Solutions of a constrained Hermitian matrix-valued function optimization problem with applications

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Abstract. Let $f(X) = (XC + D)M(XC + D)^* - G$ be a given nonlinear Hermitian matrix-valued function with $M = M^*$ and $G = G^*$, and assume that the variable matrix X satisfies the consistent linear matrix equation $XA = B$. This paper shows how to characterize the semi-definiteness of $f(X)$ subject to all solutions of $XA = B$. As applications, a standard method is obtained for finding analytical solutions X_0 of $X_0 A = B$ such that the matrix inequality $f(X) \succcurlyeq f(X_0)$ or $f(X) \preccurlyeq f(X_0)$ holds for all solutions of $XA = B$. The whole work provides direct access, as a standard example, to a very simple algebraic treatment of the constrained Hermitian matrix-valued function and the corresponding semi-definiteness and optimization problems.

1 Introduction

Throughout this paper, $\mathbb{C}^{m \times n}$ stands for the collection of all $m \times n$ complex matrices, and $\mathbb{C}_H^m$ stands for the set of all $m \times m$ complex Hermitian matrices. The symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, the rank and the range (column space) of a matrix $A \in \mathbb{C}^{m \times n}$, respectively. I_m denotes the identity matrix of order m. The Moore–Penrose generalized inverse of A, denoted by $A^\dagger$, is defined to be the unique solution X satisfying the four matrix equations $AGA = A$, $GAG = G$, $(AG)^* = AG$, and $(GA)^* = GA$. Further, let E_A and F_A stand for $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$, which satisfy $E_A = F_{A^*}$ and $F_A = E_{A^*}$. The two symbols $i_+(A)$ and $i_-(A)$ for $A \in \mathbb{C}_H^m$, called the positive and negative inertias of A, stand for the number of the positive and negative eigenvalues of A counted with multiplicities, respectively, both which satisfy $r(A) = i_+(A) + i_-(A)$. For brief, we use $i_\pm(A)$ to denote the both numbers. Two $A, B \in \mathbb{C}_H^m$ are said to satisfy the inequalities $A \succ B$, $A \succcurlyeq B$, $A \prec B$, and $A \preccurlyeq B$ in the Löwner partial ordering if $A - B$ is positive definite, positive semi-definite, negative definite, and negative semi-definite respectively. It is well known that the Löwner partial ordering is a surprisingly strong and useful property on Hermitian matrices. For more issues about connections between the inertias and the Löwner partial ordering of Hermitian matrices, as well as specific applications of the matrix inertias and Löwner partial ordering in statistics; see, e.g., [13, 15].

The optimal rank and inertia problems of Hermitian matrix-valued functions are the problems of finding the largest and smallest ranks and inertias of the Hermitian matrix-valued functions over some feasible matrix sets. A matrix-valued function is a map between the two matrix spaces $\mathbb{C}^{m \times n}$ and $\mathbb{C}^{p \times q}$, which can generally be written as

$$Y = f(X) \ \text{ for } \ Y \in \mathbb{C}^{m \times n} \ \text{ and } \ X \in \mathbb{C}^{p \times q}, \tag{1.1}$$

or briefly, $f : \mathbb{C}^{p \times q} \to \mathbb{C}^{m \times n}$. Mappings between matrix spaces can be constructed arbitrarily from ordinary operations of given matrices and variable matrices, but linear and nonlinear Hermitian matrix-valued function with a single variable matrix were widely used and extensively studied from theoretical and applied points of view. One of the simplest forms of the nonlinear Hermitian matrix-valued function $f(X)$ in (1.1) is given by

$$\begin{aligned} f(X) &= (XC + D)M(XC + D)^* - G \\ &= XCMC^*X^* + XCMD^* + DMC^*X^* + DMD^* - G, \end{aligned} \tag{1.2}$$

where $C \in \mathbb{C}^{p \times m}$, $D \in \mathbb{C}^{n \times m}$, $G \in \mathbb{C}_H^n$, and $M \in \mathbb{C}_H^m$ are given, $X \in \mathbb{C}^{n \times p}$ is a variable matrix. If $n = 1$, then (1.2) becomes a scalar function for the row vector X. Further, we assume that the variable matrix $X \in \mathbb{C}^{n \times p}$ is the solution of a consistent linear matrix equation

$$XA = B, \tag{1.3}$$

where $A \in \mathbb{C}^{p \times q}$ and $B \in \mathbb{C}^{n \times q}$ are given. Eq. (1.2) subject to (1.3) becomes a constrained symmetric quadratic matrix-valued function if all matrices in (1.2) and (1.3) are replaced by real matrices. Nonlinear Hermitian matrix-valued functions with the form of $f(X)$ in (1.2) occur widely in matrix theory and

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applications, while many problems in matrix theory and applications can reduce to certain cases of (1.2) subject to (1.3) and their optimization problems. For example, the minimization of (1.2) subject to (1.3) in the Löwner partial ordering for the real matrix case and its applications in parametric quadratic programming and statistical analysis were approached in [3, 14]. Formulas for calculating the rank and inertia of a special case of $f_1(X) = XMX^* - G$ subject to (1.3) and their applications were given in [17]. It should be pointed out that the reduced form $f_1(X)$ and (1.2) are not necessarily equivalent, because we can choose $CMC^* = 0$ and $CMD^* \neq 0$. In this case, (1.2) is linear for both X and X^*, but $f_1(X)$ with $M \neq 0$ is always nonlinear for both X and X^*. Some recent work on the applications of minimization of (1.2) subject to (1.3) for the real matrix case in deriving best linear unbiased predictors/estimiators of all unknown parameters under linear random-effects models was given in [21, 22].

In order to establish a unified optimization theory of (1.2) subject to (1.3), this paper aims at solving the following three fundamental problems:

(I) Derive analytical formulas for calculating the maximum and minimum ranks and inertias of $f(X)$ in (1.2) when X runs over $\mathbb{C}^{n \times p}$.

