tomography (pCT)*, Journal of Parallel and Distributed Computing **73** (2013), 1605–1612.
- [30] B. F. Svaiter, *On weak convergence of the Douglas-Rachford method*, SIAM Journal on Control and Optimization **49** (2011), 280–287.