On the diameter of lattice polytopes

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December 24, 2015

Abstract

In this paper we show that the diameter of a d-dimensional lattice polytope in $[0,k]^n$ is at most $\left\lfloor \left(k - \frac{1}{2}\right) d \right\rfloor$. This result implies that the diameter of a d-dimensional half-integral polytope is at most $\left\lfloor \frac{3}{2} d \right\rfloor$. We also show that for half-integral polytopes the latter bound is tight for any d.

1 Introduction

The 1-*skeleton* of a polyhedron P is the graph whose nodes are the vertices of P, and that has an edge joining two nodes if and only if the corresponding vertices of P are adjacent on P. Given vertices u, v of P, the *distance* $\delta^P(u,v)$ between u and v is the length of a shortest path connecting u and v on the 1-skeleton of P. We may write $\delta(u,v)$ instead of $\delta^P(u,v)$ when the polyhedron we are referring to is clear from the context. The *diameter* $\delta(P)$ of P is the smallest number that bounds the distance between any pair of vertices of P.

In this paper, we investigate the diameter of *lattice polytopes*, i.e. polytopes whose vertices are integral. Lattice polytopes play a crucial role in discrete optimization and integer programming problems, where the variables are constrained to assume integer values. Our goal is to define a bound on the diameter of a lattice polytope P, that depends on the dimension of P and on the parameter $k = \max\{||x - y||_\infty : x, y \in P\}$, in order to apply such bound to classes of polytopes for which k is known to be small. A similar approach has been followed by Bonifas et al. [4], who showed that the diameter of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is bounded by a polynomial that depends on n and on the parameter Δ, defined as the largest absolute value of any sub-determinant of A. Note that, while Δ is related to the external description of P, k is related to its internal description. However, both Δ and k are in general not polynomial in n and in the number of the facet-defining inequalities of P.

For $k \in \mathbb{N}$, a $(0,k)$-*polytope* $P \subseteq \mathbb{R}^n$ is a lattice polytope contained in $[0,k]^n$. Naddef [10] showed that the diameter of a d-dimensional $(0,1)$-polytope is at most d, and this bound is tight for the hypercube $[0,1]^d$. Kleinschmidt and Onn [8] extended this result by proving that the diameter of a d-dimensional $(0,k)$-polytope cannot exceed kd. However, their bound is not tight for $k \geq 2$.

Our main contribution is establishing an upper bound for the diameter of a d-dimensional $(0,k)$-polytope, which refines the bound by Kleinschmidt and Onn.

Theorem 1. *For $k \geq 2$, the diameter of a d-dimensional $(0,k)$-polytope is at most $\left\lfloor \left(k - \frac{1}{2}\right) d \right\rfloor$.*

The proof of Theorem 1 is elementary, as it combines an induction argument with basic tools from linear programming and polyhedral theory. Our proof is also constructive, since it shows how to build a path between two given vertices of P, whose length does not exceed our bound.

For $(0,2)$-polytopes, we show that the upper bound given in Theorem 1 is tight for any d.

Corollary 2. *The diameter of a d-dimensional $(0,2)$-polytope is at most $\left\lfloor \frac{3}{2} d \right\rfloor$. Moreover, for any natural number d, there exists a d-dimensional $(0,2)$-polytope attaining this bound.*

The lower bound of Corollary 2 follows by an easy construction based on the cartesian product of polytopes of dimension one and two. It is well-known that, given two polytopes P_1 and P_2, their cartesian product $P_1 \times P_2$ satisfies $\delta(P_1 \times P_2) = \delta(P_1) + \delta(P_2)$. Now, let $H_1 = [0,2]$ and $H_2 = \text{conv}\{(0,0),(1,0),(0,1),(2,1),(1,2),(2,2)\}$. For even d, let $H_d = (H_2)^{d/2}$, and for odd d, let $H_d = H_{d-1} \times H_1$. Thus for all $d \in \mathbb{N}$, H_d is a d-dimensional $(0,2)$-polytope, with $\delta(H_d) = \left\lfloor \frac{3}{2} d \right\rfloor$.

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Corollary 2 has important implications for the diameter of half-integral polytopes. *Half-integral polytopes* are polytopes whose vertices have components in $\{0, \frac{1}{2}, 1\}$, and they are affinely equivalent to $(0,2)$-polytopes. The class of half-integral polytopes is very rich, as many half-integral polytopes appear in the literature as relaxations of $(0,1)$-polytopes arising from combinatorial optimization problems. In some cases, while the $(0,1)$-polytope defined as the convex hull of the feasible solutions to the combinatorial problem has exponentially many facets, there is a linear relaxation, defined by a polynomial number of constraints, that yields a half-integral polytope.

