Euler Polytopes and Convex Matroid Optimization

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December 20, 2015

Abstract

Del Pia and Michini recently improved the upper bound of kd due to Kleinschmidt and Onn for the largest possible diameter of the convex hull of a set of points in dimension d whose coordinates are integers between 0 and k. We introduce *Euler polytopes* which include a family of lattice polytopes with diameter $(k+1)d/2$, and thus reduce the gap between the lower and upper bounds. In addition, we highlight connections between Euler polytopes and a parameter studied in convex matroid optimization and strengthen the lower and upper bounds for this parameter.

Keywords: Euler polytopes, convex matroid optimization, zonotopes, diameter of lattice polytopes

1 Introduction

The convex hull of integer-valued points is called a lattice polytope and, if all the vertices are drawn from $\{0, 1, \ldots, k\}^d$, it is refereed to as a lattice (d, k)-polytope. For simplicity, we only consider full dimensional lattice (d, k)-polytopes. Let $\delta(d, k)$ be the maximum possible edge-diameter over all lattice (d, k)-polytopes. Naddef [11] showed in 1989 that $\delta(d, 1) = d$, Kleinschmidt and Onn [9] generalized this result in 1992 showing that $\delta(d, k) \leq kd$, before Del Pia and Michini [3] recently strengthened the upper bound to $\delta(d, k) \leq kd - \lceil d/2 \rceil$ and showed that $\delta(d, 2) = \lfloor 3d/2 \rfloor$. Del Pia and Michini conclude their paper noting that the current lower bound for $\delta(d, k)$ is of order $k^{2/3}d$ and ask whether the gap between the lower and upper bounds could be closed, or at least reduced. The order $k^{2/3}d$ lower bound for $\delta(d, k)$ is a direct

1

consequence of the determination of $\delta(2,k)$ which was investigated independently in the early nineties by Thiele [13], Balog and Bárány [2], and Acketa and Žunić [1]. In Section 2, we introduce a family of zonotopes which includes lattice polytopes whose diameter achieves $\delta(2,k)$ in dimension 2 and achieves $(k+1)d/2$ for infinitely many d for each odd k. A lower bound of $kd/2 - k^2/4$ for $\delta(d,k)$ is derived for any d and k, reducing the gap between the lower and upper bounds for $\delta(d,k)$. We call *Euler polytopes* the introduced family of zonotopes due to the link with the Euler totient function described in Section 2.2. The Euler polytopes are also of interest for convex matroid optimization and, along with another family of lattice polytopes having a large number of vertices, we strengthen the established upper and lower bounds for a parameter used in convex matroid optimization in Section 3.

2 Lattice polytopes with large diameter

2.1 A family of lattice polytopes arising as zonotopes

Given a finite $X \subset \mathbb{R}^d$, let $\sum X = \sum\{x : x \in X\}$ with $\sum \emptyset = 0$, and let $zone(G) = \mathrm{conv}(\sum X : X \subseteq G)$ denote the zonotope generated by G. For $x \in \mathbb{Z}^d$, let $\gcd(x)$ be the largest integer dividing all entries of x, and let $\succ$ be the lexicographic order on $\mathbb{R}^d$ with $x \succ y$ if the first nonzero entry of $x - y$ is positive. For $q = \infty$ or a positive integer, and d,p positive integers, we consider the zonotope generated by the set $G_q(d,p) = \{x \in \mathbb{Z}^d : \|x\|_q \leq p, \ \gcd(x) = 1, \ x \succ 0\}$. The considered zonotope is denoted by $Z_q(d,p)$, and called an *Euler polytope* due to the link with the Euler totient function highlighted in Proposition 2.4.

