On the Polyhedral Structure of Two-Level Lot-Sizing Problems with Supplier Selection

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Abstract

In this paper, we study a two-level lot-sizing problem with supplier selection (LSS). This *NP-hard* problem arises in different production planning and supply chain management applications. We first present a dynamic programming algorithm for LSS that is polynomial when the number of plants is fixed. We use this algorithm to describe the convex hull of solutions to the problem in an extended space of variables. We then investigate a smaller "multi-commodity" extended formulation, show that it has fractional vertices, and present a family of valid inequalities to strengthen it. Next, we explore the polyhedral structure of the formulation of the problem with traditional variables. We derive several families of strong valid inequalities for this formulation, and give conditions under which they are facet-defining. We prove that these new families of facet-defining inequalities are sufficient to describe the convex hull of the problem with two plants and two periods. Finally, we show numerically that incorporating these valid inequalities within a cut-and-branch framework leads to significant improvement in computation.

1 Introduction and literature review

We consider a supplier with P production plants and a single customer whose (positive) deterministic demand for each time period is known. Customer demand may be produced in any of the plants but must be satisfied without backlogging. In this setting, it is possible to carry inventory at both the customer location and at production plants. The supplier has control over the timing of shipments which leads to the coordination of production and transportation decisions. We seek to determine an optimal production and transportation schedule that minimizes fixed and variable production costs, as well as inventory holding and transportation costs. We assume that production and shipment capacities in each period are sufficiently large to satisfy the entire demand. We also assume that beginning and ending inventory levels are zero across the supply chain. The above problem, which we coin *lot-sizing with supplier selection* (LSS), arises in vendor managed inventory, requirements planning with substitutions, and multi-item multi-facility supply chain planning. We elaborate on these applications next.

Vendor managed inventory (VMI) is a supply chain initiative where the supplier is authorized to manage the inventory of agreed-upon SKUs at retail locations. This approach, where inventory across multiple echelons is managed collectively, has long attracted the interest of the supply chain research community; see Çetinkaya and Lee [2000]. Successful practical applications of this idea have shown that (*i*) economies of scale can be achieved through shipment and production consolidation, (*ii*) bullwhip effect can be reduced through integration of information systems, (*iii*) stockout frequency can be reduced, and (*iv*) cost savings for the entire system are possible.

Requirements planning with substitutions is a production planning problem that arises from the desire to exploit flexibility in bill of materials. Specifically, the problem is a two-stage production planning problem where it is possible to use substitute components or subassemblies produced by an upstream stage to meet demand in each period at the downstream stage; see Balakrishnan and Geunes [2000]. Requirements planning with substitutions reduces to LSS, if we view substitute parts as different suppliers, and assume that a single part is required for each end product.

Cross-facility capacity management is crucial in high technology industries with high capital investment and short life cycles. The problem can be formulated as a multi commodity network flow model for multi-product, multi-facility production planning; see Wu and Golbasi [2004]. The single-product multi-facility substructure, which corresponds to LSS, is identified in Wu and Golbasi [2004] as a main source of difficulty in solving the problem. Similarly, LSS arises as a substructure of virtually all multi-item multi-facility deterministic production planning problems.

In this paper, we study LSS from a polyhedral perspective. Because strong inequalities generated for this common (sub)structure can be used as cutting planes for all models that contain it, our results can be used in the solution of other multi-level multi-facility lot-sizing problems. We next review lot-sizing literature, focusing on polyhedral studies that are most related to our work.

Many variants of the lot-sizing problem have been studied in the past. Most early studies of lot-sizing focus on single echelon problems. Building on an extended-formulation for facility location given by Krarup and Bilde [1977], Barany et al. [1984] give a linear description of the convex hull of the single-item uncapacitated lot-sizing problem. Convex hull descriptions or strong valid inequalities have been derived for variants of lot-sizing that allow backlogging (Küçükyavuz and Pochet [2009], Pochet and Wolsey [1988]), capacities (Atamtürk and Küçükyavuz [2005], Atamtürk and Muñoz [2004], Pochet and Wolsey [1993]) and stochastic demand (Guan et al. [2006a], Guan et al. [2006b]), among others. We refer interested readers to Pochet and Wolsey [2006] for an extensive discussion.

Polyhedral studies of multi-echelon lot-sizing problems have received comparably less interest in the literature. Gaglioppa et al. [2008] give valid inequalities for a variant of the problem where assembly structures are complex. Recent results on serial assembly are obtained in Melo and Wolsey [2010]. In particular, the authors give a convex hull description for the two-echelon case in a higher dimension through the use of dynamic programming. Zhang et al. [2012] extend this work by incorporating intermediate demand and considering multiple echelons. The authors derive a family of facet-defining inequalities for the problem and propose a hierarchy of formulations whose strengths are compared both theoretically and numerically. Zhao and Klabjan [2012] study a supplier selection extension of the traditional lot-sizing problem where at each period a lot-size and a subset of suppliers must be selected. The authors provide a convex hull description for the uncapacitated case. Macaron [2005] studies algorithms and valid inequalities for LSS for the special case where there are two plants. In particular, a dynamic programming algorithm is presented and valid inequalities are introduced. The results we derive in this paper are for the extension of the problem where P plants are available for production. Further, a more comprehensive study of formulations, and families of facet-defining inequalities is undertaken.

In Section 2, we introduce the problem and describe a mixed integer programming (MIP) formulation with traditional variables. We then make observations about the structure of optimal solutions. We use these observations in Section 3 to give a dynamic programming algorithm for the solution of the problem. This algorithm is polynomial when the number of production plants is fixed. We then use the dynamic programming recursion to develop a linear description of the convex hull of solutions to the problem in a higher dimensional space. We also describe an alternative formulation that is smaller in size but has fractional vertices. We present a family of valid inequalities to strengthen it. In Section 4,

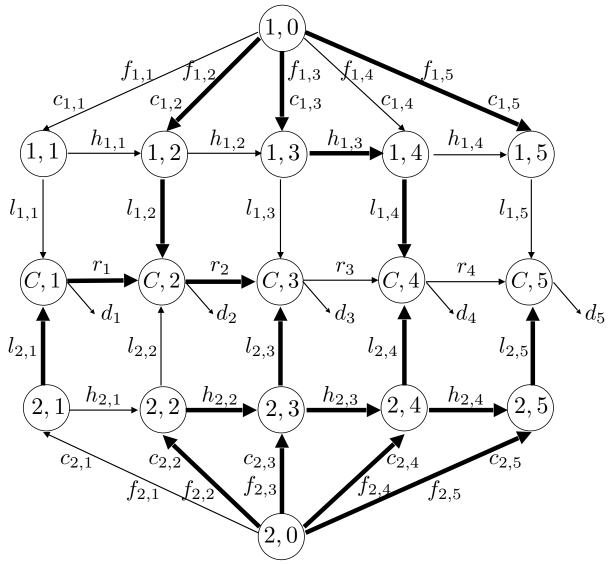


Figure 1: Instance of LSS with $P = 2$ and $T = 5$.

we study the polyhedral structure of a formulation of the problem with traditional variables. We identify several families of facet-defining inequalities. In Section 7.8, we prove that the families of inequalities we derive are sufficient to describe the convex hull of the problem with two plants and two periods. In Section 5, we evaluate the effectiveness of these inequalities inside of a branch-and-cut algorithm. We conclude the paper in Section 6.

2 Mathematical formulation

For $p \in \mathcal{P} := \{1, \ldots, P\}$ and $t \in \mathcal{T} := \{1, \ldots, T\}$, we let $f_{p,t}$ and $c_{p,t}$ be the fixed and variable cost of production at plant p in period t, respectively. We also let $h_{p,t}$ be the variable inventory holding cost, $l_{p,t}$ be the variable transportation cost for plant p in period t, and r_t be the variable inventory holding cost at the customer level in period t. A network representation of an instance where $P = 2$ and $T = 5$ is given in Figure 1. Let $\mathcal{T}_0 = \mathcal{T} \bigcup \{0\}$. In this representation, we include nodes (p, t) for $p \in \mathcal{P}$ and $t \in \mathcal{T}_0$ as well as nodes (C, t) for $t \in \mathcal{T}$. An arc from node $(p, 0)$ to any other node (p, t) represents a production decision at plant p in period t. Moreover an arc from node (p, t) to node (C, t) represents a shipment decision from plant p in period t. The remaining arcs represent inventory holding decisions.

A formulation of the problem is given by

$$\min \sum_{p\in\mathcal{P}} \sum_{t\in\mathcal{T}} (f_{p,t}y_{p,t} + c_{p,t}x_{p,t} + h_{p,t}s_{p,t} + l_{p,t}v_{p,t}) + \sum_{t\in\mathcal{T}} r_t\sigma_t \tag{1}$$

$$s.t. \quad s_{p,t-1} + x_{p,t} = v_{p,t} + s_{p,t} \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{2}$$

$$\sigma_{t-1} + \sum_{p\in\mathcal{P}} v_{p,t} = d_t + \sigma_t \qquad \forall t \in \mathcal{T} \tag{3}$$

$$x_{p,t} \leqslant d_{tT} y_{p,t} \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{4}$$

$$y_{p,t} \leqslant 1 \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{5}$$

$$\sigma \in \mathbb{R}_{+}^{T}, s, v, x \in \mathbb{R}_{+}^{PT}, y \in \mathbb{Z}_{+}^{PT}, \tag{6}$$

where, for $p \in \mathcal{P}$ and $t \in \mathcal{T}$, $x_{p,t}$ is the quantity produced at plant p in period t, $y_{p,t}$ is the corresponding setup variable, $v_{p,t}$ is the quantity shipped from plant p in period t, and $\sigma_t/s_{p,t}$ are the levels of inventory at the end of period t for the customer/plant p, respectively.

We can eliminate the variables σ_t and $s_{p,t}$ from this formulation by using the relations $\sigma_t = \left(\sum_{p\in\mathcal{P}} \sum_{i=1}^{t} v_{p,i}\right) - d_{1t}$ and $s_{p,t} = \sum_{i=1}^{t} (x_{p,i} - v_{p,i})$, where we define $d_{1t} = \sum_{i=1}^{t} d_i$ for $t \in \mathcal{T}$. After performing these substitutions, (1)-(6) can be written as

$$\min \sum_{p\in\mathcal{P}} \sum_{t\in\mathcal{T}} (f_{p,t} y_{p,t} + \gamma_{p,t} x_{p,t} + \delta_{p,t} v_{p,t}) - C \tag{7}$$

$$s.t. \quad \sum_{i=1}^{t} x_{p,i} \geqslant \sum_{i=1}^{t} v_{p,i} \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T}\backslash\{T\} \tag{8}$$

$$\sum_{i\in\mathcal{T}} x_{p,i} = \sum_{i\in\mathcal{T}} v_{p,i} \qquad \forall p \in \mathcal{P} \tag{9}$$

$$\sum_{p\in\mathcal{P}} \sum_{i=1}^{t} v_{p,i} \geqslant d_{1t} \qquad \forall t \in \mathcal{T}\backslash\{T\} \tag{10}$$

$$\sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{T}} v_{p,i} = d_{1T} \tag{11}$$

$$x_{p,t} \leqslant d_{tT} y_{p,t} \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{12}$$

$$y_{p,t} \leqslant 1 \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{13}$$

$$v, x \in \mathbb{R}_{+}^{PT}, y \in \mathbb{Z}_{+}^{PT}, \tag{14}$$

where $C = \sum_{t\in\mathcal{T}} r_t d_{1t}$, $\gamma_{p,t} = c_{p,t} + \sum_{i=t}^{T} h_{p,i}$ and $\delta_{p,t} = l_{p,t} + \sum_{i=t}^{T} (r_i - h_{p,i})$, for $p \in \mathcal{P}$ and for $t \in \mathcal{T}$. In the remainder of this paper, we refer to (7)-(14) as the *natural formulation* of LSS. We next make some observations about the structure of optimal solutions to LSS.

Observation 2.1. When $P \geqslant 2$, LSS might not have an optimal solution that can be described through *regeneration intervals*, *i.e.*, time intervals with no inventory entering or leaving the interval, and production in the first period satisfying demand in all periods of the interval; see Pochet and Wolsey [2006] for a discussion.

Observation 2.1 implies that there is a fundamental difference between classical lot-sizing problems and the problem we study. To illustrate it, consider the instance given in Figure 1 where the bold arcs have very large fixed and variable production, shipment and/or inventory holding costs compared to those of the light arcs. The unique optimal solution is to produce $d_1 + d_3 + d_4$ at plant 1 in period 1, d_2 at plant 2 in period 1, and d_5 at plant 1 in period 4. In this solution, the production for plant 1 does not exhibit regeneration intervals and there is production at both plants in the same period. We note however that each period demand is satisfied by the production of a single plant. This solution characteristic can be shown to hold more generally, as we discuss next.

By adding an artificial source node that connects to nodes $(p, 0)$ for $p \in \mathcal{P}$ with cost 0, LSS reduces to a single source uncapacitated fixed charge network flow problem. Therefore, the extreme point solutions of LSS correspond to trees rooted at the artificial source node with arcs not in the tree carrying zero flow; see Balakrishnan and Geunes [2000]. It follows that no more than a single arc among those entering a node can carry positive flow. The following result ensues.

Proposition 2.1. *There is an optimal solution to LSS such that*

(i) $\sigma_{t-1}\left(\sum_{p\in\mathcal{P}} v_{p,t}\right) = 0$ *for* $t \in \mathcal{T}$,

(ii) $v_{p,t}v_{\hat{p},t} = 0$ *for* $t \in \mathcal{T}$ *and* $p, \hat{p} \in \mathcal{P}$,

(iii) If (p,t) *is a production node then* $x_{p,t} = \sum_{i\in\tau_p(t)} d_i$ *where* $\tau_p(t) \subseteq \{t, \ldots, T\}$. *Moreover*

(a) for $\hat{t} > t$ *and* $x_{p,\hat{t}} = \sum_{i\in\tau_p(\hat{t})} d_i$ *we have that* $i < j$ *for each* $i \in \tau_p(t)$ *and* $j \in \tau_p(\hat{t})$,

(b) for $\hat{p} \neq p$ *and* $x_{\hat{p},t} = \sum_{i\in\tau_{\hat{p}}(t)} d_i$ *we have that* $\tau_p(t) \bigcap \tau_{\hat{p}}(t) = \varnothing$.

Similar observations have been made by Balakrishnan and Geunes [2000] and Wu and Golbasi [2004].

3 Dynamic programming algorithm and extended formulations

In this section we describe a dynamic programming algorithm to solve LSS. This algorithm can be used to derive an extended formulation that provides the convex hull of integer solutions; see 7.2. We then describe alternative formulations that are smaller in size but have more variables than the natural formulation. We show that these formulations are not ideal, and propose a family of valid inequalities to strengthen them.

3.1 Dynamic programming algorithm

We first argue that we should not expect to find a polynomial algorithm to solve LSS as this problem is *NP-hard*. Wu and Golbasi [2004] prove that the single-item multi-facility supply chain planning problem is *NP-hard*. Their proof involves a reduction from the facility location problem which is known to be *NP-hard*. The only difference between LSS and the single-item multi-facility supply chain planning problem given in Wu and Golbasi [2004] is that, in LSS, it is possible to hold inventory at the customer level. We therefore adapt the proof given by Wu and Golbasi [2004] to show that LSS is *NP-hard* by reduction from the facility location problem. Consider an instance of the facility location problem with a set $\mathcal{P}$ of facilities and a set $\mathcal{T}$ of customers. We denote the fixed cost of opening facility p by K_p and the cost of assigning customer t to facility p by $a_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$. We next reduce this instance to one of LSS. This LSS instance has a set $\mathcal{P}$ of plants, and a set $\mathcal{T}$ of time periods. We let $f_{p,t} = K_p$, $l_{p,t} = a_{p,t}$, $h_{p,t} = 0$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$, $d_t = 1$, $r_t = \max_{p\in\mathcal{P}} \max_{\tau\in\mathcal{T}}\{a_{p,\tau}\} + 1$ for $t \in \mathcal{T}$, $c_{p,1} = 0$ and $c_{p,t} = 1$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}\backslash\{1\}$. Consider any feasible solution to the facility location problem where facility p is assigned to the subset of customers τ_p of $\mathcal{T}$. Then a feasible solution of same objective value can be obtained for LSS by producing the demand of periods τ_p in plant p in period 1, and by shipping each demand when it is due. Similarly, take any optimal solution to the LSS instance. It is easy to see that it must be such that $x_{p,t} = 0$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}\backslash\{1\}$. It follows that $x_{p,1} = \sum_{i\in\tau_p} d_i$ for some $\tau_p \subseteq \mathcal{T}$ and $v_{p,i} = d_i$ for $i \in \tau_p$. It can be readily verified that the facility location solution where customers τ_p are assigned to plant p, has the same objective value as this solution. This shows that LSS is *NP-hard*.

We next describe an $O\left(PT^{P+1}\right)$ dynamic programming algorithm to solve LSS. A similar algorithm to solve LSS with nonnegative setup costs is given in Balakrishnan and Geunes [2000]. Although, these two algorithms have the same running time, the version we present here is different in terms of how stages are defined and connected. We present this version as it makes it easier to interpret the dual variables that arise when deriving an extended formulation based on the dynamic programming recursion.

In the dynamic program presented below, we assume that there are no holding costs in the system as the transformation presented before (7)-(14) allows for these costs to be incorporated into shipment and production costs. In the following, we use $\gamma_{p,t}$ for production costs and $\delta_{p,t}$ for shipment costs. We also define $f^+_{p,t} = \max\{f_{p,t}, 0\}$ and $f^-_{p,t} = \min\{f_{p,t}, 0\}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$.

Observation 3.1. If the optimal value of LSS with fixed costs $f^+_{p,t}$ is z^*, then the optimal value of LSS with fixed costs $f_{p,t}$ is $z^{\text{opt}} = z^* + f^-$, where $f^- = \sum_{p\in\mathcal{P}} \sum_{t\in\mathcal{T}} f^-_{p,t}$.

We next present our dynamic programming algorithm for LSS with costs $f^+_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$. As a preprocessing step, we calculate for each production period j and demand period $t \geqslant j$, the time period at which items produced in period j to meet the demand in period t are shipped from facility p to the customer. This can be done by solving a shortest path problem between nodes (p, j) and (C, t) in the network described in Section 2 using the modified shipment costs $\delta_{p,t}$. Denote by $H^p(j, t)$ the length of this shortest path. It is clear that $H^p(j, t) = \min\{\delta_{p,j}, \ldots, \delta_{p,t}\}$. To reduce the amount of computation needed, we may calculate $H^p(j, t)$, for $p \in \mathcal{P}$, for $j \in \mathcal{T}$ and for $t \in \{j, \ldots, T\}$ using the recursion $H^p(j, t) = \min\{H^p(j, t-1), \delta_{p,t}\}$ where $H^p(j, j-1) = \infty$ for $j \in \mathcal{T}$. It follows that the time needed to compute all relevant values of $H^p(j, t)$ is bounded above by $O\left(PT^2\right)$ assuming that $\delta_{p,t}$ has been pre-computed. Similarly, it can be verified that the amount of computation required in calculating $\delta_{p,t}$ is bounded above by $O(PT)$.

For $t \in \mathcal{T}$ and $\mathbf{i} = \left(i^1, \ldots, i^P\right) \in \mathcal{T}_0^P$, we let $G(t, \mathbf{i})$ be the minimum cost of satisfying the entire demand up to and including that of period t, with the additional requirement that d_t is produced with setup $\mathbf{i}$, *i.e.*, from plant p in period i^p for some $p \in \mathcal{P}$ and that all production in plant p occurs no later than period i^p. In particular, we take $i^p = 0$ as representing the situation where plant p does not produce during the time window $\{1, \ldots, t\}$. By definition $G(0, \mathbf{0}) = 0$ and $G(t, \mathbf{0}) = \infty$ for $t \in \mathcal{T}$. For a given setup vector $\mathbf{i}$, we define $\mathcal{P}^+(\mathbf{i}) = \{p \in \mathcal{P} \mid i^p > 0\}$ as the set of plants from which demand may be satisfied. For $t \in \mathcal{T}_0$, we define $\mathcal{I}(t) = \{\mathbf{i} \in \mathcal{T}_0^P \mid i^p \leqslant t, \forall p \in \mathcal{P}\}$ as the set of setup vectors compatible with the production of d_{1t}. Moreover, for $t \in \mathcal{T}_0$ and $\mathbf{i} \in \mathcal{I}(t)$, we let $F(t, \mathbf{i}) = \{p \in \mathcal{P} \mid i^p = t\}$, that is the set of plants for which a setup is performed in period t. We also define $\mathcal{K}(t, \mathbf{i}) = \{\mathbf{j} \in \mathcal{I}(t-1) \mid j^p \leqslant i^p - 1 \text{ for } p \in F(t, \mathbf{i}) \text{ and } j^p = i^p \text{ for } p \in \mathcal{P}\backslash F(t, \mathbf{i})\}$ as the set of predecessors of vector $\mathbf{i} \in \mathcal{I}(t)$ in period $t \in \mathcal{T}$.

Then, for $t \in \mathcal{T}$ and $\mathbf{i} \in \mathcal{I}(t)$, we calculate $G(t, \mathbf{i})$ recursively as

$$G(t, \mathbf{i}) = \min_{\mathbf{j}\in\mathcal{K}(t,\mathbf{i})} G(t-1, \mathbf{j}) + \sum_{p\in F(t,\mathbf{i})} f^+_{p,t} + \min_{p\in\mathcal{P}^+(\mathbf{i})} \{\gamma_{p,i^p} + H^p(i^p, t)\} d_t. \tag{15}$$

Recursion (15) states that the minimum cost of producing d_{1t} with setup vector $\mathbf{i}$ is the sum of the minimum cost of producing d_{1t-1} with a compatible setup vector $\mathbf{j}$, the fixed cost required for making the transition from vector $\mathbf{j}$ in period $t-1$ to vector $\mathbf{i}$ in period t, and the minimum cost of producing d_t with setup vector $\mathbf{i}$. Observe that, if $\mathbf{j} \in \mathcal{K}(t, \mathbf{i})$ is such that $\mathbf{i} = \mathbf{j}$ then we do not need to perform a new setup in period t since $F(t, \mathbf{i}) = \varnothing$.

Now, define $G(T)$ as the minimum cost of satisfying the demand up to and including period T. It is clear that

$$G(T) = \min_{\mathbf{i}\in\mathcal{I}(T)} G(T, \mathbf{i}) + f^-. \tag{16}$$

The following result is proven in 7.1.

Theorem 3.1. *The dynamic programming recursions given by* (15)-(16) *solve LSS in time* $O\left(P(2T)^{P+1}\right)$.

In particular, this algorithm is asymptotically faster than that described by Macaron [2005] that has a running time $O(T^4)$ for the special case where $P = 2$. Balakrishnan and Geunes [2000] present a different $O(P(2T)^{P+1})$ algorithm, while Wu and Golbasi [2004] describe an algorithm with running time $O(P^{T+1} T \log T)$.

Using the dynamic programming recursions (15)-(16), it is possible to create an ideal formulation of LSS; see 7.2. The number of variables it uses however is exponential. We next investigate other formulations that use a smaller number of variables.

3.2 Multi-commodity formulation

In this section, we present multi-commodity formulations for LSS. We define $\phi_{p,t,t',t''}$ to be the amount produced at plant p in period t shipped in period t' to satisfy the demand in period t''. We write

$$\min \quad \sum_{p \in \mathcal{P}} \left[\sum_{t \in \mathcal{T}} f_{p,t} y_{p,t} + \sum_{t \in \mathcal{T}} \sum_{t'=t}^{T} \sum_{t''=t'}^{T} \left(\gamma_{p,t} + \delta_{p,t'} \right) \phi_{p,t,t',t''} \right] - C \tag{17}$$

$$s.t. \quad \sum_{p \in \mathcal{P}} \sum_{t=1}^{t''} \sum_{t'=t}^{t''} \phi_{p,t,t' t''} = d_{t''} \qquad \forall t'' \in \mathcal{T} \tag{18}$$

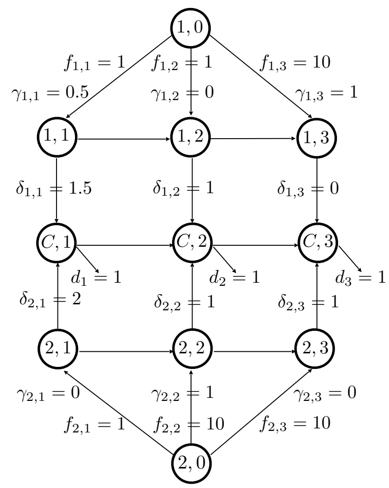
$$\sum_{t'=t}^{t''} \phi_{p,t,t',t''} \leqslant d_{t''} y_{p,t} \qquad \forall p \in \mathcal{P}, \forall t'' \in \mathcal{T}, \forall t \leqslant t'' \tag{19}$$

$$y_{p,t} \leqslant 1 \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{20}$$

$$\phi \in \mathbb{R}_+^{\frac{PT^3+3PT^2+2PT}{6}}, y \in \mathbb{Z}_+^{PT}. \tag{21}$$

In Section 5, we show empirically that this formulation provides a strong linear programming relaxation for LSS. Since it has $O(PT^3)$ variables, it becomes less appealing to use in practice as the number of periods increases. Observe however that, for the problem we study, we do not need to keep track of the time period at which shipments occur since they can be deduced from the production time and the time period at which demand is used by solving a shortest path problem. This observation applies because there are no fixed costs and capacities related to shipment variables. It allows us to derive a smaller formulation that we describe next. Let $\phi_{p,t,t'}$ be the amount produced at plant p in period t to satisfy the demand in period t'. Recall that we defined $H^p(j,t)$ as the length of the shortest path between nodes (p, j) and (C, t) in the network described in Section 2. Using this optimal shipment cost, which can be precomputed in time $O\left(PT^2\right)$, we can formulate LSS as

$$\min \quad \left\{ \sum_{p \in \mathcal{P}} \left[\sum_{t \in \mathcal{T}} f_{p,t} y_{p,t} + \sum_{t' \in \mathcal{T}} \sum_{t=1}^{t'} \left(\gamma_{p,t} + H^p(t,t') \right) \phi_{p,t,t'} \right] - C \;\middle|\; (\phi, y) \in Q^{MC}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] \right\}, \tag{22}$$



$H^1(j,t)$			
$j \backslash t$	1	2	3
1	1.5	1	0
2	-	1	0
3	-	-	0

$H^2(j,t)$			
$j \backslash t$	1	2	3
1	2	1	1
2	-	1	1
3	-	-	1

Figure 2: Problem with modified costs $\gamma_{p,t}$ and $\delta_{p,t}$ and minimum shipment costs $H^p\,(j,t)$

where $Q^{\text{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = R^{\text{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] \bigcap \left(\mathbb{R}_+^{\frac{PT^2+PT}{2}} \times \mathbb{Z}^{PT}\right)$ and

$$R^{\text{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = \left\{ (\phi; y) \in \mathbb{R}_+^{\frac{PT^2+PT}{2}} \times \mathbb{R}^{PT} \;\middle|\; \begin{array}{lll} \sum_{p\in\mathcal{P}} \sum_{t=1}^{t'} \phi_{p,t,t'} = d_{t'} & \forall t' \in \mathcal{T} & (23a) \\ \phi_{p,t,t'} \leqslant d_{t'} y_{p,t} & \forall p \in \mathcal{P}, \forall t' \in \mathcal{T}, \forall t \leqslant t' & (23b) \\ y_{p,t} \leqslant 1 & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & (23c) \end{array} \right\}.$$

In the above formulation, we avoid the introduction of the extra index t'', thereby reducing the number of variables from $O(PT^3)$ to $O(PT^2)$. We denote the convex hull of integer solutions of this formulation as $Q^{I,\text{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = \text{conv}(Q^{\text{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}])$. When sets $\mathcal{P}$, $\mathcal{T}$ and the vector $\mathbf{d}$ are clear from the context, we use the shorthand notation $Q^{I,\text{MC}}$.

The following example shows that (23a)-(23c) is not an ideal formulation for LSS. A similar derivation shows that (17)-(21) is not an ideal formulation for LSS either.

Example 3.1. Consider an instance with 2 plants and 3 periods for which the costs $f_{p,t}$, $\gamma_{p,t}$ and $\delta_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$, and demand d_t for $t \in \mathcal{T}$ are presented in Figure 2.

The solution $(\bar{\phi}, \bar{y})$ where $\bar{y}_{1,1} = \bar{y}_{1,2} = \bar{y}_{2,1} = \frac{1}{2}$ and $\bar{\phi}_{1,1,1} = \bar{\phi}_{1,1,3} = \bar{\phi}_{1,2,2} = \bar{\phi}_{1,2,3} = \bar{\phi}_{2,1,1} = \bar{\phi}_{2,1,2} = \frac{1}{2}$ and all other variables are set to 0 is a feasible solution to $R^{\text{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}]$. We next consider the dual problem. We associate variables u_t for $t \in \mathcal{T}$ to (23a), $\omega_{p,t,t'}$ for $p \in \mathcal{P}$, $t \in \mathcal{T}$ and $t' \in \{t, \ldots, T\}$ to (23b) and $v_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$ to (23c). The solution $(\bar{u}, \bar{\omega}, \bar{v})$ where $\bar{u}_1 = 2.75$, $\bar{u}_2 = 1.25$, $\bar{u}_3 = 0.75$, $\bar{\omega}_{1,1,1} = \bar{\omega}_{1,2,3} = \bar{\omega}_{2,1,1} = 0.75$, $\bar{\omega}_{1,1,3} = \bar{\omega}_{1,2,2} = \bar{\omega}_{2,1,2} = \bar{\omega}_{2,2,2} = \bar{\omega}_{2,2,3} = 0.25$, $\bar{\omega}_{1,3,3} = \bar{\omega}_{2,3,3} = 5$ and where all other variables are set to 0 is dual feasible. Moreover, for every strictly positive variable in the primal (resp. dual) the corresponding dual (resp. primal) constraint is tight. This shows that $(\bar{\phi}, \bar{y})$ and $(\bar{u}, \bar{\omega}, \bar{v})$ are (strictly) complementary and that $(\bar{\phi}, \bar{y})$ is a fractional extreme point of $R^{\text{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}]$.

The fractional extreme point of Example 3.1 is cut off by the valid inequality

$$\frac{\phi_{1,2,2} + \phi_{2,1,2}}{d_2} + \frac{\phi_{1,1,3} + \phi_{1,2,3}}{d_3} \leqslant y_{1,1} + y_{1,2} + y_{2,1}. \qquad (24)$$

We next introduce a family of valid inequalities for $Q^{I,MC}$ that generalize (24). For each element i of a subset $\mathcal{I}$ of periods $\mathcal{I} \subseteq \mathcal{T}\backslash\{1\}$, we select a nonempty subset V_i of plant-period pairs that can be used to produce d_i, *i.e.*, $V_i \neq \varnothing$ and $V_i \subseteq \{(p,t) \mid p \in \mathcal{P}, t \in \{1,\ldots,i\}\}$. Given these sets, we let $\mathcal{I}_{p,1}$ be the subset of $\mathcal{I}$ for which $(p,1)$ is selected among the plant-period pairs of V_i, *i.e.*, $\mathcal{I}_{p,1} = \{i \in \mathcal{I} \mid (p,1) \in V_i\}$. We say that $U \subseteq \{(p,t) \in V \mid t \geqslant 2\}$ is a *covering* of a subset $\mathcal{S}$ of $\mathcal{I}$ if $U \bigcap V_i \neq \varnothing$ for all $i \in \mathcal{S}$. In the remainder, we assume that there exists a covering of $\mathcal{I}$. In other words, we assume for each $i \in \mathcal{I}$ that $V_i \supseteq \{(p,t)\}$ for some $p \in \mathcal{P}$ and $t \in \mathcal{T}\backslash\{1\}$.

Given these sets, we define the *covering inequality* as

$$\sum_{i\in\mathcal{I}} \sum_{(p,t)\in V_i} \frac{\phi_{p,t,i}}{d_i} \leqslant \sum_{(p,t)\in V \mid t\geqslant 2} y_{p,t} + \sum_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1}, \tag{25}$$

where $V = \bigcup_{i\in\mathcal{I}} V_i$. In particular, (24) is of the form (25) with $\mathcal{I} = \{2,3\}$, $V_2 = \{(1,2),(2,1)\}$, $V_3 = \{(1,1),(1,2)\}$, $V = \{(1,1),(2,1),(1,2)\}$, $\mathcal{I}_{1,1} = \{3\}$, and $\mathcal{I}_{2,1} = \{2\}$. The proof of the following result is given in 7.3.

Theorem 3.2. *Covering inequality* (25) *is valid for* $Q^{I,MC}$ *if*

(i) The size of a minimal covering of $\mathcal{I}$ *is at least* $|\mathcal{I}| - \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}|$.

(ii) For all $p \in \mathcal{P}$ *the size of a minimal covering of* $\mathcal{I}\backslash\mathcal{I}_{p,1}$ *is at least* $|\mathcal{I}| - |\mathcal{I}_{p,1}|$.

Theorem 3.2 implies that (24) is valid for the set of Example 3.1, as $U_1 = \{(1,2)\}$ is a covering of $\mathcal{I}$, $U_2 = \{(1,2)\}$ is a covering of $|\mathcal{I}|\backslash\{3\}$, $U_3 = \{(1,2)\}$ is a covering of $|\mathcal{I}|\backslash\{2\}$, and $|\mathcal{I}_{1,1}| = |\mathcal{I}_{2,1}| = 1$.