(II) Establish necessary and sufficient conditions for

$$f(X) \succcurlyeq 0 \ \text{ or } \ f(X) \preccurlyeq 0 \ \text{ subject to } \ XA = B \tag{1.4}$$

to hold in the Löwner partial ordering, respectively;

(III) Give analytical solution X_0 of $XA = B$ such that

$$f(X) \succcurlyeq f(X_0) \ \text{ or } \ f(X) \preccurlyeq f(X_0) \ \text{ subject to } \ XA = B \tag{1.5}$$

to hold in the Löwner partial ordering, respectively.

To appreciate the importance of this research, it is helpful to consider a covariance problem in statistic inference. Let y be a random vector with expectation and covariance matrix as follows

$$E(y) = \mu, \ \ Cov(y) = \Sigma,$$

and let $\mathcal{S}$ be certain set consisting of linear estimators generated from y as follows

$$\mathcal{S} = \{Ly + b\}$$

where b is a given random or non-random vector, L is an arbitrary matrix satisfying certain restriction, say, $E(Ly + b) = 0$. A fundamental optimization problem on the given set $\mathcal{S}$ is to find $L_0 y + b \in \mathcal{S}$ that minimizes the objective covariance matrix of $Ly + b \in \mathcal{S}$ in the Löwner partial ordering, i.e., to find $L_0 y + b \in \mathcal{S}$ such that

$$Cov(L_0 y + b) \preccurlyeq Cov(Ly + b) \ \text{ holds for all } \ Ly + b \in \mathcal{S}.$$

The $Cov(Ly + b)$ is equivalent to a symmetric quadratic matrix-valued function for L of the form in (1.2). Analytical solutions to optimization problems in the Löwner sense have been the most desirable objects of study in both mathematics and applications. In particular, once analytical solution to (1.5) is obtained, we can use the solution as an effective tool in the derivation of exact algebraic expressions of the well-known best linear unbiased predictors/estimators of unknown parameters under linear regression models.

It should be pointed out that the best-known Lagrangian method is not available for solving (1.2) and (1.5), because the optimal criteria in (1.5) are defined from the Löwner partial ordering instead of scalar functions of matrices like traces or norms of matrices. In this instance, we can use matrix rank and inertia formulas instead of the Lagrangian method, and establish a standard algebraic process to solve (1.2) and (1.5).

We next present some known results on the solution of linear matrix equation, and matrix rank/inertia formulas.

Lemma 1.1 ([12]). *The linear matrix equation $AX = B$ is consistent if and only if $r[A, B] = r(A)$, or equivalently, $AA^\dagger B = B$. In this case, the general solution of $AX = B$ can be written in the following parametric form $X = A^\dagger B + (I - A^\dagger A)U$, where U is an arbitrary matrix.*

Lemma 1.2. *Let $A, B \in \mathbb{C}^{m \times n}$, or $A, B \in \mathbb{C}_{\mathrm{H}}^m$. Then, the following assertions hold.*

(a) *$A = B$ if and only if $r(A - B) = 0$.*

(b) *$A \succ B$ $(A \prec B)$ if and only if $i_+(A - B) = m$ $(i_-(A - B) = m)$.*

(c) $A \succcurlyeq B$ $(A \preccurlyeq B)$ if and only if $i_-(A - B) = 0$ $(i_+(A - B) = 0)$.

(d) *For two given Hermitian matrix sets $\mathcal{S}$ and $\mathcal{T}$, $A \succcurlyeq B$ $(A \preccurlyeq B)$ holds for all $A \in \mathcal{S}$ and $B \in \mathcal{T}$ if and only if $\max_{A \in \mathcal{S}, B \in \mathcal{T}} i_-(A - B) = 0$ $(\max_{A \in \mathcal{S}, B \in \mathcal{T}} i_+(A - B) = 0)$.*

The assertions in Lemma 1.2 directly follow from the definitions of rank/inertia, definiteness, and semi-definiteness of (Hermitian) matrices. These assertions show that if certain expansion formulas for calculating ranks/inertias of differences of (Hermitian) matrices are established, we can use them to characterize the corresponding matrix equalities and inequalities. This fact reflects without doubt the most exciting values of ranks/inertias in matrix analysis and applications, and thus it is really necessary to produce numerous matrix rank/inertia formulas from the theoretical and applied points of view.

Lemma 1.3 ([11]). *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, and $C \in \mathbb{C}^{l \times n}$. Then,*

$$r[A, \, B] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{1.6}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C F_A) = r(C) + r(A F_C), \tag{1.7}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{1.8}$$

Lemma 1.4 ([15]). *Let $A \in \mathbb{C}_{\mathrm{H}}^m$, $B \in \mathbb{C}_{\mathrm{H}}^n$, $Q \in \mathbb{C}^{m \times n}$, and assume that $P \in \mathbb{C}^{m \times m}$ is nonsingular. Then,*

$$i_\pm(PAP^*) = i_\pm(A) \quad \text{(Sylvester's law of inertia)}, \tag{1.9}$$

$$i_\pm(A^\dagger) = i_\pm(A), \quad i_\pm(-A) = i_\mp(A), \tag{1.10}$$

$$i_\pm \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_\pm(A) + i_\pm(B), \tag{1.11}$$

$$i_+ \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = i_- \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q). \tag{1.12}$$

Lemma 1.5 ([15]). *Let $A \in \mathbb{C}_{\mathrm{H}}^m$ and $B \in \mathbb{C}^{m \times n}$, and $C \in \mathbb{C}_{\mathrm{H}}^n$. Then,*

$$i_\pm \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} = r(B) + i_\pm(E_B A E_B), \tag{1.13}$$

$$i_\pm \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = i_\pm(A) + i_\pm \begin{bmatrix} 0 & E_A B \\ B^* E_A & C - B^* A^\dagger B \end{bmatrix}. \tag{1.14}$$

In particular,

$$A \succcurlyeq 0 \Rightarrow i_+ \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} = r[A, \, B] \text{ and } i_- \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} = r(B), \tag{1.15}$$

and

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \succcurlyeq 0 \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A), \ A \succcurlyeq 0, \text{ and } C - B^* A^\dagger B \succcurlyeq 0. \tag{1.16}$$