Definition 2.1 (**Euler Polytopes**).
The Euler polytope $Z_q(d,p)$ of norm q, dimension d, and order p, is the zonotope:

$$Z_q(d,p) = \mathrm{zone}(\{x \in \mathbb{Z}^d : \|x\|_q \leq p, \ \gcd(x) = 1, \ x \succ 0\}).$$

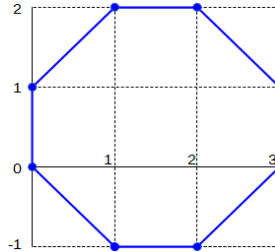To illustrate Definition 2.1, we list a few example of Euler polytopes:

Figure 1: $Z_1(2,2)$

(i) $G_1(2,2) = \{(0,1),(1,0),(1,1),(1,-1)\}$ and $Z_1(2,2)$ is the octagon whose vertices are $\{(0,0),(0,1),(1,2),(2,2),(3,1),(3,0),(2,-1),(1-1)\}$, and, up to translation, a lattice $(2,3)$-polytope, see Figure 1.

(ii) $Z_\infty(3,1)$ has 13 generators and is, up to translation, a lattice $(3,9)$-polytope, and thus has diameter 13 and 96 vertices. Note that $Z_\infty(3,1)$ is the *truncated small rhombicuboctahedron*, see Figure 2 for an illustration available on wikipedia [15], and is the Minkowski sum of a cube, a *truncated octahedron*, and a *rhombic dodecahedron*, see for instance Eppstein's webpage [4].
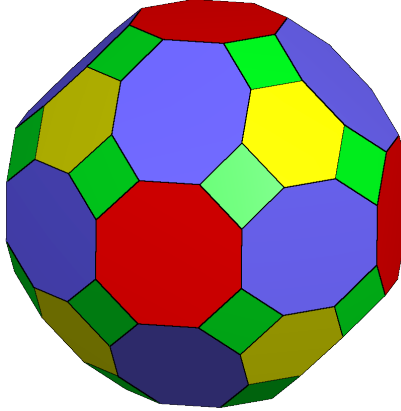

Figure 2: $Z_\infty(3,1)$ is homothetic to the truncated small rhombicuboctahedron

(iii) $Z_1(d,1)$ has d generators and is, up to translation, the d-dimensional $\{0,1\}$-cube; and thus has diameter d and 2^d vertices.

We recall in Proposition 2.2 some properties of zonotopes and refer to the books of Fukuda [5], Grünbaum [7], and Ziegler [16] for polytopes and, in particular, for zonotopes.

Proposition 2.2. *Let $\delta(P)$ and $f_0(P)$ denote respectively the diameter and the number of vertices of a polytope P, and let $m(Z)$ denotes the number of generators of a zonotope Z. The diameter of a zonotope Z satisfies $\delta(Z) \leq m(Z)$, and this inequality is satisfied with equality if no pair of generators of Z are linearly dependent. The number of vertices of a zonotope Z satisfies $f_0(Z) \leq 2 \sum_{i=0}^{i=d-1} \binom{d-1}{i}$.*

Note that since no pair of generators of $Z_q(d,p)$ are collinear, Proposition 2.2 implies that $\delta(Z_q(d,p)) = m(Z_q(d,p))$.

Observation 2.3. *Let Z be a zonotope generated by integer-valued generators m^j : $j = 1, \ldots, m(Z)$. The zonotope Z is, up to translation, a lattice (d, k)-polytope with*

$$k \leq \max_{i=1,2,\ldots,d} \sum_{j=1}^{j=m(Z)} |m_i^j|.$$

To illustrate the introduced Euler zonotopes, Table 1 provides the number of vertices of $Z_1(d, p)$ followed by its diameter $\delta(Z_1(d, p))$ and the k of this, up to translation, lattice (d, k)-polytope which were computed for small d and p. For instance, the entry $48\,(9, 5)$ for $(d, p) = (3, 2)$ in Table 1 indicates that $Z_1(3, 2)$ has 48 vertices, diameter 9, and is, up to translation, a lattice $(3, 5)$-polytope. Note that $Z_1(3, 2)$ is the *truncated cuboctahedron* – which is also called *great rhombicuboctahedron* – see Figure 3 for an illustration available on wikipedia [14], and is the Minkowski sum of an octahedron and a cuboctahedron, see for instance Eppstein's webpage [4].