Verifying that conditions (i)-(ii) hold is not trivial since finding a minimal covering is in general a difficult optimization problem. We next introduce a subfamily of inequalities whose construction guarantees that conditions (i)-(ii) hold.

In the set of covering inequalities defined above, let $\mathcal{I} = \{i,j\} \subseteq \mathcal{T}\backslash\{1\}$, where we assume without loss of generality that $i < j$. Let $V_i \subseteq \{(p,t) \mid p \in \mathcal{P}, t \in \{1,\ldots,i\}\}$ and $V_j \subseteq \{(p,t) \mid p \in \mathcal{P}, t \in \{1,\ldots,j\}\}$ be such that $V_i\backslash\bigcup_{p\in\mathcal{P}}\{(p,1)\} = V_j\backslash\bigcup_{p\in\mathcal{P}}\{(p,1)\} = W \neq \varnothing$. Note that $W \bigcap \{(p,T)\} = \varnothing$ for $p \in \mathcal{P}$, since only d_T can be produced in period T. We then write the *two-covering* inequality as

$$\sum_{(p,t)\in V_i} \frac{\phi_{p,t,i}}{d_i} + \sum_{(p,t)\in V_j} \frac{\phi_{p,t,j}}{d_j} \leqslant \sum_{(p,t)\in W} y_{p,t} + \sum_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1}, \tag{26}$$

where $\mathcal{I}_{p,1} \subseteq \{i,j\}$ for $p \in \mathcal{P}$. By Theorem 3.2, (26) is valid for LSS, if $|\mathcal{I}_{p,1}| \geqslant 1$ for $p \in \mathcal{P}$, since $U = \{(p,t)\}$ for any $(p,t) \in W$ is a covering of $\mathcal{I}$. We next identify conditions under which (26) is facet-defining for LSS; see 7.4 for a proof.

Theorem 3.3. *A valid two-covering inequality* (26) *is facet-defining for* $Q^{I,MC}$, $|\mathcal{I}_{p,1}| = 1$ *for* $p \in \mathcal{P}$ *and* $\bigcup_{p\in\mathcal{P}} \mathcal{I}_{p,1} = \{i,j\}$.

In particular, Theorem 3.3 shows that (24) defines a facet of $Q^{\mathrm{I,MC}}$ when $P = 2$ and $T \geqslant 3$. We remark that the proof of Theorem 3.3 is given under the following assumption.

Assumption 3.1. *All demands are strictly positive,* i.e., $\mathbf{d} \in \mathbb{R}^T_{++}$.

This assumption is common in the study of lot-sizing problems [Barany et al., 1984; Zhang et al., 2012] and streamlines proofs. We impose Assumption 3.1 in the remainder of this paper.

4 Polyhedral study of the natural formulation of LSS

In this section, we derive several families of facet-defining inequalities for the convex hull of the natural formulation of LSS.

4.1 Preliminaries

To start, we first establish the dimension of the convex hull of mixed integer solutions to LSS. We also prove that most inequalities of its formulation are facet-defining. For $\mathbf{d} \in \mathbb{R}^T_+$, we define $R^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}]$ as

$$\left\{ (x;v;y) \in \mathbb{R}^{3PT} \;\middle|\; \begin{array}{lll} \sum_{i=1}^t x_{p,i} \geqslant \sum_{i=1}^t v_{p,i} & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \backslash \{T\} & (27a) \\ \sum_{i \in \mathcal{T}} x_{p,i} = \sum_{i \in \mathcal{T}} v_{p,i} & \forall p \in \mathcal{P} & (27b) \\ \sum_{p \in \mathcal{P}} \sum_{i=1}^t v_{p,i} \geqslant d_{1t} & \forall t \in \mathcal{T} \backslash \{T\} & (27c) \\ \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{T}} v_{p,i} = d_{1T} & & (27d) \\ x_{p,t} \leqslant d_{tT} y_{p,t} & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & (27e) \\ \sum_{p \in \mathcal{P}} y_{p,1} \geqslant 1 & & (27f) \\ y_{p,t} \leqslant 1 & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & (27g) \\ x_{p,t} \geqslant 0 & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & (27h) \\ v_{p,t} \geqslant 0 & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & (27i) \end{array} \right\}.$$

We do not include the constraints $y_{p,t} \geqslant 0$ in the formulation of R as they are implied by (27e) and (27h). We are interested in the set of mixed integer solutions of $R^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}]$. For this reason, we define $Q^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = R^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] \bigcap \left(\mathbb{R}^{2PT} \times \mathbb{Z}^{PT}\right)$ and $Q^{I,\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = \text{conv}(Q^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}])$. When the sets $\mathcal{P}$, $\mathcal{T}$ and the vector $\mathbf{d}$ are clear from the context, we use the simpler notation R, Q and Q^I. It is simple to verify that Q^I is a polyhedron.

In the remainder of this section, we establish some basic polyhedral results about Q^I. We use $(x;v;y)$ as a shorthand notation for the vector $(x_1, x_2, \ldots, x_P; v_1, v_2, \ldots, v_P; y_1, y_2, \ldots, y_P)$ in Q^I (or Q) where vectors x_p, v_p and y_p are the production, shipment and setup variables of plant p, for $p \in \mathcal{P}$. Components of these vectors are the variables $x_{p,t}$, $v_{p,t}$ and $y_{p,t}$ corresponding to time period t, for $t \in \mathcal{T}$. Finally, given a vector $z \in \mathbb{R}^n$, we use the notation $z_{\looparrowright[z_i=\theta]}$ to represent the vector obtained by replacing its component z_i (if it exists) by the value θ. To streamline notation, we write $(z_{\looparrowright[z_i=\theta]})_{\looparrowright[z_j=\gamma]}$ as $z_{\looparrowright[z_i=\theta, z_j=\gamma]}$.

We next determine the dimension of Q^I, which is proven in 7.5. We consider the cases where $P \geqslant 2$ and $P = 1$ separately. Knowing the dimension of Q^I when $P = 1$ is useful in subsequent derivations.

Theorem 4.1. *When $P \geqslant 2$, $\dim\left(Q^I\right) = 3PT - (P+1)$. When $P = 1$, $\dim\left(Q^I\right) = 3T - 3$.*

When sets $\mathcal{P}$ and $\mathcal{T}$ are clear from the context, we often make use of the set $\mathbb{Q}_{\mathbb{P},\mathbb{T}}[\mathbf{d}]$ defined as

$$\left\{ (x;v;y) \in Q[\mathbf{d} \circ \mathbf{1}_{\mathbb{T}}] \;\middle|\; \begin{array}{ll} x_{p,t} = 0,\ v_{p,t} = 0,\ y_{p,t} = 0 & \forall t \in \bar{\mathbb{T}}, \forall p \in \mathcal{P} \\ x_{p,t} = 0,\ v_{p,t} = 0,\ y_{p,t} = 0 & \forall t \in \mathcal{T}, \forall p \in \bar{\mathbb{P}} \end{array} \right\}$$

where $\mathbb{P} \subseteq \mathcal{P}$, $\mathbb{T} \subseteq \mathcal{T}$, $\mathbf{d} \in \mathbb{R}^T_+$, $\mathbf{1}_{\mathbb{T}}$ is the indicator vector of $\mathbb{T}$, $a \circ b$ represents the componentwise product of conforming vectors a and b, $\bar{\mathbb{T}} = \mathcal{T} \backslash \mathbb{T}$ and $\bar{\mathbb{P}} = \mathcal{P} \backslash \mathbb{P}$. We also define $\mathbb{Q}^I_{\mathbb{P},\mathbb{T}}[\mathbf{d}] = \text{conv}(\mathbb{Q}_{\mathbb{P},\mathbb{T}}[\mathbf{d}])$. It is clear that $\mathbb{Q}^I_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = Q^I[\mathbf{d}]$. To illustrate the use of the notation introduced before, consider $(x^1; y^1; v^1)$ to be a solution of $\mathbb{Q}_{\{1\},\{1,\ldots,t\}}[\mathbf{d}]$ and $(x^2; y^2; v^2)$ to be a solution of $\mathbb{Q}_{\mathcal{P}\backslash\{1\},\{t+1,\ldots,T\}}[\mathbf{d}]$ for some $t \in \mathcal{T}$.

Then $(x^1; y^1; v^1) + (x^2; y^2; v^2)$ is a solution of Q in which the demand for the first t periods is satisfied by plant 1, while the demand of later periods is satisfied by the other plants.

Observe that, after removing from $\mathbb{Q}_{\mathbb{P},\mathbb{T}}[\mathbf{d}]$ all variables that are fixed to zero, we obtain a problem of the same form as Q, with a possibly different number of plants and time periods. This observation, combined with Theorem 4.1, leads to the following result.

Corollary 4.1. *Assume that $\mathbf{d}_{\mathbb{T}}$, the vector obtained from $\mathbf{d}$ by only keeping those components with indices in $\mathbb{T}$, is strictly positive. Then* $\dim(\mathbb{Q}_{\mathbb{P},\mathbb{T}}[\mathbf{d}]) = 3|\mathbb{P}||\mathbb{T}| - (|\mathbb{P}| + 1)$ *if* $|\mathbb{P}| \geqslant 2$ *and* $\dim(\mathbb{Q}_{\mathbb{P},\mathbb{T}}[\mathbf{d}]) = 3|\mathbb{T}| - 3$ *if* $|\mathbb{P}| = 1$.

For the remainder of this section we make the following streamlining assumption.

Assumption 4.1. $T \geqslant 2$.

We provide a minimal linear description of $Q^I_{\mathcal{P},\mathcal{T}}$ for the case where $T = 1$ in Section 7.8. We prove the following result in 7.5. It shows that the inequalities in the description of R are, for the most part, facet-defining for Q^I.

Theorem 4.2. *(i) For $p \in \mathcal{P}$ and $t \in \mathcal{T}\backslash\{T\}$, (27a) defines a facet of Q^I.*

(ii) For $t \in \mathcal{T}\backslash\{T\}$, (27c) defines a facet of Q^I.

(iii) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (27e) defines a facet of Q^I if and only if (i) $P \geqslant 2$ or (ii) $P = 1$ and $t \neq 1$.

(iv) Inequality (27f) defines a facet of Q^I if and only if $P \geqslant 2$.

(v) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (27g) defines a facet of Q^I if and only if (i) $P \geqslant 2$ or (ii) $P = 1$ and $t \geqslant 2$.

(vi) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (27h) defines a facet of Q^I if and only if $t \geqslant 2$.

(vii) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (27i) defines a facet of Q^I if and only if (a) $2 \leqslant t \leqslant T - 1$ or (b) $P \geqslant 3$ and $t = 1$.

We mention that, in the traditional uncapacitated lot-sizing polytope, inequality $x_1 \leqslant d_{1T} y_1$ is not facet-defining. This characteristic remains when $P = 1$ but does not hold anymore when $P \geqslant 2$.

4.2 Families of non-trivial facet-defining inequalities

In this section we present families of non-trivial facet-defining inequalities for Q^I. We first give in Theorem 4.3, a family of inequalities that bound from above the amount of production occurring at a plant in successive periods. To explain these inequalities, we observe that valid inequalities $\sum_{i=t}^T v_{p,i} \geqslant \sum_{i=t}^T x_{p,i}$ and $d_{(j+1)T} \geqslant \sum_{i=j+1}^T v_{p,i}$ can be respectively obtained by subtracting (27b) from (27a) written for $t - 1$, and by relaxing to a single plant the inequality obtained by subtracting (27c) with $t = j$ from (27d). Summing these two inequalities (in the case where $j \geqslant t$) yields the valid inequality $\sum_{i=t}^T x_{p,i} \leqslant \sum_{i=t}^j v_{p,i} + d_{(j+1)T}$, which is the basis for ensuing strengthened inequality (28). The following result is proven in 7.6.

Theorem 4.3. *The production upper bound inequality*

$$\sum_{i=t}^{k} x_{p,i} \leqslant \sum_{i=t}^{j} v_{p,i} + d_{(j+1)T} \sum_{i=t}^{k} y_{p,i} \tag{28}$$

is valid for Q^I when $1 \leqslant t \leqslant k \leqslant j \leqslant T-1$. Furthermore, such a valid inequality is facet-defining for Q^I if and only if $P \geqslant 2$ or $P = 1$ and $t \geqslant 2$.

Note that one might conjecture that the requirement $k \in \{t, \ldots, j\}$ in the definition of (28) is too stringent and could be replaced with $k \in \{t, \ldots, T\}$. It follows from an argument similar to that used in the proof of Theorem 4.3 that (28) is indeed valid when $k \in \{j+1, \ldots, T\}$. However, when $k > j$, the corresponding inequality is not facet-defining for Q^I. In fact, the inequality obtained as an equal weight conic combination of (28) when $k = j$ and (27e) for $i = j+1, \ldots, k$ dominates it as $d_{(j+1)T} \geqslant d_{iT}$ for $i = j+1, \ldots, k$.

Example 4.1. Consider an instance of LSS where $P = 2$, $T = 5$, and $(d_1, d_2, d_3, d_4, d_5) = (5, 7, 8, 9, 4)$. It follows from Theorem 4.3 that

$$\begin{aligned}
&x_{1,1} \leqslant v_{1,1} + 28y_{1,1}, && \text{where } (p,t,j,k) = (1,1,1,1)\\
&x_{2,2} + x_{2,3} \leqslant v_{2,2} + v_{2,3} + 13(y_{2,2} + y_{2,3}), && \text{where } (p,t,j,k) = (2,2,3,3)\\
&x_{1,1} \leqslant v_{1,1} + v_{1,2} + 21y_{1,1}, && \text{where } (p,t,j,k) = (1,1,2,1)\\
&x_{2,2} + x_{2,3} \leqslant v_{2,2} + v_{2,3} + v_{2,4} + 4\left(y_{2,2} + y_{2,3}\right), && \text{where } (p,t,j,k) = (2,2,4,3)
\end{aligned}$$

are facet-defining for Q^I.

Inequality (28) states that, if production occurs at plant p during the time window $\{t, \ldots, k\}$, then the quantity produced is smaller than the total shipment from plant p during time window $\{t, \ldots, j\}$, plus the total demand leftover after time period j.

The non-trivial inequalities presented above involve a single plant. Next we present facet-defining inequalities for Q^I that are similar to (l, S) inequalities [Barany et al., 1984] and involve all plants. To describe them, we first define $L = \{1, \ldots, l\}$ for $l \in \mathcal{T}$. For $p \in \mathcal{P}$, we let $\left(S^p \bigcup \bar{S}^p \bigcup V^p \bigcup \bar{V}^p\right)$ be a partition of L. For each such partition and for $i \in \{1, \ldots, l+1\}$, we define

$$\sigma_p(i) = \begin{cases} \max\left\{j \mid \{i, i+1, \ldots, j\} \subseteq \left(V^p \bigcup \bar{V}^p\right)\right\} & \text{if } i \in \bar{V}^p,\\ i-1 & \text{otherwise.}\end{cases}$$

In particular, note that $\sigma_p(l+1) = l$.

Given this notation, we refer to inequalities of the form

$$\sum_{p\in\mathcal{P}} \left[\sum_{i\in S^p} x_{p,i} + \sum_{i\in\bar{S}^p} d_{il}y_{p,i} + \sum_{i\in V^p} v_{p,i} + \sum_{i\in\bar{V}^p} \left(v_{p,i} + d_{(\sigma_p(i)+1)l}y_{p,i}\right)\right] \geqslant d_{1l} \tag{29}$$

as (l, S, V) inequalities. In (29) we refer to the variables associated with V^p and $\bar{V}^p$ as *shipment terms* and those associated with S^p and $\bar{S}^p$ as *production terms*. Observe that for $i \in \bar{V}^p$, $\sigma_p(i)$ refers to the largest index j for which all terms from period i to period j are shipment terms.

For $i \in \bar{V}^p$, if $\sigma_p(i) = l$, then $v_{p,i} + d_{(\sigma_p(i)+1)l}y_{p,i} = v_{p,i}$ as we assume that $d_{(l+1)l} = 0$. Therefore, including i in V^p would yield the same inequality, and we may pose the following assumption.

Assumption 4.2. *$\sigma_p(i) < l$ for $i \in \bar{V}^p$ and $p \in \mathcal{P}$.*

We next argue that, under some mild assumptions, (29) is valid for Q^I. The following lemma helps in establishing this result. It uses the ensuing notation.

For $(x;v;y) \in Q$ and $p \in \mathcal{P}$, we define

$$I^p[y] = \{j \in L \mid j \in (\bar{S}^p \bigcup \bar{V}^p) \text{ and } y_{p,j} = 1\}$$

and

$$t^p[y] = \begin{cases} l+1 & \text{if} \quad I^p[y] = \varnothing, \\ \min\{j \mid j \in I^p[y]\} & \text{if} \quad I^p[y] \neq \varnothing. \end{cases}$$

We also define, $\ell^p(x;v;y) = \sum_{i \in S^p} x_{p,i} + \sum_{i \in \bar{S}^p} d_{il} y_{p,i} + \sum_{i \in V^p} v_{p,i} + \sum_{i \in \bar{V}^p} \big(v_{p,i} + d_{(\sigma_p(i)+1)l} y_{p,i}\big)$, which is the part of (29) related to plant p. Finally, we define $LHS(x;v;y)$, or simply LHS, to be $\sum_{p \in \mathcal{P}} \ell^p(x;v;y)$, *i.e.*, to be the left-hand-side value of (29) for point $(x;v;y)$.

Lemma 4.1. *Let $(x;v;y)$ be a feasible solution to Q and let $p \in \mathcal{P}$. Define $s_p = \sigma_p(t^p[y])$.*

(a) If there exists $\kappa^p \in \{0, \ldots, l\}$ such that $\{1, \ldots, \kappa^p\} \subseteq (S^p \bigcup \bar{S}^p \bigcup \bar{V}^p)$ and $\{\kappa^p+1, \ldots, l\} \subseteq (V^p \bigcup \bar{V}^p)$, then

$$\ell^p(x;v;y) \geqslant \sum_{i=1}^{s_p} v_{p,i} + d_{(s_p+1)l}.$$

(b) If there exists $s \leqslant s_p$ and $\kappa^p \in \{0, \ldots, s\}$ such that $\{1, \ldots, \kappa^p\} \subseteq (S^p \bigcup \bar{S}^p \bigcup \bar{V}^p)$ and $\{\kappa^p+1, \ldots, s\} \subseteq (V^p \bigcup \bar{V}^p)$, then

$$\ell^p(x;v;y) \geqslant \sum_{i=1}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]},$$

where $y_{p,t^p[y]}$ is taken to be zero if $I^p[y] = \varnothing$.

Proof. We start with the proof of (a). First we observe that, for $i \leqslant t^p[y] - 1$

(i) $d_{il} y_{p,i} = x_{p,i}$ if $i \in \bar{S}^p$,

(ii) $v_{p,i} + d_{(\sigma_p(i)+1)l} y_{p,i} \geqslant x_{p,i}$ if $i \in \bar{V}^p$,

(iii) $v_{p,i} + d_{(\sigma_p(i)+1)l} y_{p,i} = v_{p,i}$ if $i \in \bar{V}^p$,

since $y_{p,i} = 0 = x_{p,i}$ under this assumption. There are several cases.

First assume that $I^p[y] = \varnothing$. In this case $t^p[y] = l+1$ and $s^p = l$. We write that

$$\ell^p(x;v;y) \geqslant \sum_{i=1}^{\kappa^p} x_{p,i} + \sum_{i=\kappa^p+1}^{l} v_{p,i} \geqslant \sum_{i=1}^{l} v_{p,i}$$

where the first inequality is obtained by applying (i) for $i \in \bar{S}^p$, (ii) for $i \in \bar{V}^p$ with $i \leqslant \kappa^p$, and (iii) for $i \in \bar{V}^p$ with $i \geqslant \kappa^p + 1$, while the second inequality holds because of (27a) or (27b). This proves the results as $d_{(s_p+1)l} = 0$ in this case.

Second, assume that $I^p[y] \neq \varnothing$. If $t^p[y] \in \bar{S}^p$, then $s_p = t^p[y] - 1$ and $\kappa^p \geqslant s_p + 1$. We write that

$$\ell^p(x;v;y) \geqslant \sum_{i=1}^{s^p} x_{p,i} + d_{(s_p+1)l} y_{p,(s_p+1)} \geqslant \sum_{i=1}^{s_p} v_{p,i} + d_{(s_p+1)l},$$

where the first inequality is obtained by applying (i) for $i \in \bar{S}^p$ with $i \leqslant s_p$, (ii) for $i \in \bar{V}^p$ with $i \leqslant s_p$, and by lower bounding all terms with $i \geqslant s_p + 2$ by zero, while the second inequality follows from (27a) or (27b) and the fact that $y_{p,(s_p+1)} = 1$.

If $t^p[y] \in \bar{V}^p$, then $s_p \geqslant t^p[y]$ and $\{t^p[y], \ldots, s_p\} \subseteq \left(V^p \bigcup \bar{V}^p\right)$. Using Assumption 4.2, we have that $\kappa_p \geqslant s_p + 1$ since $\{t^p[y], \ldots, s_p\} \in \left(V^p \bigcup \bar{V}^p\right)$. We write that

$$\ell^p(x; v; y) \geqslant \sum_{i=1}^{t^p[y]-1} x_{p,i} + v_{p,t^p[y]} + d_{(s_p+1)l} y_{p,t^p[y]} + \sum_{i=t^p[y]+1}^{s_p} v_{p,i} \geqslant \sum_{i=1}^{s_p} v_{p,i} + d_{(s_p+1)l}$$

where the first inequality is obtained by applying (i) for $i \in \bar{S}^p$ with $i \leqslant t^p[y] - 1$, (ii) for $i \in \bar{V}^p$ with $i \leqslant t^p[y] - 1$, lower bounding all $i \in (V^p \bigcup \bar{V}^p)$ with $t^p[y] + 1 \leqslant i \leqslant s_p$ by $v_{p,i}$, and lower bounding all terms with $i \geqslant s_p + 1$ by 0, while the second inequality follows from (27a) or (27b) and the fact that $y_{p,t^p[y]} = 1$.

We next prove (b). There are two cases.

Assume first that $t^p[y] \geqslant s$. Then $y_{p,i} = 0$ for $i \in (\bar{S}^p \bigcup \bar{V}^p)$ such that $i \leqslant s$. As $\kappa^p \leqslant s$, we write

$$\ell^p(x; v; y) \geqslant \sum_{i=1}^{\kappa^p} x_{p,i} + \sum_{i=\kappa_p+1}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]} \geqslant \sum_{i=1}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]}$$

where the first inequality holds by applying (i) for $i \in \bar{S}^p$ with $i \leqslant \kappa^p$, (ii) for $i \in \bar{V}^p$ with $i \leqslant \kappa^p$, and (iii) for $i \in \bar{V}^p$ with $\kappa^p + 1 \leqslant i \leqslant s$, and by lower bounding all terms involving $i \geqslant s + 1$ with $i \neq t^p[y]$ by zero, while the second inequality holds because of (27a) or (27b).

Assume second that $t^p[y] \leqslant s$. Clearly $t^p[y] \in \bar{V}^p$. Assume not, then $t^p[y] \in \bar{S}^p$. This would then imply that $s \leqslant s_p = t^p[y] - 1 \leqslant s - 1$, a contradiction. It follows that $\{t^p[y], \ldots, s\} \subseteq \left(V^p \bigcup \bar{V}^p\right)$ and therefore, we may assume that $\kappa^p \leqslant t^p[y] - 1$. We write

$$\ell^p(x; v; y) \geqslant \sum_{i=1}^{\kappa^p} x_{p,i} + \sum_{i=\kappa^p+1}^{t^p[y]-1} v_{p,i} + \sum_{i=t^p[y]}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]} \geqslant \sum_{i=1}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]},$$

where the first inequality holds by applying (i) for $i \in \bar{S}^p$ with $i \leqslant \kappa^p$, (ii) for $i \in \bar{V}^p$ with $i \leqslant \kappa^p$, and (iii) for $i \in \bar{V}^p$ with $\kappa^p + 1 \leqslant i \leqslant t^p[y] - 1$, by lower bounding all terms $i \in \bar{V}^p$ with $t^p[y] + 1 \leqslant i \leqslant s$ by $v_{p,i}$, and by lower bounding all terms with $i \geqslant s + 1$ by zero, while the second inequality holds because of (27a) or (27b). □

We next give necessary and sufficient conditions for (l, S, V) inequalities (29) to be valid for Q^I.

Theorem 4.4. *An (l, S, V) inequality (29) is valid for Q^I if and only if one of the following conditions is satisfied*

(a) $1 \in \bigcup_{p\in\mathcal{P}} (S^p \bigcup V^p)$ *and, for each* $p \in \mathcal{P}$*, there exists* $k^p \in \{0, \ldots, l\}$ *such that* $\{1, \ldots, k^p\} \subseteq (S^p \bigcup \bar{S}^p \bigcup \bar{V}^p)$ *and* $\{k^p + 1, \ldots, l\} \subseteq (V^p \bigcup \bar{V}^p)$*,*

(b) $1 \in \bigcap_{p\in\mathcal{P}} \left(\bar{S}^p \bigcup \bar{V}^p\right)$ *and, for each* $p \in \mathcal{P}$*, there exists* $k^p \in \{0, \ldots, \sigma'\}$ *such that* $\{1, \ldots, k^p\} \subseteq \left(S^p \bigcup \bar{S}^p \bigcup \bar{V}^p\right)$ *and* $\{k^p + 1, \ldots, \sigma'\} \subseteq (V^p \bigcap \bar{V}^p)$*, where* $\sigma' = \max_{p\in\mathcal{P}}\{\sigma_p(1)\}$*.*

Proof. Let $(x;v;y)$ be any feasible solution to Q. As before, define $s_p = \sigma_p(t^p[y])$ for $p \in \mathcal{P}$. Let $s^* = \min_{p\in\mathcal{P}} s_p$ and let $p^* \in \mathcal{P}$ be such that $s^* = s_{p^*}$.

We start by showing that (29) is valid under (a). We write

$$LHS(x;v;y) = \sum_{p\in\mathcal{P}} \ell^p(x;v;y) \geqslant \sum_{p\in\mathcal{P}}\sum_{i=1}^{s_p} v_{p,i} + \sum_{p\in\mathcal{P}} d_{(s_p+1)l} \geqslant \sum_{p\in\mathcal{P}}\sum_{i=1}^{s^*} v_{p,i} + d_{(s^*+1)l} \geqslant d_{1l}$$

where the first inequality follows from Lemma 4.1(a) using $\kappa^p = k^p$, the second is obtained by lower bounding terms $v_{p,i}$ by zero for $p \neq p^*$ and $i \in \{s^*+1,\ldots,s_p\}$ and by lower bounding $d_{(s_p+1)l}$ by zero for $p \neq p^*$, while the third holds because of (27c) or (27d).

We now show that (29) is valid under assumption (b). Since $1 \in \bigcap_{p\in\mathcal{P}}(\bar{S}^p \bigcup \bar{V}^p)$, then $I^p[y] \neq \varnothing$ and $t^p[y] = 1$ for some $p \in \mathcal{P}$ as $\sum_{p\in\mathcal{P}} y_{p,1} \geqslant 1$. It follows that $s^* \leqslant \max_{p\in\mathcal{P}} \sigma_p(1) = \sigma'$. We write that

$$\begin{aligned} LHS(x;v;y) = \sum_{p\in\mathcal{P}} \ell^p(x;v;y) &\geqslant \sum_{p\in\mathcal{P}}\sum_{i=1}^{s^*} v_{p,i} + \sum_{p\in\mathcal{P}} d_{(s_p+1)l} y_{p,t^p[y]} \\ &\geqslant d_{1s^*} + d_{(s_{p^*}+1)l} \geqslant d_{1l} \end{aligned} \tag{30}$$

where the first inequality is obtained by applying Lemma 4.1(b) with $s = s^* \leqslant s_p$ for each $p \in \mathcal{P}$ and $\kappa^p = \min\{k^p, s\}$, the second inequality holds because of (27c) and the fact that $I^{p^*}[y] \neq \varnothing$, and the third because $s_{p^*} = s^*$.

Next we prove that if (a) or (b) are not satisfied then it is possible to construct a feasible solution of Q that violates (29). Let π be an index of $\mathcal{P}$ such that $j < i$ where $j \in V^\pi$ and $i \in \left(S^\pi \bigcup \bar{S}^\pi\right)$ and, in the case of (b), $i, j \leqslant \sigma'$. It the latter case, it must then be that $\sigma' > 1$. For condition (a), let ρ be an index of $\mathcal{P}$ such that $1 \in (S^\rho \bigcup V^\rho)$. Then if $\pi \neq \rho$, we construct the vector $(\tilde{x};\tilde{v};\tilde{y})$ with nonzero components $(\tilde{x}_\pi, \tilde{v}_\pi, \tilde{y}_\pi) = (d_{jl}e_j, (d_{jl} - d_i)\, e_j + d_i e_i, e_j)$ and $(\tilde{x}_\rho, \tilde{v}_\rho, \tilde{y}_\rho) = (d_{1(j-1)}e_1 + d_{(l+1)T}e_{l+1}, d_{1(j-1)}e_1 + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$. If $\pi = \rho = p$, we construct $(\hat{x};\hat{v};\hat{y})$ where $\hat{x}_p = \tilde{x}_\pi + \tilde{x}_\rho$, $\hat{v}_p = \tilde{v}_\pi + \tilde{v}_\rho$, and $\hat{y}_p = \tilde{y}_\pi + \tilde{y}_\rho$ if $j \geqslant 2$, and $\hat{y}_p = \tilde{y}_\rho$ if $j = 1$. For condition (b), we choose index $\rho \in \mathcal{P}$ such that $\sigma_\rho(1) = \sigma'$. In this case, it must be that $\pi \neq \rho$, since $\{1,\ldots,\sigma'\} \subseteq \bar{V}^\rho$ for plant ρ and $j \in V^\pi$ and $j \leqslant \sigma'$ for plant π. We then construct the vector $(\bar{x};\bar{v};\bar{y})$ with nonzero components $(\bar{x}_\pi, \bar{v}_\pi, \bar{y}_\pi) = (d_{j\sigma'}e_j, (d_{j\sigma'} - d_i)e_j + d_i e_i, e_j)$ and $(\bar{x}_\rho, \bar{v}_\rho, \bar{y}_\rho) = ((d_{1j-1} + d_{(\sigma'+1)T})e_1, d_{1j-1}e_1 + d_{(\sigma'+1)T}e_{\sigma'+1}, e_1)$. For all aforementioned points, the left-hand-side of (29) is equal to $d_{1l} - d_i$ whereas the right-hand-side is d_{1l}, proving the result. $\square$

When $1 \in \bigcup_{p\in\mathcal{P}}(S^p \bigcup V^p)$, the condition of Theorem 4.4 specifies that, in order for (29) to be valid, all production terms must occur before terms in V^p, for each plant $p \in \mathcal{P}$. Assumption 4.2 then implies that valid (l,S,V) inequalities can always be written in a way that terms in V^p only show up in periods $\{k^p+1,\ldots,l\}$ for a suitable $k^p \in \{0,\ldots,l\}$. In the remainder of this paper, we assume that (l,S,V) inequalities are written in this way.

When $1 \in \bigcap_{p\in\mathcal{P}}(\bar{S}^p \bigcup \bar{V}^p)$, the validity condition of Theorem 4.4 is less restrictive. It states that, for (29) to be valid, it suffices that all production terms occur before terms in V^p, for each plant $p \in \mathcal{P}$ within the time window $\{1,\ldots,\sigma'\}$. In particular, the ordering of production and shipment terms after σ' is arbitrary. Moreover, when $1 \in \bigcap_{p\in\mathcal{P}}(\bar{S}^p \bigcup \bar{V}^p)$ we have that $\sigma_p(1) < l$ for $p \in \mathcal{P}$ because of Assumption 4.2. We therefore assume without loss of generality that $\sigma' < l$ in the remainder of this section.

Example 4.2. Consider an instance of LSS where $P = 2$, $T = 5$, and $(d_1, d_2, d_3, d_4, d_5) = (5, 7, 8, 9, 4)$. It follows from Theorem 4.4 that

$$5 \leqslant x_{1,1} + v_{2,1} \tag{31}$$
$$l = 1, S^1 = \{1\}, V^2 = \{1\},$$
$$12 \leqslant x_{1,1} + 7y_{1,2} + (v_{2,1} + 7y_{2,1}) + 7y_{2,2} \tag{32}$$
$$l = 2, S^1 = \{1\}, \bar{S}^1 = \{2\}, \bar{S}^2 = \{2\}, \bar{V}^2 = \{1\},$$
$$20 \leqslant 20y_{1,1} + x_{1,2} + x_{1,3} + v_{2,1} + v_{2,2} + v_{2,3} \tag{33}$$
$$l = 3, \bar{S}^1 = \{1\}, S^1 = \{2, 3\}, V^2 = L,$$
$$29 \leqslant 29y_{1,1} + (v_{1,2} + 9y_{1,2}) + (v_{1,3} + 9y_{1,3}) + x_{1,4} + \sum_{i=1}^{4} v_{2,i} \tag{34}$$
$$l = 4, S^1 = \{4\}, \bar{S}^1 = \{1\}, \bar{V}^1 = \{2, 3\}, V^2 = L,$$
$$33 \leqslant (v_{1,1} + 28y_{1,1}) + x_{1,2} + x_{1,3} + 13y_{1,4} + x_{1,5} + \sum_{i=1}^{5} v_{2,i} \tag{35}$$
$$l = 5, S^1 = \{2, 3, 5\}, \bar{S}^1 = \{4\}, \bar{V}^1 = \{1\}, V^2 = L,$$

are valid inequalities for Q^I.