Lemma 1.6 ([18]). *Let $A \in \mathbb{C}_{\mathrm{H}}^m$, $B \in \mathbb{C}^{m \times n}$, and $C \in \mathbb{C}_{\mathrm{H}}^n$ be given, and $X \in \mathbb{C}^{n \times m}$ be a variable matrix. Then, the maximum partial inertias of $XAX^* + XB + B^*X^* + C$ are given by*

$$\max_{X \in \mathbb{C}^{n \times m}} i_\pm (XAX^* + XB + B^*X^* + C) = \min \left\{ n, \ i_\pm \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right\}. \tag{1.17}$$

Hence,

$$XAX^* + XB + B^*X^* + C \succcurlyeq 0 \text{ for all } X \in \mathbb{C}^{n \times m} \Leftrightarrow \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \succcurlyeq 0,$$

$$XAX^* + XB + B^*X^* + C \preccurlyeq 0 \text{ for all } X \in \mathbb{C}^{n \times m} \Leftrightarrow \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \preccurlyeq 0.$$

There do exist matrices $X \in \mathbb{C}^{n \times m}$ satisfying the two formulas in (1.17), while the constructions of the matrices were described or formulated in [18].

2 Semi-definiteness of Hermitian matrix-valued functions

We first establish two formulas for calculating the maximum partial inertias of $f(X)$ in (1.2).

Lemma 2.1. *Let $f(X)$ be as given in (1.2), and let $J = \begin{bmatrix} DMD^* - G & DMC^* \\ CMD^* & CMC^* \end{bmatrix}$. Then,*

$$\max_{X \in \mathbb{C}^{n \times p}} i_\pm [f(X)] = \min \{n, \quad i_\pm(J)\}. \tag{2.1}$$

Hence, the following results hold.

(a) $f(X) \succcurlyeq 0$ *holds for all* $X \in \mathbb{C}^{n \times p}$ *if and only if* $J \succcurlyeq 0$.

(b) $f(X) \preccurlyeq 0$ *holds for all* $X \in \mathbb{C}^{n \times p}$ *if and only if* $J \preccurlyeq 0$.

(c) $f(X) = 0$ *holds for all* $X \in \mathbb{C}^{n \times p}$ *if and only if* $J = 0$.

Proof. Expanding the $f(X)$ in (1.2) gives

$$
\begin{aligned}
f(X) &= (XC + D)M(XC + D)^* - G \\
&= XCMC^*X^* + XCMD^* + DMC^*X^* + DMD^* - G.
\end{aligned} \tag{2.2}
$$

Applying (1.17) to (2.2) yields (2.1). Note from Lemma 1.2(d) that $f(X) \succcurlyeq 0$ for all $X \in \mathbb{C}^{n \times p}$ is equivalent to $\max_{X \in \mathbb{C}^{n \times p}} i_- [f(X)] = i_-(J) = 0$, i.e., $J \succcurlyeq 0$ by Lemma 1.2(c), thus proving (a). Result (b) can be shown similarly. Combining (a) and (b) leads to (c). $\qquad \square$

From Lemma 2.1, we further obtain the following results on the maximum partial inertias and semi-definiteness of $f(X)$ in (1.2) subject to (1.3).

Theorem 2.2. *Let $f(X)$ be as given in (1.2), and assume that $XA = B$ is consistent. Also, let*

$$
J = \begin{bmatrix} DMD^* - G & DMC^* & B \\ CMD^* & CMC^* & -A \\ B^* & -A^* & 0 \end{bmatrix}.
$$

Then,

$$\max_{XA=B} i_\pm[f(X)] = \min \{n, \quad i_\pm(J) - r(A)\}. \tag{2.3}$$

Hence, the following results hold.

(a) $f(X) \succcurlyeq 0$ *holds for all solutions of* $XA = B$ *if and only if* $i_-(J) = r(A)$.

(b) $f(X) \preccurlyeq 0$ *holds for all solutions of* $XA = B$ *if and only if* $i_+(J) = r(A)$.

(c) $f(X) = 0$ *holds for all solutions of* $XA = B$ *if and only if* $r(J) = 2r(A)$.

Proof. From Lemma 1.1, $XA = B$ is consistent if and only if $BA^\dagger A = B$. In this case, the general solution is $X = BA^\dagger + UE_A$, where $U \in \mathbb{C}^{n \times p}$ is arbitrary. Substituting it into $f(X)$ in (1.2) gives

$$f(X) = \left(UE_AC + BA^\dagger C + D\right)M\left(UE_AC + BA^\dagger C + D\right)^* - G,$$

which is a new Hermitian matrix-valued function with respect to U. Thus, we obtain from (1.17) that

$$
\begin{aligned}
\max_{XA=B} i_\pm[f(X)] &= \max_{U \in \mathbb{C}^{n \times p}} i_\pm \left[(UE_AC + H)M(UE_AC + H)^* - G\right] \\
&= \min \left\{n, \quad i_\pm \begin{bmatrix} HMH^* - G & HMC^*E_A \\ E_ACMH^* & E_ACMC^*E_A \end{bmatrix}\right\},
\end{aligned} \tag{2.4}
$$

where $H = BA^\dagger C + D$. Applying (1.13) and simplifying by Lemma 1.4 to the block matrix in (2.4), we

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further obtain

$$
\begin{aligned}
i_\pm &\begin{bmatrix} HMH^* - G & HMC^*E_A \\ E_ACMH^* & E_ACMC^*E_A \end{bmatrix} \\
&= i_\pm \begin{bmatrix} (BA^\dagger C + D)M(BA^\dagger C + D)^* - G & (BA^\dagger C + D)MC^*E_A \\ E_ACM(BA^\dagger C + D)^* & E_ACMC^*E_A \end{bmatrix} \\
&= i_\pm \begin{bmatrix} (BA^\dagger C + D)M(BA^\dagger C + D)^* - G & BA^\dagger CMC^* + DMC^* & 0 \\ CMC^*(BA^\dagger)^* + CMD^* & CMC^* & A \\ 0 & A^* & 0 \end{bmatrix} - r(A) \\
&= i_\pm \begin{bmatrix} DMD^* - G & DMC^* & -B \\ CMD^* & CMC^* & A \\ -B^* & A^* & 0 \end{bmatrix} - r(A) \\
&= i_\pm \begin{bmatrix} DMD^* - G & DMC^* & B \\ CMD^* & CMC^* & -A \\ B^* & -A^* & 0 \end{bmatrix} - r(A) = i_\pm(J) - r(A). \tag{2.5}
\end{aligned}
$$

Substituting (2.5) into (2.4) gives (2.3). Setting the right-hand side of (2.3) equal to zero, we obtain the results in (a) and (b) by Lemma 1.2(d). Combining (a) and (b) and applying $i_\pm(J) \geqslant r(A)$, we obtain (c). $\qquad\square$

The previous results can also be applied to compare two Hermitian matrix-valued functions of the same size in the Löwner sense.