	$Z_1(d, p)$	1	2	p 3	4
	2	4 (2,1)	8 (4,3)	16 (8,9)	24 (12,17)
d	3	8 (3,1)	48 (9,5)	336 (25,21)	1248 (49,53)
	4	16 (4,1)	384 (16,7)	15360 (56,37)	203904 (136,117)

Table 1: Number of vertices (diameter, integer range) of $Z_1(d, p)$
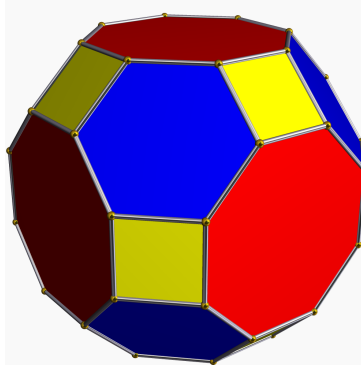

Figure 3: $Z_1(3, 2)$ is homothetic to the truncated cuboctahedron

2.2 Tighter bounds for diameter of lattice polytopes

Finding lattice polygons with the largest diameter; that is, to determine $\delta(2,k)$, was investigated independently in the early nineties by Thiele [13], Balog and Bárány [2], and Acketa and Žunić [1]. This question can be also found in Ziegler's book [16] as Exercise 4.15. This result is summarized in Proposition 2.4 which states that, up to translation, $Z_1(2,p)$ is a lattice $(2,k)$-polytope with $k = \sum_{n=1}^{n=p} n\phi(n)$ where $\phi(n)$ denotes the Euler totient function counting positive integers less of equal to n and relatively prime with n. Note that $\phi(1)$ is set to 1. In addition, Proposition 2.4 states that $\delta(2,k) = \delta(Z_1(2,p))$.

Proposition 2.4. *The zonotope $Z_1(2,p)$ is, up to translation, a lattice $(2,k)$-polytope with $k = \sum_{n=1}^{n=p} n\phi(n)$ where $\phi(n)$ denotes the Euler totient function. In addition, the diameter of $Z_1(2,p)$ is $2\sum_{n=1}^{n=p} \phi(n)$ and satisfies $\delta(Z_1(2,p)) = \delta(2,k)$. Thus, $\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3}\log k)$.*

Note that a lattice polygon is in bijection with a set of integer-valued vectors adding to zero and such that no pair of vectors are positive multiple of each other. Such set of vectors forms a $(2,k)$-polytope with $2k$ being the maximum between the sum of the norms of the first coordinates of the vectors and the sum of the norms of the second coordinates of the vectors. Then, for $k = \sum_{n=1}^{n=p} n\phi(n)$, one can show that $\delta(2,k)$ is achieved uniquely by a translation of $Z_1(2,p)$. For $k \neq \sum_{n=1}^{n=p} n\phi(n)$, $\delta(2,k)$ is achieved by the zonotope generated by the union of $G_1(2,p)$ and an appropriate subset of $G_1(2,p+1) \setminus G_1(2,p)$ for an appropriate p. For the order of $\sum n\phi(n)$ and $\sum \phi(n)$ we refer to [8]. The first values of $\delta(2,k)$ are given in Table 2.

p of $Z_1(2,p)$	1	2							3								4
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\delta(2,k)$	2	3	4	4	5	6	6	7	8	8	8	9	10	10	10	11	12

Table 2: Relation between $Z_1(2,p)$ and $\delta(2,k)$

While Proposition 2.4 states that $Z_1(2,p)$ maximizes the diameter among lattice polygons, Theorem 2.6, which is implied by Lemma 2.5, shows that $Z_1(d,2)$ yields a strengthening of the lower bound for $\delta(d,k)$.