Inequalities (32), (33) and (34) are facet-defining while (31) and (35) are not. Inequality (31) is dominated by $5 \leqslant v_{1,1} + v_{2,1}$ which is of the form (27c). Also, (35) is dominated by an equal weight conic combination of the inequalities $x_{1,4} \leqslant 13y_{1,4}$ and $x_{1,1} \leqslant v_{1,1} + 28y_{1,1}$, which are shown to be facet-defining in Theorems 4.2 (iii) and 4.3, and equalities $\sum_{i \in \mathcal{T}} v_{1,i} = \sum_{i \in \mathcal{T}} x_{1,i}$ and $33 = \sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{T}} v_{p,i}$.

We next present necessary and sufficient conditions for (29) to be facet-defining for Q^I.

Theorem 4.5. *For $P \geqslant 2$, let (29) be a valid inequality for Q^I that does not define the same face of Q^I as (27f). Then (29) is facet-defining for Q^I if and only if the following conditions are satisfied:*

(a) $1 \in \bigcup_{p \in \mathcal{P}} (S^p \bigcup V^p)$,

(b) $l < T$,

(c) $V^p = L$ or $\left(\bar{S}^p \bigcup \bar{V}^p\right) \neq \varnothing$ for each $p \in \mathcal{P}$.

Proof. We denote by F the face of Q^I that (29) defines. For the direct implication, assume that (29) is facet-defining for Q^I.

First, assume by contradiction that $1 \notin \bigcup_{p \in \mathcal{P}} (S^p \bigcup V^p)$, *i.e.*, $1 \in \bigcap_{p \in \mathcal{P}} (\bar{S}^p \bigcup \bar{V}^p)$. Let F' be the face of Q^I that (27f) induces. We claim that $F' \supseteq F$, thereby showing that (29) is either not facet-defining or induces that same face as (27f). Assume for a contradiction that $F' \nsupseteq F$. Then there exists $(x; v; y) \in F$ such that $\sum_{p \in \mathcal{P}} y_{p,1} \geqslant 2$. Let $p^* \in \text{argmin}_{p \in \mathcal{P}} \sigma_p(t^p[y])$. There exists $\bar{p} \in \mathcal{P} \backslash \{p^*\}$ such that $y_{\bar{p},1} = 1$. Inequality (30) then implies that $LHS(x; v; y) \geqslant d_{1l} + d_{(\sigma_{\bar{p}}(1)+1)l}$. Since $\sigma_{\bar{p}}(1) < l$ by Assumption 4.2, we conclude that $(x; v; y) \notin F$, the desired contradiction.

Second, assume for a contradiction that $l = T$. Because of Assumption 4.2, and the validity assumption (a) of Theorem 4.4, we must have that $\{i, \ldots, \sigma_p(i)\} \in \bar{V}^p$ with $\sigma_p(i) < l$ for each $i \in \bar{V}^p$. Define for $p \in \mathcal{P}$, $\mathcal{I}^p = \{i \in \bar{V}^p \mid i - 1 \notin \bar{V}^p\}$ and $k^p = \max\{i \mid i \notin V^p\}$. In the preceding definition, we take $k^p = 0$ if $V^p = L$. Then (29) can be expressed as an equal weight conic combination of the valid inequalities

(27e) for $i \in \bar{S}^p$ and $p \in \mathcal{P}$, (28) with $t = i$ and $j = k = \sigma_p(i)$, for $p \in \mathcal{P}$ and $i \in \mathcal{I}^p$, (27a) with $t = k^p$ for $p \in \mathcal{P}$, and (27d), after exchanging the right-hand-side and left-hand-side terms of (27a) and (27d). If $k^p \geqslant 1$ for some $p \in \mathcal{P}$, then the above derivation uses at least one of (27e) or (28), and (27a). Because the corresponding inequalities define distinct facets of Q^I, we conclude that (29) is not facet-defining for Q^I. If $k^p = 0$ for all $p \in \mathcal{P}$, then (28) reduces to $\sum_{p\in\mathcal{P}} \sum_{i=1}^{T} v_{p,i} \geqslant d_{1T}$. This inequality is always satisfied at equality as shown by (27d).

Finally, assume that $V^\pi \neq L$ and $\left(\bar{S}^\pi \bigcup \bar{V}^\pi\right) = \varnothing$ for some index $\pi \in \mathcal{P}$. Then using the validity condition (a) of Theorem 4.4, the contribution of plant π to the left-hand-side of (29) is $\sum_{i=1}^{k^\pi} x_{\pi,i} + \sum_{i=k^\pi+1}^{l} v_{\pi,i}$ for some $k^\pi \in \{1,\ldots,l\}$. Consider now the inequality (29) obtained by changing the partition $(S^\pi, \bar{S}^\pi, V^\pi, \bar{V}^\pi)$ of L where $\bar{S}^\pi = \bar{V}^\pi = \varnothing$ to $(\varnothing, \varnothing, L, \varnothing)$. The original inequality can be obtained as an equal weight conic combination of this new inequality and (27a) with $t = k^\pi$. Since these inequalities define distinct faces of Q^I, then (29) is not facet-defining.

We next show that conditions (a)-(c) are sufficient for (29) to be facet-defining for Q^I. To this end, we present $3PT - (P+1)$ affinely independent solutions in $(F \bigcap Q)$.

For $p \in \mathcal{P}$, define $m_p = \min\{t \mid t \in (\bar{S}^p \bigcup \bar{V}^p)\}$ where we assume $m_p = l+1$ if $(\bar{S}^p \bigcup \bar{V}^p) = \varnothing$. Further, let $m = \min_{p\in\mathcal{P}} m_p$. For the remainder of this proof, we select π to be an index in $\mathcal{P}$ for which $1 \in (S^\pi \bigcup V^\pi)$. Such index exists because of (a). In the solutions presented below, if $p = \pi$, we sum the solution vectors presented for plants p and π.

First, assume that $m = l+1$. Then, by assumption (c), $V^p = L$ for all $p \in \mathcal{P}$. Then (29) reduces to $\sum_{p\in\mathcal{P}} \sum_{i=1}^{l} v_{p,i} \geqslant d_{1l}$, which is shown to be facet-defining in Theorem 4.2 (ii).

Second, assume that $1 < m < l+1$. For this case we select π to be equal to p, although we express the points as a function of π, to help streamline notation when discussing the later case where $m = 1$. We define $\mathbf{d}' = \mathbf{d}_{\hookrightarrow[d_{m-1}=d_{(m-1)l}]}$. By Theorem 4.1 and Corollary 4.1, there are affinely independent solutions $(\hat{x}^s; \hat{v}^s; \hat{y}^s)$ for $s = 1,\ldots,\varsigma := 3P(m-1) - P$ in $\mathbb{Q}^I_{\mathcal{P},\{1,\ldots,m-1\}}[\mathbf{d}']$ and $(\acute{x}^r; \acute{v}^r; \acute{y}^r)$ for $r = 1,\ldots,\varrho := 3P(T-l) - P$ in $\mathbb{Q}^I_{\mathcal{P},\{l+1,\ldots,T\}}[\mathbf{d}]$. We construct the affinely independent vectors $(\bar{\bar{x}}^s; \bar{\bar{v}}^s; \bar{\bar{y}}^s)$ and $(\tilde{\bar{x}}^r; \tilde{\bar{v}}^r; \tilde{\bar{y}}^r)$ for $s \in \{1,\ldots,\varsigma\}$ and $r \in \{2,\ldots,\varrho\}$ defined as

(i) $(\bar{\bar{x}}^s; \bar{\bar{v}}^s; \bar{\bar{y}}^s) = (\hat{x}^s; \hat{v}^s; \hat{y}^s) + \left(\acute{x}^1; \acute{v}^1; \acute{y}^1\right)$,

(ii) $(\tilde{\bar{x}}^r; \tilde{\bar{v}}^r; \tilde{\bar{y}}^r) = \left(\hat{x}^1; \hat{v}^1; \hat{y}^1\right) + (\acute{x}^r; \acute{v}^r; \acute{y}^r)$.

Note that these points belong to F as $\{1,\ldots,m-1\}$ is either completely included in S^p or completely included in V^p for each $p \in \mathcal{P}$ by condition (c), and the validity assumption (a) of Theorem 4.4.

Next, we proceed in two steps. In the first we construct $3P(l-m+1)$ points, and in the second we construct the remaining P. The first $3P(l-m+1)$ points are obtained by creating three affinely independent solutions for each $i \in \{m,\ldots,l\}$ and for each $p \in \mathcal{P}$.

First, for each $i \in \bar{V}^p \bigcap \{m,\ldots,l\}$, we construct the vectors $\left(\check{x}^{p,i}; \check{v}^{p,i}; \check{y}^{p,i}\right)$, $\left(\mathring{x}^{p,i}; \mathring{v}^{p,i}; \mathring{y}^{p,i}\right)$, and $\left(\dddot{x}^{p,i}; \dddot{v}^{p,i}; \dddot{y}^{p,i}\right)$ with nonzero components

(iii) $(\check{x}^{p,i}_p, \check{v}^{p,i}_p, \check{y}^{p,i}_p) = (d_{il}e_i + d_{(l+1)T}e_{l+1}, d_{i\sigma_p(i)}e_i + d_{(\sigma_p(i)+1)l}e_{\sigma_p(i)+1} + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$,

$(\check{x}^{p,i}_\pi, \check{v}^{p,i}_\pi, \check{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,

(iv) $(\mathring{x}^{p,i}_p, \mathring{v}^{p,i}_p, \mathring{y}^{p,i}_p) = (d_{iT}e_i, d_{i\sigma_p(i)}e_i + d_{(\sigma_p(i)+1)T}e_{\sigma_p(i)+1}, e_i)$,

$(\mathring{x}^{p,i}_\pi, \mathring{v}^{p,i}_\pi, \mathring{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,

(v) $(\dddot{x}_p^{p,i}, \dddot{v}_p^{p,i}, \dddot{y}_p^{p,i}) = (d_{(\sigma_p(i)+1)l}e_i + d_{(l+1)T}e_{l+1}, d_{(\sigma_p(i)+1)l}e_{\sigma_p(i)+1} + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$,

$(\dddot{x}_\pi^{p,i}, \dddot{v}_\pi^{p,i}, \dddot{y}_\pi^{p,i}) = (d_{1\sigma_p(i)}e_1, d_{1\sigma_p(i)}e_1, e_1)$.

Vectors (iii)-(v) belong to F by definition of $\sigma_p(i)$. They satisfy all but one of the equalities $\frac{x_{p,i}}{d_{(\sigma_p(i)+1)l}} + \left(1 - \frac{d_{iT}}{d_{(\sigma_p(i)+1)l}}\right) \frac{v_{p,i}}{d_{i\sigma_p(i)}} = y_{p,i}$, $x_{p,i} = v_{p,i} + d_{(\sigma_p(i)+1)l}y_{p,i}$, and $v_{p,i} = d_{i\sigma_p(i)}y_{p,i}$ that are satisfied by all solutions in families (i)-(ii) as well as other points of the form (iii)-(v). Because $l < T$ by condition (b), each equality is violated by exactly one of these vectors, showing that they are affinely independent from each other and from all previous solutions.

Second, for each $i \in S^p \bigcap \{m, \ldots, l\}$, we construct vectors $(\dot{x}^{p,i}; \dot{v}^{p,i}; \dot{y}^{p,i})$, $(\hat{x}^{p,i}; \hat{v}^{p,i}; \hat{y}^{p,i})$, and $(\tilde{x}^{p,i}; \tilde{v}^{p,i}; \tilde{y}^{p,i})$ with nonzero components

(vi) $(\dot{x}_p^{p,i}, \dot{v}_p^{p,i}, \dot{y}_p^{p,i}) = (d_{(l+1)T}e_{l+1}, d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$,

$(\dot{x}_\pi^{p,i}, \dot{v}_\pi^{p,i}, \dot{y}_\pi^{p,i}) = (d_{1l}e_1, d_{1l}e_1, e_1)$,

(vii) $(\hat{x}_p^{p,i}, \hat{v}_p^{p,i}, \hat{y}_p^{p,i}) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1(i-1)}e_1 + d_{il}e_i + d_{(l+1)T}e_{l+1}, e_1 + e_i + e_{l+1})$,

(viii) $(\tilde{x}_p^{p,i}, \tilde{v}_p^{p,i}, \tilde{y}_p^{p,i}) = (d_{il}e_i + d_{(l+1)T}e_{l+1}, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$,

$(\tilde{x}_\pi^{p,i}, \tilde{v}_\pi^{p,i}, \tilde{y}_\pi^{p,i}) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$.

Vectors (vi)-(viii) belong to F since $i \geqslant m > 1$ and $i \in S^p$ imply that $1 \in S^p$ because of assumption (a) of Theorem 4.4 and the definition of m. If $(i-1) \in \bar{V}^p$, we define $r = \min\{k \in \{m, \ldots, l\} \mid \{k, \ldots, i-1\} \in \bar{V}^p\}$. Otherwise, we let $r = i$. Vectors (vi)-(viii) satisfy all but one of the equalities $v_{p,i} = d_{il}y_{p,i} + \sum_{j=r}^{i-1}(x_{p,j} - v_{p,j})$, $\sum_{j=r}^{i} x_{p,j} = \sum_{j=r}^{i} v_{p,j}$, and $x_{p,i} = 0$ that are satisfied by all previous solutions, as well as other points of the form (vi)-(viii). Moreover, each equality is violated by exactly one of these vectors, showing that they are affinely independent from each other and from all vectors presented earlier.

Third, for each $i \in \bar{S}^p \bigcap \{m, \ldots, l\}$, we construct vectors $(\check{x}^{p,i}; \check{v}^{p,i}; \check{y}^{p,i})$, $(\ddot{x}^{p,i}; \ddot{v}^{p,i}; \ddot{y}^{p,i})$ and $(\tilde{x}^{p,i}; \tilde{v}^{p,i}; \tilde{y}^{p,i})$ with nonzero components

(ix) $(\check{x}_p^{p,i}, \check{v}_p^{p,i}, \check{y}_p^{p,i}) = (d_{iT}e_i, d_{iT}e_i, e_i)$,

$(\check{x}_\pi^{p,i}, \check{v}_\pi^{p,i}, \check{y}_\pi^{p,i}) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,

(x) $(\ddot{x}_p^{p,i}, \ddot{v}_p^{p,i}, \ddot{y}_p^{p,i}) = (d_{iT}e_i, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i)$,

$(\ddot{x}_\pi^{p,i}, \ddot{v}_\pi^{p,i}, \ddot{y}_\pi^{p,i}) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,

(xi) $(\tilde{x}_p^{p,i}, \tilde{v}_p^{p,i}, \tilde{y}_p^{p,i}) = (d_{il}e_i + d_{(l+1)T}e_{l+1}, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$,

$(\tilde{x}_\pi^{p,i}, \tilde{v}_\pi^{p,i}, \tilde{y}_\pi^{p,i}) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$.

For r defined as in the case where $i \in S^p$, each of these vectors satisfy all but one of the equalities $v_{p,i} = d_{il}y_{p,i} + \sum_{j=r}^{i-1}(x_{p,j} - v_{p,j})$, $\sum_{j=r}^{i} x_{p,j} = \sum_{j=r}^{i} v_{p,j}$ and $x_{p,i} = d_{iT}y_{p,i}$ that are satisfied by all previous solutions, as well as other points of the form (ix)-(xi), because $l < T$ by condition (b). This shows that these vectors are affinely independent from each other and from all solutions presented earlier.

Fourth, for each $i \in V^p \bigcap \{m, \ldots, l\}$, we construct vectors $(\acute{x}^{p,i}, \acute{v}^{p,i}, \acute{y}^{p,i})$, $(\ddot{x}^{p,i}; \ddot{v}^{p,i}; \ddot{y}^{p,i})$, and $(\tilde{x}^{p,i}; \tilde{v}^{p,i}; \tilde{y}^{p,i})$ with nonzero components

(xii) $(\acute{x}^{p,i}_p;\acute{v}^{p,i}_p;\acute{y}^{p,i}_p) = (d_{(l+1)T}e_{l+1}, d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$,

$(\acute{x}^{p,i}_\pi;\acute{v}^{p,i}_\pi;\acute{y}^{p,i}_\pi) = (d_{1l}e_1, d_{1l}e_1, e_1)$,

(xiii) $(\ddot{x}^{p,i}_p, \ddot{v}^{p,i}_p, \ddot{y}^{p,i}_p) = (d_{iT}e_i, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i)$,

$(\ddot{x}^{p,i}_\pi, \ddot{v}^{p,i}_\pi, \ddot{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,

(xiv) $(\tilde{x}^{p,i}_p, \tilde{v}^{p,i}_p, \tilde{y}^{p,i}_p) = (d_{il}e_i + d_{(l+1)T}e_{l+1}, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$,

$(\tilde{x}^{p,i}_\pi, \tilde{v}^{p,i}_\pi, \tilde{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$.

These vectors satisfy all but one of the equalities $v_{p,i} = d_{il}y_{p,i}$, $x_{p,i} = v_{p,i}$ and $d_{il}x_{p,i} = d_{iT}v_{p,i}$ that are satisfied by all previous solutions, as well as other points of the form (xii)-(xiv). Since $l < T$ by condition (b), each equality is violated by exactly one of these vectors, showing that they are affinely independent from each other and from all solutions presented earlier.

Finally, we construct one additional point for each plant $p \in \mathcal{P}$. We observe that all previous solutions satisfy, for each $p \in \mathcal{P}$, the equality $\sum_{i\in \bar{S}^p \bigcap\{m,\dots,l\}} \sum_{j=r_i}^{i} (x_{p,j} - v_{p,j}) + \sum_{i\in V^p \bigcap\{m,\dots,l\}} (x_{p,i} - v_{p,i}) + \sum_{i=l+1}^{T} (x_{p,i} - v_{p,i}) = 0$ where $r_i = \min\{k \in \{m, \dots, l\} \mid \{k, \dots, i-1\} \in \bar{V}^p\}$ if $(i-1) \in \bar{V}^p$ and $i \notin \bar{V}^p$ and $r_i = i$, otherwise. We next present a family of affinely independent vectors that satisfy all but one of these equalities. Assume first that $1 \in S^p$. We consider two subcases. If $\bar{S}^p = \varnothing$ then we let j^p be any index in $\bar{V}^p$. Such index exists because of condition (c). We construct $(\bar{x}^p;\bar{v}^p;\bar{y}^p)$ with nonzero components

(xv) $(\bar{x}^p_p, \bar{v}^p_p, \bar{y}^p_p) = (d_{1(j^p-1)}e_1 + d_{j^pT}e_{j^p}, d_{1(j^p-1)}e_1 + d_{j^p\sigma_p(j^p)}e_{j^p} + d_{(\sigma_p(j^p)+1)l}e_{\sigma_p(j^p)+1}$

$+d_{(l+1)T}e_{l+1}, e_1 + e_{j^p})$.

Otherwise, we let j^p be any index in $\bar{S}^p$ and define $(\bar{x}^p;\bar{v}^p;\bar{y}^p)$ as follows

(xv$'$) $(\bar{x}^p_p, \bar{v}^p_p, \bar{y}^p_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1(j^p-1)}e_1 + d_{j^pl}e_{j^p} + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$.

On the other hand, if $1 \in V^p$, we construct $(\grave{x}^p, \grave{v}^p, \grave{y}^p)$ with nonzero components

(xv$''$) $(\grave{x}^p_p, \grave{v}^p_p, \grave{y}^p_p) = (d_{1T}e_1, d_{1l}e_1 + d_{(l+1)T}e_{l+1}, e_1)$.

This concludes the proof for the case where $m > 1$. The proof for the case where $m = 1$ is given in 7.7. □

Example 4.3. Consider an instance of LSS where $P = 2$, $T = 5$ and $(d_1, d_2, d_3, d_4, d_5) = (5, 7, 8, 9, 4)$. It follows from Theorem 4.5 that

$$5 \leqslant 5y_{1,1} + v_{2,1} \tag{36}$$
$$l = 1, \bar{S}^1 = L, V^2 = L,$$
$$12 \leqslant x_{1,1} + 7y_{1,2} + (v_{2,1} + 7y_{2,1}) + 7y_{2,2} \tag{37}$$
$$l = 2, S^1 = \{1\}, \bar{S}^1 = \{2\}, \bar{S}^2 = \{2\}, \bar{V}^2 = \{1\},$$
$$20 \leqslant x_{1,1} + (v_{1,2} + 8y_{1,2}) + x_{1,3} + v_{2,1} + v_{2,2} + v_{2,3} \tag{38}$$
$$l = 3, S^1 = \{1, 3\}, \bar{V}^1 = \{2\}, V^2 = L,$$
$$29 \leqslant x_{1,1} + 24y_{1,2} + v_{1,3} + v_{1,4} + x_{2,1} + v_{2,2} + 9y_{2,2} + v_{2,3} + 9y_{2,3} + x_{2,4} \tag{39}$$
$$l = 4, S^1 = \{1\}, \bar{S}^1 = \{2\}, V^1 = \{3, 4\}, S^2 = \{1, 4\}, \bar{V}^2 = \{2, 3\},$$

are facet-defining for Q^I. Theorem 4.5 also shows that inequalities (32)-(34) are facet-defining for Q^I. Inequality (31) is not facet-defining for Q^I since it violates condition (c) for $p = 1$. Similarly, (35) is not facet-defining for Q^I since it violates condition (b).

When (29) involves only production terms, we obtain the following result.

Corollary 4.2. *Inequality*

$$\sum_{p\in\mathcal{P}} \left[\sum_{i\in S^p} x_{p,i} + \sum_{i\in L\setminus S^p} d_{il} y_{p,i} \right] \geqslant d_{1l} \tag{40}$$

is facet-defining for Q^I if and only if

(a) $1 \in \bigcup_{p\in\mathcal{P}} S^p$,

(b) $l < T$,

(c) $\bar{S}^p \neq \varnothing$ *for* $p \in \mathcal{P}$.

Observe that inequalities (40) are similar to the traditional (l, S) inequalities of Barany et al. [1984] both in structure, and in the conditions under which they are facet-defining.

In 7.8, we show that the families of inequalities derived in this section are sufficient to describe the convex hull of LSS when T and P are small.

5 Computational Results

In this section, we investigate the effectiveness of formulations and families of inequalities we derived in the solution of instances of LSS.

5.1 Separation algorithms

First we note that the number of production inequalities (28) is bounded above by $O(PT^3)$. They can therefore be trivially separated in polynomial time. For this reason, we focus only on the separation of the (l, S, V) inequalities (29) which are exponential in number. In particular, we present an $O\left(PT^4\right)$ algorithm that, given a fractional solution $(x^*; v^*; y^*)$, outputs a violated (l, S, V) inequality if one exists. Observe that when validity conditions of Theorem 4.4 are applied, (l, S, V) inequalities can be written in the form

$$\sum_{p\in\mathcal{P}} \left[\sum_{i\in S^p} x_{p,i} + \sum_{i\in \bar{S}^p} d_{il} y_{p,i} + \sum_{i\in \bar{V}^p} \left(v_{p,i} + d_{(\sigma_p(i)+1)l} y_{p,i} \right) + \sum_{i=k_p}^{l} v_{p,i} \right] \geqslant d_{1l}, \tag{41}$$

where $k_p \leqslant l+1$ and S^p, $\bar{S}^p$ and $\bar{V}^p$ form a partition of $\{1, \ldots, k_p\}$ for $p \in \mathcal{P}$. Therefore, we must decide for each $i \in L = \{1, \ldots, l\}$ and each plant $p \in \mathcal{P}$, whether i belongs to S^p, $\bar{S}^p$, $\bar{V}^p$ or $V^p = \{k_p, \ldots, l\}$. Similar to the separation of (l, S) inequalities, see [Barany et al., 1984], we minimize for each plant $p \in \mathcal{P}$ and $l \in \mathcal{T}$ the left-hand-side of (41). In order to do so, we solve a collection of shortest path problems on networks $G_l^p = (N_l^p, A_l^p)$ for $p \in \mathcal{P}$ and for $l \in \{2, \ldots, T-1\}$. A graphical representation of one such network is given in Figure 3.

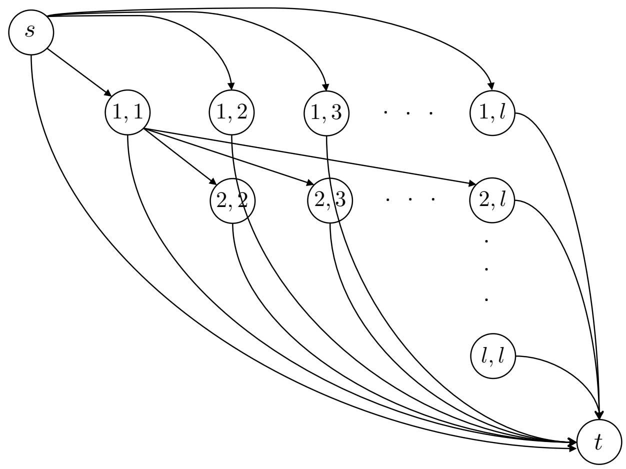


Figure 3: Network for separation of (l, S, V) inequalities.

To construct network G_l^p, we create nodes (i, j) for all $i \in \{1, \ldots, l\}$ and $j \in \{i, \ldots, l\}$ together with nodes $(0, 0)$ and $(l+1, l+1)$. To streamline notation, we refer to node $(0, 0)$ as s and to node $(l+1, l+1)$ as t. Node $(i, j) \in N_l^p \backslash \{s, t\}$ represents the decision to include the entire time window $[i, j]$ in one of the sets S^p, $\bar{S}^p$ or $\bar{V}^p$. We include an arc from each node $(i, j) \in N_l^p \backslash \{t\}$ to any other node $(j+1, k)$ to represent the decision of assigning time periods $[i, j]$ to a common set among $S^p, \bar{S}^p$ and $\bar{V}^p$ while assigning periods $[j+1, k]$ to a (possibly different) common set among $S^p, \bar{S}^p$ and $\bar{V}^p$. The cost of such an arc is equal to $\min\left\{\sum_{i=j+1}^{k} x_{p,i}^*, \sum_{i=j+1}^{k} d_{il} y_{p,i}^*, \sum_{i=j+1}^{k}\left(v_{p,i}^* + d_{(k+1)l} y_{p,i}^*\right)\right\}$. Moreover, we connect each node $(i, j) \in N_l^p \backslash \{t\}$ to node t to model the decision of assigning the entire time window $[j+1, l]$ to V^p. The cost of the corresponding arc is $\sum_{i=j+1}^{l} v_{p,i}^*$. It is easy to verify that G_l^p is acyclic.

It can now be easily verified that any path from s to t in network G_l^p represents a partition of L into subintervals. Further, the shortest path from s to t computes such a partition for which the component of the left-hand-side of (41) related to plant p is computed exactly and is minimized. Repeating the shortest path computation for each $p \in \mathcal{P}$ allows us to determine whether an (l, S, V) inequality with $L = \{1, \ldots, l\}$ is violated. To obtain a maximally violated (l, S, V) inequality given a fractional solution $(x^*; v^*; y^*)$, it suffices to enumerate all values of $l \in \{2, \ldots, T-1\}$. This leads to Algorithm 1.

Algorithm 1 Separation algorithm for (l, S, V) inequalities.

```
for l = 2, ..., T − 1 do
    for p ∈ P do
      Create network G_l^p = (N_l^p, A_l^p).
      Find a shortest path on G_l^p. Let α_l^p be the length of this shortest path.
    end for
    if Σ_{p∈P} α_l^p < d_{1l} then output (S^p, S̄^p, V^p, V̄^p) for p ∈ P.
    end if
end for
```

We next analyze the running time of Algorithm 1. First recall that the shortest path problem on an acyclic graph can be solved in time $O(m+n)$ where m is the number of arcs in the network and n is the number of nodes. Since $G_l^p = (N_l^p, A_l^p)$ is acyclic and $|N_l^p| \leqslant |A_l^p|$, it suffices to derive a bound on the cardinality of A_l^p to estimate the worst-case running time of the algorithm. For fixed l, the number of nodes in G_l^p is bounded above by $O(l^2)$. Moreover, the degree of each node is bounded above by $O(l)$. It follows that the total number of arcs in G_l^p is $O(l^3)$. Finding a shortest path on G_l^p therefore requires $O(l^3)$ computation. Since the solution of a shortest path problem is required for each $l \in \{2, \dots, T-1\}$ and for each $p \in \mathcal{P}$, we conclude that the amount of computation required by the separation algorithm is bounded above by $O(PT^4)$.

Remark 1. The arc cost calculations required for creating G_l^p need to be performed efficiently in order to achieve the $O(PT^4)$ bound. In particular, before creating G_l^p, we define $D_t = \sum_{i=1}^t d_i$, $Y_{p,t} = \sum_{i=1}^t y_{p,i}^*$ and $V_{p,t} = \sum_{i=1}^t v_{p,i}^*$. Because $D_t = D_{t-1} + d_t$, $Y_{p,t} = Y_{p,t-1} + y_{p,t}^*$ and $V_{p,t} = V_{p,t-1} + v_{p,t}^*$ for $t > 1$, computing values D_t for $t \in \mathcal{T}$ and values $Y_{p,t}$ and $V_{p,t}$ for $p \in \mathcal{P}$ and for $t \in \mathcal{T}$ requires an amount of computation that is bounded above by $O(PT)$. Once these values are calculated, the cost of arc $((i,j),(j+1,k))$ can be computed from the cost of arc $((i,j),(j+1,k-1))$ by adding $x_{p,k}^*$ to the first term, $(D_l - D_{k-1})\, y_{p,k}^*$ to the second term, and $v_{p,k}^* + (D_l - D_k)\, y_{p,k}^* - d_k(Y_{p,k-1} - Y_{p,j})$ to the third term. Moreover the cost of arc $((i,j),t)$ can be obtained as $(V_{p,l} - V_{p,j})$. We conclude that all arc costs can be calculated using an amount of computation that is bounded above by $O(1)$, resulting in a total effort of $O(l^3)$.

Although Algorithm 1 is polynomial, it is relatively expensive to run in practice. For this reason, we implement a heuristic based on the following observation.

Remark 2. If we lower bound $\sigma_p(i)$ by i for $p \in \mathcal{P}$ and for $i \in \bar{V}^p$, (l,S,V) inequalities become

$$\sum_{p\in\mathcal{P}} \left[\sum_{i\in S^p} x_{p,i} + \sum_{i\in\bar{S}^p} d_{il} y_{p,i} + \sum_{i\in\bar{V}^p} \left(v_{p,i} + d_{(i+1)l} y_{p,i} \right) + \sum_{i=k_p}^{l} v_{p,i} \right] \geqslant d_{1l}. \tag{42}$$

Because $d_{(\sigma_p(i)+1)l} \leqslant d_{(i+1)l}$, (2) is valid but not necessarily facet-defining for LSS. When applying Algorithm 1 to separate (42), the amount of computation required can be bounded above by $O\left(PT^2\right)$. This is due to the fact that when building G_l^p, it is sufficient to include nodes (i,i) for $i \in L \bigcup \{0, l+1\}$. Therefore the shortest path cost on G_l^p can be calculated in $O(l)$ for fixed $l \in L$ and $p \in \mathcal{P}$ since the degree of each node in the graph is bounded above by 2. Since $|\mathcal{P}| = P$ and $|L| = O(T)$, we obtain the desired result. If a violated inequality is found using this heuristic, it is easy to verify that the (l,S,V) inequality based on $\left(S^p, \bar{S}^p, V^p, \bar{V}^p\right)$ is also valid and violated. This provides us with a faster procedure to heuristically separate (l,S,V) inequalities.

5.2 Experimental setup and instance generation

We present two sets of computational experiments in Section 5.3. The first set is composed of uncapacitated LSS instances with $T = 25$. We use these instances to evaluate the strength of the different formulations introduced in this paper and of the facet-defining inequalities we developed. The second set is comprised of capacitated instances of LSS with number of periods T chosen randomly in the interval $[70, 100]$. We use these instances to evaluate the strength of (l,S,V) inequalities inside of a cut-and-branch framework. Together these experiments show that our results provide computational improvements for both LSS instances as well as instances of other problems for which LSS is a relaxation.

We generate random instances as follows. We use the ratio of maximum fixed production cost to maximum variable production cost, δ, as the main lever to produce instances with different characteristics. We set the maximum production cost, c_{max}, to 100 and then calculate the maximum fixed cost as $f_{max} = \delta c_{max}$. To create fixed and variable production costs we first generate a random number $\alpha_{p,t}$

uniformly in the interval $[0, 1]$, and then calculate $c_{p,t} = \alpha_{p,t} c_{max}$ and $f_{p,t} = (1 - \alpha_{p,t}) f_{max}$. This choice of parameters tends to generate harder instances by creating less obvious tradeoffs between periods with small fixed cost and large production cost and periods with large fixed cost and small production cost. Customer demand and inventory cost for each period are generated using a discrete uniform distribution in the intervals $[50, 100]$ and $[5, 20]$, respectively. For each plant, we generate transportation and inventory holding costs using a discrete uniform distribution with values in the interval $[5, 20]$.

For uncapacitated instances, we set the production and shipment capacity of each plant in each time period to $\sum_{t \in \mathcal{T}} d_t$. For capacitated instances we generate a common production and shipment capacity for all plants and time periods. This number is randomly selected in the interval $[1.5\bar{d}, 1.75\bar{d}]$ where $\bar{d} = \frac{\sum_{t \in \mathcal{T}} d_t}{T}$ is the average demand.