Corollary 2.3. *Let*

$$
\begin{aligned}
f_1(X) &= (XC_1 + D_1)M_1(XC_1 + D_1)^* - G_1, \\
f_2(X) &= (XC_2 + D_2)M_2(XC_2 + D_2)^* - G_2,
\end{aligned}
$$

where $C_1 \in \mathbb{C}^{p \times m_1}$, $C_2 \in \mathbb{C}^{p \times m_2}$, $D_1 \in \mathbb{C}^{n \times m_1}$, $D_2 \in \mathbb{C}^{n \times m_2}$, $G_1,\ G_2 \in \mathbb{C}_H^n$, $M_1 \in \mathbb{C}_H^{m_1}$ and $M_2 \in \mathbb{C}_H^{m_2}$ are given. Also, let

$$
J = \begin{bmatrix} D_1M_1D_1^* - D_2M_2D_2^* - G_1 + G_2 & D_1M_1C_1^* - D_2M_2C_2^* \\ C_1M_1D_1^* - C_2M_2D_2^* & C_1M_1C_1^* - C_2M_2C_2^* \end{bmatrix}.
$$

Then, the following results hold.

(a) $f_1(X) \succcurlyeq f_2(X)$ *holds for all $X \in \mathbb{C}^{n \times p}$ if and only if $J \succcurlyeq 0$.*

(b) $f_1(X) \preccurlyeq f_2(X)$ *holds for all $X \in \mathbb{C}^{n \times p}$ if and only if $J \preccurlyeq 0$.*

(c) $f_1(X) = f_2(X)$ *holds for all $X \in \mathbb{C}^{n \times p}$ if and only if $J = 0$.*

Proof. It is easy to see that the difference

$$
\begin{aligned}
&f_1(X) - f_2(X) \\
&= (XC_1 + D_1)M_1(XC_1 + D_1)^* - (XC_2 + D_2)M_2(XC_2 + D_2)^* - (G_1 - G_2) \\
&= [XC_1 + D_1,\ XC_2 + D_2] \begin{bmatrix} M_1 & 0 \\ 0 & -M_2 \end{bmatrix} \begin{bmatrix} (XC_1 + D_1)^* \\ (XC_2 + D_2)^* \end{bmatrix} - (G_1 - G_2) \\
&= (X[C_1,\ C_2] + [D_1,\ D_2]) \begin{bmatrix} M_1 & 0 \\ 0 & -M_2 \end{bmatrix} (X[C_1,\ C_2] + [D_1,\ D_2])^* - (G_1 - G_2)
\end{aligned}
$$

is a special form of $f(X)$ in (1.2) again. So that (a) and (b) follow from Lemma 2.1. Combining (a) and (b) leads to (c). $\qquad\square$

Corollary 2.4. *Let*

$$
\begin{aligned}
f_1(X_1) &= (X_1C_1 + D_1)M_1(X_1C_1 + D_1)^* - G_1, \\
f_2(X_2) &= (X_2C_2 + D_2)M_2(X_2C_2 + D_2)^* - G_2,
\end{aligned}
$$

where $C_1 \in \mathbb{C}^{p_1 \times m_1}$, $C_2 \in \mathbb{C}^{p_2 \times m_2}$, $D_1 \in \mathbb{C}^{n \times m_1}$, $D_2 \in \mathbb{C}^{n \times m_2}$, $G_1,\ G_2 \in \mathbb{C}_H^n$, $M_1 \in \mathbb{C}_H^{m_1}$, and $M_2 \in \mathbb{C}_H^{m_2}$ are given, and $X_1 \in \mathbb{C}^{n \times p_1}$ and $X_2 \in \mathbb{C}^{n \times p_2}$ are variable matrices. Also let

$$
J = \begin{bmatrix} D_1M_1D_1^* - D_2M_2D_2^* - G_1 + G_2 & D_1M_1C_1^* & -D_2M_2C_2^* \\ C_1M_1D_1^* & C_1M_1C_1^* & 0 \\ -C_2M_2D_2^* & 0 & -C_2M_2C_2^* \end{bmatrix}.
$$

Then, the following results hold.

(a) $f_1(X_1) \succcurlyeq f_2(X_2)$ holds for all $X_1 \in \mathbb{C}^{n \times p_1}$ and $X_2 \in \mathbb{C}^{n \times p_2}$ if and only if $J \succcurlyeq 0$.

(b) $f_1(X_1) \preccurlyeq f_2(X_2)$ holds for all $X_1 \in \mathbb{C}^{n \times p_1}$ and $X_2 \in \mathbb{C}^{n \times p_2}$ if and only if $J \preccurlyeq 0$.

(c) $f_1(X_1) = f_2(X_2)$ holds for all $X_1 \in \mathbb{C}^{n \times p_1}$ and $X_2 \in \mathbb{C}^{n \times p_2}$ if and only if $J = 0$.