Lemma 2.5. *The zonotope $Z_1(d,2)$ is, up to translation, a lattice $(d, 2d-1)$-polytope with diameter d^2. Thus, $\delta(d, 2d-1) \geq d^2$ for any d.*

Proof. We first note that the number of generators of $Z_1(d,2)$ is d^2. The generators of $Z_1(d,2)$ are $\{-1,0,1\}$-valued : d permutations of $(1,0,\ldots,0)$, $\binom{d}{2}$ permutations of $(1,1,0,\ldots,0)$, and $\binom{d}{2}$ permutations of $(1,-1,0,\ldots,0)$. Thus, by proposition 2.2,

$\delta(Z_1(d, 2)) = d^2$. As the sum of the the norm of the i-th coordinate of the d^2 generators of $Z_1(d, 2)$ is $2d - 1$, $Z_1(d, 2)$ is, up to translation, a lattice $(d, 2d - 1)$-polytope by Observation 2.3. $\qquad\square$

Theorem 2.6. *For positive integers d and k,*

   (i) $\delta(d, k) \geq (k + 1)d/2$ *for odd k and d a multiple of $(k + 1)/2$,*

  (ii) $\delta(d, k) > (k + 1)d/2 - (k + 1)^2/4$ *for odd k,*

 (iii) $\delta(d, k) > kd/2 - k^2/4$.

Proof. Given two polytopes $P^1 \subset \mathbb{R}^{d_1}$ and $P^2 \subset \mathbb{R}^{d_2}$, the cartesian product of P^1 and P^2 is the polytope $P^1 \times P^2 = \{(x, y) \in \mathbb{R}^{d_1 + d_2} : x \in P^1, y \in P^2\}$. One can observe that $P^1 \times P^2$ is a $(d_1 + d_2)$ dimensional polypope with diameter $\delta(P^1 \times P^2) = \delta(P^1) + \delta(P^2)$. In particular, the cartesian product of $Z_1(d, 2)$ by itself α times is a lattice $(\alpha d, 2d - 1)$-polytope with diameter αd^2. Thus, $\delta(\alpha d, 2d - 1) \geq \alpha d^2$ for any positive integers α and d; which implies item (i) by setting $2d - 1 = k$. Since $\delta(d, k)$ is nondecreasing with d and $\lfloor d/\alpha \rfloor \leq d/\alpha$ for any positive integer α, setting $\alpha = (k + 1)/2$ yields $\delta(d, k) \geq \delta(\lfloor 2d/(k + 1) \rfloor (k + 1)/2, k)$ for odd k. Applying item (i) gives $\delta(d, k) \geq \lfloor 2d/(k + 1) \rfloor (k + 1)^2/4$ for odd k. Applying $\lfloor d/\alpha \rfloor > d/\alpha - 1$ with $\alpha = (k + 1)/2$ yields item (ii). For even k, applying item (ii) for $k - 1$ and $\delta(d, k) \geq \delta(d, k - 1)$, gives $\delta(d, k) \geq \lfloor 2d/k \rfloor k^2/4$ for even k. Applying $\lfloor d/\alpha \rfloor > d/\alpha - 1$ with $\alpha = k/2$ yields item (iii) and completes the proof. $\qquad\square$

Note that considering cartesian products of polygons achieving $\delta(2, k)$ can slightly strengthen Theorem 2.6 but without improving the leading term $kd/2$.

As noted in the comments following Proposition 2.4, $\delta(2, k)$ is achieved by zonotopes. Similarly, the other known values of $\delta(d, k)$; that is, $\delta(1, k) = 1$, $\delta(d, 1) = d$, $\delta(d, 2) = \lfloor 3d/2 \rfloor$, and $\delta(3, 3) = 6$, are also achieved by zonotopes. In addition, remark that among all zonotopes which are lattice $(d, 2d - 1)$-polytopes, a translation of $Z_1(d, 2)$ maximizes the number of linearity independent generators. These observations motivate Conjecture 2.7.

Conjecture 2.7. *The largest diameter $\delta(d, k)$ among all lattice (d, k)-polytopes is achieved by a zonotope; $\delta(d, 2d - 1) = d^2$ and is uniquely achieved, up to translation, by $Z_1(d, 2)$; and $\delta(d, k) \leq \lfloor (k + 1)d/2 \rfloor$ for any d and k.*

3   Lattice polytopes with many vertices

3.1   Convex matroid optimization and Euler zonotopes

Call $S \subset \{0, 1\}^n$ a *matroid* if it is the set of the indicators of bases of a matroid over $\{1, \ldots, n\}$. For a $d \times n$ matrix W, let $WS = \{Wx : x \in S\}$, and let $\mathrm{conv}(WS) =$