5.3 Results

		CPX		*LSVH*			*LSVE*			*PUB*			*MC3*	*MC4*
P.δ	*RGap*	*Gap*	*Cuts*	*Gap*	*Cuts*		*Gap*	*Cuts*		*Gap*	*Cuts*		*Gap*	*Gap*
					U	*C*		*U*	*C*		*U*	*C*		
2.25	25.8	1.0	24	0.1	49	18	0.0	29	19	1.0	5	24	0.0	0.0
2.50	35.6	4.6	28	0.1	112	16	0.0	70	20	4.1	12	28	0.0	0.0
2.75	43.9	7.5	33	0.2	148	14	0.0	84	21	7.5	17	33	0.0	0.0
2.100	48.6	8.2	31	0.0	196	11	0.0	80	18	7.5	14	32	0.0	0.0
3.25	29.8	2.2	34	0.0	88	22	0.0	49	26	2.2	12	34	0.0	0.0
3.50	45.5	8.7	40	0.0	235	16	0.0	132	21	8.6	24	40	0.0	0.0
3.75	48.9	10.6	44	0.6	303	11	0.0	198	16	10.7	20	42	0.0	0.0
3.100	54.7	14.6	41	1.1	362	9	0.0	247	15	14.3	30	41	0.0	0.0
4.25	34.4	4.7	43	0.4	178	22	0.1	137	23	4.7	19	41	0.0	0.0
4.50	47.4	10.0	50	0.8	331	14	0.2	234	22	9.7	30	50	0.0	0.0
4.75	52.2	12.9	48	1.5	345	12	0.0	214	17	12.7	26	49	0.0	0.0
4.100	58.6	15.5	52	0.5	490	9	0.0	299	16	15.1	30	52	0.0	0.0
5.25	39.8	6.1	49	0.5	198	24	0.1	184	24	6.0	21	47	0.0	0.0
5.50	50.4	12.0	52	0.9	355	16	0.2	249	21	12.0	34	52	0.0	0.0
5.75	56.5	16.3	59	2.6	430	14	1.2	403	15	16.2	42	59	0.0	0.0
5.100	60.6	18.4	57	4.0	458	15	2.4	411	17	18.1	31	56	0.0	0.0

Table 1: Strength of different formulations on Instance Set 1.

In this section we present the results of our computational experiments. We perform all computations using IBM ILOG CPLEX version 12.5, on a Windows machine with Windows 7 Professional running a 64-bit x86 processor equipped with 2.83 GHz Intel Core2 Quad Q9500 chips and 4GB RAM.

In our first set of experiments, we consider the relative optimality gap at the root node right before branching. The gap reported is calculated as $100\frac{z^* - \bar{z}}{z^*}$ where z^* is objective function value of the optimal solution and $\bar{z}$ is the optimal value of the LP relaxation at the the root node right before branching. We summarize our results in Table 1. We investigate the following: the natural formulation with default CPLEX cuts (CPX), the natural formulation strengthened with (l, S, V) inequalities separated using the heuristic separation algorithm and CPLEX cuts ($LSVH$), the natural formulation strengthened with (l, S, V) inequalities separated using the exact separation algorithm and CPLEX cuts ($LSVE$), the natural formulation strengthened with production upper bound inequalities and CPLEX cuts (PUB), and the multi-commodity formulation with three-index and four-index variables (*MC3* and *MC4*, respectively). For this experiment, we create 10 uncapacitated instances of LSS for each combination of the parameters $P \in \{2, 3, 4, 5\}$ and $\delta \in \{25, 50, 75, 100\}$. The initial average relative gap of the relaxation for each choice of parameters is reported under the header *RGap*. The average relative optimality gap at the root node right before branching and the number of cuts added to the relaxation are reported under the headers *Gap* and *Cuts* (when applicable, the number of user cuts is reported under U while the number of CPLEX cuts is reported under C), respectively. Table 1 shows that the natural formulation strengthened with (l, S, V) cuts separated using the exact separation algorithm solved LSS at the root node for most instances considered. This suggests that (l, S, V) cuts provide a good approximation of the convex hull of LSS. Using the exact separation algorithm in practice might however be too time-consuming. Table 1 shows that (l, S, V) cuts separated using the heuristic separation algorithm are effective at reducing the initial optimality gap. In fact, on average, heuristically separated (l, S, V) inequalities close 98% of the optimality gap that cannot be closed using CPLEX cuts alone. For all of the instances presented in Table 1 both multi-commodity formulations have zero initial optimality gap. This is the reason we do not report the number of cuts generated for multi-commodity formulations. This suggests that multi-commodity formulations are strong in practice. In fact, our computational experience is that the multi-commodity formulation is effective in solving uncapacitated LSS instances even as the number of plants increases. Its use however becomes less appealing as additional restrictions, such as capacities are introduced.

We next explore the effectiveness of (l, S, V) cuts in solving capacitated instances of LSS. Similar to before, we create 10 capacitated instances of LSS for each choice of the parameters $P \in \{2, 3, 4, 5\}$ and $\delta \in \{125, 250\}$. In Table 2, we report average results across these 10 instances. We compare the solution performance of the natural formulation with default CPLEX cuts (CPX), to the natural formulation strengthened with both heuristically separated (l, S, V) cuts and default CPLEX cuts ($LSVH$), and the multi-commodity formulation with default CPLEX cuts (*MC4*). We do not use the multi-commodity formulation with three-index variables since it cannot be used in the presence of shipment capacities. Moreover, we choose not to investigate the effectiveness of production upper bound cuts since our results in Table 1 suggest that they are not as strong as default CPLEX cuts.

In our implementation, we first derive a set of violated (l, S, V) cuts by using the callback functionalities of CPLEX. In this framework, our polynomial-time heuristic separation routine of Section 5.1 is invoked each time a fractional solution of the root relaxation is found. If it returns violated cuts, up to 10 of them are added to the cut pool and the relaxation is solved again. This procedure continues until the improvement in gap becomes less than 0.1% over three consecutive iterations or when the separation routine is visited more than 50 times. When the callback routine is terminated, we calculate the slack of each cut discovered. We then create a new model containing the initial formulation and the cuts we generated that have zero slack. We solve the resulting model with CPLEX in its default settings.

In Table 2, we report the results obtained by solving each model using CPLEX in its default settings with a time limit of 3600 seconds. We set the relative optimality gap tolerance to 0.1%. In Table 2, we denote the average relative optimality gap reported by CPLEX at the root node right before branching as *RGap*, the average total solution time (including time spent in the separation routine) in seconds as *Time*, the average number of nodes in the branch-and-bound tree as *NNodes*, the average relative optimality gap when CPLEX stops as *Gap*, the average number of (l, S, V) cuts added to the formulation as *LCuts*, the average number of CPLEX cuts added to the formulation as *CCuts*, and the number of instances that are solved to 0.1% optimality as *#Opt*.

Table 2 shows that when $\delta = 125$, adding (l, S, V) cuts to the natural formulation reduced the number of nodes by 54% on average, the solution time by 51% on average, and the strengthened formulation was

		$\delta = 125$			$\delta = 250$		
		CPX	*LSVH*	*MC4*	*CPX*	*LSVH*	*MC4*
$P = 2$	*RGap*	2.21	1.27	2.59	1.44	1.13	2.01
	Time	1241	609	3625	10	6	2788
	NNodes	2803693	1220881	68670	17497	5152	50186
	Gap	0.11	0.10	0.58	0.10	0.10	0.23
	LCuts	0	20	0	0	10	0
	CCuts	111	92	496	62	47	429
	#Opt	8	9	0	10	10	3
$P = 3$	*RGap*	1.81	1.23	3.84	1.24	0.88	1.78
	Time	1064	378	3649	62	53	2737
	NNodes	1483977	491653	44444	61332	53707	28730
	Gap	0.11	0.10	0.53	0.10	0.10	0.25
	LCuts	0	16	0	0	7	0
	CCuts	123	108	659	72	58	588
	#Opt	9	10	0	10	10	5
$P = 4$	*RGap*	1.61	1.19	3.79	1.42	1.04	5.23
	Time	1477	691	3699	34	24	3404
	NNodes	1267324	481330	32115	24588	17493	25804
	Gap	0.11	0.10	0.48	0.10	0.10	0.42
	LCuts	0	17	0	0	11	0
	CCuts	140	121	818	78	76	731
	#Opt	7	9	0	10	10	1
$P = 5$	*RGap*	1.60	1.09	6.36	1.47	0.91	11.37
	Time	2449	1629	3604	148	222	3000
	NNodes	1947505	1142436	24299	109602	174253	19755
	Gap	0.16	0.13	0.47	0.10	0.10	0.34
	LCuts	0	19	0	0	13	0
	CCuts	146	139	992	85	80	817
	#Opt	5	7	1	10	10	3

Table 2: Cut-and-branch performance for Instance Set 2.

able to solve more instances to optimality within the time limit. These result illustrate the computational potential of using (l, S, V) cuts in solving capacitated LSS instances.

Our results also show that the multi-commodity formulation was unable to solve most of the instances considered within the time limit when $\delta = 125$. It was able to solve less than 50% of the instances within the time limit when $\delta = 250$. When $\delta = 250$, adding (l, S, V) cuts to the natural formulation reduced the number of nodes in most instances. The difference in solution times was small, however, because the solution time for these instances using default CPLEX cuts is under 180 seconds on average.

We conclude from these experiments that for uncapacitated instances both the natural formulation with (l, S, V) cuts and multi-commodity formulations provide strong relaxations of LSS. Our results in Table 2 show that when capacitated instances are considered the natural formulation strengthened with (l, S, V) cuts outperforms both the multi-commodity formulation and the natural formulation strengthened with default CPLEX cuts only.

6 Conclusion

In this paper, we study a two-level lot-sizing problem with supplier selection (LSS). This *NP-hard* problem arises, either naturally or as a relaxation, in different production planning and supply chain management applications. We present an $O\left(PT^{P+2}\right)$ dynamic programming algorithm for the solution of this problem, and use the associated recursion to obtain a convex hull description of the underlying set in an extended space. We describe a multi-commodity formulation that has an empirically strong relaxation but has a large number of variables and does have fractional extreme points. We present valid inequalities to strengthen this formulation. For the traditional formulation, we propose two new families of facet-defining inequalities that can be separated in polynomial time. We prove that these inequalities are sufficient to describe the convex hull of the problem when $P = 2$ and $T = 2$. We finally demonstrate that these inequalities can be successfully used to reduce the solution times of branch-and-cut algorithms.

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7 Appendix

7.1 Proof of Theorem 3.1

Theorem 3.1. *The dynamic programming recursions given by* (15)-(16) *solve LSS in time* $O\left(P(2T)^{P+1}\right)$.

Proof. Similar to the proof given by Balakrishnan and Geunes [2000], we first describe a network on which solving the shortest path problem corresponds to executing (15)-(16). Let $G(0)$ be the source, $G(T)$ be the destination and $G(t, \mathbf{i})$ for $t \in \mathcal{T}$ and $\mathbf{i} \in \mathcal{I}(t)$ be the transshipment nodes of this network. For $t \in \mathcal{T}$ we denote all nodes $G(t, \mathbf{i})$ for $\mathbf{i} \in \mathcal{I}(t)$ as layer t. In this network, there is an arc between each node $G(T, \mathbf{i})$ for $\mathbf{i} \in \mathcal{I}(T)$ and the node $G(T)$ with cost f^-. Moreover, there is an arc between nodes $G(t-1, \mathbf{j})$ and $G(t, \mathbf{i})$ for $t \in \mathcal{T}$, $\mathbf{i} \in \mathcal{I}(t)$ and $\mathbf{j} \in \mathcal{K}(t, \mathbf{i})$. Since each component i^p for $p \in \mathcal{P}$ of setup vector $\mathbf{i}$ either stays the same or is updated by a new setup in each period, the outdegree of each node is 2^P. There are $O((t+1)^P)$ nodes in each layer $t \in \mathcal{T}_0$ and therefore $2^P O(t^P)$ arcs between layers t and $t+1$. This leads to a total of $O(2^P \sum_{t=0}^{T-1}(t+1)^P) = 2^P O(T^{P+1}) = O((2T)^{P+1})$ arcs. The cost of any such arc is determined using (15) which requires a computation bounded above by $O(P)$ for fixed $\mathbf{i} \in \mathcal{I}(t)$ and $\mathbf{j} \in \mathcal{K}(t, \mathbf{i})$. By construction, this network is acyclic and therefore finding a shortest path only requires the inspection of arc costs once. We conclude that the running time of the algorithm is $O(P(2T)^{P+1})$. □

7.2 Dynamic programming extended formulation

In this section, we study the extended formulation induced by the dynamic programming algorithm described in Section 3.1. To this end, we let $\chi_{p,t,\mathbf{i}} = \sum_{p' \in F(t,\mathbf{i})} f^+_{p',t} + \{\gamma_{p,i^p} + H^p\,(i^p, t)\}d_t$ for $t \in \mathcal{T}$, $\mathbf{i} \in \mathcal{I}(t)$, and $p \in \mathcal{P}^+(\mathbf{i})$, that is the cost of producing d_t at plant p with setup vector $\mathbf{i}$. We then write (15) and (16) as the set of linear constraints (45) and (46), respectively. We obtain the formulation

$$\max \quad G(T) - G(0, \mathbf{0}) \tag{43}$$

$$s.t. \quad G(0, \mathbf{0}) = 0 \tag{44}$$

$$G(t, \mathbf{i}) - G(t-1, \mathbf{j}) \leqslant \chi_{p,t,\mathbf{i}} \qquad \forall t \in \mathcal{T}, \forall \mathbf{i} \in \mathcal{I}(t), \forall p \in \mathcal{P}^+(\mathbf{i}), \forall \mathbf{j} \in \mathcal{K}(t, \mathbf{i}) \tag{45}$$

$$G(T) - G(T, \mathbf{i}) \leqslant f^- \qquad \forall \mathbf{i} \in \mathcal{I}(T) \tag{46}$$

$$G(T), G(t, \mathbf{i}) \in \mathbb{R} \qquad \forall t \in \mathcal{T}, \forall \mathbf{i} \in \mathcal{I}(t), \tag{47}$$

where the sets $\mathcal{P}^+(\mathbf{i})$, $\mathcal{I}(t)$ and $\mathcal{K}(t, \mathbf{i})$ are as defined previously. This formulation has $O\left(T^{P+1}\right)$ variables and $O\left(P(2T)^{P+1}\right)$ constraints.

Next we derive the linear programming dual of (43)-(47). We define the set $\mathcal{L}(t, \mathbf{i}) = \{\mathbf{j} \in \mathcal{I}(t+1) \mid \mathbf{i} \in \mathcal{K}(t+1, \mathbf{j})\}$ as the set of successor setup vectors of vector $\mathbf{i} \in \mathcal{I}(t)$ in period t. We associate the variables $u_{p,t,\mathbf{i},\mathbf{j}}$ for $t \in \mathcal{T}$, $\mathbf{i} \in \mathcal{I}(t)$, $p \in \mathcal{P}^+(\mathbf{i})$, and $\mathbf{j} \in \mathcal{K}(t, \mathbf{i})$ with constraints (45) and variables $\omega_{\mathbf{i}}$ for $\mathbf{i} \in \mathcal{I}(T)$ with constraints (46). We then write

$$\min \quad \sum_{t \in \mathcal{T}} \sum_{\mathbf{i} \in \mathcal{I}(t)} \sum_{p \in \mathcal{P}^+(\mathbf{i})} \chi_{p,t,\mathbf{i}} \sum_{\mathbf{j} \in \mathcal{K}(t,\mathbf{i})} u_{p,t,\mathbf{i},\mathbf{j}} + f^- \sum_{\mathbf{i} \in \mathcal{I}(T)} \omega_{\mathbf{i}} \tag{48}$$

$$s.t. \quad \sum_{\mathbf{i} \in \mathcal{I}(T)} \omega_{\mathbf{i}} = 1 \tag{49}$$

$$- \omega_{\mathbf{i}} + \sum_{p \in \mathcal{P}^+(\mathbf{i})} \sum_{\mathbf{j} \in \mathcal{K}(T,\mathbf{i})} u_{p,T,\mathbf{i},\mathbf{j}} = 0 \qquad \forall \mathbf{i} \in \mathcal{I}(T) \tag{50}$$

$$- \sum_{\mathbf{k} \in \mathcal{L}(t,\mathbf{i})} \sum_{p \in \mathcal{P}^+(\mathbf{k})} u_{p,t+1,\mathbf{k},\mathbf{i}} + \sum_{\mathbf{j} \in \mathcal{K}(t,\mathbf{i})} \sum_{p \in \mathcal{P}^+(\mathbf{i})} u_{p,t,\mathbf{i},\mathbf{j}} = 0 \qquad \forall t \in \mathcal{T} \backslash \{T\}, \forall \mathbf{i} \in \mathcal{I}(t) \tag{51}$$

$$u \in \mathbb{R}_+^{\sum_{t \in \mathcal{T}} \sum_{\mathbf{i} \in \mathcal{I}(t)} |\mathcal{P}^+(\mathbf{i})| \times |\mathcal{K}(t,\mathbf{i})|}, \omega \in \mathbb{R}_+^{|\mathcal{I}(T)|}. \tag{52}$$

This formulation has $O\left(T^{P+1}\right)$ constraints and $O\left(P(2T)^{P+1}\right)$ variables. Note that this formulation does not contain the constraint corresponding to the primal variable $G\left(0, \mathbf{0}\right)$ since it is redundant. By using constraints (49) and (50), we equivalently obtain

$$\min \sum_{t \in \mathcal{T}} \sum_{\mathbf{i} \in \mathcal{I}(t)} \sum_{p \in \mathcal{P}^+(\mathbf{i})} \chi_{p,t,\mathbf{i}} \sum_{\mathbf{j} \in \mathcal{K}(t,\mathbf{i})} u_{p,t,\mathbf{i},\mathbf{j}} + f^- \tag{53}$$

$$s.t. \sum_{\mathbf{k} \in \mathcal{L}(t,\mathbf{i})} \sum_{p \in \mathcal{P}^+(\mathbf{k})} u_{p,t+1,\mathbf{k},\mathbf{i}} - \sum_{\mathbf{j} \in \mathcal{K}(t,\mathbf{i})} \sum_{p \in \mathcal{P}^+(\mathbf{i})} u_{p,t,\mathbf{i},\mathbf{j}} = 0 \qquad \forall t \in \mathcal{T} \backslash \{T\}, \forall \mathbf{i} \in \mathcal{I}(t) \tag{54}$$

$$\sum_{\mathbf{i} \in \mathcal{I}(T)} \sum_{p \in \mathcal{P}^+(\mathbf{i})} \sum_{\mathbf{j} \in \mathcal{K}(T,\mathbf{i})} u_{p,T,\mathbf{i},\mathbf{j}} = 1 \tag{55}$$

$$u \in \mathbb{R}_+^{\sum_{t \in \mathcal{T}} \sum_{\mathbf{i} \in \mathcal{I}(t)} |\mathcal{P}^+(\mathbf{i})| \times |\mathcal{K}(t,\mathbf{i})|}. \tag{56}$$

In the above formulation, $u_{p,t,\mathbf{i},\mathbf{j}}$ is equal to 1 if d_t is produced at plant p with vector $\mathbf{i}$ after producing d_{t-1} with vector $\mathbf{j}$ and 0 otherwise. With this interpretation, it is clear that the dual objective is to minimize the total cost. Constraints (54) ensure the continuity of setup vectors between subsequent production periods and, constraint (55) ensures that the entire demand d_{1T} is satisfied.

Next, we relate the dual variables to the original variables of the problem. In particular,

$$x_{p,t} = \sum_{t'=t}^{T} \sum_{\mathbf{i} \in \mathcal{I}(t') | i^p = t} \sum_{\mathbf{j} \in \mathcal{K}(t',\mathbf{i})} u_{p,t',\mathbf{i},\mathbf{j}} \, d_{t'} \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{57}$$

$$v_{p,t} = \sum_{t'=1}^{t} \eta_{p,t,t',t''} \left(\sum_{t''=t}^{T} \sum_{\mathbf{i} \in \mathcal{I}(t'') | i^p = t'} \sum_{\mathbf{j} \in \mathcal{K}(t'',\mathbf{i})} u_{p,t'',\mathbf{i},\mathbf{j}} \, d_{t''} \right) \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T}, \tag{58}$$

where we define $\eta_{p,t,t',t''}$ to be 1 if the shortest path between t' and t'' ($t' \leqslant t \leqslant t''$) for plant p goes through arc $(p,t) - (C,t)$ and to be 0 otherwise. Moreover for $p \in \mathcal{P}$ and $t \in \mathcal{T}$, we have that $y_{p,t} = 1$ if $f_{p,t} \leqslant 0$ and that $y_{p,t} = \sum_{\mathbf{i} \in \mathcal{I}(t) | i^p = t} \sum_{\mathbf{j} \in \mathcal{K}(t,\mathbf{i})} u_{p,t,\mathbf{i},\mathbf{j}}$ if $f_{p,t} > 0$, which we write as

$$1 \geqslant y_{p,t} \geqslant \sum_{\mathbf{i} \in \mathcal{I}(t) | i^p = t} \sum_{\mathbf{j} \in \mathcal{K}(t,\mathbf{i})} u_{p,t,\mathbf{i},\mathbf{j}} \qquad \forall p \in \mathcal{P}, \forall t \in \mathcal{T}. \tag{59}$$

With these additions, we obtain the formulation:

$$\min \sum_{p \in P} \sum_{t \in \mathcal{T}} [\gamma_{p,t} x_{p,t} + f_{p,t} y_{p,t} + \delta_{p,t} v_{p,t}] \tag{60}$$

$$s.t. \quad (54) - (55), (57) - (59) \tag{61}$$

$$u \in \mathbb{R}_+^{\sum_{t \in \mathcal{T}} \sum_{\mathbf{i} \in \mathcal{I}(t)} |\mathcal{P}^+(\mathbf{i})| \times |\mathcal{K}(t,\mathbf{i})|}, v, x \in \mathbb{R}_+^{PT}, y \in \mathbb{Z}_+^{PT}. \tag{62}$$

This formulation has an optimal solution where $y_{p,t} \in \{0,1\}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$, and $x_{p,t}$ and $v_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$ satisfy the constraints of LSS.

7.3 Proof of Theorem 3.2

Theorem 3.2. *Covering inequality* (25) *is valid for* $Q^{I,MC}$ *if*

(i) The size of a minimal covering of $\mathcal{I}$ *is at least* $|\mathcal{I}| - \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}|$.

(ii) For all $p \in \mathcal{P}$ *the size of a minimal covering of* $\mathcal{I}\backslash\mathcal{I}_{p,1}$ *is at least* $|\mathcal{I}| - |\mathcal{I}_{p,1}|$.

Proof. We first argue that condition (i) implies that for any non-empty subset $\mathcal{S}$ of $\mathcal{I}$ such that $|\mathcal{S}| > \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}|$, the size of a minimal covering of $\mathcal{S}$ should be at least $|\mathcal{S}| - \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}|$. Assume not, then there exists a covering U of $\mathcal{S}$, with $|U| \leqslant |\mathcal{S}| - \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| - 1$. For each $i \in \mathcal{I}\backslash\mathcal{S}$, select $(p,t) \in V_i$ such that $t \geqslant 2$, and include it in U. Such element exists since we assume that there exists a covering of $\mathcal{I}$. The set obtained is a covering of $\mathcal{I}$ of size at most $|\mathcal{I}| - \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| - 1$, which is the desired contradiction. A similar argument shows that, for each $p \in \mathcal{P}$, condition (ii) implies that for any nonempty subset $\mathcal{S}$ of $\mathcal{I}$, the size of a minimal covering of $\mathcal{S}\backslash\mathcal{I}_{p,1}$ should be at least $|\mathcal{S}| - |\mathcal{I}_{p,1} \bigcap \mathcal{S}|$.

Consider any feasible solution (y, ϕ) of LSS. Let $\mathcal{I}^* = \{i \in \mathcal{I} \mid \phi_{p,t,i} > 0 \text{ for some } (p,t) \in V_i\}$. Clearly, the left-hand-side of (25) is less than or equal to $|\mathcal{I}^*|$ since (23a) implies that $\sum_{(p,t)\in V_i} \frac{\phi_{p,t,i}}{d_i} \leqslant 1$ for each $i \in \mathcal{I}$. Moreover, for each $i \in \mathcal{I}^*$ there exists $(p,t) \in V_i$ such that $y_{p,t} = 1$. Let $U = \{(p,t) \in V \mid y_{p,t} = 1\}$ and $\bar{\mathcal{P}} = \{p \in \mathcal{P} \mid (p,1) \in U\}$. We consider two cases.

First, assume that $\bar{\mathcal{P}} \neq \varnothing$. If $|\mathcal{I}^*| \leqslant \sum_{p\in\bar{\mathcal{P}}} |\mathcal{I}_{p,1}|$ then (25) is valid since $y_{p,1} = 1$ for $p \in \bar{\mathcal{P}}$. Therefore, we assume that $|\mathcal{I}^*| > \sum_{p\in\bar{\mathcal{P}}} |\mathcal{I}_{p,1}|$. For some $\bar{p} \in \bar{\mathcal{P}}$, let $U_{\bar{p}} = U\backslash\bigcup_{p\in\bar{\mathcal{P}}\backslash\{\bar{p}\}}\{(p,1)\}$. Then, $U^* = U_{\bar{p}}\backslash\{(\bar{p},1)\}$ is nonempty and is a covering of $(\mathcal{I}^*\backslash\bigcup_{p\in\bar{\mathcal{P}}\backslash\{\bar{p}\}} \mathcal{I}_{p,1})\backslash\mathcal{I}_{\bar{p},1}$. By condition (ii), we must have that $|U^*| \geqslant |\mathcal{I}^*\backslash\bigcup_{p\in\bar{\mathcal{P}}\backslash\{(\bar{p},1)\}} \mathcal{I}_{p,1}| - |(\mathcal{I}^*\backslash\bigcup_{p\in\bar{\mathcal{P}}\backslash\{\bar{p}\}} \mathcal{I}_{p,1}) \bigcap \mathcal{I}_{\bar{p},1}|$. Therefore, we have that $\sum_{(p,t)\in V|t\geqslant 2} y_{p,t} + \sum_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1} \geqslant |\mathcal{I}^*\backslash\bigcup_{p\in\bar{\mathcal{P}}\backslash\{(\bar{p},1)\}} \mathcal{I}_{p,1}| - |(\mathcal{I}^*\backslash\bigcup_{p\in\bar{\mathcal{P}}\backslash\{\bar{p}\}} \mathcal{I}_{p,1}) \bigcap \mathcal{I}_{\bar{p},1}| + \sum_{p\in\bar{\mathcal{P}}} |\mathcal{I}_{p,1}| \geqslant |\mathcal{I}^*| - \sum_{p\in\bar{\mathcal{P}}\backslash\{\bar{p}\}} |\mathcal{I}_{p,1}| - |\mathcal{I}_{\bar{p},1}| + \sum_{p\in\bar{\mathcal{P}}} |\mathcal{I}_{p,1}| \geqslant |\mathcal{I}^*|$, where the first inequality holds because $y_{p,1} = 1$ for $p \in \bar{\mathcal{P}}$, and the second holds because $(\mathcal{I}^* \bigcap \mathcal{I}_{p,1}) \subseteq \mathcal{I}_{p,1}$ for $p \in \bar{\mathcal{P}}$.

Second, assume that $\bar{\mathcal{P}} = \varnothing$. Then U is a covering of $\mathcal{I}^*$. We assume that $|U| \leqslant |\mathcal{I}^*|$ since otherwise (25) is valid as $y_{p,t} = 1$ for $(p,t) \in U$. Moreover, we assume that $|\mathcal{I}^*| > \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}|$ since otherwise (25) is valid as combining (23a) for $t' = 1$ and (23b) for $t' = 1$ and $p \in \mathcal{P}$ implies that $\sum_{p\in\mathcal{P}} y_{p,1} \geqslant 1$, and therefore $\sum_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1} \geqslant \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| \geqslant |\mathcal{I}^*|$. Then condition (i) implies that the size of U is at least $|\mathcal{I}^*| - \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}|$. Therefore, we have that $\sum_{(p,t)\in V|t\geqslant 2} y_{p,t} + \sum_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1} \geqslant |\mathcal{I}^*| - \min_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| + \sum_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1} \geqslant |\mathcal{I}^*|$, where the last inequality holds because $\sum_{p\in\mathcal{P}} y_{p,1} \geqslant 1$ as explained previously. □

7.4 Proof of Theorem 3.3

We first present the following preliminary result that is useful in proving Theorem 3.3.

Proposition 7.1. *When* $P \geqslant 2$, $\dim\left(Q^{I,MC}\right) = \frac{PT^2+PT}{2} + (P-1)T$.

Proof. There are $\frac{PT^2+PT}{2} + PT$ variables in the formulation of $Q^{I,\mathrm{MC}}$, and every feasible solution of $Q^{I,\mathrm{MC}}$ satisfies the T linearly independent equalities (23a). This shows that $\dim\left(Q^{I,\mathrm{MC}}\right) \leqslant \frac{PT^2+PT}{2} +$

$(P-1)T$. To prove equality, we present $\frac{PT^2+PT}{2} + (P-1)T + 1$ affinely independent points in $Q^{I,\mathrm{MC}}$. First, we construct the vector $(\bar{y}, \bar{\phi})$ with nonzero components

(i) $\bar{y}_{p,t} = 1$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$ and $\bar{\phi}_{1,1,t'} = d_{t'}$ for $t' \in \mathcal{T}$.

Second, we construct the vectors $(\dot{y}^{(\bar{p},\bar{t}),t'}; \dot{\phi}^{(\bar{p},\bar{t}),t'})$ for each $(\bar{p}, \bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash\{(1,1)\}$ and $t' \in \{\bar{t}, \ldots, T\}$ with nonzero components

(ii) $\dot{y}_{p,t} = 1$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$, $\dot{\phi}_{1,1,t''} = d_{t''}$ for $t'' \in \mathcal{T}\backslash\{t'\}$, and $\dot{\phi}_{\bar{p},\bar{t},t'} = d_{t'}$.

These $\frac{PT(T+1)}{2} - T$ vectors (ii) are affinely independent from each other and from previous solutions since each vector has a nonzero component that is equal to zero in previous solutions.

Third, we construct the vectors $(\tilde{y}^{(\bar{p},\bar{t})}; \tilde{\phi}^{(\bar{p},\bar{t})})$ for each $(\bar{p}, \bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash\{(1,1)\}$, with nonzero components

(iii) $\tilde{y}_{p,t} = 1$ for $(p,t) \in (\mathcal{P} \times \mathcal{T})\backslash\{(\bar{p}, \bar{t})\}$ and $\tilde{\phi}_{1,1,t'} = d_{t'}$ for $t' \in \mathcal{T}$.

Vectors (iii) are affinely independent from each other and from previously presented solutions since they each violate exactly one of the equalities $y_{p,t} = 1$ for $(p,t) \in \mathcal{P} \times \mathcal{T}\backslash\{(1,1)\}$ which are satisfied by all previous solutions. Additionally, we construct the vector $(\breve{y}, \breve{\phi})$ with nonzero components

(iv) $\breve{y}_{2,1} = 1$ and $\breve{\phi}_{2,1,t'} = d_{t'}$ for $t' \in \mathcal{T}$.

This vector violates the equality $y_{1,1} = 1$ that is satisfied by all previous solutions, and is therefore affinely independent from vectors (i)-(iii). □

Theorem 3.3. *A valid two-covering inequality* (26) *is facet-defining for* $Q^{I,MC}$*,* $|\mathcal{I}_{p,1}| = 1$ *for* $p \in \mathcal{P}$ *and* $\bigcup_{p \in \mathcal{P}} \mathcal{I}_{p,1} = \{i, j\}$*.*

Proof. First, using the assumption that $|\mathcal{I}_{p,1}| = 1$ for $p \in \mathcal{P}$, we rewrite (26) as

$$\sum_{(p,t) \in W} \frac{\phi_{p,t,i}}{d_i} + \sum_{(p,t) \in W} \frac{\phi_{p,t,j}}{d_j} + \sum_{p \in \mathcal{P}_i} \frac{\phi_{p,1,i}}{d_i} + \sum_{p \in \mathcal{P}_j} \frac{\phi_{p,1,j}}{d_j} \leqslant \sum_{(p,t) \in W} y_{p,t} + \sum_{p \in \mathcal{P}} y_{p,1}, \tag{63}$$

where $\mathcal{P}_i = \{p \in \mathcal{P} \mid \mathcal{I}_{p,1} = \{i\}\} \neq \varnothing$, $\mathcal{P}_j = \{p \in \mathcal{P} \mid \mathcal{I}_{p,1} = \{j\}\} \neq \varnothing$, $\mathcal{P}_i \bigcap \mathcal{P}_j = \varnothing$, and $\mathcal{P}_i \bigcup \mathcal{P}_j = \mathcal{P}$.

Consider any face of $\mathrm{conv}(Q^{I,\mathrm{MC}})$ that contains the points of $Q^{I,\mathrm{MC}}$ that also satisfy (63) at equality. Denote any inequality that defines this face as

$$\sum_{p \in \mathcal{P}} \sum_{t \in \mathcal{T}} \sum_{t'=t}^{T} \alpha_{p,t,t'} \phi_{p,t,t'} + \sum_{p \in \mathcal{P}} \sum_{t \in \mathcal{T}} \beta_{p,t} y_{p,t} = \delta. \tag{64}$$

We show next that (64) is in fact a scalar multiple of (63) after using the equalities of LSS. In the remainder, we let $V = \bigcup_{p \in \mathcal{P}} \{(p,1)\} \bigcup W$ and assume, without loss of generality, that $\mathcal{I}_{1,1} = \{i\}$ and $\mathcal{I}_{2,1} = \{j\}$, *i.e.*, $1 \in \mathcal{P}_i$ and $2 \in \mathcal{P}_j$.