Proof. Note that

$$
\begin{aligned}
f_1&(X_1) - f_2(X_2) \\
&= X_1 C_1 M_1 C_1^* X_1^* + X_1 C_1 M_1 D_1^* + D_1 M_1 C_1^* X_1^* + D_1 M_1 D_1^* - G_1 \\
&\quad - X_2 C_2 M_2 C_2^* X_2^* - X_2 C_2 M_2 D_2^* - D_2 M_2 C_2^* X_2^* - D_2 M_2 D_2^* + G_2 \\
&= [\, X_1,\ X_2 \,] \begin{bmatrix} C_1 M_1 C_1^* & 0 \\ 0 & -C_2 M_2 C_2^* \end{bmatrix} \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} + [\, X_1,\ X_2 \,] \begin{bmatrix} C_1 M_1 D_1^* \\ -C_2 M_2 D_2^* \end{bmatrix} \\
&\quad + [\, D_1 M_1 C_1^*,\ -D_2 M_2 C_2^* \,] \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} + D_1 M_1 D_1^* - D_2 M_2 D_2^* - G_1 + G_2.
\end{aligned}
$$

Applying Lemma 2.1 to this matrix-valued function yields (a) and (b). Combining (a) and (b) leads to (c). $\qquad\square$

Corollary 2.5. *Let $f_1(X)$ and $f_2(X)$ be as given in Corollary 2.3, and assume that $XA = B$ is consistent, where $A \in \mathbb{C}^{p \times q}$ and $B \in \mathbb{C}^{n \times q}$ are given. Also, let*

$$
J = \begin{bmatrix} D_1 M_1 D_1^* - D_2 M_2 D_2^* - G_1 + G_2 & D_1 M_1 C_1^* - D_2 M_2 C_2^* & B \\ C_1 M_1 D_1^* - C_2 M_2 D_2^* & C_1 M_1 C_1^* - C_2 M_2 C_2^* & -A \\ B^* & -A^* & 0 \end{bmatrix}.
$$

Then, the following results hold.

(a) $f_1(X) \succcurlyeq f_2(X)$ *holds for all solutions of $XA = B$ if and only if $i_-(J) = r(A)$.*

(b) $f_1(X) \preccurlyeq f_2(X)$ *holds for all solutions of $XA = B$ if and only if $i_+(J) = r(A)$.*

(c) $f_1(X) = f_2(X)$ *holds for all solutions of $XA = B$ if and only if $r(J) = 2r(A)$.*

Corollary 2.6. *Let*

$$
\begin{aligned}
f_1(X_1) &= (\, X_1 C_1 + D_1 \,) M_1 (\, X_1 C_1 + D_1 \,)^* - G_1 \quad \text{s.t.} \quad X_1 A_1 = B_1, \\
f_2(X_2) &= (\, X_2 C_2 + D_2 \,) M_2 (\, X_2 C_2 + D_2 \,)^* - G_2 \quad \text{s.t.} \quad X_2 A_2 = B_2,
\end{aligned}
$$

and assume that both $X_1 A_1 = B_1$ and $X_2 A_2 = B_2$ are consistent, respectively, where $A_1 \in \mathbb{C}^{p_1 \times q_1}$, $A_2 \in \mathbb{C}^{p_2 \times q_2}$, $B_1 \in \mathbb{C}^{n \times q_1}$, $B_2 \in \mathbb{C}^{n \times q_2}$, $C_1 \in \mathbb{C}^{p_1 \times m_1}$, $C_2 \in \mathbb{C}^{p_2 \times m_2}$, $D_1 \in \mathbb{C}^{n \times m_1}$, $D_2 \in \mathbb{C}^{n \times m_2}$, $G_1, G_2 \in \mathbb{C}_H^n$, $M_1 \in \mathbb{C}_H^{m_1}$, and $M_2 \in \mathbb{C}_H^{m_2}$ are given, and $X_1 \in \mathbb{C}^{n \times p_1}$ and $X_2 \in \mathbb{C}^{n \times p_2}$ are variable matrices. Also, let

$$
J = \begin{bmatrix} D_1 M_1 D_1^* - D_2 M_2 D_2^* - G_1 + G_2 & D_1 M_1 C_1^* & -D_2 M_2 C_2^* & B_1 & B_2 \\ C_1 M_1 D_1^* & C_1 M_1 C_1^* & 0 & -A_1 & 0 \\ -C_2 M_2 D_2^* & 0 & -C_2 M_2 C_2^* & 0 & -A_2 \\ B_1^* & -A_1^* & 0 & 0 & 0 \\ B_2^* & 0 & -A_2^* & 0 & 0 \end{bmatrix}.
$$

Then, the following results hold.

(a) $f_1(X_1) \succcurlyeq f_2(X_2)$ *holds for all solutions of $X_1 A_1 = B_1$ and $X_2 A_2 = B_2$ if and only if $i_-(J) = r(A_1) + r(A_2)$.*

(b) $f_1(X_1) \preccurlyeq f_2(X_2)$ *holds for all all solutions of $X_1 A_1 = B_1$ and $X_2 A_2 = B_2$ if and only if $i_+(J) = r(A_1) + r(A_2)$.*

(c) $f_1(X_1) = f_2(X_2)$ *holds for all all solutions of $X_1 A_1 = B_1$ and $X_2 A_2 = B_2$ if and only if $r(J) = 2r(A_1) + 2r(A_2)$.*

The previous results can be applied to the perturbation analysis of $f(X)$ in (1.2). For instance, let

$$
\begin{aligned}
f_1(X) &= (XC + D)M(XC + D)^* - G, \\
f_2(X) &= [X(C + \delta C) + (D + \delta D)](M + \delta M)[X(C + \delta C) + (D + \delta D)]^* \\
&\quad - (G + \delta G),
\end{aligned}
$$

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or

$$f_1(X) = (XC_1 + D_1)M_1(XC_1 + D_1)^* - G_1,$$

$$f_2(X) = (X[\,C_1,\,C_2\,] + [\,D_1,\,D_2\,]) \begin{bmatrix} M_1 & N \\ N^* & M_2 \end{bmatrix} (X[\,C_1,\,C_2\,] + [\,D_1,\,D_2\,])^* - G_2.$$

Then, equalities and inequalities between $f_1(X)$ and $f_2(X)$ can be established from Corollary 2.3(a)–(c).

3 Solutions of some optimization problems

In light of the results in Section 2, we are able to derive exact algebraic solutions to (1.5).