There are several classes of polytopes that are known to be half-integral, such as the fractional matching polytope and the fractional stable set polytope [2], the linear relaxation of the boolean quadric polytope and the rooted semimetric polytope [12] (see also [14] and [9]). An interesting class of half-integral polytopes arises from totally dual half-integral systems, such as the fractional stable matching polytope [1, 6], and the fractional matroid matching polytope [13, 7].

The rest of the paper is devoted to proving Theorem 1.

2 Proof of main result

In order to bound the diameter of a non full-dimensional $(0,k)$-polytope $P \subseteq \mathbb{R}^n$, we define the *projection of P onto the i-coordinate hyperplane* as the polytope

$$\{\bar{x} \in \mathbb{R}^{n-1} : \exists\, x \in P \text{ with } x_j = \bar{x}_j \text{ for } j = 1, \dots, i-1,\ x_j = \bar{x}_{j-1} \text{ for } j = i+1, \dots, n\}.$$

That is, we simply drop the i-th coordinate from all vectors in P. Since integral vectors are mapped into integral vectors, the next lemma follows from Theorem 3.3 in [11].

Lemma 1. *Let P be a d-dimensional $(0,k)$-polytope in $\mathbb{R}^n$ with $d \geq 1$. Then there exists a full-dimensional $(0,k)$-polytope in $\mathbb{R}^d$ with the same 1-skeleton as P.*

For $d, k \in \mathbb{N}$, we define δ_k^d to be the maximum possible diameter of a $(0,k)$-polytope of dimension at most d, i.e.

$$\delta_k^d = \max\{\delta(P) : P \text{ is a } (0,k)\text{-polytope of dimension at most } d\}.$$

Note that the maximum in the definition of δ_k^d always exists. In fact, it follows from Lemma 1 that the number of vertices of a d-dimensional $(0,k)$-polytope is at most $(k+1)^d$, thus also its diameter is upper bounded by $(k+1)^d$, which is a number independent on the dimension of the ambient space of P. Moreover, for fixed k, the value δ_k^d is clearly non-decreasing in d.

We now present some lemmas that will be used to prove Theorem 1. These results follow by applying the ideas introduced by Kleinschmidt and Onn in [8]. The next lemma shows how to bound the distance $\delta(u, F)$ between a vertex u of a lattice polytope P and a face F of P, that is defined as $\delta(u, F) = \min\{\delta(u, v) : v \text{ is a vertex of } F\}$. We say that two vertices u, v of a polytope are *neighbors* if $\delta(u, v) = 1$. We denote by e^i, for $i = 1, \dots, n$, the i-th vector of the standard basis of $\mathbb{R}^n$.

Lemma 2. *Let P be a lattice polytope, and let u be a vertex of P. Let c be an integral vector, $\gamma = \min\{cx : x \in P\}$, and $F = \{x \in P : cx = \gamma\}$. Then $\delta(u, F) \leq cu - \gamma$.*

Proof. We show that there exists a vertex v of F such that $\delta(u, v) \leq cu - \gamma$. We prove this statement by induction on the integer value $cu - \gamma \geq 0$. The statement is trivial for $cu - \gamma = 0$, as we can set $v = u$. Assume $cu - \gamma \geq 1$. Since F is nonempty, there exists a neighbor u' of u with $cu' < cu$ (see, e.g., [5]). The integrality of c, u' and u, implies $cu' \leq cu - 1$. As $cu' - \gamma \leq cu - \gamma - 1$, by the induction hypothesis there exists a vertex v of F such that $\delta(u', v) \leq cu' - \gamma$. Therefore $\delta(u, v) \leq \delta(u, u') + \delta(u', v) \leq 1 + cu' - \gamma \leq cu - \gamma$. □

Given two vertices u and v and a face F of a lattice polytope P, we have $\delta(u, v) \leq \delta(u, F) + \delta(v, F) + \delta(F)$. By applying Lemma 2 to both u and v, we obtain an upper bound on $\delta(u, v)$ that depends on F:

Lemma 3. *Let P be a lattice polytope, and let u, v be vertices of P. Let c be an integral vector, $\gamma = \min\{cx : x \in P\}$, and $F = \{x \in P : cx = \gamma\}$. Then $\delta(u, v) \leq \delta(F) + cu + cv - 2\gamma$.*

Let P be a $(0,k)$-polytope in $\mathbb{R}^n$ and let $l = \min\{x_i : x \in P\}$ and $h = \max\{x_i : x \in P\}$ for some $i \in \{1, \dots, n\}$. We can bound the distance between any two vertices u and v of P by bounding their distances from the faces $L = \{x \in P : x_i = l\}$ and $H = \{x \in P : x_i = h\}$. If $u_i + v_i \leq l + h$, Lemma 3 applied with $F = L$, $c = e^i$ and $\gamma = l$ implies $\delta(u, v) \leq \delta(L) + (h - l)$. If $u_i + v_i \geq l + h$, Lemma 3 applied with $F = H$, $c = -e^i$ and $\gamma = -h$ implies $\delta(u, v) \leq \delta(H) + (h - l)$. Since L and H are $(0,k)$-polytopes of dimension at most $n-1$, we have that both $\delta(L)$ and $\delta(H)$ are at most δ_k^{n-1}.