First, we construct the vectors $(\bar{y}^p; \bar{\phi}^p)$ for $p \in \mathcal{P}$ with nonzero components

(i) $\bar{y}_{p,1} = 1$, $\bar{\phi}_{p,1,t'} = d_{t'}$ for $t' \in \mathcal{T}$.

Second, we construct the vectors $(\tilde{y}^{(\bar{p},\bar{t})}, \tilde{\phi}^{(\bar{p},\bar{t})})$ for $(\bar{p}, \bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V$, with nonzero components

(ii) $\tilde{y}_{1,1} = 1$, $\tilde{\phi}_{1,1,t'} = d_{t'}$ for $t' \in \mathcal{T}$, and $\tilde{y}_{\bar{p},\bar{t}} = 1$.

If $\mathcal{T}\backslash\{1,i,j\} \neq \varnothing$, we additionally construct the vectors $(\ddot{y}^{(\bar{p},\bar{t}),p,t'}; \ddot{\phi}^{(\bar{p},\bar{t}),p,t'})$ for $p \in \mathcal{P}$, $t' \in \mathcal{T}\backslash\{1,i,j\}$ and $\{(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V \mid \bar{t} \leqslant t'\}$ with nonzero components

(iii) $\ddot{y}_{p,1} = 1$, $\ddot{\phi}_{p,1,t''} = d_{t''}$ for $t'' \in \mathcal{T}\backslash\{t'\}$, $\ddot{y}_{\bar{p},\bar{t}} = 1$ and $\ddot{\phi}_{\bar{p},\bar{t},t'} = d_{t'}$.

We remark that if $\mathcal{T}\backslash\{1,i,j\} = \varnothing$, then there are no terms in (64) involving t'.

Subtracting (64) written for the vector (i) with $p = 1$ from (64) written for vectors (ii), we obtain $\beta_{\bar{p},\bar{t}} = 0$ for $(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V$. Similarly, for $p \in \mathcal{P}$, subtracting (64) written for the vector (i) from (64) written for vectors (iii) and using the identity $\beta_{\bar{p},\bar{t}} = 0$, we obtain $\alpha_{\bar{p},\bar{t},t'} = \alpha_{p,1,t'}$ for $t' \in \mathcal{T}\backslash\{1,i,j\}$ and $\{(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V \mid \bar{t} \leqslant t'\}$. Moreover, if $\{(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V \mid \bar{t} \leqslant t'\} \neq \varnothing$, then we obtain $\alpha_{p,1,t'} = \alpha_{1,1,t'}$ for $t' \in \mathcal{T}\backslash\{1,i,j\}$.

Third, we construct the vectors $(\dddot{y}^{(\bar{p},\bar{t}),p}; \dddot{\phi}^{(\bar{p},\bar{t}),p})$ for $p \in \mathcal{P}_i$ and for $\{(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V \mid \bar{t} \leqslant j\}$ with nonzero components

(iv) $\dddot{y}_{p,1} = 1$, $\dddot{\phi}_{p,1,t'} = d_{t'}$ for $t' \in \mathcal{T}\backslash\{j\}$, $\dddot{y}_{\bar{p},\bar{t}} = 1$ and $\dddot{\phi}_{\bar{p},\bar{t},j} = d_j$.

For $p \in \mathcal{P}_i$, subtracting (64) written for the vector (i) from (64) written for vectors (iv), we obtain $\alpha_{\bar{p},\bar{t},j} = \alpha_{p,1,j}$ for $\{(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V \mid \bar{t} \leqslant j\}$ as $\beta_{\bar{p},\bar{t}} = 0$. Since $1 \in \mathcal{P}_i$ by assumption, we conclude that $\alpha_{\bar{p},\bar{t},j} = \alpha_{1,1,j}$ for $\{(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V \mid \bar{t} \leqslant j\}$.

Fourth, we construct the vectors $(\acute{y}^{(\bar{p},\bar{t}),p}, \acute{\phi}^{(\bar{p},\bar{t}),p})$ for $p \in \mathcal{P}_j$ and for $\{(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V \mid \bar{t} \leqslant i\}$ with nonzero components

(v) $\acute{y}_{p,1} = 1$, $\acute{\phi}_{p,1,t'} = d_{t'}$ for $t' \in \mathcal{T}\backslash\{i\}$, $\acute{y}_{\bar{p},\bar{t}} = 1$ and $\acute{\phi}_{\bar{p},\bar{t},i} = d_i$.

For $p \in \mathcal{P}_j$, subtracting (64) written for the vector (i) from (64) written for vectors (v), we obtain $\alpha_{\bar{p},\bar{t},i} = \alpha_{p,1,i}$ for $\{(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V \mid \bar{t} \leqslant i\}$ as $\beta_{\bar{p},\bar{t}} = 0$. Since $2 \in \mathcal{P}_j$ by assumption, we conclude that $\alpha_{\bar{p},\bar{t},i} = \alpha_{2,1,i}$ for $\{(\bar{p},\bar{t}) \in (\mathcal{P} \times \mathcal{T})\backslash V \mid \bar{t} \leqslant i\}$.

Fifth, for $(\bar{p},\bar{t}) \in W$, we construct the vectors $(\grave{y}^{(\bar{p},\bar{t}),p}, \grave{\phi}^{(\bar{p},\bar{t}),p})$ for $p \in \mathcal{P}$, with nonzero components

(vi) $\grave{y}_{p,1} = 1$, $\grave{\phi}_{p,1,t'} = d_{t'}$ for $t' \in \mathcal{T}\backslash\{i,j\}$, $\grave{y}_{\bar{p},\bar{t}} = 1$, $\grave{\phi}_{\bar{p},\bar{t},i} = d_i$, and $\grave{\phi}_{\bar{p},\bar{t},j} = d_j$.

If $\{\bar{t},\ldots,T\}\backslash\{i,j\} \neq \varnothing$, we additionally construct the vectors $(\hat{y}^{(\bar{p},\bar{t}),p,t'}, \hat{\phi}^{(\bar{p},\bar{t}),p,t'})$ for $p \in \mathcal{P}$ and for $t' \in \{\bar{t},\ldots,T\}\backslash\{i,j\}$ with nonzero components

(vii) $\hat{y}_{p,1} = 1$, $\hat{\phi}_{p,1,t''} = d_{t''}$ for $t'' \in \mathcal{T}\backslash\{i,j,t'\}$, $\hat{y}_{\bar{p},\bar{t}} = 1$, $\hat{\phi}_{\bar{p},\bar{t},i} = d_i$, $\hat{\phi}_{\bar{p},\bar{t},j} = d_j$, and $\hat{\phi}_{\bar{p},\bar{t},t'} = d_{t'}$.

We remark that if $\{\bar{t},\ldots,T\}\backslash\{i,j\} = \varnothing$, then there are no terms in (64) involving t'.

For $p \in \mathcal{P}$, subtracting (64) written for the vector (vi) from (64) written for vectors (vii), we obtain $\alpha_{\bar{p},\bar{t},t'} = \alpha_{p,1,t'}$ for $(\bar{p},\bar{t}) \in W$ and $t' \in \{\bar{t},\ldots,T\}\backslash\{i,j\}$. If $\{\bar{t},\ldots,T\}\backslash\{i,j\} \neq \varnothing$, we conclude that $\alpha_{p,1,t'} = \alpha_{1,1,t'}$ for $p \in \mathcal{P}\backslash\{1\}$ and $t' \in \{\bar{t},\ldots,T\}\backslash\{i,j\}$.

Sixth, we construct the vectors $(\breve{y}^{(\bar{p},\bar{t}),p}, \breve{\phi}^{(\bar{p},\bar{t}),p})$ for each $(\bar{p},\bar{t}) \in W$ and $p \in \mathcal{P}$. If $p \in \mathcal{P}_i$ then the nonzero components are given as

(viii) $\breve{y}_{p,1} = 1$, $\breve{\phi}_{p,1,t'} = d_{t'}$ for $t' \in \mathcal{T}\backslash\{j\}$, $\breve{y}_{\bar{p},\bar{t}} = 1$, and $\breve{\phi}_{\bar{p},\bar{t},j} = d_j$,

and if $p \in \mathcal{P}_j$ then the nonzero components are given as

(viii)$'$ $\breve{y}_{p,1} = 1$, $\breve{\phi}_{p,1,t'} = d_{t'}$ for $t' \in \mathcal{T}\backslash\{i\}$, $\breve{y}_{\bar{p},\bar{t}} = 1$, and $\breve{\phi}_{\bar{p},\bar{t},i} = d_i$.

For $p \in \mathcal{P}_i$, subtracting (64) written for the vector (vi) from (64) written for vectors (viii), we obtain $\alpha_{\bar{p},\bar{t},i} = \alpha_{p,1,i}$ for $(\bar{p},\bar{t}) \in W$. Since by assumption $W \neq \varnothing$ and $1 \in \mathcal{P}_i$, we conclude $\alpha_{p,1,i} = \alpha_{1,1,i}$ for $p \in \mathcal{P}_i \backslash \{1\}$. Similarly, for $p \in \mathcal{P}_j$, subtracting (64) written for the vector (vi) from (64) written for vectors (viii)$'$, we obtain $\alpha_{\bar{p},\bar{t},j} = \alpha_{p,1,j}$ for $(\bar{p},\bar{t}) \in W$. Since by assumption $W \neq \varnothing$ and $2 \in \mathcal{P}_j$, we conclude $\alpha_{p,1,j} = \alpha_{2,1,j}$ for $p \in \mathcal{P}_j \backslash \{2\}$.

Next, we construct the vectors $(\dot{y}^{p,\bar{p}}, \dot{\phi}^{p,\bar{p}})$ for each $p \in \mathcal{P}_i$ and for each $\bar{p} \in \mathcal{P}_j$ with nonzero components

(ix) $\dot{y}_{p,1} = 1$, $\dot{\phi}_{p,1,t'} = d_{t'}$ for $t' \in \mathcal{T} \backslash \{1, j\}$, $\dot{y}_{\bar{p},1} = 1$, $\dot{\phi}_{\bar{p},1,1} = d_1$ and $\dot{\phi}_{\bar{p},1,j} = d_j$,

and the vectors $(\dot{y}^{p,\bar{p}}, \dot{\phi}^{p,\bar{p}})$ for each $p \in \mathcal{P}_i$ and for each $\bar{p} \in \mathcal{P}_j$ with nonzero components

(ix)$'$ $\dot{y}_{\bar{p},1} = 1$, $\dot{\phi}_{\bar{p},1,t'} = d_{t'}$ for $t' \in \mathcal{T} \backslash \{1, i\}$, $\dot{y}_{p,1} = 1$, $\dot{\phi}_{p,1,1} = d_1$ and $\dot{\phi}_{p,1,i} = d_i$.

We remark that, if $\mathcal{T} \backslash \{1, i, j\} \neq \varnothing$ then for each $t' \in \mathcal{T} \backslash \{1, i, j\}$ we have that (p, t) such that $2 \leqslant t \leqslant t'$ for $p \in \mathcal{P}$ belongs either to $(\mathcal{P} \times \mathcal{T}) \backslash V$ or to W. Therefore, identities $\alpha_{p,1,t'} = \alpha_{1,1,t'}$ for $p \in \mathcal{P} \backslash \{1\}$ and $t' \in \mathcal{T} \backslash \{1, i, j\}$ can be obtained using either vectors (i)-(iii) or vectors (vi)-(vii).

By subtracting (64) written for the vector (ix) from (64) written for the vector (ix)$'$ with fixed $p \in \mathcal{P}_i$ and $\bar{p} \in \mathcal{P}_j$, and using the identities $\alpha_{p,1,t'} = \alpha_{1,1,t'}$ for $p \in \mathcal{P} \backslash \{1\}$ and $t' \in \mathcal{T} \backslash \{1, i, j\}$ when $\mathcal{T} \backslash \{1, i, j\} \neq \varnothing$, we obtain $\alpha_{p,1,1} = \alpha_{\bar{p},1,1}$ for $p \in \mathcal{P}_i$ and $\bar{p} \in \mathcal{P}_j$. Therefore, we conclude that $\alpha_{p,1,1} = \alpha_{1,1,1}$ for $p \in \mathcal{P} \backslash \{1\}$.

Next, for $\bar{p} \in \mathcal{P}_i \backslash \{1\}$, subtracting (64) written for the vector (i) with $p = 1$ from (64) written for the vector (i) with $p = \bar{p}$, and using the identities $\alpha_{\bar{p},1,t'} = \alpha_{1,1,t'}$ for $\bar{p} \in \mathcal{P} \backslash \{1\}$ and $t' \in \mathcal{T} \backslash \{i, j\}$, along with the identities $\alpha_{\bar{p},1,i} = \alpha_{1,1,i}$ for $\bar{p} \in \mathcal{P}_i \backslash \{1\}$, we obtain $\alpha_{\bar{p},1,j} - \alpha_{1,1,j} = \beta_{\bar{p},1} - \beta_{1,1}$ for $\bar{p} \in \mathcal{P}_i \backslash \{1\}$. Similarly, for $\bar{p} \in \mathcal{P}_j \backslash \{2\}$, subtracting (64) written for the vector (i) with $p = 2$ from (64) written for the vector (i) with $p = \bar{p}$, and using the identities $\alpha_{\bar{p},1,t'} = \alpha_{1,1,t'}$ for $\bar{p} \in \mathcal{P} \backslash \{1\}$ and $t' \in \mathcal{T} \backslash \{i, j\}$, along with the identities $\alpha_{\bar{p},1,j} = \alpha_{2,1,j}$ for $\bar{p} \in \mathcal{P}_j \backslash \{2\}$, we obtain $\alpha_{\bar{p},1,i} - \alpha_{2,1,i} = \beta_{\bar{p},1} - \beta_{2,1}$ for $\bar{p} \in \mathcal{P}_j \backslash \{2\}$.

Moreover, for $\bar{p} \in \mathcal{P} \backslash \{1\}$ and for some $(\dot{p}, \dot{t}) \in W$, subtracting (64) written for the vector (vi) with $p = 1$ from (64) written for the vector (vi) with $p = \bar{p}$, and using the identities $\alpha_{\bar{p},1,t'} = \alpha_{1,1,t'}$ for $t' \in \mathcal{T} \backslash \{i, j\}$, we obtain $\beta_{\bar{p},1} = \beta_{1,1}$ for $\bar{p} \in \mathcal{P} \backslash \{1\}$. Using these relations along with those obtained in the previous paragraph, we conclude that $\alpha_{p,1,j} = \alpha_{1,1,j}$ for $p \in \mathcal{P}_i \backslash \{1\}$, and that $\alpha_{p,1,i} = \alpha_{2,1,i}$ for $p \in \mathcal{P}_j \backslash \{2\}$.

Using the relations obtained and the equalities (23a), we can write (64) as

$$\sum_{(p,t) \in V} \beta_{p,t} y_{p,t} + (\alpha_{1,1,i} - \alpha_{2,1,i}) \sum_{(p,t) \in W \bigcup \{(p,1) | p \in \mathcal{P}_i\}} \phi_{p,t,i} + (\alpha_{2,1,j} - \alpha_{1,1,j}) \sum_{(p,t) \in W \bigcup \{(p,1) | p \in \mathcal{P}_j\}} \phi_{p,t,j} = \delta', \tag{65}$$

where $\delta' = \delta - \sum_{t' \in \mathcal{T} \backslash \{i,j\}} \alpha_{1,1,t'} d_{t'} - \alpha_{2,1,i} d_i - \alpha_{1,1,j} d_j$. Subtracting (65) written for the vector (i) with $p = 1$ from (65) written for the vector (viii) with $p = 1$ and $(\bar{p}, \bar{t}) \in W$, we obtain $(\alpha_{2,1,j} - \alpha_{1,1,j}) = -\frac{\beta_{\bar{p},\bar{t}}}{d_j}$ for $(\bar{p}, \bar{t}) \in W$. Since $W \neq \varnothing$ by assumption, we conclude that $(\alpha_{2,1,j} - \alpha_{1,1,j}) = -\frac{\beta}{d_j}$, where $\beta = \beta_{\bar{p},\bar{t}}$ for $(\bar{p}, \bar{t}) \in W$. Similarly, using (65) written for the vector (i) with $p = 2$ and the vector (viii)$'$ with $p = 2$ and $(\bar{p}, \bar{t}) \in W$, we obtain $(\alpha_{1,1,i} - \alpha_{2,1,i}) = -\frac{\beta_{\bar{p},\bar{t}}}{d_i}$ for $(\bar{p}, \bar{t}) \in W$. Since $W \neq \varnothing$ by assumption, we conclude that $(\alpha_{1,1,i} - \alpha_{2,1,i}) = -\frac{\beta}{d_i}$.

Additionally, using the relations $\beta_{1,1} = \beta_{p,1}$ for $p \in \mathcal{P} \backslash \{1\}$, we can write (65) as

$$\sum_{p \in \mathcal{P}} \beta_{1,1} y_{p,1} + \sum_{(p,t) \in W} \beta y_{p,t} - \frac{\beta}{d_i} \sum_{(p,t) \in W \bigcup \{(p,1) | p \in \mathcal{P}_i\}} \phi_{p,t,i} - \frac{\beta}{d_j} \sum_{(p,t) \in W \bigcup \{(p,1) | p \in \mathcal{P}_j\}} \phi_{p,t,j} = \delta'. \tag{66}$$

Finally, using (66) written for the vector (i) with $p = 1$ and the vector (ix) with $p = 1$ and $\bar{p} = 2$, we obtain $\beta_{1,1} - \beta = \delta'$ and $2\beta_{1,1} - 2\beta = \delta'$, respectively. We therefore conclude that $\beta_{1,1} = \beta$ and $\delta' = \beta_{1,1} - \beta = 0$. This is the desired result, since (63) can be obtained from (66) by dividing every term by β. ☐

7.5 Proof of Theorems 4.1 and 4.2

The following lemma is useful in proving Theorems 4.1 and 4.2.

Lemma 7.1. *Let $P \geqslant 2$. Define $\mathcal{H} = \{(x; v; y) \in \mathbb{R}^{3PT} \,|\, y_{2,1} = 1\}$. Then* $\dim\left(\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}} \bigcap \mathcal{H}\right) = 3PT - (P+1) - 3T$.

Proof. When $P = 2$, the result follows directly from Corollary 4.1 as $y_{2,1} = 1$ in all feasible solutions of $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$. When $P \geqslant 3$, Corollary 4.1 shows that we can construct affinely independent points $(\dot{x}^r; \dot{v}^r; \dot{y}^r)$ of $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ for $r = 1, \ldots, \rho := 3PT - 3T - P + 1$. Consider now the collection of points $(\dot{x}^r; \dot{v}^r; \dot{y}^r)_{\looparrowright[y_{2,1}=1]}$. These points clearly belong to $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ and $\mathcal{H}$. Because a single component was modified, this family of vectors must contain a subcollection of at least $\rho - 1$ affinely independent vectors. Further, because $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ is not contained in $\mathcal{H}$ as $|\mathcal{P}\setminus\{1\}| \geqslant 2$, every subcollection of affinely independent vectors in $\mathcal{H}$ cannot have more than $\rho - 1$ members. ☐

Theorem 4.1. *When $P \geqslant 2$, $\dim\left(Q^I\right) = 3PT - (P+1)$. When $P = 1$, $\dim\left(Q^I\right) = 3T - 3$.*

Proof. First, we assume that $P \geqslant 2$. Every feasible solution of Q^I satisfies the $(P+1)$ linearly independent equalities (27b) and (27d). This shows that $\dim\left(Q^I\right) \leqslant 3PT - (P+1)$. To prove that $\dim\left(Q^I\right) \geqslant 3PT - (P+1)$, we present $3PT - P$ affinely independent points in Q^I. For $i \in \mathcal{T}$, $j \in \mathcal{T}\setminus\{1\}$, $p \in \mathcal{P}$ and $q \in \mathcal{P}\setminus\{1\}$, we construct the vectors $(\tilde{x}^{p,i}; \tilde{v}^{p,i}; \tilde{y}^{p,i})$, $(\hat{x}^{p,j}; \hat{v}^{p,j}; \hat{y}^{p,j})$, $(\dot{x}^{p,j}; \dot{v}^{p,j}; \dot{y}^{p,j})$, $(\check{x}^q; \check{v}^q; \check{y}^q)$, and $(\ddot{x}; \ddot{v}; \ddot{y})$, whose nonzero components are

(i) $(\tilde{x}^{p,i}_p, \tilde{v}^{p,i}_p, \tilde{y}^{p,i}_p) = (d_{1T}e_1, \sum_{k=1}^{i-1} d_k e_k + d_{iT}e_i, e_1)$,

(ii) $(\hat{x}^{p,j}_p, \hat{v}^{p,j}_p, \hat{y}^{p,j}_p) = (d_{1T}e_1, d_{1T}e_1, e_1 + e_j)$,

(iii) $(\dot{x}^{p,j}_p, \dot{v}^{p,j}_p, \dot{y}^{p,j}_p) = (\sum_{k=1}^{j-1} d_k e_k + d_{jT}e_j, \sum_{k=1}^{j-1} d_k e_k + d_{jT}e_j, \sum_{k=1}^{j} e_k)$,

(iv) $(\check{x}^q_1, \check{v}^q_1, \check{y}^q_1) = (d_1 e_1, d_1 e_1, e_1)$, $(\check{x}^q_q, \check{v}^q_q, \check{y}^q_q) = (d_{2T}e_1, d_{2T}e_1, e_1)$,

(v) $(\ddot{x}_1, \ddot{v}_1, \ddot{y}_1) = (\mathbf{0}, \mathbf{0}, e_1)$, $(\ddot{x}_2, \ddot{v}_2, \ddot{y}_2) = (d_{1T}e_1, d_{1T}e_1, e_1)$.

It is easily verified that these points belong to Q^I and that families (i)-(iii) of points are affinely independent since each new point that is added has one nonzero entry where all previous vectors had a zero entry. Further these three families of vectors satisfy the constraints $\sum_{i=1}^{T} x_{p,i} = d_{1T} y_{p,1}$ for $p \in \mathcal{P}\setminus\{1\}$. Since the points in family (iv) satisfy all but one of these constraints, and each of these constraints is violated by exactly one such point, we conclude that the points (iv) are affinely independent, and are also affinely independent from the points of the previous three families. Finally, since all points in families (i)-(iv) satisfy the constraint $\sum_{p\in\mathcal{P}\setminus\{1\}} \sum_{i\in\mathcal{T}} x_{p,i} = d_{2T} \sum_{p\in\mathcal{P}\setminus\{1\}} y_{p,1} + d_1(1 - y_{1,1})$, but vector (v) does not, we conclude that all $3PT - P$ proposed points are affinely independent.

Second, we assume that $P = 1$. In this case, production must occur in the first period. It follows that $y_{1,1} = 1$ in all feasible solutions to Q^I. Together with the two other linearly independent equalities (27b) and (27d), we conclude that $\dim\left(Q^I\right) \leqslant 3T - 3$. Now for $i \in \mathcal{T}$ and $j \in \mathcal{T}\setminus\{1\}$, consider the points $(\tilde{x}^{1,i};\tilde{v}^{1,i};\tilde{y}^{1,i})$, $(\hat{x}^{1,j};\hat{v}^{1,j};\hat{y}^{1,j})$ and $(\dot{x}^{1,j};\dot{v}^{1,j};\dot{y}^{1,j})$ where we drop the subscript since $p=1$ is clear from the context. It is easily verified that these points belong to Q^I. This proves the result since these $3T-2$ points were previously argued to be affinely independent. □

Theorem 4.2. *(i) For $p \in \mathcal{P}$ and $t \in \mathcal{T}\backslash\{T\}$, (27a) defines a facet of Q^I.*

(ii) For $t \in \mathcal{T}\backslash\{T\}$, (27c) defines a facet of Q^I.

(iii) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (27e) defines a facet of Q^I if and only if (i) $P \geqslant 2$ or (ii) $P=1$ and $t \neq 1$.

(iv) Inequality (27f) defines a facet of Q^I if and only if $P \geqslant 2$.

(v) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (27g) defines a facet of Q^I if and only if (i) $P \geqslant 2$ or (ii) $P=1$ and $t \geqslant 2$.

(vi) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (27h) defines a facet of Q^I if and only if $t \geqslant 2$.

(vii) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (27i) defines a facet of Q^I if and only if (a) $2 \leqslant t \leqslant T-1$ or (b) $P \geqslant 3$ and $t=1$.

Proof of Theorem 4.2 (i). We assume without loss of generality that $p=1$. Inequality (27a) is valid for Q^I since it belongs to the formulation of R. We denote by F the face of Q^I it induces. It is clear that $F \neq Q^I$ since the vector $(\mathring{x};\mathring{v};\mathring{y})$ with nonzero components $(\mathring{x}_1,\mathring{v}_1,\mathring{y}_1)=(d_{1T}e_1, d_{1t}e_1+d_{(t+1)T}e_{t+1}, e_1)$ belongs to Q^I but not to F.

Consider first the case where $P \geqslant 2$. We construct $3PT-(P+1)$ affinely independent points in F. It follows from Lemma 7.1 that we can find affinely independent solutions

(i) $(\grave{x}^j;\grave{v}^j;\grave{y}^j)$,

in $\mathbb{Q}_{\mathcal{P}\backslash\{1\},\mathcal{T}}[\mathbf{d}]$ for $j=1,\ldots,3PT-P-3T$ with $\grave{y}^j_{2,1}=1$. Further, it follows from Corollary 4.1 that we can find affinely independent solutions $(\bar{x}^r,\bar{v}^r,\bar{y}^r)$ in $\mathbb{Q}_{\{1\},\{1,\ldots,t\}}[\mathbf{d}]$ for $r=1,\ldots,\varrho:=3t-2$, and affinely independent solutions $(\bar{\bar{x}}^s,\bar{\bar{v}}^s,\bar{\bar{y}}^s)$ in $\mathbb{Q}_{\{1\},\{t+1,\ldots,T\}}[\mathbf{d}]$ for $s=1,\ldots,\varsigma:=3(T-t)-2$. Given these points, we construct for $s=1,\ldots,\varsigma$ and $r=2,\ldots,\varrho$

(ii) $(\dot{x}^s;\dot{v}^s;\dot{y}^s)=((\bar{x}^1;\bar{v}^1;\bar{y}^1)+(\bar{\bar{x}}^s;\bar{\bar{v}}^s;\bar{\bar{y}}^s))_{\circlearrowright[y_{2,1}=1]}$,

(iii) $(\breve{x}^r;\breve{v}^r;\breve{y}^r)=((\bar{x}^r;\bar{v}^r;\bar{y}^r)+(\bar{\bar{x}}^1;\bar{\bar{v}}^1;\bar{\bar{y}}^1))_{\circlearrowright[y_{2,1}=1]}$.

It is clear from their construction that vectors in families (i)-(iii) are affinely independent. Further they satisfy the constraints $\sum_{j=1}^t x_{1,j}=d_{1t}y_{1,1}$, $\sum_{j=t+1}^T x_{1,j}=d_{(t+1)T}y_{1,t+1}$, $y_{1,1}=y_{1,t+1}$, and $y_{2,1}=1$. Next, we construct the vectors $(\breve{x};\breve{v};\breve{y})$, $(\ddot{x};\ddot{v};\ddot{y})$, $(\acute{x};\acute{v};\acute{y})$ and $(\tilde{x};\tilde{v};\tilde{y})$ with nonzero components

(iv) $(\breve{x}_1,\breve{v}_1,\breve{y}_1)=(d_{(t+1)T}e_{t+1}, d_{(t+1)T}e_{t+1}, e_1+e_{t+1})$, $(\breve{x}_2,\breve{v}_2,\breve{y}_2)=(d_{1t}e_1,d_{1t}e_1,e_1)$,

(v) $(\ddot{x}_1,\ddot{v}_1,\ddot{y}_1)=(d_{1t}e_1,d_{1t}e_1,e_1+e_{t+1})$, $(\ddot{x}_2,\ddot{v}_2,\ddot{y}_2)=(d_{(t+1)T}e_{t+1},d_{(t+1)T}e_{t+1},e_1+e_{t+1})$,

(vi) $(\acute{x}_1,\acute{v}_1,\acute{y}_1)=(d_{1t}e_1,d_{1t}e_1,e_1)$, $(\acute{x}_2,\acute{v}_2,\acute{y}_2)=(d_{(t+1)T}e_{t+1},d_{(t+1)T}e_{t+1},e_1+e_{t+1})$,

(vii) $(\tilde{x}_1,\tilde{v}_1,\tilde{y}_1)=(d_{1t}e_1+d_{(t+1)T}e_{t+1}, d_{1t}e_1+d_{(t+1)T}e_{t+1}, e_1+e_{t+1})$.

Since the four points (v)-(vii) satisfy all but one of the equalities satisfied by previous solutions, and each equality is violated by exactly one such point, we conclude that they are affinely independent from each other, and are also affinely independent from all previously described vectors. These points prove that the dimension of F is as desired.

Consider second the case where $P=1$. Similar to before, we construct, for $s=1,\ldots,\varsigma$ and for $r=2,\ldots,\varrho$, the affinely independent vectors

(i) $(\dot{x}^s;\dot{v}^s;\dot{y}^s) = ((\bar{x}^1;\bar{v}^1;\bar{y}^1) + (\bar{\bar{x}}^s;\bar{\bar{v}}^s;\bar{\bar{y}}^s))$,

(ii) $(\check{x}^r;\check{v}^r;\check{y}^r) = ((\bar{x}^r;\bar{v}^r;\bar{y}^r) + (\bar{\bar{x}}^1;\bar{\bar{v}}^1;\bar{\bar{y}}^1))$,

together with the two vectors $(\ddot{x};\ddot{v};\ddot{y})$ and $(\breve{x};\breve{v};\breve{y})$ defined as

(iii) $(\ddot{x};\ddot{v};\ddot{y}) = (d_{1T}e_1; d_{1T}e_1; e_1 + e_{t+1})$,

(iv) $(\breve{x};\breve{v};\breve{y}) = (d_{1T}e_1; d_{1T}e_1; e_1)$.

Since vector (iii) violates $\sum_{j=t+1}^{T} x_{1,j} = d_{(t+1)T} y_{1,t+1}$ and vector (iv) violates $y_{1,t+1} = 1$ that are satisfied by solutions in families (i)-(ii), we conclude that they are affinely independent from each other and from the vectors presented earlier. □

Proof of Theorem 4.2 (ii). Inequality (27c) is valid for Q^I since it belongs to the formulation of R. We denote by F the face of Q^I it induces. It is clear that $F \neq Q^I$ since the vector $(\mathring{x};\mathring{v};\mathring{y})$ with nonzero components $(\mathring{x}_1,\mathring{v}_1,\mathring{y}_1) = (d_{1T}e_1, d_{1T}e_1, e_1)$ belongs to Q^I but not to F.

Assume first that $P \geqslant 2$. Using Corollary 4.1 we can find affinely independent solutions $(\breve{x}^s;\breve{v}^s;\breve{y}^s)$ in $\mathbb{Q}_{\mathcal{P},\{1,\ldots,t\}}[\mathbf{d}]$ for $s = 1,\ldots,\varsigma := 3Pt - P$ and, affinely independent solutions $(\acute{x}^r;\acute{v}^r;\acute{y}^r)$ in $\mathbb{Q}_{\mathcal{P},\{t+1,\ldots,T\}}[\mathbf{d}]$ for $r = 1,\ldots,\varrho := 3P(T-t) - P$. Clearly, the vectors

(i) $(\dot{x}^s;\dot{v}^s;\dot{y}^s) = (\breve{x}^s;\breve{v}^s;\breve{y}^s) + \left(\acute{x}^1;\acute{v}^1;\acute{y}^1\right)$

(ii) $(\hat{x}^r;\hat{v}^r;\hat{y}^r) = \left(\breve{x}^1;\breve{v}^1;\breve{y}^1\right) + (\acute{x}^r;\acute{v}^r;\acute{y}^r)$

for $s = 1,\ldots,\varsigma$ and $r = 2,\ldots,\varrho$ are affinely independent points of F. Further, they satisfy the equalities $\sum_{i=1}^{t} x_{p,i} = \sum_{i=1}^{t} v_{p,i}$ for $p \in \mathcal{P}$. We next construct, for $p \in \mathcal{P}$, the vectors $(\check{x}^p;\check{v}^p;\check{y}^p)$ whose non-zero components are

(iii) $(\check{x}^p_p,\check{v}^p_p,\check{y}^p_p) = (d_{1(t+1)}e_1 + d_{(t+2)T}e_{t+2}, d_{1t}e_1 + d_{t+1}e_{t+1} + d_{(t+2)T}e_{t+2}, e_1 + e_{t+2})$.

When $t = T-1$, these points should be taken as $(\check{x}^p_p,\check{v}^p_p,\check{y}^p_p) = (d_{1T}e_1, d_{1(T-1)}e_1 + d_Te_T, e_1)$, for $p \in \mathcal{P}$. Since these points satisfy all but one of equalities $\sum_{i=1}^{t} x_{p,i} = \sum_{i=1}^{t} v_{p,i}$ for $p \in \mathcal{P}$, and each equality is violated by exactly one such point, we conclude that all $3PT - (P+1)$ vectors presented above are affinely independent.