$W\mathrm{conv}(S)$ be the projection to $\mathbb{R}^d$ of $\mathrm{conv}(S)$ by W. Given a convex function $f : \mathbb{R}^d \to \mathbb{R}$, *convex matroid optimization* deals with maximizing the composite function $f(Wx)$ over S; that is, $\max \{f(Wx) : x \in S\}$, and is concerned with the $\mathrm{conv}(WS)$; that is, the projection of the set of the feasible points. The maximization problem can also be interpreted as a problem of multicriteria optimization, where each row of W gives a linear criterion $W_i x$ and f compromises these criteria. Thus, W is called the *criteria* matrix or *weight* matrix. The projection polytope $\mathrm{conv}(WS)$ and its vertices play a key role in solving the maximization problem as, for any convex function f, there is an optimal solution x whose projection $y = Wx$ is a vertex of $\mathrm{conv}(WS)$. In particular, the enumeration of all vertices of $\mathrm{conv}(WS)$ enables to compute the optimal objective value by picking that vertex attaining the optimal value $f(y) = f(Wx)$. Thus, it suffices that f is presented by a *comparison oracle* that, queried on vectors $y, z \in \mathbb{R}^d$, asserts whether or not $f(y) < f(z)$. *Coarse* criteria matrices; that is, W whose entries are small or in $\{0, 1, \ldots, p\}$, are of particular interest. In multicriteria combinatorial optimization, this case corresponds to the weight $W_{i,j}$ attributed to element j of the ground set $\{1, \ldots, n\}$ under criterion i being a small or in $\{0, 1, \ldots, p\}$ for all i, j. In the reminder of Section 3, we only consider $\{0, 1, \ldots, p\}$-valued W. We refer to Melamed and Onn [10], and references therein, for convex integer optimization and, in particular, for convex matroid optimization.

The *normal cone* of a polytope $P \subset \mathbb{R}^n$ at its vertex v is the relatively open cone of the linear functions $h \in \mathbb{R}^n$ uniquely maximized over P at v. A polytope P is a *refinement* of a polytope Q if the normal cone of P at every vertex of P is contained in the normal cone of Q at some vertex of Q.

Proposition 3.1. *For positive integers d, p, n, a matroid $S \subset \{0, 1\}^n$, and a $d \times n$ criteria matrix W with entries in $\{0, 1, \ldots, p\}$, $Z_\infty(d, p)$ is a refinement of $\mathrm{conv}(WS)$. Thus, the maximum number $m(d, p)$ of vertices of $\mathrm{conv}(WS)$ is independent of n, S, and W. In addition, $m(d, p)$ is at most the number of vertices of $Z_\infty(d, p)$.*

Proof. For a matroid $S \subset \{0, 1\}^n$, any edge of $\mathrm{conv}(S)$ is parallel to the difference $\mathbf{1}_i - \mathbf{1}_j$ between a pair of unit vectors in $\mathbb{R}^n$, see [10, 12]. Therefore, any edge of the projection $\mathrm{conv}(WS)$ by W is parallel to the difference $W^i - W^j$ between a pair of columns of W which belongs to $\{0, \pm 1, \cdots \pm p\}^d$. Hence, the zonotope generated by $G = \{x \in \mathbb{Z}^d : \|x\|_\infty \le p\}$ is a refinement of $\mathrm{conv}(WS)$, see [6, 12]. Note that $G_\infty(d, p)$ is a maximal subset of G without pair of linearly dependent elements. Thus, $Z_\infty(d, p)$ is homothetic to $\mathrm{zone}(G)$ and hence a refinement of $\mathrm{conv}(WS)$. Therefore, the maximum number of vertices of $\mathrm{conv}(WS)$ is independent of n, S, and W. This number is denoted by $m(d, p)$ and is at most the number of vertices of $Z_\infty(d, p)$. $\square$

While Proposition 3.1 shows that the number of vertices of the zonotope generated by $G_\infty(d, p)$ provides an upper bound for $m(d, p)$, Proposition 3.2 shows that the number

of vertices of the zonotope $Z_\infty^+(d,p)$ generated by the nonnegative elements of $G_\infty(d,p)$ provides a lower bound for $m(d,p)$.