Theorem 3.1. *Let $f(X)$ be as given in (1.2). Then, the following results hold.*

(a) *There exists an $X_0 \in \mathbb{C}^{n \times p}$ such that*

$$f(X) \succcurlyeq f(X_0) \tag{3.1}$$

holds for all $X \in \mathbb{C}^{n \times p}$ if and only if

$$CMC^* \succcurlyeq 0 \quad and \quad \mathscr{R}(CMD^*) \subseteq \mathscr{R}(CMC^*). \tag{3.2}$$

In this case, $f(X)$ can be decomposed as

$$f(X) = (XCMC^* + DMC^*)(CMC^*)^\dagger (XCMC^* + DMC^*)^* \\ + DMD^* - DMC^*(CMC^*)^\dagger CMD^* - G. \tag{3.3}$$

The matrix X_0 satisfying (3.1) is determined by the following consistent matrix equation

$$X_0 CMC^* + DMC^* = 0, \tag{3.4}$$

and the general expression of X_0 and the corresponding minimum $f(X_0)$ are given by

$$X_0 = \operatorname*{arg\,min}_{X \in \mathbb{C}^{n \times p}} f(X) = -DMC^*(CMC^*)^\dagger + V[I_p - (CMC^*)(CMC^*)^\dagger], \tag{3.5}$$

$$f(X_0) = \min_{X \in \mathbb{C}^{n \times p}} f(X) = DMD^* - DMC^*(CMC^*)^\dagger CMD^* - G, \tag{3.6}$$

where $V \in \mathbb{C}^{n \times p}$ is arbitrary. In particular, X_0 satisfying (3.1) is unique if and only if $r(CMC^*) = p$. In this case,

$$f(X) = (XCMC^* + DMC^*)(CMC^*)^{-1}(XCMC^* + DMC^*)^* \\ + DMD^* - DMC^*(CMC^*)^{-1}CMD^* - G, \tag{3.7}$$

and

$$\operatorname*{arg\,min}_{X \in \mathbb{C}^{n \times p}} f(X) = -DMC^*(CMC^*)^{-1}, \tag{3.8}$$

$$\min_{X \in \mathbb{C}^{n \times p}} f(X) = DMD^* - DMC^*(CMC^*)^{-1}CMD^* - G. \tag{3.9}$$

(b) *There exists an $X_0 \in \mathbb{C}^{n \times p}$ such that*

$$f(X) \preccurlyeq f(X_0) \tag{3.10}$$

holds for all $X \in \mathbb{C}^{n \times p}$ if and only if

$$CMC^* \preccurlyeq 0 \quad and \quad \mathscr{R}(CMD^*) \subseteq \mathscr{R}(CMC^*). \tag{3.11}$$

In this case, $f(X)$ can be decomposed as

$$f(X) = (XCMC^* + DMC^*)(CMC^*)^\dagger (XCMC^* + DMC^*)^* \\ + DMD^* - DMC^*(CMC^*)^\dagger CMD^* - G. \tag{3.12}$$

The matrix X_0 satisfying (3.10) is determined by the following consistent matrix equation

$$X_0 CMC^* + DMC^* = 0, \tag{3.13}$$

and the general expression of X_0 and the corresponding $f(X_0)$ are given by

$$X_0 = \underset{X \in \mathbb{C}^{n \times p}}{\arg\max} f(X) = -DMC^*(CMC^*)^\dagger + V\left[I_p - (CMC^*)(CMC^*)^\dagger\right], \tag{3.14}$$

$$f(X_0) = \max_{X \in \mathbb{C}^{n \times p}} f(X) = DMD^* - DMC^*(CMC^*)^\dagger CMD^* - G, \tag{3.15}$$

where $V \in \mathbb{C}^{n \times p}$ is arbitrary. In particular, X_0 satisfying (3.10) is unique if and only if $r(CMC^*) = p$. In this case,

$$f(X) = (XCMC^* + DMC^*)(CMC^*)^{-1}(XCMC^* + DMC^*)^*$$
$$+ DMD^* - DMC^*(CMC^*)^{-1}CMD^* - G, \tag{3.16}$$

and

$$\underset{X \in \mathbb{C}^{n \times p}}{\arg\max} f(X) = -DMC^*(CMC^*)^{-1}, \tag{3.17}$$

$$\max_{X \in \mathbb{C}^{n \times p}} f(X) = DMD^* - DMC^*(CMC^*)^{-1}CMD^* - G. \tag{3.18}$$

Proof. For any X, $X_0 \in \mathbb{C}^{n \times p}$, the difference $f(X) - f(X_0)$ is

$$f(X) - f(X_0) = (XC + D)M(XC + D)^* - (X_0C + D)M(X_0C + D)^*. \tag{3.19}$$

Applying (2.1) to (3.19) yields

$$\max_{X \in \mathbb{C}^{n \times p}} i_-\left[f(X) - f(X_0)\right]$$
$$= \min\left\{n, \quad i_-\begin{bmatrix} DMD^* - (X_0C + D)M(X_0C + D)^* & DMC^* \\ CMD^* & CMC^* \end{bmatrix}\right\}. \tag{3.20}$$

Setting the right-hand side of (3.20) equal to zero, we see from Lemma 1.2(d) that (3.1) holds if and only if

$$i_-\begin{bmatrix} DMD^* - (X_0C + D)M(X_0C + D)^* & DMC^* \\ CMD^* & CMC^* \end{bmatrix} = 0,$$

that is,

$$\begin{bmatrix} DMD^* - (X_0C + D)M(X_0C + D)^* & DMC^* \\ CMD^* & CMC^* \end{bmatrix} \succcurlyeq 0,$$

which, by (1.16), is equivalent to

$$CMC^* \succcurlyeq 0, \quad \mathscr{R}(CMD^*) \subseteq \mathscr{R}(CMC^*), \tag{3.21}$$

$$DMD^* - (X_0C + D)M(X_0C + D)^* - DMC^*(CMC^*)^\dagger CMD^* \succcurlyeq 0. \tag{3.22}$$

We obtain from (3.21) that $(CMC^*)^\dagger \succcurlyeq 0$, $(CMC^*)(CMC^*)^\dagger CMD^* = CMD^*$, and $DMC^*(CMC^*)^\dagger(CMC^*) = DMC^*$. So that it is easy to verify

$$(XCMC^* + DMC^*)(CMC^*)^\dagger(XCMC^* + DMC^*)^*$$
$$= XCMC^*X^* + XCMD^* + DMC^*X^* + DMC^*(CMC^*)^\dagger CMD^*$$
$$= (XC + D)M(XC + D)^* - DMD^* + DMC^*(CMC^*)^\dagger CMD^*.$$