Assume second that $P = 1$. As before, $(\dot{x}^s;\dot{v}^s;\dot{y}^s)$ and $(\hat{x}^r;\hat{v}^r;\hat{y}^r)$ for $s = 1,\ldots,\varsigma := 3t-2$ and $r = 2,\ldots,\varrho := 3(T-t)-2$ are affinely independent vectors of F. Additionally, we construct the vectors $(\bar{x};\bar{v};\bar{y})$ and $(\ddot{x};\ddot{v};\ddot{y})$ where the nonzero components are

(iii) $(\bar{x};\bar{v};\bar{y}) = (d_{1T}e_1; d_{1t}e_1 + d_{(t+1)T}e_{t+1}; e_1 + e_{t+1})$

(iv) $(\ddot{x};\ddot{v};\ddot{y}) = (d_{1T}e_1; d_{1t}e_1 + d_{(t+1)T}e_{t+1}; e_1)$.

Vector (iii) violates $\sum_{i=t+1}^{T} x_{1,i} = d_{(t+1)T} y_{1,t+1}$ while vector (iv) violates $y_{1,t+1} = 1$. Since these equalities are satisfied by all other points we conclude that these vectors are affinely independent from each other, and from the vectors presented previously. This concludes the proof. □

Proof of Theorem 4.2 (iii). When $P = 1$ and $t = 1$ all solutions that satisfy (27e) at equality are such that $x_{1,j} = 0$ for $j = 2,\ldots,T$, showing that (27e) is not facet-defining.

For the reverse direction, assume without loss of generality that $p = 1$. Inequality (27e) is valid for Q^I since it belongs to the formulation of R. We denote by F the face of Q^I it induces. It is clear that $F \neq Q^I$ since the vector $(\mathring{x};\mathring{v};\mathring{y})$ with nonzero components $(\mathring{x}_1,\mathring{v}_1,\mathring{y}_1) = (0,0,e_t)$ and $(\mathring{x}_2,\mathring{v}_2,\mathring{y}_2) =$

$(d_{1T}e_1, d_{1T}e_1, e_1)$ when $P \geqslant 2$, and $(\mathring{x}_1, \mathring{v}_1, \mathring{y}_1) = (d_{1T}e_1, d_{1T}e_1, e_1 + e_t)$ when $P = 1$ belongs to Q^I but not to F.

First consider the case where $P \geqslant 2$. There are two cases.

Assume first that $t \geqslant 2$. By Lemma 7.1, we can construct affinely independent points

(i) $(\hat{x}^s; \hat{v}^s; \hat{y}^s)$

for $s = 1, \ldots, \varsigma := 3PT - P - 3T$ in $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ where $\hat{y}^s_{2,1} = 1$. We define $\mathbf{d}' = \mathbf{d}_{\looparrowright[d_{t-1}=d_{(t-1)t}]}$. By Corollary 4.1, we can construct affinely independent points

(ii) $(\dot{x}^r; \dot{v}^r; \dot{y}^r)_{\looparrowright[y_{2,1}=1]}$

where $(\dot{x}^r; \dot{v}^r; \dot{y}^r) \in \mathbb{Q}_{\{1\},\mathcal{T}\setminus\{t\}}[\mathbf{d}']$ for $r = 1, \ldots, \varrho := 3T - 5$. These points satisfy the equalities $\sum_{i=1}^T x_{1,i} = d_{1T}y_{1,1}$, $v_{1,t} = x_{1,t}$, $y_{1,t} = 0$ and $y_{2,1} = 1$. We next construct four additional points in F that each violate exactly one of these equalities. The desired points are $(\bar{x}; \bar{v}; \bar{y})$, $(\breve{x}; \breve{v}; \breve{y})$, $(\check{x}; \check{v}; \check{y})$ and $(\ddot{x}; \ddot{v}; \ddot{y})$ with nonzero components

(iii) $(\bar{x}_1, \bar{v}_1, \bar{y}_1) = (\mathbf{0}, \mathbf{0}, e_1)$, $(\bar{x}_2, \bar{v}_2, \bar{y}_2) = (d_{1T}e_1, d_{1T}e_1, e_1)$,

(iv) $(\breve{x}_1, \breve{v}_1, \breve{y}_1) = (d_{1T}e_1, d_{1(t-1)}e_1 + d_{tT}e_t, e_1)$, $(\breve{x}_2, \breve{v}_2, \breve{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(v) $(\check{x}_1, \check{v}_1, \check{y}_1) = (d_{1t-1}e_1 + d_{tT}e_t, d_{1t-1}e_1 + d_{tT}e_t, e_1 + e_t)$, $(\check{x}_2, \check{v}_2, \check{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(vi) $(\ddot{x}_1, \ddot{v}_1, \ddot{y}_1) = (d_{1T}e_1, d_{1T}e_1, e_1)$.

Assume next that $t = 1$. In addition to the points

(i) $(\hat{x}^s; \hat{v}^s; \hat{y}^s)_{\looparrowright[y_{1,2}=1]}$,

for $s = 1, \ldots, \varsigma$, where $(\hat{x}^s; \hat{v}^s; \hat{y}^s)$ is as described above, we construct, using Corollary 4.1 the points

(ii) $(\dot{x}^r; \dot{v}^r; \dot{y}^r)_{\looparrowright[x_{2,1}=d_1, v_{2,1}=d_1, y_{2,1}=1]}$

where $(\dot{x}^r; \dot{v}^r; \dot{y}^r) \in \mathbb{Q}_{\{1\},\{2,\ldots,T\}}[\mathbf{d}]$ for $r = 1, \ldots, \rho := 3T - 5$. These points satisfy the equalities $y_{1,2} = 1$, $y_{2,1} = 1$, $y_{1,1} + y_{2,1} = 1$. Next, we construct the points $(\breve{x}; \breve{v}; \breve{y})$, $(\ddot{x}; \ddot{v}; \ddot{y})$ and $(\check{x}; \check{v}; \check{y})$ with nonzero components

(iii) $(\breve{x}_2, \breve{v}_2, \breve{y}_2) = (d_{1T}e_1, d_{1T}e_1, e_1)$,

(iv) $(\ddot{x}_1, \ddot{v}_1, \ddot{y}_1) = (d_{1T}e_1, d_{1T}e_1, e_1 + e_2)$,

(v) $(\check{x}_1, \check{v}_1, \check{y}_1) = (d_{1T}e_1, d_{1T}e_1, e_1 + e_2)$, $(\check{x}_2, \check{v}_2, \check{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$.

Points (i)-(v) satisfy the equality $\sum_{i=2}^T x_{1,i} = \sum_{i=2}^T v_{1,i}$. Additionally, we construct the point $(\bar{x}; \bar{v}; \bar{y})$ with nonzero components

(vi) $(\bar{x}_1, \bar{v}_1, \bar{y}_1) = (d_{1T}e_1, d_1e_1 + d_{2T}e_2, e_1)$.

Finally, we consider the case where $P = 1$. We have that $t > 1$. Similar to the case where $P \geqslant 2$ and $t \geqslant 2$, we can construct the affinely independent points

(i) $(\dot{x}^r; \dot{v}^r; \dot{y}^r)$,

for $r = 1, \ldots, \varrho = 3T - 5$ in $\mathbb{Q}_{\{1\},\mathcal{T}\setminus\{t\}}[\mathbf{d}']$. These points satisfy the equalities $x_{1,t} = v_{1,t}$ and $y_{1,t} = 0$. Consider now the points $(\bar{x}; \bar{v}; \bar{y})$ and $(\check{x}; \check{v}; \check{y})$ defined as

(ii) $(\bar{x}; \bar{v}; \bar{y}) = (d_{1T}e_1; d_{1(t-1)}e_1 + d_{tT}e_t; e_1)$,

(iii) $(\check{x};\check{v};\check{y}) = (d_{1(t-1)}e_1 + d_{tT}e_t; d_{1(t-1)}e_1 + d_{tT}e_t; e_1 + e_t)$.

Vector (ii) violates $x_{1,t} = v_{1,t}$ while vector (iii) violates $y_{1,t} = 0$. We conclude that these vectors are affinely independent from each other and from vectors presented previously. $\square$

Proof of Theorem 4.2 (iv). Inequality (27f) is valid for Q^I since it belongs to the formulation of R. We denote by F the face of Q^I it induces. Assume first that $P \geqslant 2$. It is clear that $F \neq Q^I$ since the vector $(\mathring{x};\mathring{v};\mathring{y})$ with nonzero components $\mathring{x}_1 = d_{1T}e_1$, $\mathring{v}_1 = d_{1T}e_1$, and $\mathring{y} = e$ belongs to Q^I but not to F.

Consider the points $\left(\tilde{x}^{p,i};\tilde{v}^{p,i};\tilde{y}^{p,i}\right)$, $\left(\hat{x}^{p,j};\hat{v}^{p,j};\hat{y}^{p,j}\right)$, and $\left(\dot{x}^{p,j};\dot{v}^{p,j};\dot{y}^{p,j}\right)$ where $p \in \mathcal{P}$, $i \in \mathcal{T}$ and $j \in \mathcal{T}\backslash\{1\}$ whose non-zero components are given as

(i) $(\tilde{x}^{p,i}_p, \tilde{v}^{p,i}_p, \tilde{y}^{p,i}_p) = (d_{1T}e_1, \sum_{k=1}^{i-1} d_k e_k + d_{iT}e_i, e_1)$,

(ii) $(\hat{x}^{p,j}_p, \hat{v}^{p,j}_p, \hat{y}^{p,j}_p) = (d_{1T}e_1, d_{1T}e_1, e_1 + e_j)$,

(iii) $(\dot{x}^{p,j}_p, \dot{v}^{p,j}_p, \dot{y}^{p,j}_p) = (\sum_{k=1}^{j-1} d_k e_k + d_{jT}e_j, \sum_{k=1}^{j-1} d_k e_k + d_{jT}e_j, \sum_{k=1}^{j} e_k)$.

These points satisfy the equalities $\sum_{i=1}^{T} x_{p,i} = d_{1T}y_{p,1}$ for $p \in \mathcal{P}\backslash\{1\}$. Additionally, we construct the vectors $(\check{x}^p;\check{v}^p;\check{y}^p)$ for $p \in \mathcal{P}\backslash\{1\}$ whose non-zero elements are

(iv) $(\check{x}^p_1, \check{v}^p_1, \check{y}^p_1) = (d_1e_1, d_1e_1, e_1)$, $(\check{x}^p_p, \check{v}^p_p, \check{y}^p_p) = (d_{2T}e_2, d_{2T}e_2, e_2)$.

Vectors in family (iv) satisfy all but one of the equalities presented previously. Moreover, each equality is violated by one such point, thereby showing that these vectors are affinely independent.

Now assume that $P = 1$, (27f) reduces to $y_{1,1} \geqslant 1$. This inequality is satisfied at equality by all feasible solutions to Q^I. It is therefore not facet-defining for Q^I. $\square$

Proof of Theorem 4.2 (v). When $P = 1$ and $t = 1$ (27g) is satisfied at equality by all solutions to Q^I, showing that it is not facet-defining for Q^I.

For the reverse direction, assume without loss of generality that $p = 1$. Inequality (27g) is valid for Q^I since it belongs to the formulation of R. It is clear that $F \neq Q^I$ since the vector $(\mathring{x};\mathring{v};\mathring{y})$ with nonzero components $(\mathring{x}_2, \mathring{v}_2, \mathring{y}_2) = (d_{1T}e_1, d_{1T}e_1, e_1)$ when $P = 2$ and $(\mathring{x}_1, \mathring{v}_1, \mathring{y}_1) = (d_{1T}e_1, d_{1T}e_1, e_1)$ when $P = 1$ belongs to Q^I but not to F. Let $(\bar{x}^r;\bar{v}^r;\bar{y}^r)$ be affinely independent vectors for $r = 1, \ldots, \rho := \dim\left(Q^I\right) + 1$ in Q^I. Construct the vectors $(\dot{x}^r;\dot{v}^r;\dot{y}^r) = (\bar{x}^r;\bar{v}^r;\bar{y}^r)_{\looparrowright[y_{p,t}=1]}$. These vectors belong to Q^I. Further, since only one component of each vector is changed, we conclude that this set of vectors contains a subcollection of at least $\rho - 1$ affinely independent vectors. Moreover, since $F \neq Q^I$, every subcollection of affinely independent vectors in F cannot have more than $\rho - 1$ members. This concludes the proof. $\square$

Proof of Theorem 4.2 (vi). Assume that $t = 1$. If $P = 1$, then $x_{1,1} \geqslant d_1 > 0$, showing that (27h) defines an empty face of Q^I. If $P \geqslant 2$, then (27h) is not facet-defining for Q^I as any solution of Q^I that satisfies (27h) at equality must also satisfy $v_{1,1} = 0$. This equality is linearly independent from the equalities of Q since we have assumed that $T \geqslant 2$.

Assume now that $t \geqslant 2$. We assume without loss of generality that $p = 1$. We denote by F the face of Q^I defined by (27h). It is clear that $F \neq Q^I$ since the vector $(\mathring{x};\mathring{v};\mathring{y})$ with nonzero components $(\mathring{x}_1, \mathring{v}_1, \mathring{y}_1) = (d_{1t-1}e_1 + d_{tT}e_t, d_{1t-1}e_1 + d_{tT}e_t, e_1 + e_t)$ belongs to Q^I but not to F. We first consider the case where $P \geqslant 2$. By Lemma 7.1, there are affinely independent points $(\acute{x}^s;\acute{v}^s;\acute{y}^s)$ for $s = 1, \ldots, \varsigma := 3PT - P - 3T$ in $\mathbb{Q}_{\mathcal{P}\backslash\{1\},\mathcal{T}}[\mathbf{d}]$ for which $\acute{y}^s_{2,1} = 1$. We then construct the solutions

(i) $(\dot{x}^s; \dot{v}^s; \dot{y}^s) = (\acute{x}^s; \acute{v}^s; \acute{y}^s)_{\hookrightarrow[y_{1,t}=1]}$.

Next, we define $\mathbf{d}' = \mathbf{d}_{\hookrightarrow[d_{t-1}=d_{(t-1)t}]}$. By Corollary 4.1, there are affinely independent solutions $(\tilde{x}^i; \tilde{v}^i; \tilde{y}^i)$ for $i = 1, \ldots, \eta := 3T - 5$ in $\mathbb{Q}_{\{1\},\mathcal{T}\setminus\{t\}}[\mathbf{d}']$. Construct

(ii) $(\ddot{x}^i; \ddot{v}^i; \ddot{y}^i) = (\tilde{x}^i; \tilde{v}^i; \tilde{y}^i)_{\hookrightarrow[y_{2,1}=1, y_{1,t}=1]}$.

Observe that solutions $(\dot{x}^s; \dot{v}^s; \dot{y}^s)$ and $(\ddot{x}^i; \ddot{v}^i; \ddot{y}^i)$ satisfy $y_{2,1} = 1$, $y_{1,t} = 1$, $v_{1,t} = 0$, and $\sum_{i=1}^{T} x_{1,i} = d_{1T} y_{1,1}$. Next, we construct $(\hat{x}; \hat{v}; \hat{y})$, $(\check{x}; \check{v}; \check{y})$, $(\breve{x}; \breve{v}; \breve{y})$ and $(\dddot{x}; \dddot{v}; \dddot{y})$ with nonzero components

(iii) $(\hat{x}_1, \hat{v}_1, \hat{y}_1) = (d_{1T} e_1, d_{1T} e_1, e_1 + e_t)$,

(iv) $(\check{x}_1, \check{v}_1, \check{y}_1) = (d_{1T} e_1, d_{1T} e_1, e_1)$, $(\check{x}_2, \check{v}_2, \check{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(v) $(\breve{x}_1, \breve{v}_1, \breve{y}_1) = (d_{1T} e_1, d_{1(t-1)} e_1 + d_{tT} e_t, e_1 + e_t)$, $(\breve{x}_2, \breve{v}_2, \breve{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(vi) $(\dddot{x}_1, \dddot{v}_1, \dddot{y}_1) = (d_{1(t-1)} e_1, d_{1(t-1)} e_1, e_1 + e_t)$, $(\dddot{x}_2, \dddot{v}_2, \dddot{y}_2) = (d_{tT} e_1, d_{tT} e_1, e_1)$.

Since solutions (iii)-(vi) satisfy all but one of the above equalities, and each equality is violated by exactly one such solution, we conclude that vectors (i)-(vi) are affinely independent.

Next we consider the case where $P = 1$. As above, the vectors $(\tilde{x}^i; \tilde{v}^i; \tilde{y}^i)_{\hookrightarrow[y_{1,t}=1]}$ for $i = 1, \ldots, \eta = 3T-5$ belong to F. We additionally construct the points $(\check{x}; \check{v}; \check{y})$ and $(\breve{x}; \breve{v}; \breve{y})$ with nonzero components

(i) $(\check{x}_1, \check{v}_1, \check{y}_1) = (d_{1T} e_1, d_{1T} e_1, e_1)$,

(ii) $(\breve{x}_1, \breve{v}_1, \breve{y}_1) = (d_{1T} e_1, d_{1t-1} e_1 + d_{tT} e_t, e_1 + e_t)$,

which can be verified to be affinely independent from each other and from the previously described points by considering the equalities $y_{1,t} = 1$ and $v_{1,t} = 0$. This concludes the proof. □

Proof of Theorem 4.2 (vii). We assume without loss of generality that $p = 1$. When $P = 1$ and $t = 1$, there is no feasible solution that satisfies (27i) at equality. When $P = 2$ and $t = 1$, all feasible solutions that satisfy (27i) at equality must also satisfy $y_{2,1} = 1$ because of (27a), (27c), and (27e). These two observations show that (27i) is not facet-defining for Q^I when $P \leqslant 2$ and $t = 1$. On the other hand, when $t = T$, all feasible solutions that satisfy (27i) at equality must satisfy $x_{1,T} = 0$, thereby showing that it is not facet-defining for Q^I.

For the reverse direction, we first observe that (27i) is valid for Q^I as it belongs to the formulation of R. We denote by F the face of Q^I it induces. Clearly, $F \neq Q^I$ since the vector $(\mathring{x}; \mathring{v}; \mathring{y})$ with nonzero components $(\mathring{x}_1, \mathring{v}_1, \mathring{y}_1) = (d_{1T} e_1, d_{1(t-1)} e_1 + d_{tT} e_t, e_1)$ belongs to Q^I but not to F.

We first consider the case where $P \geqslant 2$ and $2 \leqslant t \leqslant T - 1$. By Lemma 7.1, there exists affinely independent points $(\acute{x}^r; \acute{v}^r; \acute{y}^r)$ for $r = 1, \ldots, \rho := 3PT - P - 3T$ in $\mathbb{Q}_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ where $\acute{y}^r_{2,1} = 1$. We construct vectors

(i) $(\dot{x}^r; \dot{v}^r; \dot{y}^r) = (\acute{x}^r; \acute{v}^r; \acute{y}^r)_{\hookrightarrow[y_{1,t}=1]}$.

Next, we define $\mathbf{d}' = \mathbf{d}_{\hookrightarrow[d_{t-1}=d_{(t-1)t}]}$. By Corollary 4.1, we construct affinely independent solutions

(ii) $(\bar{x}^s; \bar{v}^s; \bar{y}^s) = (\tilde{x}^s; \tilde{v}^s; \tilde{y}^s)_{\hookrightarrow[y_{2,1}=1, y_{1,t}=1]}$

where $(\tilde{x}^s; \tilde{v}^s; \tilde{y}^s) \in \mathbb{Q}_{\{1\},\mathcal{T}\setminus\{t\}}[\mathbf{d}']$ for $s = 1, \ldots, \varsigma := 3T - 5$. These solutions satisfy the equalities $\sum_{i=1}^{T} x_{1,i} = d_{1T} y_{1,1}$, $x_{1,t} = 0$, $y_{1,t} = 1$, $y_{2,1} = 1$. Additionally, we construct the vectors $(\dddot{x}; \dddot{v}; \dddot{y})$, $(\check{x}; \check{v}; \check{y})$, $(\hat{x}; \hat{v}; \hat{y})$ and $(\ddot{x}; \ddot{v}; \ddot{y})$ with nonzero components

(iii) $(\ddot{x}_1, \ddot{v}_1, \ddot{y}_1) = (d_{1t}e_1, d_{1t}e_1, e_1 + e_t)$, $(\ddot{x}_2, \ddot{v}_2, \ddot{y}_2) = (d_{(t+1)T}e_1, d_{(t+1)T}e_1, e_1)$,

(iv) $(\check{x}_1, \check{v}_1, \check{y}_1) = (d_{1t}e_1 + d_{(t+1)T}e_t, d_{1t}e_1 + d_{(t+1)T}e_{t+1}, e_1 + e_t)$, $(\check{x}_2, \check{v}_2, \check{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(v) $(\hat{x}_1, \hat{v}_1, \hat{y}_1) = (d_{1T}e_1, d_{1T}e_1, e_1)$, $(\hat{x}_2, \hat{v}_2, \hat{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(vi) $(\dddot{x}_1, \dddot{v}_1, \dddot{y}_1) = (d_{1T}e_1, d_{1T}e_1, e_1 + e_t)$.

These vectors satisfy all but one of the equalities presented above, and each equality is violated by exactly one of the vectors. We conclude that these vectors are affinely independent from each other and from vectors presented previously, thereby proving the result.

Next we consider the case where $P = 1$ and $2 \leqslant t \leqslant T - 1$. Similar to before, there are affinely independent points $(\tilde{x}^s; \tilde{v}^s; \tilde{y}^s)$ in $\mathbb{Q}_{\{1\},\mathcal{T}\setminus\{t\}}[\mathbf{d}']$ for $s = 1, \ldots, \varsigma$. We construct the vectors

(i) $(\bar{x}^s; \bar{v}^s; \bar{y}^s) = (\tilde{x}^s; \tilde{v}^s; \tilde{y}^s)_{\looparrowright[y_{1,t}=1]}$.

These solutions satisfy $x_{1,t} = 0$ and $y_{1,t} = 1$. Consider now the vectors $(\check{x}; \check{v}; \check{y})$ and $(\hat{x}; \hat{v}; \hat{y})$ with nonzero components

(ii) $(\check{x}; \check{v}; \check{y}) = (d_{1t}e_1 + d_{(t+1)T}e_t; d_{1t}e_1 + d_{(t+1)T}e_{t+1}; e_1 + e_t)$,

(iii) $(\hat{x}; \hat{v}; \hat{y}) = (d_{1T}e_1; d_{1T}e_1; e_1)$.

Vector $(\check{x}; \check{v}; \check{y})$ violates $x_{1,t} = 0$ whereas vector $(\hat{x}; \hat{v}; \hat{y})$ violates $y_{1,t} = 1$, showing that these solutions are affinely independent.

Finally, we consider the case where $P \geqslant 3$ and $t = 1$. Similar to above, we construct affinely independent points

(i) $(\dot{x}^r; \dot{v}^r; \dot{y}^r) = (\acute{x}^r; \acute{v}^r; \acute{y}^r)_{\looparrowright[y_{1,1}=1]}$

for $r = 1, \ldots, \rho$ where $(\acute{x}^r; \acute{v}^r; \acute{y}^r) \in \mathbb{Q}_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$, and $\acute{y}^r_{2,1} = 1$. By Corollary 4.1, we construct affinely independent solutions

(ii) $(\bar{x}^s; \bar{v}^s; \bar{y}^s) = (\tilde{x}^s; \tilde{v}^s; \tilde{y}^s)_{\looparrowright[y_{1,1}=1, x_{2,1}=d_1, v_{2,1}=d_1, y_{2,1}=1]}$

where $(\tilde{x}^s; \tilde{v}^s; \tilde{y}^s) \in \mathbb{Q}_{\{1\},\mathcal{T}\setminus\{1\}}[\mathbf{d}]$ for $s = 1, \ldots, \varsigma$. These solutions satisfy the equalities $\sum_{i=1}^{T} x_{1,i} = d_{2T}y_{1,2}$, $x_{1,1} = 0$, $y_{1,1} = 1$, and $y_{2,1} = 1$. We next construct the vectors $(\ddot{x}; \ddot{v}; \ddot{y})$, $(\check{x}; \check{v}; \check{y})$, $(\hat{x}; \hat{v}; \hat{y})$, and $(\dddot{x}; \dddot{v}; \dddot{y})$ with nonzero components

(iii) $(\ddot{x}_1, \ddot{v}_1, \ddot{y}_1) = (\mathbf{0}, \mathbf{0}, e_1 + e_2)$, $(\ddot{x}_2, \ddot{v}_2, \ddot{y}_2) = (d_{1T}e_1, d_{1T}e_1, e_1)$,

(iv) $(\check{x}_1, \check{v}_1, \check{y}_1) = (d_{2T}e_1, d_{2T}e_2, e_1 + e_2)$, $(\check{x}_2, \check{v}_2, \check{y}_2) = (d_1e_1, d_1e_1, e_1)$,

(v) $(\hat{x}_2, \hat{v}_2, \hat{y}_2) = (d_{1T}e_1, d_{1T}e_1, e_1)$,

(vi) $(\dddot{x}_1, \dddot{v}_1, \dddot{y}_1) = (\mathbf{0}, \mathbf{0}, e_1)$, $(\dddot{x}_3, \dddot{v}_3, \dddot{y}_3) = (d_{1T}e_1, d_{1T}e_1, e_1)$.

These vectors satisfy all but one of the equalities given above and each equality is violated by exactly one of them. We conclude that vectors (i)-(vi) are affinely independent. This concludes the proof. □

7.6 Proof of Theorem 4.3

Theorem 4.3. *The production upper bound inequality*

$$\sum_{i=t}^{k} x_{p,i} \leqslant \sum_{i=t}^{j} v_{p,i} + d_{(j+1)T} \sum_{i=t}^{k} y_{p,i} \tag{28}$$

is valid for Q^I when $1 \leqslant t \leqslant k \leqslant j \leqslant T-1$. Furthermore, such a valid inequality is facet-defining for Q^I if and only if $P \geqslant 2$ or $P = 1$ and $t \geqslant 2$.

Proof. We assume without loss of generality that $p = 1$. Consider $t \in \mathcal{T}\backslash\{T\}$, $k \in \{t, \ldots, T-1\}$ and $j \in \{k, \ldots, T-1\}$. We first argue that (28) is valid for Q^I under these assumptions. When $y_{1,i} = 0$ for all $i \in \{t, \ldots, k\}$, (28) reduces to $0 \leqslant \sum_{i=t}^{j} v_{1,i}$, which is valid for Q^I as $Q^I \subseteq \mathbb{R}_+^{3PT}$. If $y_{1,i} = 1$ for some $i \in \{t, \ldots, k\}$ then (28) is dominated by $\sum_{i=t}^{T} x_{p,i} \leqslant \sum_{i=t}^{j} v_{p,i} + d_{(j+1)T}$. We denote by F the face of Q^I that (28) induces. Clearly, $F \neq Q^I$ as the vector $(\mathring{x}; \mathring{v}; \mathring{y})$ with nonzero components $(\mathring{x}_1, \mathring{v}_1, \mathring{y}_1) = (\mathbf{0}, \mathbf{0}, e_t)$; $(\mathring{x}_2, \mathring{v}_2, \mathring{y}_2) = (d_{1T}e_1, d_{1T}e_1, e_1)$ when $P \geqslant 2$ and $(\mathring{x}_1, \mathring{v}_1, \mathring{y}_1) = (d_{1T}e_1, d_{1T}e_1, \sum_{i=1}^{t} e_i)$ when $P = 1$, belongs to Q^I but not to F.

Next we prove that (28) is facet-defining for Q^I under the stated assumptions. There are three cases.

For the first case, assume $P \geqslant 2$ and $t > 1$. Then using Lemma 7.1, there are affinely independent solutions

(i) $(\tilde{x}^i; \tilde{v}^i; \tilde{y}^i)$

for $i = 1, \ldots, \rho := 3PT - P - 3T$ in $\mathbb{Q}^I_{\mathcal{P}\backslash\{1\}, \mathcal{T}}[\mathbf{d}]$ for which $\tilde{y}^i_{2,1} = 1$. We next define $\mathbf{d}' = \mathbf{d}_{\looparrowright[d_{t-1} = d_{(t-1)j}]}$. By Corollary 4.1, there are affinely independent points $(\acute{x}^s; \acute{v}^s; \acute{y}^s)$ for $s = 1, \ldots, \varsigma := 3T - 3(j-t) - 5$ in $\mathbb{Q}_{1, \mathcal{T}\backslash\{t, \ldots, j\}}[\mathbf{d}']$. We construct

(ii) $(\dot{x}^s; \dot{v}^s; \dot{y}^s) = (\acute{x}^s; \acute{v}^s; \acute{y}^s)_{\looparrowright[y_{2,1}=1]}$.

These solutions belong to F as $x_{1,i} = v_{1,i} = y_{1,i} = 0$ for $i = t, \ldots, j$. Next we construct for $i = t, \ldots, j$ and $r = t+1, \ldots, j$, vectors $\left(\breve{x}^i; \breve{v}^i; \breve{y}^i\right)$, and $(\bar{x}^r; \bar{v}^r; \bar{y}^r)$, with non-zero components

(iii) $(\breve{x}^i_1, \breve{v}^i_1, \breve{y}^i_1) = (d_{1(t-1)}e_1 + d_{tT}e_t, d_{1(t-1)}e_1 + \sum_{q=t}^{i-1} d_q e_q + d_{ij}e_i + d_{(j+1)T}e_{j+1}, e_1 + e_t)$, $(\breve{x}^i_2, \breve{v}^i_2, \breve{y}^i_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(iv) $(\bar{x}^r_1, \bar{v}^r_1, \bar{y}^r_1) = (d_{1j}e_1 + d_{(j+1)T}e_r, d_{1j}e_1 + d_{(j+1)T}e_{j+1}, e_1 + e_r)$, $(\bar{x}^r_2, \bar{v}^r_2, \bar{y}^r_2) = (\mathbf{0}, \mathbf{0}, e_1)$.

These vectors are affinely independent from each other and from those of families (i) and (ii) since each added vector has a nonzero component where all previous points had a zero component. Additionally, we construct vectors $(\check{x}^m; \check{v}^m; \check{y}^m)$ and $(\hat{x}^n; \hat{v}^n; \hat{y}^n)$ for $m = t+1, \ldots, k$, and $n = k+1, \ldots, j$, and vectors $(\dddot{x}; \dddot{v}; \dddot{y})$, $(\ddot{x}; \ddot{v}; \ddot{y})$ and $(\grave{x}; \grave{v}; \grave{y})$ with nonzero components

(v) $(\check{x}^m_1, \check{v}^m_1, \check{y}^m_1) = (d_{1(m-1)}e_1 + d_{mT}e_m, d_{1(m-1)}e_1 + d_{mj}e_m + d_{(j+1)T}e_{j+1}, e_1 + e_m)$,

$(\check{x}^m_2, \check{v}^m_2, \check{y}^m_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(vi) $(\hat{x}^n_1, \hat{v}^n_1, \hat{y}^n_1) = (\breve{x}^t_1, \breve{v}^t_1, \breve{y}^t_1) + (0, 0, e_n)$, $(\hat{x}^n_2, \hat{v}^n_2, \hat{y}^n_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(vii) $(\dddot{x}_1, \dddot{v}_1, \dddot{y}_1) = (d_{1j}e_1 + d_{(j+1)T}e_t, d_{1j}e_1 + d_{(j+1)T}e_{j+1}, e_1 + e_t)$, $(\dddot{x}_2, \dddot{v}_2, \dddot{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(viii) $(\ddot{x}_1, \ddot{v}_1, \ddot{y}_1) = (d_{tT}e_t, d_{tj}e_t + d_{(j+1)T}e_{j+1}, e_t)$, $(\ddot{x}_2, \ddot{v}_2, \ddot{y}_2) = (d_{1(t-1)}e_1, d_{1(t-1)}e_1, e_1)$,

(ix) $(\grave{x}_1, \grave{v}_1, \grave{y}_1) = (d_{1T}e_1, d_{1T}e_1, e_1)$.

Observe that, all vectors in families (i)-(iv) satisfy the equalities $x_{1,i} = d_{(j+1)T}y_{1,i}$ for $i = t+1, \ldots, j$, $x_{1,t} = d_{tT}y_{1,t}$, $\sum_{i=1}^{T} x_{1,i} = d_{1T}y_{1,1}$ and $y_{2,1} = 1$. Vectors (v)-(ix) satisfy all but one of these equalities and each equality is violated by only one such point. Therefore we conclude that these vectors are affinely independent from vectors (i)-(iv), and from each other.

For the second case, assume that $P \geqslant 2$ and $t = 1$. Similar to above, there exists affinely independent solutions $(\tilde{x}^i; \tilde{v}^i; \tilde{y}^i)$ for $i = 1, \ldots, \rho$ in $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ for which $\tilde{y}^i_{2,1} = 1$. We construct the vectors

(i) $(\bar{\bar{x}}^i; \bar{\bar{v}}^i; \bar{\bar{y}}^i) = (\tilde{x}^i; \tilde{v}^i; \tilde{y}^i)_{\hookrightarrow[y_{1,j+1}=1]}$.