Let $G_q^+(d,p) = G_q(d,p) \cap \mathbb{Z}_+^d$ where $\mathbb{Z}_+ = \{0,1,\dots\}$, and let $Z_q^+(d,p) = \mathrm{zone}(G_q^+(d,p))$. For instance, $G_\infty^+(2,2) = \{(0,1),(1,0),(1,1),(1,2),(2,1)\}$ and $Z_\infty^+(2,2)$ is the decagon whose vertices are $\{(0,0),(0,1),(1,0),(1,3),(2,4),(3,1),(4,2),4,5),(5,4),(5,5)\}$, see Figure 4.
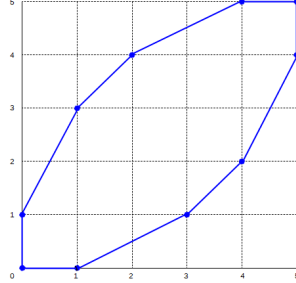

Figure 4: $Z_\infty^+(2,2)$

Proposition 3.2. *For positive integers d and p, there exist a positive integer n, a matroid $S \subset \{0,1\}^n$, and a $d \times n$ criteria matrix W with entries in $\{0,1,\dots,p\}$ such that $\mathrm{conv}(WS) = Z_\infty^+(d,p)$. Thus, the maximum number $m(d,p)$ of vertices of $\mathrm{conv}(WS)$ is at least the number of vertices of $Z_\infty^+(d,p)$.*

Proof. Let m denotes the number of vertices of $Z_\infty^+(d,p)$, and let W be the $d \times 2m$ matrix whose first m columns are the elements of $G_\infty^+(d,p)$, say, ordered lexicographically, and last m columns consist of zeros. Let S be the (set of indicators of bases of the) uniform matroid U_{2m}^m – that is, S consists of all vectors in $\{0,1\}^{2m}$ with exactly m zeros and m ones. One can easily check that $WS = \{\sum X : X \subseteq Z_\infty^+(d,p)\}$ and thus, $\mathrm{conv}(WS) = \mathrm{zone}(G_\infty^+(d,p)) = Z_\infty^+(d,p)$. $\square$

Theorem 3.3, which simply combines Propositions 3.1 and 3.2, highlights the roles played by Euler zonotopes to bound $m(d,p)$.

Theorem 3.3. *For positive integers d,p,n, a matroid $S \subset \{0,1\}^n$, and a $d \times n$ criteria matrix W with entries in $\{0,1,\dots,p\}$, the maximum number $m(d,p)$ of vertices of $\mathrm{conv}(WS)$ satisfies $f_0(Z_\infty^+(d,p)) \le m(d,p) \le f_0(Z_\infty(d,p))$.*

Tables 3 and 4 provide the number of vertices of $Z_\infty^+(d,p)$, respectively $Z_\infty(d,p)$, followed by the diameter and the k of these, up to translation, lattice (d,k)-polytopes for small d and p. For instance, the entry 96 $(13,9)$ for $(d,p) = (3,1)$ in Table 4 indicates that $Z_\infty(3,1)$ has 96 vertices, diameter 13, and is, up to translation, a lattice $(3,9)$-polytope.

$Z_\infty^+(d,p)$	p			
	1	2	3	4
2	6(3,2)	10 (5,5)	18 (9,14)	26(13,26)
d 3	32 (7,4)	212 (19,19)	1418 (49,76)	4916 (91,184)
4	370 (15,8)	27778 (65,65)	(225,344)	(529,1064)

Table 3: Number of vertices (diameter, integer range) of $Z_\infty^+(d,p)$

$Z_\infty(d,p)$	p			
	1	2	3	4
2	8 (4,3)	16 (8,9)	32 (16,27)	48 (24,51)
d 3	96 (13,9)	1248 (49,57)	10944 (145,249)	43680 (289,633)
4	5376 (40,27)	(272,321)	(1120,1923)	(2928,6459)