Substituting this formula into (1.2) leads to (3.3). Since

$$(XCMC^* + DMC^*)(CMC^*)^\dagger(XCMC^* + DMC^*)^* \succcurlyeq 0$$

under (3.2), we see from (3.3) that

$$f(X) \succcurlyeq DMD^* - DMC^*(CMC^*)^\dagger CMD^* - G \tag{3.23}$$

for all $X \in \mathbb{C}^{n \times p}$. Setting the first term in (3.3) equal to null yields

$$(X_0CMC^* + DMC^*)(CMC^*)^\dagger(X_0CMC^* + DMC^*)^* = 0. \tag{3.24}$$

Applying a trivial fact $PQQ^*P^* = 0 \Leftrightarrow PQQ^* = 0 \Leftrightarrow PQ = 0$ and $(CMC^*)^\dagger \succcurlyeq 0$ to (3.24) yields $(X_0CMC^* + DMC^*)(CMC^*)^\dagger = 0$. Further, post-multiplying CMC^* to both sides of the equality and simplifying by the second condition in (3.21) leads to (3.4). Eq.(3.5) follows from Lemma 1.1, while (3.6) follows from (3.23). Result (b) can be shown by a similar approach. □

Theorem 3.2. *Let $f(X)$ be as given in (1.2), and assume that $XA = B$ is consistent. Also, let $K = \begin{bmatrix} CMC^* & A \\ A^* & 0 \end{bmatrix}$ and $H = BA^\dagger C + D$. Then, the following results hold.*

(a) *There exists a solution X_0 of $XA = B$ such that*

$$f(X) \succcurlyeq f(X_0) \tag{3.25}$$

holds for all solutions of $XA = B$ if and only if

$$E_A CMC^* E_A \succcurlyeq 0 \quad and \quad \mathscr{R}(E_A CMH^*) \subseteq \mathscr{R}(E_A CMC^* E_A), \tag{3.26}$$

or equivalently,

$$i_-(K) = r(A) \quad and \quad \mathscr{R}\begin{bmatrix} CMD^* \\ -B^* \end{bmatrix} \subseteq \mathscr{R}(K). \tag{3.27}$$

In this instance, $f(X)$ can be decomposed as

$$\begin{aligned}
&f(X) \\
&= (XCMC^* E_A + DMC^* E_A)(E_A CMC^* E_A)^\dagger (XCMC^* E_A + DMC^* E_A)^* \\
&\quad + HMH^* - HMC^*(E_A CMC^* E_A)^\dagger CMH^* - G.
\end{aligned} \tag{3.28}$$

The matrix X_0 satisfying (3.25) is determined by the following consistent matrix equation

$$X_0[\, A, \, CMC^* E_A \,] = [\, B, \, -DMC^* E_A \,]. \tag{3.29}$$

In this case,

$$\begin{aligned}
\operatorname*{argmin}_{XA=B} f(X) &= [\, B, \, -DMC^* E_A \,][\, A, \, CMC^* E_A \,]^\dagger \\
&\quad + V[\, I_p - [\, A, \, CMC^* E_A \,][\, A, \, CMC^* E_A \,]^\dagger \,], \tag{3.30} \\
\min_{XA=B} f(X) &= HMH^* - HMC^*(E_A CMC^* E_A)^\dagger CMH^* - G, \tag{3.31}
\end{aligned}$$

where $V \in \mathbb{C}^{n \times p}$ is arbitrary.

(b) *There exists a solution X_0 of $XA = B$ such that*

$$f(X) \preccurlyeq f(X_0) \tag{3.32}$$

holds for all solutions of $XA = B$ if and only if

$$E_A CMC^* E_A \preccurlyeq 0 \quad and \quad \mathscr{R}(E_A CMH^*) \subseteq \mathscr{R}(E_A CMC^* E_A), \tag{3.33}$$

or equivalently,

$$i_+(K) = r(A) \quad and \quad \mathscr{R}\begin{bmatrix} CMD^* \\ -B^* \end{bmatrix} \subseteq \mathscr{R}(K). \tag{3.34}$$

In this case, $f(X)$ can be decomposed as

$$\begin{aligned}
&f(X) \\
&= (XCMC^* E_A + DMC^* E_A)(E_A CMC^* E_A)^\dagger (XCMC^* E_A + DMC^* E_A)^* \\
&\quad + HMH^* - HMC^*(E_A CMC^* E_A)^\dagger CMH^* - G.
\end{aligned} \tag{3.35}$$

The matrix X_0 satisfying (3.32) is determined by the following consistent matrix equation

$$X_0[\, A, \, CMC^* E_A \,] = [\, B, \, -DMC^* E_A \,]. \tag{3.36}$$

In this case,

$$\begin{aligned}
\operatorname*{argmax}_{XA=B} f(X) &= [\, B, \, -DMC^* E_A \,][\, A, \, CMC^* E_A \,]^\dagger \\
&\quad + V[\, I_p - [\, A, \, CMC^* E_A \,][\, A, \, CMC^* E_A \,]^\dagger \,], \tag{3.37} \\
\max_{XA=B} f(X) &= HMH^* - HMC^*(E_A CMC^* E_A)^\dagger CMH^* - G, \tag{3.38}
\end{aligned}$$

where $V \in \mathbb{C}^{n \times p}$ is arbitrary.