By Corollary 4.1, there are affinely independent solutions $(\acute{x}^s; \acute{v}^s; \acute{y}^s)$ for $s = 1, \ldots, 3(T-j) - 2$ in $\mathbb{Q}_{\{1\},\{j+1,\ldots,T\}}[\mathbf{d}]$. Then we construct the solutions

(ii) $(\hat{x}; \hat{v}; \hat{y}) + (\acute{x}^s; \acute{v}^s; \acute{y}^s)$

where $(\hat{x}; \hat{v}; \hat{y})$ is a vector with nonzero components $(\hat{x}_2, \hat{v}_2, \hat{y}_2) = (d_{1j}e_1, d_{1j}e_1, e_1)$. These solutions belong to F as $x_{1,i} = v_{1,i} = y_{1,i} = 0$ for $i = 1, \ldots, j$. Next we construct for $i = 1, \ldots, j$, and $r = 2, \ldots, j$, the vectors $\left(\breve{x}^i, \breve{v}^i; \breve{y}^i\right)$, and $(\dot{x}^r; \dot{v}^r; \dot{y}^r)$ with nonzero components

(iii) $\left(\breve{x}^i_1, \breve{v}^i_1, \breve{y}^i_1\right) = (d_{1T}e_1, \sum_{q=1}^{i-1} d_q e_q + d_{ij}e_i + d_{(j+1)T}e_{j+1}, e_1 + e_{j+1})$, $\left(\breve{x}^i_2, \breve{v}^i_2, \breve{y}^i_2\right) = (\mathbf{0}, \mathbf{0}, e_1)$,

(iv) $(\dot{x}^r_1, \dot{v}^r_1, \dot{y}^r_1) = (d_{(j+1)T}e_r, d_{(j+1)T}e_{j+1}, e_r + e_{j+1})$, $(\dot{x}^r_2, \dot{v}^r_2, \dot{y}^r_2) = (d_{1j}e_1, d_{1j}e_1, e_1)$.

Vectors in families (iii) and (iv) are affinely independent from each other, and from previously described vectors since each of them has a nonzero component where previous vectors have a zero component. Next, we construct for $m = 2, \ldots, k$ and $n = k+1, \ldots, j$ vectors $(\bar{x}^m; \bar{v}^m; \bar{y}^m)$, and $(\grave{x}^n; \grave{v}^n; \grave{y}^n)$ and vectors $(\ddot{x}; \ddot{v}; \ddot{y})$, $(\check{x}; \check{v}; \check{y})$ and $(\ddot{x}; \ddot{v}; \ddot{y})$ with nonzero components

(v) $(\bar{x}^m_1, \bar{v}^m_1, \bar{y}^m_1) = (d_{mT}e_m, d_{mj}e_m + d_{(j+1)T}e_{j+1}, e_m + e_{j+1})$,

$(\bar{x}^m_2, \bar{v}^m_2, \bar{y}^m_2) = (d_{1(m-1)}e_1, d_{1(m-1)}e_1, e_1)$

(vi) $(\grave{x}^n_1, \grave{v}^n_1, \grave{y}^n_1) = \left(\breve{x}^1_1, \breve{v}^1_1, \breve{y}^1_1\right) + (\mathbf{0}, \mathbf{0}, e_n)$, $(\grave{x}^n_2, \grave{v}^n_2, \grave{y}^n_2) = (\mathbf{0}, \mathbf{0}, e_1)$,

(vii) $(\ddot{x}_1, \ddot{v}_1, \ddot{y}_1) = (d_{(j+1)T}e_1, d_{(j+1)T}e_{j+1}, e_1 + e_{j+1})$, $(\ddot{x}_2, \ddot{v}_2, \ddot{y}_2) = (d_{1j}e_1, d_{1j}e_1, e_1)$,

(viii) $(\check{x}_1, \check{v}_1, \check{y}_1) = (d_{1T}e_1, d_{1j}e_1 + d_{(j+1)T}e_{j+1}, e_1 + e_{j+1})$,

(ix) $(\ddot{x}_1, \ddot{v}_1, \ddot{y}_1) = (d_{1T}e_1, d_{1j}e_1 + d_{(j+1)T}e_{j+1}, e_1)$, $(\ddot{x}_2, \ddot{v}_2, \ddot{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$.

Observe that, every solution in families (i)-(iv) satisfy the equalities $x_{1,i} = d_{(j+1)T}y_{1,i}$ for $i = 2, \ldots, j$, $x_{1,1} = d_{1T}y_{1,1}$, $y_{2,1} = 1$, and $y_{1,j+1} = 1$. Vectors (v)-(ix) satisfy all but one of these equalities and each equality is violated exactly once by one of vectors (v)-(ix). Therefore we conclude that vectors (i)-(ix) are affinely independent.

For the third case, assume that $P = 1$ and $t \geqslant 2$, we construct vectors by removing all components corresponding to plant 2 from the points created for the case where $P \geqslant 2$ and $t \geqslant 2$. In particular, $(\dot{x}^s; \dot{v}^s; \dot{y}^s)$, $\left(\breve{x}^i; \breve{v}^i; \breve{y}^i\right)$, $(\bar{x}^r; \bar{v}^r; \bar{y}^r)$, $(\check{x}^m; \check{v}^m; \check{y}^m)$, $(\hat{x}^n; \hat{v}^n; \hat{y}^n)$, and $(\ddot{x}; \ddot{v}; \ddot{y})$ for $s = 1, \ldots, \varsigma$, $i = t, \ldots, j$, $r = t+1, \ldots, j$, $m = t+1, \ldots, k$, and $n = k+1, \ldots, j$. These points belong to F and are affinely independent thereby showing that F is a facet.

Finally, when $P = 1$ and $t = 1$, (28) is not facet-defining for Q^I. In fact, every solution in the face of Q^I defined by (28) must satisfy $\sum_{i=1}^{k} x_{1,i} = d_{1T}$ as $\sum_{i=1}^{k} x_{1,i} = \sum_{i=1}^{j} v_{1,i} + d_{j+1T} \sum_{i=1}^{k} y_{1,i}$, $\sum_{i=1}^{k} y_{1,i} \geqslant 1$, and $\sum_{i=1}^{j} v_{1,i} \geqslant d_{1j}$. This concludes the proof. □

7.7 Proof of Theorem 4.5 (case $m = 1$)

Theorem 4.5. *For $P \geqslant 2$, let (29) be a valid inequality for Q^I that does not define the same face of Q^I as (27f). Then (29) is facet-defining for Q^I if and only if the following conditions are satisfied:*

(a) $1 \in \bigcup_{p\in\mathcal{P}} (S^p \bigcup V^p)$,

(b) $l < T$,

(c) $V^p = L$ *or* $\left(\bar{S}^p \bigcup \bar{V}^p\right) \neq \varnothing$ *for each* $p \in \mathcal{P}$.

Proof. Assume that $m = 1$. By Theorem 4.1, there are affinely independent solutions $(\acute{x}^r; \acute{v}^r; \acute{y}^r)$ for $r = 1, \ldots, \varrho$ in $\mathbb{Q}^I_{\mathcal{P},\{l+1,\ldots,T\}}[\mathbf{d}]$. Then, we construct the vectors

(i) $(\bar{\bar{x}}^r, \bar{\bar{v}}^r, \bar{\bar{y}}^r) = (\tilde{x}; \tilde{v}; \tilde{y}) + (\acute{x}^r; \acute{v}^r; \acute{y}^r)$

where $(\tilde{x}; \tilde{v}; \tilde{y})$ has nonzero components $(\tilde{x}_\pi, \tilde{v}_\pi, \tilde{y}_\pi) = (d_{1l}e_1, d_{1l}e_1, e_1)$ where $\pi \in \mathcal{P}$ is an index for which $1 \in (S^\pi \bigcup V^\pi)$. By using the same construction as in the case where $m > 1$, three affinely independent solutions can be obtained for all periods $i \in \{2, \ldots, l\}$ and for all plants $p \in \mathcal{P}$, yielding $3P(l-1)$ solutions in addition to those given in (i). In constructing these solutions, if $1 \in \bar{V}^p$, we modify point (vii) by replacing its shipment component by $\hat{v}_p = d_{1\sigma_p(1)}e_1 + d_{(\sigma_p(1)+1)(i-1)}e_{\sigma_p(1)+1} + d_{il}e_i + d_{(l+1)T}e_{l+1}$ if $i > \sigma_p(1) + 1$ (no change is necessary when $i = \sigma_p(1) + 1$). These solutions belong to F by definition of $\sigma_p(1)$. Moreover, for $i \in (S^p \bigcup \bar{S}^p)$, we redefine $r = \min\{k \in \{2, \ldots, l\} \mid \{k, \ldots, i-1\} \in \bar{V}^p\}$ if $(i-1) \in \bar{V}^p$ and $r = i$ otherwise. After these modifications, it is easy to verify that the points described above are still independent, as equalities used to argue independence do not involve the first period, and therefore when $\pi = p$ they continue to hold.

Below, we present three additional solutions for each plant $p \in \mathcal{P}\backslash\{\pi\}$. We distinguish four cases depending on whether $1 \in S^p, \bar{S}^p, V^p$ or $\bar{V}^p$. Note that because these solutions pertain to different plants, and the proposed equalities are homogenous and involve a single plant, it is enough to verify independence from points (i)-(xiv) of the previous case.

First, if $1 \in S^p$, we construct vectors $(\tilde{x}^p; \tilde{v}^p; \tilde{y}^p)$, and $(\dot{x}^p; \dot{v}^p; \dot{y}^p)$ with nonzero components

(ii) $(\tilde{x}^p_p, \tilde{v}^p_p, \tilde{y}^p_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1l}e_1 + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$,

(iii) $(\dot{x}^p_p, \dot{v}^p_p, \dot{y}^p_p) = (d_{(l+1)T}e_{l+1}, d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$,

$(\dot{x}^p_\pi, \dot{v}^p_\pi, \dot{y}^p_\pi) = (d_{1l}e_1, d_{1l}e_1, e_1)$.

We observe that (ii) violates the equality $(x_{p,1} - d_{1T}y_{p,1}) + \sum_{i\in S^p\bigcap\{2,\ldots,l\}} \frac{d_{(l+1)T}}{d_{il}} \sum_{j=r_i}^{i}(v_{p,j} - x_{p,j}) = 0$ that is satisfied by all previous solutions. Moreover, (iii) violates the equality $x_{p,1} = d_{1l}y_{p,1}$ that is satisfied by all previous solutions and (ii). We next consider two cases to construct a third point for plant p. If $\bar{S}^p = \varnothing$ then we let $j^p = \min\{i \mid i \in \bar{V}^p\}$, which exists because of condition (c). We construct $(\bar{x}^p; \bar{v}^p; \bar{y}^p)$ with nonzero components

(iv) $(\bar{x}^p_p, \bar{v}^p_p, \bar{y}^p_p) = (d_{1(j^p-1)}e_1 + d_{j^pT}e_{j^p}, d_{1(j^p-1)}e_1 + d_{j^p\sigma_p(j^p)}e_{j^p} + d_{(\sigma_p(j^p)+1)l}e_{\sigma_p(j^p)+1}$
$+d_{(l+1)T}e_{l+1}, e_1 + e_{j^p})$.

Otherwise, we let j^p be an index in $\bar{S}^p$ and modify $(\bar{x}^p; \bar{v}^p; \bar{y}^p)$ as follows

(iv′) $(\bar{x}^p_p, \bar{v}^p_p, \bar{y}^p_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1(j^p-1)}e_1 + d_{j^pl}e_{j^p} + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$.

We observe that all solutions presented previously satisfy the equality $\sum_{i\in\bar{S}^p\bigcap\{2,\ldots,l\}}\sum_{j=r_i}^{i}(x_{p,j}-v_{p,j})+\sum_{i\in V^p\bigcap\{2,\ldots,l\}}(x_{p,i}-v_{p,i})+\sum_{i=l+1}^{T}(x_{p,i}-v_{p,i})=0$ which is violated by (iv) and (iv$'$). Therefore all points presented for plant p are affinely independent from each other and from previous solutions.

Second, if $1\in\bar{S}^p$, we construct the vectors $(\tilde{x}^p;\tilde{v}^p;\tilde{y}^p)$, $(\check{x}^p;\check{v}^p;\check{y}^p)$, and $(\breve{x}^p;\breve{v}^p;\breve{y}^p)$ with nonzero components

(v) $(\tilde{x}_p^p,\tilde{v}_p^p,\tilde{y}_p^p)=(d_{1l}e_1+d_{(l+1)T}e_{l+1},d_{1l}e_1+d_{(l+1)T}e_{l+1},e_1+e_{l+1})$,

(vi) $(\check{x}_p^p,\check{v}_p^p,\check{y}_p^p)=(d_{1T}e_1,d_{1T}e_1,e_1)$,

(vii) $(\breve{x}_p^p,\breve{v}_p^p,\breve{y}_p^p)=(d_{1T}e_1,d_{1l}e_1+d_{(l+1)T}e_{l+1},e_1)$.

Vector (v) violates the equality $(x_{p,1}-d_{1T}y_{p,1})+\sum_{i\in S^p\bigcap\{2,\ldots,l\}}\frac{d_{(l+1)T}}{d_{il}}\sum_{j=r_i}^{i}(v_{p,j}-x_{p,j})=0$ and vector (vi) violates the equality $x_{p,1}=d_{1l}y_{p,1}$ that are satisfied by all previous solutions. Finally vector (vii) violates the equality $\sum_{i\in\bar{S}^p\bigcap\{2,\ldots,l\}}\sum_{j=r_i}^{i}(x_{p,j}-v_{p,j})+\sum_{i\in V^p\bigcap\{2,\ldots,l\}}(x_{p,i}-v_{p,i})+\sum_{i=l+1}^{T}(x_{p,i}-v_{p,i})=0$ that is satisfied by all previous solutions as well as (v) and (vi). Therefore all points presented for plant p are affinely independent from each other and from previous solutions.

Third, if $1\in V^p$ then $\{1,\ldots,l\}\in V^p$ by condition (c). We construct the vectors $(\dot{x}^p;\dot{v}^p;\dot{y}^p)$, $(\tilde{x}^p;\tilde{v}^p;\tilde{y}^p)$, and $(\breve{x}^p;\breve{v}^p;\breve{y}^p)$ with nonzero components

(viii) $(\dot{x}_p^p;\dot{v}_p^p;\dot{y}_p^p)=(d_{(l+1)T}e_{l+1},d_{(l+1)T}e_{l+1},e_1+e_{l+1})$,

$(\dot{x}_\pi^p;\dot{v}_\pi^p;\dot{y}_\pi^p)=(d_{1l}e_1,d_{1l}e_1,e_1)$,

(ix) $(\tilde{x}_p^p,\tilde{v}_p^p,\tilde{y}_p^p)=(d_{1l}e_1+d_{(l+1)T}e_{l+1},d_{1l}e_1+d_{(l+1)T}e_{l+1},e_1+e_{l+1})$,

(x) $(\breve{x}_p^p,\breve{v}_p^p,\breve{y}_p^p)=(d_{1T}e_1,d_{1l}e_1+d_{(l+1)T}e_{l+1},e_1)$.

Vector (viii) violates the equality $v_{p,1}=d_{1l}y_{p,1}$ whereas (ix) violates the equality $x_{p,1}=0$ that are satisfied by all previous solutions. Moreover, vector (x) violates the equality $x_{p,1}=v_{p,1}$ that is satisfied by all previous solutions, showing that these solutions are affinely independent from each other and from all solutions presented earlier.

Fourth, if $1\in\bar{V}^p$, we construct the vectors $(\bar{x}^p,\bar{v}^p,\bar{y}^p)$, and $(\mathring{x}^p;\mathring{v}^p;\mathring{y}^p)$ with nonzero components

(xi) $(\bar{x}_p^p,\bar{v}_p^p,\bar{y}_p^p)=(d_{1T}e_1,d_{1\sigma_p(1)}e_1+d_{(\sigma_p(1)+1)l}e_{\sigma_p(1)+1}+d_{(l+1)T}e_{l+1},e_1)$,

(xii) $(\mathring{x}_p^p,\mathring{v}_p^p,\mathring{y}_p^p)=(d_{(\sigma_p(1)+1)l}e_1+d_{(l+1)T}e_{l+1},d_{(\sigma_p(1)+1)l}e_{\sigma_p(1)+1}+d_{(l+1)T}e_{l+1},e_1+e_{l+1})$,

$(\mathring{x}_\pi^p,\mathring{v}_\pi^p,\mathring{y}_\pi^p)=(d_{1\sigma_p(1)}e_1,d_{1\sigma_p(1)}e_1,e_1)$.

These points belong to F by definition of $\sigma_p(1)$. Vector (xi) violates the equality $x_{p,1}=v_{p,1}+d_{(\sigma_p(1)+1)l}y_{p,1}$, and vector (xii) violates $v_{p,1}=d_{1\sigma_p(1)}y_{p,1}$ that are satisfied by all previous solutions. Finally, we construct one additional vector based on two cases. On the one hand, if $S^p=\varnothing$, we construct the vector $(\ddot{x}^p,\ddot{v}^p,\ddot{y}^p)$ with nonzero components

(xiii) $(\ddot{x}_p^p,\ddot{v}_p^p,\ddot{y}_p^p)=(d_{1l}e_1+d_{(l+1)T}e_{l+1},d_{1\sigma_p(1)}e_1+d_{(\sigma_p(1)+1)l}e_{\sigma_p(1)+1}+d_{(l+1)T}e_{l+1},e_1+e_{l+1})$.

This vector violates the equality $\frac{x_{p,1}}{d_{(\sigma_p(1)+1)l}}+\left(1-\frac{d_{1T}}{d_{(\sigma_p(1)+1)l}}\right)\frac{v_{p,1}}{d_{1\sigma_p(1)}}=y_{p,1}$, that is satisfied by all previous solutions.

On the other hand if $S^p\neq\varnothing$, we consider two cases. If $(\sigma_p(1)+1)\in S^p$, we construct the vector $(\ddot{x}^p,\ddot{v}^p,\ddot{y}^p)$ with nonzero components

(xiii$'$) $(\ddot{x}_p^p,\ddot{v}_p^p,\ddot{y}_p^p)=(d_{1T}e_1,d_{1\sigma_p(1)}e_1+d_{(\sigma_p(1)+1)T}e_{\sigma_p(1)+1},e_1)$.

We observe that this vector violates the equality $\sum_{i\in\bar{S}^p\bigcap\{2,\ldots,l\}}\sum_{j=r_i}^{i}(x_{p,j}-v_{p,j})+\sum_{i=l+1}^{T}(x_{p,i}-v_{p,i})+\sum_{i\in V^p\bigcap\{2,\ldots,l\}}(x_{p,i}-v_{p,i})+(x_{p,1}-v_{p,1}-d_{(\sigma_p(1)+1)l}y_{p,1})=0$, that is satisfied by all previous solutions. Second, if $(\sigma_p(1)+1)\in\bar{S}^p$, we let $j^p=\min\{i\mid i\in S^p\}>1$, we construct the vector $(\ddot{x}^p,\ddot{v}^p,\ddot{y}^p)$ with nonzero components

(xiii$''$) $(\ddot{x}^p_p,\ddot{v}^p_p,\ddot{y}^p_p)=(d_{1T}e_1, d_{1\sigma_p(1)}e_1+d_{(\sigma_p(1)+1)(j^p-1)}e_{\sigma_p(1)+1}+d_{j^pT}e_{j^p}, e_1)$.

We observe that the equality $x_{p,1}-(1+\frac{d_{(l+1)T}}{d_{1\sigma_p(1)}})v_{p,1}-d_{(\sigma_p(1)+1)l}y_{p,1}+\sum_{i\in S^p}\frac{d_{(l+1)T}}{d_{il}}\sum_{j=r_i}^{i}(v_{p,i}-x_{p,i})=0$ that is satisfied by all previous solutions is violated by this solution. Therefore all points presented for plant p are affinely independent from each other and from previous solutions.

Finally, we present two more solutions for plant π. We let ρ be an index in $\mathcal{P}\backslash\{\pi\}$. Then, we construct the point $(\grave{x};\grave{v};\grave{y})$ with nonzero components

(xiv) $(\grave{x}_\pi,\grave{v}_\pi,\grave{y}_\pi)=(d_{(l+1)T}e_{l+1}, d_{(l+1)T}e_{l+1}, e_1+e_{l+1})$,

$(\grave{x}_\rho,\grave{v}_\rho,\grave{y}_\rho)=(d_{1l}e_1, d_{1l}e_1, e_1)$.

When $1\in\bar{V}^\rho$, we modify $(\grave{x};\grave{v};\grave{y})$ by setting $\grave{v}_\rho=d_{1\sigma_\rho(1)}e_1+d_{(\sigma_\rho(1)+1)l}e_{\sigma_\rho(1)+1}$. We let $\mathcal{P}_1=\{p\in\mathcal{P}\mid 1\in\bar{V}^p\}$, $\mathcal{P}_2=\{p\in\mathcal{P}\backslash\{\pi\}\mid 1\in S^p \text{ and } \bar{S}^p\neq\varnothing\}$, $\mathcal{P}_2'=\{p\in\mathcal{P}\backslash\{\pi\}\mid 1\in S^p \text{ and } \bar{S}^p=\varnothing\}$, $\mathcal{P}_3=\{p\in\mathcal{P}\mid 1\in\bar{S}^p\}$, and $\mathcal{P}_4=\{p\in\mathcal{P}\backslash\{\pi\}\mid 1\in V^p\}$. Vector (xiv) violates the equality $\sum_{p\in\mathcal{P}_1}\frac{v_{p,1}}{d_{1\sigma_p(1)}}+\sum_{p\in\mathcal{P}_2}\frac{x_{p,1}}{d_{1l}}+\sum_{p\in\mathcal{P}_2'}\left(\frac{x_{p,1}}{d_{1l}}+\frac{d_{j^pl}}{d_{(l+1)T}d_{1l}}\left(\sum_{i\in V^p}(v_{p,i}-x_{p,i})+\sum_{i=l+1}^{T}(v_{p,i}-x_{p,i})\right)\right)+\sum_{p\in\mathcal{P}_3}y_{p,1}+\sum_{p\in\mathcal{P}_4}\frac{v_{p,1}}{d_{1l}}+y_{\pi,1}=1$, where $j^p=\min\{i\mid i\in\bar{V}^p\}$ for $p\in\mathcal{P}_2'$. This equality is satisfied by all solutions presented previously.

To construct an additional point, we consider two cases as $1\in(S^\pi\bigcup V^\pi)$. On the one hand, if $1\in S^\pi$ and $\bar{S}^\pi\neq\varnothing$, we define $j^\pi=\min\{q\mid q\in\bar{S}^\pi\}$. Then we construct the point $(\widehat{x};\widehat{v};\widehat{y})$ with nonzero components

(xv) $(\widehat{x}_\pi,\widehat{v}_\pi,\widehat{y}_\pi)=(d_{1l}e_1+d_{(l+1)T}e_{l+1}, d_{1(j^\pi-1)}e_1+d_{j^\pi l}e_{j^\pi}+d_{(l+1)T}e_{l+1}, e_1+e_{l+1})$.

If, on the other hand, $\bar{S}^\pi=\varnothing$ then we define $j^\pi=\min\{q\mid q\in\bar{V}^\pi\}$, we construct $(\widehat{x};\widehat{v};\widehat{y})$ as follows

(xv$'$) $(\widehat{x}_\pi,\widehat{v}_\pi,\widehat{y}_\pi)=(d_{1(j^\pi-1)}e_1+d_{j^\pi T}e_{j^\pi}, d_{1(j^\pi-1)}e_1+d_{j^\pi\sigma_\pi(j^\pi)}e_{j^\pi}+d_{(\sigma_\pi(j^\pi)+1)l}e_{\sigma_\pi(j^\pi)+1}+d_{(l+1)T}e_{l+1}, e_1+e_{j^\pi})$.

Index $j^\pi\in(\bar{S}^p\bigcup\bar{V}^p)$ exists because of (c).

On the other hand, if $1\in V^\pi$ then we construct the point $(\dddot{x};\dddot{v};\dddot{y})$ with nonzero components

(xv$''$) $(\dddot{x}_\pi,\dddot{v}_\pi,\dddot{y}_\pi)=(d_{1T}e_1, d_{1l}e_1+d_{(l+1)T}e_{l+1}, e_1)$.

Vectors (xv),(xv$'$), and (xv$''$) violate the equality $\sum_{i\in V^\pi\bigcap\{2,\ldots,l\}}(x_{\pi,i}-v_{\pi,i})+\sum_{i=l+1}^{T}(x_{\pi,i}-v_{\pi,i})+\sum_{i\in\bar{S}^\pi\bigcap\{2,\ldots,l\}}\sum_{j=r_i}^{i}(x_{\pi,j}-v_{\pi,j})=0$ that is satisfied by all solutions presented previously. This concludes the proof. □

7.8 Complete convex hull descriptions

In this section, we show that the families of inequalities described in Section 4 are sufficient to describe the convex hull of LSS when T and P are small. When $P=T=1$, the set reduces to the single point $(x_{1,1},v_{1,1},y_{1,1})=(d_1,d_1,1)$. For this reason we next consider sets where $P+T\geqslant 3$.

Theorem 7.1. *The convex hull of* $Q_{P,1}[\mathbf{d}]$*, for* $\mathbf{d} \in \mathbb{R}_{++}$ *is given by*

$$Q^* = \left\{ (x;v;y) \in \mathbb{R}^P_+ \times \mathbb{R}^{2P} \;\middle|\; \begin{array}{lll} x_{p,1} = v_{p,1} & \forall p \in \mathcal{P} & (67a) \\ \sum_{p\in\mathcal{P}} x_{p,1} = d_1 & & (67b) \\ x_{p,1} \leqslant d_1 y_{p,1} & \forall p \in \mathcal{P} & (67c) \\ y_{p,1} \leqslant 1 & \forall p \in \mathcal{P} & (67d) \end{array} \right\}.$$

Proof. It is clear that $Q_{P,1}[\mathbf{d}] \subseteq Q^*$. It follows that $Q^I_{P,1}[\mathbf{d}] \subseteq Q^*$ since Q^* is a polyhedron. To prove the reverse inclusion, we verify that the extreme points of Q^* belong to $Q_{P,1}[\mathbf{d}]$. Consider the linear programs

$$(\mathcal{LP}) \quad \max\left\{ \sum_{p\in\mathcal{P}} (\alpha_{p,1}x_{p,1} + \gamma_{p,1}v_{p,1} + \beta_{p,1}y_{p,1}) \;\middle|\; (x;v;y) \in Q^* \right\}.$$

Because of (67a), we may assume that $\gamma_{p,1} = 0$ for all $p \in \mathcal{P}$. We show next that $(\mathcal{LP})$ has an optimal solution that belongs to $Q_{P,1}[\mathbf{d}]$ for all choices of $\alpha_{p,1}$ and $\beta_{p,1}$. To this end, we present both a primal feasible solution $(x^*;x^*;y^*)$ and a dual feasible solution $(s^*;t^*;u^*;w^*)$ with identical objective values. In the expression $(s^*;t^*;u^*;w^*)$, s^*, t^*, u^*, and w^* are dual variables to (67a)-(67d), respectively, and therefore, $s^* \in \mathbb{R}^p$, $t^* \in \mathbb{R}$, while u^* and $w^* \in \mathbb{R}^p_+$. To describe these points, we define $\mathcal{P}^+ = \{p \in \mathcal{P} \mid \beta_{p,1} \geqslant 0\}$. We consider two cases.

Assume first that $\mathcal{P}^+ \neq \varnothing$. In this case, select $\pi \in \text{argmax}\{\alpha_{p,1} \mid p \in \mathcal{P}^+\}$, and define $\mathcal{P}^> = \{p \in \mathcal{P}\backslash\mathcal{P}^+ \mid \alpha_{p,1}d_1 + \beta_{p,1} > \alpha_\pi d_1\}$. There are two subcases. Assume first that $\mathcal{P}^> = \varnothing$. In this case, we let $x^*_{\pi,1} = d_1$, $x^*_{p,1} = 0$ for $p \in \mathcal{P}\backslash\{\pi\}$, $y^*_{p,1} = 1$ for $p \in \mathcal{P}^+$ and $y^*_{p,1} = 0$ for $\mathcal{P}\backslash\mathcal{P}^+$. This solution of $Q_{P,1}[\mathbf{d}]$ has objective value $z^1 = \alpha_{\pi,1}d_1 + \sum_{p\in\mathcal{P}^+}\beta_{p,1}$. For the dual solution, we let $s^*_p = 0$ for $p \in \mathcal{P}$, $t^* = \alpha_\pi$, $u^*_p = 0$ for $p \in \mathcal{P}^+$, $u^*_p = -\frac{\beta_{p,1}}{d_1}$ for $p \in \mathcal{P}\backslash\mathcal{P}^+$, $w^*_p = \beta_{p,1}$ for $p \in \mathcal{P}^+$ and $w^*_p = 0$ for $p \in \mathcal{P}\backslash\mathcal{P}^+$. This solution can be verified to be dual feasible with objective value z^1. Assume second that $\mathcal{P}^> \neq \varnothing$. Let $\rho \in \text{argmax}\{\alpha_{p,1}d_1 + \beta_{p,1} \mid p \in \mathcal{P}\backslash\mathcal{P}^+\}$. In this case, we let $x^*_{\rho,1} = d_1$, $x^*_{p,1} = 0$ for $p \in \mathcal{P}\backslash\{\rho\}$, $y^*_{p,1} = 1$ for $p \in \mathcal{P}^+ \bigcup\{\rho\}$ and $y^*_{p,1} = 0$ for $p \in \mathcal{P}\backslash(\mathcal{P}^+\bigcup\{\rho\})$. This solution of $Q_{P,1}[\mathbf{d}]$ has objective value $z^2 = \alpha_{\rho,1}d_1 + \sum_{p\in\mathcal{P}^+\bigcup\{\rho\}}\beta_{p,1}$. For the dual solution, we let $s^*_p = 0$ for $p \in \mathcal{P}$, $t^* = \alpha_{\rho,1} + \frac{\beta_{\rho,1}}{d_1}$, $u^*_p = 0$ for $p \in \mathcal{P}^+$, $u^*_p = -\frac{\beta_{p,1}}{d_1}$ for $p \in \mathcal{P}\backslash\mathcal{P}^+$, $w^*_p = \beta_{p,1}$ for $p \in \mathcal{P}^+$, and $w^*_p = 0$ for $p \in \mathcal{P}\backslash\mathcal{P}^+$. This solution can be verified to be dual feasible with objective value z^2.

Assume finally that $\mathcal{P}^+ = \varnothing$. In this case, select $\pi \in \text{argmax}\{\alpha_{p,1}d_1 + \beta_{p,1} \mid p \in \mathcal{P}\}$. We let $x^*_{\pi,1} = d_1$, $x^*_{p,1} = 0$ for $p \in \mathcal{P}\backslash\{\pi\}$, $y^*_{\pi,1} = 1$, and $y^*_{p,1} = 0$ for $p \in \mathcal{P}\backslash\{\pi\}$. This solution of $Q_{P,1}[\mathbf{d}]$ has objective value $z^3 = \alpha_{\pi,1}d_1 + \beta_{\pi,1}$. For the dual solution, we let $s^*_p = 0$ for $p \in \mathcal{P}$, $t^* = \alpha_{\pi,1} + \frac{\beta_{\pi,1}}{d_1}$, $u^*_p = -\frac{\beta_{p,1}}{d_1}$ for $p \in \mathcal{P}$, $w^*_p = 0$ for $p \in \mathcal{P}$. This solution can be verified to be dual feasible with objective value z^3. □

We next describe which inequalities, among those given in Theorem 7.1 are facet-defining. Clearly the dimension of Q^* is $2P - 1$ by Theorem 4.1. To prove that (67c) is facet-defining for Q^*, we construct $2P - 1$ affinely independent vectors in the corresponding face of Q^*. These vectors are $(\bar{x}^\pi;\bar{v}^\pi;\bar{y}^\pi) = (d_1e_\pi;d_1e_\pi;e_\pi)$ for $\pi \in \mathcal{P}$, and $(\tilde{x}^\pi;\tilde{v}^\pi;\tilde{y}^\pi) = (d_1e_p;d_1e_p;e_p + e_\pi)$ for $\pi \in \mathcal{P}\backslash\{p\}$. Next we argue that (67d) is facet-defining for Q^*. We construct the vectors $(\hat{x}^\pi;\hat{v}^\pi;\hat{y}^\pi) = (d_1e_\pi;d_1e_\pi;e_\pi + e_p)$, and $(\dot{x}^\pi;\dot{v}^\pi;\dot{y}^\pi) = (d_1e_p;d_1e_p;e_p + e_\pi)$ for each $\pi \in \mathcal{P}\backslash\{p\}$ together with $(\check{x};\check{v};\check{y}) = (d_1e_p;d_1e_p;e_p)$.

Finally, an argument similar to that of Theorem 4.2 (vi) shows that the non-negativity constraint $x_{p,1} \geqslant 0$ is not facet-defining when $P \leqslant 2$. For $P \geqslant 3$, we first construct the points $(\ddot{x}^\pi;\ddot{v}^\pi;\ddot{y}^\pi) = (d_1e_\pi;d_1e_\pi;e_\pi)$ for each $\pi \in \mathcal{P}\backslash\{p\}$. We then let $\rho \neq p$ be an index of $\mathcal{P}$. We construct the points

$(\acute{x}^\pi; \acute{v}^\pi; \acute{y}^\pi) = (d_1 e_\rho; d_1 e_\rho; e_\rho + e_\pi)$ for each $\pi \in \mathcal{P}\backslash\{\rho\}$, and the point $(\mathring{x}; \mathring{v}; \mathring{y}) = (d_1 e_\sigma; d_1 e_\sigma; e_\sigma + e_\rho + e_p)$ for an index σ in $\mathcal{P}\backslash\{\rho, p\}$.

We next show that, for the case where $P = 2$ and $T = 2$, production upper bound and (l, S, V) inequalities are sufficient to obtain the convex hull.