Lemma 4. *Let P be a $(0,k)$-polytope in $\mathbb{R}^n$, and suppose that there exists $i \in \{1,\dots,n\}$ such that $x_i \in [l,h]$ for every $x \in P$. Then $\delta(P) \le \delta_k^{n-1} + (h-l)$.*

Given a d-dimensional $(0,k)$-polytope P, Kleinschmidt and Onn prove the bound $\delta(P) \le kd$ by essentially applying Lemma 1, and then Lemma 4 inductively. Therefore, their bound uses Lemma 2 only with vectors $c = \pm e^i$. To prove our refined bound, we will use Lemma 2 also with different vectors c. We are now ready to give the proof of Theorem 1.

Proof of Theorem 1. Let P be a d-dimensional $(0,k)$-polytope, with $k \ge 2$. The proof is by induction on d. The base cases are $d = 0$ and $d = 1$. The diameter of a 0-dimensional polytope is clearly zero, and the diameter of a 1-dimensional polytope is at most one, thus also bounded by $\lfloor k - \frac{1}{2} \rfloor = k-1$ since $k \ge 2$.

We now assume $d \ge 2$. Let u, v be vertices of P. By the induction hypothesis we assume that Theorem 1 is true for $(0,k)$-polytopes of dimension at most $d-1$. In particular, $\delta_k^{d-1} \le \lfloor (k-\frac{1}{2})(d-1) \rfloor$, and $\delta_k^{d-2} \le \lfloor (k-\frac{1}{2})(d-2) \rfloor$. Thus, in order to prove the inductive step, it is sufficient to show one of the following two inequalities:

$$\delta(u,v) \le \delta_k^{d-1} + k - 1, \tag{1}$$

$$\delta(u,v) \le \delta_k^{d-2} + 2k - 1. \tag{2}$$

Claim 1 *We can assume that P is full-dimensional.*

Proof of claim. By Lemma 1, there exists a full-dimensional $(0,k)$-polytope in $\mathbb{R}^d$ with the same 1-skeleton as P. $\diamond$

Claim 2 *We can assume that P intersects all facets of the hypercube $[0,k]^d$.*

Proof of claim. If there exists a facet G of the hypercube $[0,k]^d$ with $P \cap G = \emptyset$, then let $i \in \{1,\dots,d\}$ be such that $l \le x_i \le h$, with $l \ge 1$ or $h \le k-1$. By Lemma 4, $\delta(u,v) \le \delta_k^{d-1} + k - 1$, i.e. (1) is satisfied. $\diamond$

In the remainder of the paper, we will denote by k^d the d-dimensional vector with all entries equal to k.

Claim 3 *We can assume that $u + v = k^d$.*

Proof of claim. If $u+v \ne k^d$, there exists an index $i \in \{1,\dots,d\}$ such that $u_i+v_i \le k-1$ or $u_i+v_i \ge k+1$. By Lemma 3 applied with $c = e^i$ or $c = -e^i$, respectively, we obtain $\delta(u,v) \le \delta(F)+k-1$, where F is the face of P that minimizes cx. As F is a $(0,k)$-polytope of dimension at most $d-1$, we have $\delta(F) \le \delta_k^{d-1}$, therefore $\delta(u,v) \le \delta_k^{d-1} + k - 1$, i.e. (1) is satisfied. $\diamond$

Claim 4 *We can assume that $u \in \{0,k\}^d$.*

Proof of claim. Assume that u has one component u_i, $i \in \{1,\dots,d\}$, with $1 \le u_i \le k-1$. In this case we show that (2) is satisfied. Since the set $\{x \in P : x_i = 0\}$ is nonempty, there exists a neighbor s of u with $s_i < u_i$ (see, e.g., [5]). By the integrality of s and u, this implies $s_i \le u_i - 1$. Symmetrically, since the set $\{x \in P : x_i = k\}$ is nonempty, u has a neighbor t with $t_i \ge u_i + 1$. If $s_j = t_j = u_j$ for all $j \in \{1,\dots,d\}$, $j \ne i$, then by setting $\lambda = \frac{t_i - u_i}{t_i - s_i}$ we have $\lambda s + (1-\lambda)t = u$, contradicting the fact that u is a vertex of P. Thus, there exists an index $j \in \{1,\dots,d\}$ with $j \ne i$ such that either $s_j \ne u_j$ or $t_j \ne u_j$. Therefore there exists a neighbor w of u such that $w_i \ne u_i$ and $w_j \ne u_j$, for distinct indices $i, j \in \{1,\dots,d\}$ (see Fig. 1(i)).