Proof. Substituting the solution $X = BA^\dagger + UE_A$ of $XA = B$ into $f(X)$ in (1.2), we obtain

$$f(X) = (UE_AC + H)M(UE_AC + H)^* - G. \tag{3.39}$$

From Theorem 3.1, there exists a $U_0 \in \mathbb{C}^{n \times p}$ such that $f(X) \succcurlyeq f(X_0)$ holds for all $U \in \mathbb{C}^{n \times p}$ in (3.39) if and only if (3.26) holds. Note from (1.13) that

$$i_\pm(E_ACMC^*E_A) = i_\pm \begin{bmatrix} CMC^* & A \\ A^* & 0 \end{bmatrix} - r(A) = i_\pm(K) - r(A).$$

Hence, the first inequality in (3.26) is equivalent to the first inertia equality in (3.27). Applying (1.6)–(1.8) and simplifying, we obtain

$$r[\, E_ACMH^*, \, E_ACMC^*E_A \,] = r \begin{bmatrix} CM(BA^\dagger C + D)^* & CMC^* & A \\ 0 & A^* & 0 \end{bmatrix} - 2r(A)$$

$$= r \begin{bmatrix} CMD^* & CMC^* & A \\ -B^* & A^* & 0 \end{bmatrix} - 2r(A),$$

$$r(E_ACMC^*E_A) = r \begin{bmatrix} CMC^* & A \\ A^* & 0 \end{bmatrix} - 2r(A) = r(K) - 2r(A).$$

Hence, the second range inclusion in (3.27) is equivalent to

$$r \begin{bmatrix} CMD^* & CMC^* & A \\ -B^* & A^* & 0 \end{bmatrix} = r \begin{bmatrix} CMC^* & A \\ A^* & 0 \end{bmatrix},$$

that is, $\mathscr{R} \begin{bmatrix} CMD^* \\ -B^* \end{bmatrix} \subseteq \mathscr{R}(K)$, establishing the second range inclusion in (3.27). Under (3.26), (3.39) can be decomposed as

$$f(X)$$
$$= (UE_ACMC^*E_A + HMC^*E_A)(E_ACMC^*E_A)^\dagger (UE_ACMC^*E_A + HMC^*E_A)^*$$
$$+ HMH^* - HMC^*(E_ACMC^*E_A)^\dagger CMH^* - G.$$

Substituting $UE_A = X - BA^\dagger$ into the above function and simplifying yield (3.28). Setting the first term in (3.28) equal to zero leads to the linear matrix equation $XCMC^*E_A + DMC^*E_A = 0$. Combining this matrix equation with $XA = B$ yields (3.29). Result (b) can be shown similarly. $\square$

One of the useful consequences of Theorem 3.2 is given below.

Corollary 3.3. *Let $f(X)$ be as given in (1.2) with $M \succcurlyeq 0$, and assume that $XA = B$ is consistent. Then,*

$$X_0A = B \text{ and } f(X_0) = \min \Leftrightarrow X_0[\, A, \, CMC^*E_A \,] = [\, B, \, -DMC^*E_A \,],$$

where the matrix equation on the right-hand side is consistent as well. In this case, the general expression of X_0 and the corresponding $f(X_0)$ are given by

$$X_0 = \operatorname*{argmin}_{XA=B} f(X)$$
$$= [\, B, \, -DMC^*E_A \,][\, A, \, CMC^*E_A \,]^\dagger + V\left(I_p - [\, A, \, CMC^* \,][\, A, \, CMC^* \,]^\dagger\right),$$
$$f(X_0) = \min_{XA=B} f(X)$$
$$= (BA^\dagger C + D)[\, M - MC^*(E_ACMC^*E_A)^\dagger CM \,](BA^\dagger C + D)^* - G,$$

where $V \in \mathbb{C}^{n \times p}$ is arbitrary.

4 Conclusions

We studied a group of fundamental problems on ranks/inertias, equalities/inequalities, and maximization/miminization in the Löwner partial ordering of a Hermitian matrix-valued function subject to linear matrix equation by using pure algebraic operations of the given matrices in the function and restriction. These problems are clearly formulated in common matrix operations, and their solutions are presented in exact algebraic expressions, so that the results in the previous sections can be utilized as standard tools in solving many problems on Hermitian matrix-valued functions subject to linear matrix equation restrictions where the Lagrangian method is not available. It should be pointed out that all the conclusions in the previous sections are valid when replacing complex matrices with real matrices.

Linear and nonlinear complex Hermitian matrix-valued functions with single or more variable matrices can be formulated arbitrarily, and each of them is worth to investigate from theoretical and applied points of view. For instance, two extended forms of (1.2) with a single variable matrix X are given by

$$f(X) = (AXB + C)M(AXB + C)^* + D,$$
$$f(X) = DXAX^*D^* + DXB + B^*X^*D^* + C.$$

It is easy to figure out that any linear and nonlinear Hermitian matrix-valued functions have maximum/minimum possible ranks/inertias due to the finite nonnegative integer property of rank/inertia. Formulas for calculating the maximum/minimum ranks/inertias of the above two Hermitian matrix-valued functions and their applications in solving the semi-definiteness and optimization problems of the two matrix-valued functions can be established without much effort; see, e.g., [18, 23], while the corresponding results also have essential applications in statistical data analysis and inference, as well as other disciplines.

Rank/inertia of complex Hermitian (real symmetric) matrix are conceptual foundation in elementary linear algebra, and are the most significant finite nonnegative integers in reflecting intrinsic properties of matrices. There were many classic approaches on rank/inertia theory of complex Hermitian (real symmetric) matrices and their applications in the mathematical literature; see, e.g.,[1, 2, 4, 5, 6, 7, 8, 9]. Motivated by many requirements of establishing matrix equalities and inequalities, the present author reconsidered matrix rank/inertia and established in [10, 15, 16, 17, 18, 19, 23, 24] a variety of fundamental formulas for calculating maximum/minimum ranks/inertias of linear and nonlinear Hermitian matrix-valued functions. These formulas can be used to characterize various features and performances of Hermitian matrix-valued functions from many new aspects. Particularly, they can be used to derive exact algebraic solutions to the corresponding Hermitian matrix-valued function optimization problems in the Löwner partial ordering. In addition, the present author's work on matrix rank/inertia formulas also attracted much attention in the field of matrix analysis, and many results on combinations and extensions of the previous work on minmaxity of ranks/inertias of matrix-valued functions by other authors can be found in the literature. Generally speaking, we are now able to use matrix rank/inertia formulas to reveal many deep and fundamental properties of matrices and their operations, such as, establishing and simplifying various complicated matrix expressions, deriving matrix equalities/inequalities that involve generalized inverses of matrices, characterizing definiteness/semi-definiteness of Hermitian matrix-valued functions, and solving matrix-valued function optimization problems.

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