Theorem 7.2. *The convex hull of $Q_{2,2}[\mathbf{d}]$, for $\mathbf{d} \in \mathbb{R}^2_{++}$ is given by*

$$Q^* = \left\{ (x; v; y) \in \mathbb{R}^8_+ \times \mathbb{R}^4 \;\middle|\; \begin{array}{lr} v_{1,1} \leqslant x_{1,1} & (68a) \\ v_{2,1} \leqslant x_{2,1} & (68b) \\ \sum_{i=1}^2 x_{1,i} = \sum_{i=1}^2 v_{1,i} & (68c) \\ \sum_{i=1}^2 x_{2,i} = \sum_{i=1}^2 v_{2,i} & (68d) \\ \sum_{i=1}^2 (v_{1,i} + v_{2,i}) = d_{12} & (68e) \\ x_{1,1} \leqslant d_{12} y_{1,1} & (68f) \\ x_{2,1} \leqslant d_{12} y_{2,1} & (68g) \\ x_{1,2} \leqslant d_2 y_{1,2} & (68h) \\ x_{2,2} \leqslant d_2 y_{2,2} & (68i) \\ y_{1,1} \leqslant 1 & (68j) \\ y_{2,1} \leqslant 1 & (68k) \\ y_{1,2} \leqslant 1 & (68l) \\ y_{2,2} \leqslant 1 & (68m) \\ x_{1,1} \leqslant v_{1,1} + d_2 y_{1,1} & (68n) \\ x_{2,1} \leqslant v_{2,1} + d_2 y_{2,1} & (68o) \\ 1 \leqslant y_{1,1} + y_{2,1} & (68p) \\ d_1 \leqslant v_{2,1} + d_1 y_{1,1} & (68q) \\ d_1 \leqslant v_{1,1} + d_1 y_{2,1} & (68r) \\ d_1 \leqslant v_{1,1} + v_{2,1} & (68s) \end{array} \right\}.$$

Proof. We claim that $Q^* = Q^I_{2,2}[\mathbf{d}]$. It is clear that $Q_{2,2}[\mathbf{d}] \subseteq Q^*$ since the inequalities defining Q^* either belong to the definition of R or they are production upper bound or (l, S, V) inequalities. To prove the reverse implication, we show that the extreme points of Q^* have the property that $y_{p,t} \in \{0, 1\}$ for $p \in \mathcal{P}$, $t \in \mathcal{T}$, *i.e*, they belong to $Q_{2,2}[\mathbf{d}]$. Consider now any feasible solution $\bar{\delta} = (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}; v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}; y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2})$ to Q^* for which $0 < y_{p,t} < 1$ for some (p, t). We show that this solution can be expressed as a convex combination of two other solutions of Q^*. In particular, for each situation, we give Δ such that $\bar{\delta} + \epsilon\Delta$ and $\bar{\delta} - \epsilon\Delta$ are feasible to Q^* for ϵ sufficiently small.

We consider four different cases based on how many binary variables take on a fractional value.

We first assume that there is only one pair (p, t) such that $y_{p,t}$ is fractional. Without loss of generality, we assume that $p = 1$. There are two subcases based on t.

In the first subcase, we assume that $0 < y_{1,1} < 1$. We construct the desired two new feasible solutions by propagating an initial perturbation on $y_{1,1}$ to related variables. First, observe that (68p) and (68q) imply that $y_{2,1} > 0$ and $v_{2,1} > 0$. Since a single variable $y_{p,t}$ is fractional we conclude that $y_{2,1} = 1$. Because of (68b) we obtain that $x_{2,1} > 0$. Since $y_{2,1} = 1$ we know that (68p) has slack. We present in Table 3 (see Appendix) all subcases that can arise depending on which constraints are tight in the feasible solution. In this table, we list in the leftmost column all the inequality constraints (except for simple bounds) that may or may not be satisfied at equality. In particular, we do not list equality constraints, neither do we list (68p) (since this inequality is not tight in the present discussion). For each subcase, we next describe a perturbation of the point $\bar{\delta}$. We mark each constraint either with $\times$ if it is not tight at $\bar{\delta}$ in this subcase or with $\leftrightarrow$ if the perturbation affects equally both sides of inequality.

In the subsequent derivations, we provide verifications for the signs $\times$ in Table 3. However, we leave it to the reader to verify the signs $\leftrightarrow$ in Table 3 and to verify that the proposed perturbations do not violate simple bounds.

If $x_{1,1} = 0$ then $v_{1,1} = 0$ because of (68a). It follows that (68f) and (68n) have slack, as $y_{1,1} > 0$. Moreover, (68q) has slack since $v_{2,1} \geqslant d_1$ because of (68s). We let $\Delta_1 = (\mathbf{0}; \mathbf{0}; e_1)$. We may next assume $x_{1,1} > 0$. This assumption implies that $x_{2,1} < d_{12} = d_{12}y_{2,1}$ as part of total demand d_{12} is being met by $x_{1,1}$. If $v_{1,1} = 0$, we have that $0 = v_{1,1} < x_{1,1}$ and $x_{1,1} \leqslant v_{1,1} + d_2y_{1,1} < d_{12}y_{1,1}$. Therefore, (68a) and (68f) have slack. Moreover, we must have that $v_{2,1} \geqslant d_1$ because of (68s) as $v_{1,1} = 0$. It follows that (68o) and (68q) have slack as $y_{1,1} > 0$ and $y_{2,1} = 1$. If $x_{1,2} > 0$ then $y_{1,2} = 1$. It follows that (68h) has slack as $x_{1,1} > 0$ and $v_{1,1} = 0$. We let $\Delta_2 = (-d_2, d_2, 0, 0; 0, 0, 0, 0; -1, 0, 0, 0)$. If $x_{2,2} > 0$ then $y_{2,2} = 1$. It follows that (68i) has slack as $x_{1,1} > 0$ and $v_{1,1} = 0$. We let $\Delta_2 = (-d_2, 0, 0, d_2; 0, -d_2, 0, d_2; -1, 0, 0, 0)$. If $x_{1,2} = x_{2,2} = 0$ then we distinguish two cases based on whether (68s) is tight. If $v_{1,1} + v_{2,1} = d_1$, we have that $v_{2,1} = d_1$. Because $x_{1,1} \leqslant v_{1,1} + d_2y_{1,1} = d_2y_{1,1} < d_2$ and therefore $x_{2,1} = d_{12} - x_{1,1} > d_1$, we also have that $x_{2,1} > v_{2,1}$. We let $\Delta_3 = (-d_2, 0, d_2, 0; 0, -d_2, 0, d_2; -1, 0, 0, 0)$. On the other hand, if $v_{1,1} + v_{2,1} > d_1$, we let $\Delta_4 = (-d_2, 0, d_2, 0; 0, -d_2, d_2, 0; -1, 0, 0, 0)$.

As all subcases where $v_{1,1} = 0$ have been examined, we assume now that $v_{1,1} > 0$. This assumption implies that (68r) has slack as $y_{2,1} = 1$. We consider two cases based on whether (68a) is tight. First, we assume that $v_{1,1} < x_{1,1}$ (which implies $v_{1,2} > 0$). If $x_{1,2} > 0$ then $y_{1,2} = 1$. It follows that (68h) has slack as $x_{1,1} > v_{1,1}$. We let $\Delta_5 = (-d_{12}, d_2, d_1, 0; -d_1, 0, d_1, 0; -1, 0, 0, 0)$. If $x_{2,2} > 0$ then $y_{2,2} = 1$. It follows that (68i) has slack as $x_{1,1} > v_{1,1}$. We let $\Delta_5 = (-d_{12}, 0, d_1, d_2; -d_1, -d_2, d_1, d_2; -1, 0, 0, 0)$. If on the other hand $x_{1,2} = x_{2,2} = 0$ then there are two cases based on whether (68s) is tight. If $v_{1,1} + v_{2,1} = d_1$, we claim that $x_{2,1} > v_{2,1}$. In fact, when $x_{2,1} = v_{2,1}$, then $v_{2,2} = 0$ and $v_{1,1} = d_1 - x_{2,1}$. Since $x_{1,1} = d_{12} - x_{2,1}$, we then write that $v_{1,1} + d_2y_{1,1} < v_{1,1} + d_2 \leqslant d_{12} - x_{2,1} = x_{1,1}$, a contradiction to (68n). Moreover, we have that $x_{2,1} = d_{12} - x_{1,1} < d_{12} - v_{1,1} = d_1 - v_{1,1} + d_2y_{2,1} = v_{2,1} + d_2y_{2,1}$ where the second equality holds because $y_{2,1} = 1$ showing that (68o) is not tight. We let $\Delta_6 = (-d_{12}, 0, d_{12}, 0; -d_1, -d_2, d_1, d_2; -1, 0, 0, 0)$. If $d_1 < v_{1,1} + v_{2,1}$, we consider two cases based on whether (68q) is tight. If $d_1 = v_{2,1} + d_1y_{1,1}$, we have that $v_{2,1} = d_1(1 - y_{1,1})$. It follows that $x_{2,1} = d_{12} - x_{1,1} \geqslant d_{12}(1 - y_{1,1}) > v_{2,1}$. Moreover, since $x_{1,1} > v_{1,1} > d_1 - v_{2,1} = d_1y_{1,1}$, and therefore $x_{2,1} = d_{12} - x_{1,1} < d_{12} - d_1y_{1,1}$, we have that $x_{2,1} < d_{12} - d_1y_{1,1} = d_1(1 - y_{1,1}) + d_2 = v_{2,1} + d_2y_{2,1}$. We let $\Delta_7 = (-d_{12}, 0, d_{12}, 0; -d_1, -d_2, d_1, d_2; -1, 0, 0, 0)$. On the other hand, if $d_1 < v_{2,1} + d_1y_{1,1}$, we let $\Delta_8 = (-d_{12}, 0, d_{12}, 0; -d_1, -d_2, d_{12}, 0; -1, 0, 0, 0)$.

Next, we assume that $x_{1,1} = v_{1,1}$ and so (68n) has slack. There are two cases based on whether (68q) is tight. If $d_1 < v_{2,1} + d_1y_{1,1}$, we let $\Delta_9 = (-d_{12}, 0, d_{12}, 0; -d_{12}, 0, d_{12}, 0; -1, 0, 0, 0)$. If $d_1 = v_{2,1} + d_1y_{1,1}$ and if (68f) has slack, we let $\Delta_{10} = (-d_1, 0, d_1, 0; -d_1, 0, d_1, 0; -1, 0, 0, 0)$. Therefore we may assume that (68f) is tight. As a result we have that $v_{1,1} + v_{2,1} = x_{1,1} + d_1(1 - y_{1,1}) = d_{12}y_{1,1} + d_1 - d_1y_{1,1} = d_1 + d_2y_{1,1} > d_1$. If $x_{1,2} > 0$ then $y_{1,2} = 1$. It follows that (68h) has slack since $v_{1,1} + v_{2,1} > d_1$. We let $\Delta_{11} = (-d_{12}, d_2, d_1, 0; -d_{12}, d_2, d_1, 0; -1, 0, 0, 0)$. If $x_{2,2} > 0$ then $y_{2,2} = 1$. It follows that (68i) has slack since $v_{1,1} + v_{2,1} > d_1$. We let $\Delta_{11} = (-d_{12}, 0, d_1, d_2; -d_{12}, 0, d_1, d_2; -1, 0, 0, 0)$. On the other hand, if $x_{1,2} = x_{2,2} = 0$, then we have that $x_{2,1} = d_{12} - x_{1,1} = d_{12}(1 - y_{1,1})$ and therefore we conclude that $v_{2,1} = d_1(1 - y_{1,1}) < x_{2,1}$ as $d_1 < d_{12}$ and that $v_{2,2} > 0$ since $x_{2,1} > v_{2,1}$. Moreover, (68o) has slack as $x_{2,1} = d_{12}(1 - y_{1,1})$ and $v_{2,1} + d_2y_{2,1} = d_1(1 - y_{1,1}) + d_2 = d_{12}(1 - y_{1,1}) + d_2y_{1,1} = x_{2,1} + d_2y_{1,1} > x_{2,1}$. We let $\Delta_{12} = (-d_{12}, 0, d_{12}, 0; -d_{12}, 0, d_1, d_2; -1, 0, 0, 0)$.

In the second subcase we assume that $0 < y_{1,2} < 1$. We construct solutions by propagating an initial perturbation on $y_{1,2}$ to related variables. If (68h) has slack, we let $\Delta_{13} = (\mathbf{0};\mathbf{0};e_2)$. We may now assume that $x_{1,2} = d_2 y_{1,2} > 0$, which also implies that $v_{1,2} > 0$. There are two cases based on the value of $x_{2,2}$. If $x_{2,2} > 0$, then $y_{2,2} = 1$, which implies that (68i) has slack. We let $\Delta_{14} = (0,-d_2,0,d_2;0,-d_2,0,d_2;0,-1,0,0)$. If $x_{2,2} = 0$, we have that $d_1 < x_{1,1} + x_{2,1}$ as $x_{12} + x_{2,2} = d_2 y_{1,2} < d_2$. There are two cases based on whether (68s) is tight. If $v_{1,1} + v_{2,1} = d_1$ we have that either $x_{1,1} > v_{1,1}$ or $x_{2,1} > v_{2,1}$. If $x_{1,1} > v_{1,1}$ then $y_{1,1} = 1$ and therefore (68f) has slack since $x_{1,2} > 0$. Moreover, we have that (68n) has slack since $y_{1,1} = 1$ and $x_{1,2} > 0$. We let $\Delta_{15} = (d_2,-d_2,0,0;0,0,0,0;0,-1,0,0)$. If $x_{2,1} > v_{2,1}$ then $y_{2,1} = 1$ and therefore (68g) has slack since $x_{1,2} > 0$. Moreover, we have that (68o) has slack since $y_{2,1} = 1$ and $x_{1,2} > 0$. We let $\Delta_{15} = (0,-d_2,d_2,0;0,-d_2,0,d_2;0,-1,0,0)$. On the other hand, if $v_{1,1} + v_{2,1} > d_1$ then clearly $x_{1,1} > 0$ or $x_{2,1} > 0$. Further, observe that if $x_{1,1} > 0$ and $v_{1,1} = 0$ then $v_{2,1} > d_1$ and $x_{2,1} > 0$. Similarly, if $x_{2,1} > 0$ and $v_{2,1} = 0$ then $v_{1,1} > d_1$ and therefore $x_{1,1} > 0$. It follows that, there exists a plant p such that $x_{p,1} > 0$ and $v_{p,1} > 0$. If $p = 1$ then $y_{1,1} = 1$. It follows that (68f) has slack since $x_{1,2} > 0$. Moreover, we have that either $y_{2,1} = 0$ implying that $x_{2,1} = v_{2,1} = 0$ and therefore $v_{1,1} > d_1$ or that $y_{2,1} = 1$. In the first case (68r) has slack since $v_{1,1} > d_1$ and in the second case (68r) has slack since $v_{1,1} > 0$. We let $\Delta_{16} = (d_2,-d_2,0,0;d_2,-d_2,0,0;0,-1,0,0)$. If $p = 2$ then $y_{2,1} = 1$. It follows that (68g) has slack since $x_{1,2} > 0$. Moreover, we have that either $y_{1,1} = 0$ implying that $x_{1,1} = v_{1,1} = 0$ and therefore $v_{2,1} > d_1$ or that $y_{1,1} = 1$. In the first case (68q) has slack since $v_{2,1} > d_1$ and in the second case (68q) has slack since $v_{2,1} > 0$. We let $\Delta_{16} = (0,-d_2,d_2,0;0,-d_2,d_2,0;0,-1,0,0)$.

Second we assume that there are two pairs of indices (p,t) for which $y_{p,t}$ is fractional. There are three subcases based on the combinations of fractional $y_{p,t}$ values.

In the first subcase, we consider $0 < y_{1,1}, y_{2,1} < 1$. We construct solutions by propagating an initial perturbation on $y_{1,1}$ and $y_{2,1}$ to related variables. We observe that (68q) and (68r) imply that $v_{1,1} > 0$ and $v_{2,1} > 0$. It follows that $x_{1,1} > 0$ and $x_{2,1} > 0$. We distinguish three subcases based on the number of inequalities $v_{p,1} \leqslant x_{p,1}$ for $p \in \mathcal{P}$ that are not tight. First assume that $v_{1,1} < x_{1,1}$ and $v_{2,1} < x_{2,1}$. We let $\Delta_{17} = (-d_{12},0,d_{12},0;-d_1,-d_2,d_1,d_2;-1,0,1,0)$. Second assume that only one of $v_{p,1} \leqslant x_{p,1}$ for $p \in \mathcal{P}$ is tight. We assume without loss of generality that $v_{1,1} < x_{1,1}$ and $x_{2,1} = v_{2,1}$. It is clear that $x_{2,1} < v_{2,1} + d_2 y_{2,1}$ as $x_{2,1} = v_{2,1}$ and $y_{2,1} > 0$. There are two cases based on whether (68g) is tight. If $x_{2,1} = d_{12} y_{2,1}$, we have that $v_{2,1} + d_1 y_{1,1} = d_{12} y_{2,1} + d_1 y_{1,1} = d_1(y_{1,1} + y_{2,1}) + d_2 y_{2,1} \geqslant d_1 + d_2 y_{2,1} > d_1$ and $v_{1,1} + v_{2,1} = v_{1,1} + d_{12} y_{2,1} \geqslant d_1 - d_1 y_{2,1} + d_{12} y_{2,1} = d_1 + d_2 y_{2,1} > d_1$. We let $\Delta_{18} = (-d_{12},0,d_{12},0;-d_1,-d_2,d_{12},0;-1,0,1,0)$. If $x_{2,1} < d_{12} y_{2,1}$ there are two cases based on whether (68q) is tight. If $d_1 = v_{2,1} + d_1 y_{1,1}$ then we have that $x_{1,1} + x_{2,1} \leqslant d_{12} y_{1,1} + d_1(1 - y_{1,1}) = d_1 + d_2 y_{1,1} < d_{12}$ and therefore $x_{1,2} + x_{2,2} > 0$. If $x_{1,2} > 0$ then $y_{1,2} = 1$. It follows that (68h) has slack since $x_{1,1} > v_{1,1}$. We let $\Delta_{19} = (-d_{12},d_2,d_1,0;-d_1,0,d_1,0;-1,0,1,0)$. If $x_{2,2} > 0$ then $y_{2,2} = 1$. It follows that (68i) has slack since $x_{1,1} > v_{1,1}$. We let $\Delta_{19} = (-d_{12},0,d_1,d_2;-d_1,-d_2,d_1,d_2;-1,0,1,0)$. If on the other hand $d_1 < v_{2,1} + d_1 y_{1,1}$ then we assume that (68s) is tight since otherwise Δ_{18} is a feasible perturbation. It follows that $x_{1,1} + x_{2,1} = x_{1,1} + v_{2,1} \leqslant v_{1,1} + d_2 y_{1,1} + v_{2,1} = d_1 + d_2 y_{1,1} < d_{12}$ and therefore $x_{1,2} + x_{2,2} > 0$. In this case perturbations Δ_{19} are feasible depending on which one of $x_{1,2}$ and $x_{2,2}$ is positive. Third assume that $x_{1,1} = v_{1,1}$ and $x_{2,1} = v_{2,1}$. Since $x_{1,1} = v_{1,1}$, $x_{2,1} = v_{2,1}$, $y_{1,1} > 0$ and $y_{2,1} > 0$ then (68n) and (68o) have slack. There are three cases based on the number of inequalities $x_{p,1} \leqslant d_{12} y_{p,1}$ for $p \in \mathcal{P}$ that are tight. If $x_{1,1} = d_{12} y_{1,1}$ and $x_{2,1} = d_{12} y_{2,1}$ then we have that (68q) and (68r) have slack since $v_{1,1} = d_{12} y_{1,1}$ and $v_{2,1} = d_{12} y_{2,1}$ and $1 \leqslant y_{1,1} + y_{2,1}$. We let $\Delta_{20} = (-d_{12},0,d_{12},0;-d_{12},0,d_{12},0;-1,0,1,0)$. If $x_{1,1} = d_{12} y_{1,1}$ and $x_{2,1} < d_{12} y_{2,1}$ then we obtain $d_1 < d_{12} y_{1,1} + d_1 y_{2,1} = v_{1,1} + d_1 y_{2,1}$ as $1 \leqslant y_{1,1} + y_{2,1}$. We assume that (68q) is tight as otherwise Δ_{20} is a feasible perturbation. It follows that $v_{1,1} + v_{2,1} = d_{12} y_{1,1} + d_1(1 - y_{1,1}) > d_1$. Moreover, $x_{1,1} + x_{2,1} = d_{12} y_{1,1} + d_1(1 - y_{1,1}) = d_1 + d_2 y_{1,1} < d_{12}$ and therefore $x_{1,2} + x_{2,2} > 0$. If $x_{1,2} > 0$ then $y_{1,2} = 1$. It follows that (68h) has slack since $v_{1,1} + v_{2,1} > d_1$. We let $\Delta_{21} = (-d_{12},d_2,d_1,0;-d_{12},d_2,d_1,0;-1,0,1,0)$. If $x_{2,2} > 0$ then $y_{2,2} = 1$. It follows that (68i) has slack since $v_{1,1} + v_{2,1} > d_1$. We let $\Delta_{21} = (-d_{12},0,d_1,d_2;-d_{12},0,d_1,d_2;-1,0,1,0)$. Finally, if both

(68h) and (68i) have slack, we let $\Delta_{22} = (-d_1, 0, d_1, 0; -d_1, 0, d_1, 0; -1, 0, 1, 0)$.

In the second subcase, we have that $0 < y_{1,2}, y_{2,2} < 1$. We create solutions by propagating initial perturbations on $y_{1,2}$ and $y_{2,2}$ to relevant variables. If (68h) has slack, then Δ_{13} is a feasible perturbation and if (68i) has slack, we let $\Delta_{23} = (\mathbf{0}; \mathbf{0}; e_4)$. Therefore we assume that (68h) and (68i) are tight. We let $\Delta_{24} = (0, -d_2, 0, d_2; 0, -d_2, 0, d_2; 0, -1, 0, 1)$.

In the third subcase, we have that $0 < y_{p,1}, y_{p,2} < 1$ for some $p \in \mathcal{P}$. Without loss of generality, we assume that $p = 1$. Because of (68q) we have $v_{2,1} > 0$, implying in turn that $x_{2,1} > 0$ and therefore $y_{2,1} = 1$. It follows that (68p) has slack. If (68h) has slack then Δ_{13} is a feasible perturbation. Therefore, we assume that (68h) is tight. If $x_{2,2} > 0$ then $y_{2,2} = 1$ and therefore (68i) has slack as $x_{1,2} > 0$. It follows that Δ_{14} is a feasible perturbation. We assume now that $x_{2,2} = 0$. If $x_{1,1} = 0$ then $v_{1,1} = 0$ which implies that (68f) and (68n) both have slack. This implies that $v_{2,1} \geqslant d_1$ and therefore (68q) has slack. It follows that Δ_1 is a feasible perturbation. Therefore, we assume $x_{1,1} > 0$ which in turn implies that (68g) has slack. If $v_{1,1} = 0$, we have that $x_{1,1} > v_{1,1}$ and $x_{1,1} \leqslant v_{1,1} + d_2 y_{1,1} = d_2 y_{1,1} < d_{12} y_{1,1}$. Moreover, (68q) has slack since $v_{2,1} \geqslant d_1 - v_{1,1} = d_1$. We let $\Delta_{25} = (-d_2, d_2, 0, 0; 0, 0, 0, 0; -1, 1, 0, 0)$. If $v_{1,1} > 0$, we have that $v_{1,1} + d_1 y_{2,1} = v_{1,1} + d_1 > d_1$. We consider two cases depending on whether (68a) is tight. If $v_{1,1} < x_{1,1}$, we let $\Delta_{26} = (d_{12}, -d_2, -d_1, 0; d_1, 0, -d_1, 0; 1, -1, 0, 0)$. If $x_{1,1} = v_{1,1}$ then $v_{1,1} + d_2 y_{1,1} > v_{1,1} = x_{1,1}$. There are two cases based on whether (68s) is tight. If $v_{1,1} + v_{2,1} > d_1$, we let $\Delta_{27} = (d_{12}, -d_2, -d_1, 0; d_{12}, -d_2, -d_1, 0; 1, -1, 0, 0)$. If $v_{1,1} + v_{2,1} = d_1$, we have that $x_{1,1} + x_{2,1} = d_{12} - (x_{1,2} + x_{2,2}) > d_1$ and therefore $x_{1,1} + x_{2,1} - (v_{1,1} + v_{2,1}) = x_{2,1} - v_{2,1} > 0$. Moreover, since $x_{1,2} > 0$ then $x_{2,1} - v_{2,1} < d_2 = d_2 y_{2,1}$ and therefore (68o) has slack. We let $\Delta_{28} = (0, -d_2, d_2, 0; 0, -d_2, 0, d_2; 0, -1, 0, 0)$.

In the third case, we assume that $y_{p,t}$ is fractional for three pairs of indices (p, t). There are two subcases based on the combinations of variables $y_{p,t}$ that are fractional.

In the first subcase, we assume that $0 < y_{1,1}, y_{1,2}, y_{2,1} < 1$. Constraints (68q) and (68r) show that $v_{1,1} > 0$ and $v_{2,1} > 0$, implying in turn that $x_{1,1} > 0$ and $x_{2,1} > 0$. We construct feasible solutions by propagating an initial perturbation on $y_{1,2}$ to related variables. In all of the following perturbations we maintain that $y_{1,1} + y_{2,1}$ does not change. Therefore we do not check whether (68p) is tight. If (68h) has slack then Δ_{13} is a feasible perturbation. Therefore we assume that (68h) is tight. If $x_{2,2} > 0$ then $y_{2,2} = 1$. It follows that (68i) has slack as $x_{1,2} > 0$. Then Δ_{14} is a feasible perturbation. On the other hand, if $x_{2,2} = 0$, we consider four cases based on whether the inequalities $x_{p,1} \leqslant v_{p,1}$ for $p \in \mathcal{P}$ are tight. If $x_{1,1} > v_{1,1}$ and $x_{2,1} > v_{2,1}$ then Δ_{17} is feasible. If $x_{1,1} > v_{1,1}$ and $x_{2,1} = v_{2,1}$ then $v_{2,1} + d_2 y_{2,1} > v_{2,1} = x_{2,1}$. There are two cases based on whether (68g) is tight. If $x_{2,1} = d_{12} y_{2,1}$ then we have that $v_{2,1} + v_{1,1} = d_{12} y_{2,1} + v_{1,1} \geqslant d_{12} y_{2,1} + d_1(1 - y_{2,1}) = d_1 + d_2 y_{2,1} > d_1$. Moreover, $v_{2,1} + d_1 y_{1,1} = d_{12} y_{2,1} + d_1 y_{1,1} > d_1$ because of (68p). Then Δ_{18} is feasible. If $x_{2,1} < d_{12} y_{2,1}$ then we let $\Delta_{29} = (-d_{12}, d_2, d_1, 0; -d_1, 0, d_1, 0; -1, 1, 1, 0)$. If $x_{1,1} = v_{1,1}$ and $x_{2,1} > v_{2,1}$ then $v_{1,1} + d_2 y_{1,1} > v_{1,1} = x_{1,1}$. There are two cases based on whether (68f) is tight. If $x_{1,1} = d_{12} y_{1,1}$ then we have that $v_{2,1} + v_{1,1} = v_{2,1} + d_{12} y_{1,1} \geqslant d_1(1 - y_{1,1}) + d_{12} y_{1,1} = d_1 + d_2 y_{1,1} > d_1$. Moreover, $v_{1,1} + d_1 y_{2,1} = d_{12} y_{1,1} + d_1 y_{2,1} > d_1$ because of (68p). Then we let $\Delta_{30} = (d_{12}, 0, -d_{12}, 0; d_{12}, 0, -d_1, -d_2; 1, 0, -1, 0)$. On the other hand, if $x_{1,1} < d_{12} y_{1,1}$ then we let $\Delta_{31} = (d_1, d_2, -d_{12}, 0; d_1, d_2, -d_1, -d_2; 1, 1, -1, 0)$. If $x_{1,1} = v_{1,1}$ and $x_{2,1} = v_{2,1}$ then we have that $v_{1,1} + d_2 y_{1,1} > v_{1,1} = x_{1,1}$ and similarly $v_{2,1} + d_2 y_{2,1} > x_{2,1}$. There are three cases based on whether the inequalities $x_{p,1} \leqslant d_{12} y_{p,1}$ for $p \in \mathcal{P}$ are tight. We observe that $x_{1,1} \leqslant d_{12} y_{1,1}$ and $x_{2,1} \leqslant d_{12} y_{2,1}$ can not be tight simultaneously since $x_{1,2}$ is positive. If $x_{1,1} = d_{12} y_{1,1}$ and $x_{2,1} < d_{12} y_{2,1}$ then $v_{1,1} + v_{2,1} = x_{1,1} + v_{2,1} \geqslant d_{12} y_{1,1} + d_1(1 - y_{1,1}) > d_1$ and $v_{1,1} + d_1 y_{2,1} = x_{1,1} + d_1 y_{2,1} = d_{12} y_{1,1} + d_1 y_{2,1} > d_1$ because of (68p). Then, we let $\Delta_{32} = (-d_{12}, d_2, d_1, 0; -d_{12}, d_2, d_1, 0; -1, 1, 1, 0)$. If $x_{1,1} < d_{12} y_{1,1}$ and $x_{2,1} = d_{12} y_{2,1}$ then $v_{1,1} + v_{2,1} = v_{1,1} + x_{2,1} \geqslant d_1(1 - y_{2,1}) + d_{12} y_{2,1} > d_1$ and $v_{2,1} + d_1 y_{1,1} = x_{2,1} + d_1 y_{1,1} = d_{12} y_{2,1} + d_1 y_{1,1} > d_1$ because of (68p). Then, we let $\Delta_{33} = (d_1, d_2, -d_{12}, 0; d_1, d_2, -d_{12}, 0; 1, 1, -1, 0)$.

Finally, if $x_{1,1} < d_{12}y_{1,1}$ and $x_{2,1} < d_{12}y_{2,1}$ then Δ_{22} is a feasible perturbation.

In the second subcase, we assume that $0 < y_{1,1}, y_{1,2}, y_{2,2} < 1$. We construct solutions by propagating an initial perturbation on $y_{1,2}$ to related variables. If (68h) or (68i) has slack then Δ_{13} and Δ_{23} are feasible perturbations, respectively. If on the other hand (68h) and (68i) are tight then Δ_{24} is a feasible perturbation.

Finally in the fourth case, we assume that all variables $y_{p,t}$ are fractional. In this case, the perturbations Δ_{13}, Δ_{23} and Δ_{24} used in the case where $0 < y_{1,1}, y_{1,2}, y_{2,2} < 1$ can also be used since they do not rely on the value of $y_{2,1}$.

This concludes the proof that if there exists an index pair (p, t) such that $0 < y_{p,t} < 1$ in a feasible solution of Q^*, then this solution can be represented as a convex combination of two other solutions Q^*. It follows that the extreme points $(x; v; y)$ of Q^* are such that $y \in \{0, 1\}^4$. $\square$

	$x_{1,1}$ =0	$x_{1,1} > 0$													
		$v_{1,1} = 0$				$v_{1,1} > 0$									
		$x_{1,2} > 0$	$x_{2,2} > 0$	$x_{1,2} = 0 = x_{2,2}$		$v_{1,1} < x_{1,1}$					$x_{1,1} = v_{1,1}$				
						$x_{1,2} > 0$	$x_{2,2} > 0$	$x_{1,2} = 0 = x_{2,2}$			$v_{2,1} + d_1 y_{1,1} > d_1$	$v_{2,1} + d_1 y_{1,1} = d_1$			
				$v_{1,1} + v_{2,1} = d_1$	$v_{1,1} + v_{2,1} > d_1$			$v_{1,1} + v_{2,1} = d_1$	$v_{1,1} + v_{2,1} > d_1$			$x_{1,1} < d_{12} y_{1,1}$	$x_{1,1} = d_{12} y_{1,1}$		
									$d_1 = v_{2,1} + d_1 y_{1,1}$	$d_1 < v_{2,1} + d_1 y_{1,1}$			$x_{1,2} > 0$	$x_{2,2} > 0$	$x_{1,2} = 0$ $x_{2,2} = 0$
$v_{1,1} \leqslant x_{1,1}$	↔	×	×	×	×	×	×	×	×	×	↔	↔	↔	↔	↔
$v_{2,1} \leqslant x_{2,1}$	↔	↔	↔	×	↔	↔	↔	×	×	↔	↔	↔	↔	↔	×
$x_{1,1} \leqslant d_{12} y_{1,1}$	×	×	×	×	×	↔	↔	↔	↔	↔	↔	×	↔	↔	↔
$x_{2,1} \leqslant d_{12} y_{2,1}$	↔	×	×	×	×	×	×	×	×	×	×	×	×	×	×
$x_{1,2} \leqslant d_2 y_{1,2}$	↔	×	↔	↔	↔	×	↔	↔	↔	↔	↔	↔	×	↔	↔
$x_{2,2} \leqslant d_2 y_{2,2}$	↔	↔	×	↔	↔	↔	×	↔	↔	↔	↔	↔	↔	×	↔
$x_{1,1} \leqslant v_{1,1} + d_2 y_{1,1}$	×	↔	↔	↔	↔	↔	↔	↔	↔	↔	×	×	×	×	×
$x_{2,1} \leqslant v_{2,1} + d_2 y_{2,1}$	↔	×	×	×	×	↔	↔	×	×	↔	↔	↔	↔	↔	×
$d_1 \leqslant v_{2,1} + d_1 y_{1,1}$	×	×	×	×	×	↔	↔	↔	↔	×	×	↔	↔	↔	↔
$d_1 \leqslant v_{1,1} + d_1 y_{2,1}$	↔	↔	↔	↔	↔	×	×	×	×	×	×	×	×	×	×
$d_1 \leqslant v_{1,1} + v_{2,1}$	↔	↔	↔	↔	×	↔	↔	↔	×	×	↔	↔	×	×	×

Table 3: Case structure for $0 < y_{1,1} < 1$.