We assume without loss of generality that $w_i < u_i$ (if not, we can perform the change of variable $\tilde{x}_i = k - x_i$). Analogously, we assume $w_j < u_j$. As $u + v = k^d$, we have $w_i + w_j + v_i + v_j \le 2k-2$. Let $\gamma = \min\{x_i + x_j : x \in P\}$ and $F = \{x \in P : x_i + x_j = \gamma\}$. By Lemma 3 (with $c = e^i + e^j$), $\delta(w,v) \le \delta(F) + w_i + w_j + v_i + v_j - 2\gamma \le \delta(F) + 2k - 2 - 2\gamma$ (see Fig. 1(ii)).

We now show that $\delta(F) \le \delta_k^{d-2} + \gamma$. Let $\bar{F}$ be the projection of F onto the j-coordinate hyperplane. $\bar{F}$ is a $(0,k)$-polytope in $\mathbb{R}^{d-1}$ and, by Lemma 1, $\bar{F}$ has the same 1-skeleton of F. Note that, for any $x \in F$, $x_i = \gamma - x_j$ and $x_j \ge 0$ imply $x_i \le \gamma$. Therefore, $x_i \le \gamma$ for any $x \in \bar{F}$. Then, by Lemma 4, $\delta(\bar{F}) \le \delta_k^{d-2} + \gamma$, thus $\delta(F) \le \delta_k^{d-2} + \gamma$.

This implies $\delta(w,v) \le \delta_k^{d-2} + 2k - 2 - \gamma$ and, since $\gamma \ge 0$ and $\delta(u,w) = 1$, finally $\delta(u,v) \le \delta(u,w) + \delta(w,v) \le \delta_k^{d-2} + 2k - 1$, i.e. (2) is satisfied. $\diamond$

By possibly performing the change of variable $\tilde{x}_1 = k - x_1$, we can further assume without loss of generality that $u_1 = k$, and $v_1 = 0$.

Let F be the face of P defined by $F = \{x \in P : x_1 = 0\}$. F is a $(0, k)$-polytope of dimension at most $d-1$, thus $\delta(F) \leq \delta_k^{d-1}$. By Lemma 2 (with $c = e^1$), there exists a vertex u' of F such that $\delta(u, u') \leq k$. Observe that both u' and v lie in F and therefore $\delta(u', v) \leq \delta_k^{d-1}$.

If $u' = (0, u_2, \ldots, u_d)$, then u and u' are adjacent vertices of the hypercube $[0, k]^d$, implying that $\mathrm{conv}\{u, u'\}$ is an edge of $[0, k]^d$ (see Fig. 2(i)). As P is convex and it is contained in $[0, k]^d$, it follows that $\mathrm{conv}\{u, u'\}$ is also an edge of P. Therefore, $\delta(u, u') = 1$ and consequently $\delta(u, v) \leq \delta_k^{d-1} + 1$. As $k \geq 2$, it follows $\delta(u, v) \leq \delta_k^{d-1} + k - 1$, i.e. (1) is satisfied.

Thus we now assume $u' \neq (0, u_2, \ldots, u_d)$ (see Fig. 2(ii)). Then, there exists an index $i \in \{2, \ldots, d\}$ such that $u'_i + v_i \leq k - 1$ or $u'_i + v_i \geq k + 1$. We assume without loss of generality that $u'_i + v_i \leq k - 1$ (if not, we can perform the change of variable $\tilde{x}_i = k - x_i$). Let $\gamma = \min\{x_i : x \in F\}$, $F' = \{x \in F : x_i = \gamma\}$. F' is a $(0, k)$-polytope, and it has dimension at most $d - 2$ because it is contained in the intersection of the two linearly independent hyperplanes $\{x \in \mathbb{R}^d : x_1 = 0\}$ and $\{x \in \mathbb{R}^d : x_i = \gamma\}$. It follows that $\delta(F') \leq \delta_k^{d-2}$. Then, by applying Lemma 3 to the polytope F and the vertices u' and v, we have $\delta(u', v) \leq \delta(F') + u'_i + v_i \leq \delta_k^{d-2} + k - 1$. This implies $\delta(u, v) \leq \delta(u, u') + \delta(u', v) \leq \delta_k^{d-2} + 2k - 1$, i.e. (2) is satisfied. □

3 Further directions

Both our upper bound and the one by Kleinschmidt and Onn are not tight for $k \geq 3$. As an example, $\delta_3^2 = 4$, as the maximum diameter of a lattice polygon in $[0, 3]^2$ is realized by the octagon. It seems that our approach cannot be easily refined to obtain a tight upper bound for general k.

An interesting direction of research is to study the asymptotic behavior of the function δ_k^d. It is known that the maximum number of vertices of a 2-dimensional $(0, k)$-polytope is in $\Theta(k^{2/3})$ [3], which implies the asymptotically tight bound $\delta_k^2 \in \Theta(k^{2/3})$. Using cartesian products of polytopes, it follows that $\delta_k^d \in \Omega(k^{2/3} d)$. This provides an asymptotic lower bound on δ_k^d that is a fractional power with respect to k and linear in d. However, the best upper bound on δ_k^d is linear both in k and in d. In other words, there is still a significant gap between the lower and the upper bound.

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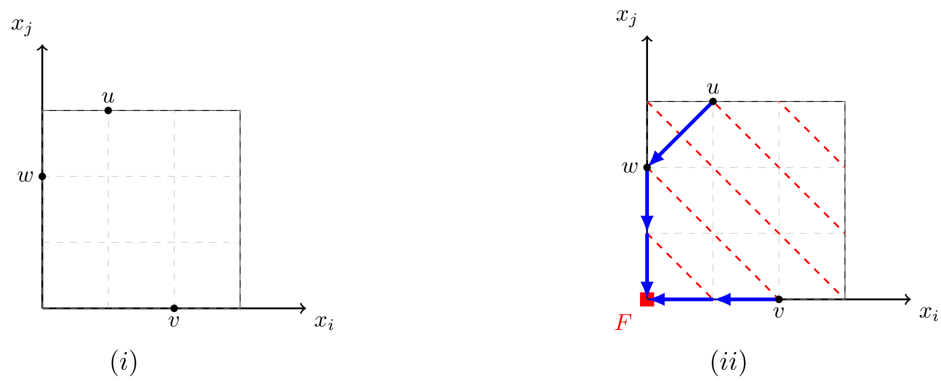

Figure 1: In Claim 4, (i) we construct a neighbor w of u with $w_i < u_i$, and $w_j < u_j$, (ii) we use Lemma 3 with $c = e^i + e^j$ to show that $\delta(w, v) \leq \delta_k^{d-2} + 2k - 2$.

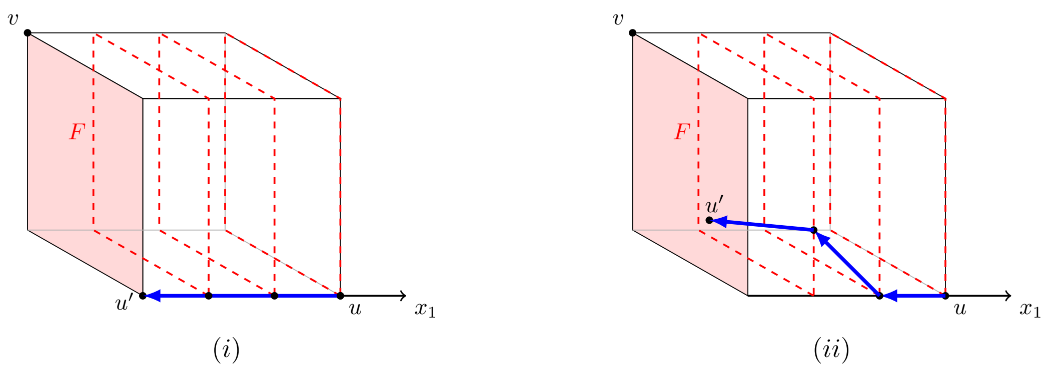

Figure 2: To bound the distance between vertices $u \in \{0, k\}^d$ with $u_1 = k$ and $v = k^d - u$, we construct a path from u to a vertex u' with $u'_1 = 0$. There are two cases: (i) $u' = (0, u_2, \ldots, u_d)$, thus $\delta(u, u') = 1$ and $\delta(u', v) \leq \delta_k^{d-1}$; (ii) $u' \neq (0, u_2, \ldots, u_d)$, thus $\delta(u, u') \leq k$ and $\delta(u', v) \leq \delta_k^{d-2} + k - 1$.