Table 4: Number of vertices (diameter, integer range) of $Z_\infty(d,p)$

Observation 3.4. *The zonotope $Z_\infty(d,1)$ has $(3^d-1)/2$ generators including $(3^{d-1}-1)/2$ generators which belong to the hyperplane $\{x \in \mathbb{R}^d : x_1 = 0\}$ and form $Z_\infty(d-1,1)$. In addition, $Z_\infty(d,1)$ is, up to translation, a lattice $(d,3^{d-1})$-polytope with diameter $(3^d-1)/2$.*

Proof. The generators of $Z_\infty(d,1)$ form all $\{-1,0,1\}$-valued d-tuples except $(0,\ldots.0)$, and without the remaining half starting with -1, thus $m(Z_\infty(d,1)) = (3^d-1)/2$ generators and, by Proposition 2.2, $\delta(Z_\infty(d,1)) = (3^d-1)/2$. On can easily check that removing the first zero of the generators of $Z_\infty(d,1)$ starting with zero yields exactly the $(3^{d-1}-1)/2$ generators of $Z_\infty(d-1,1)$. As the sum of the the norm of the i-th coordinate of the $(3^d-1)/2$ generators of $Z_\infty(d,1)$ is 3^{d-1}, $Z_\infty(d,1)$ is, up to translation, a lattice $(d,3^{d-1})$-polytope by Observation 2.3. $\square$

Note that $Z_\infty(d,1)$ is homothetic to the zonotope called $Z(d)$ in [10] where, combining Proposition 3.1, Observation 3.4, and Proposition 2.2, the upper bound of $2\sum_{i=0}^{i=d-1} \binom{(3^d-3)/2}{i}$ for $m(d,1)$ is given. Theorem 3.5 slightly strengthens the upper bound for $m(d,1)$ by exploring the structure of the generators of $Z_\infty(d,1)$.

Theorem 3.5. *For $d \geq 3$, $m(d,1) \leq 2\sum_{i=0}^{i=d-1} \binom{(3^d-3)/2}{i} - 2\binom{(3^{d-1}-3)/2}{d-1}$.*

Proof. Since $m(d,1) \leq f_0(Z_\infty(d,1))$, it is enough to show that $f_0(Z_\infty(d,1)) \leq \bar{f}(d,(3^d-1)/2) - 2\binom{(3^{d-1}-3)/2}{d-1}$ for $d \geq 3$ where $\bar{f}(d,m) = \sum_{i=0}^{i=d-1} \binom{m-1}{i}$. By duality, the number $f_0(Z)$ of vertices of a zonotope Z is equal to the number $f_{d-1}(\mathcal{A})$ of cells of the associate hyperplane arrangement $\mathcal{A}$ where each generator m^j of Z corresponds to an hyperplane h^j of $\mathcal{A}$, see [5, 16]. The upper bound of $\bar{f}(d,m)$ for $f_0(Z)$ given in

Proposition 2.2 is based on the inequality $f_{d-1}(\mathcal{A}) \leq f_{d-1}(\mathcal{A} \setminus h^j) + f_{d-1}(\mathcal{A} \cap h^j)$ for any hyperplane h^j of $\mathcal{A}$ where $\mathcal{A} \setminus h^j$ denotes the arrangement obtained by removing h^j from $\mathcal{A}$, and $\mathcal{A} \cap h^j$ denotes the arrangement obtained by intersecting $\mathcal{A}$ with h^j. Recursively applying this inequality to the arrangement $\mathcal{A}_\infty(d, 1)$ associated to $Z_\infty(d, 1)$ till the remaining $(3^{d-1} - 1)/2$ hyperplanes form a $(d-1)$-dimensional arrangement equivalent to $\mathcal{A}_\infty(d-1, 1)$ yields: $f_{d-1}(\mathcal{A}_\infty(d, 1)) \leq \bar{f}(d, (3^d - 1)/2) - \left(\bar{f}(d, (3^{d-1} - 1)/2) - \bar{f}(d-1, (3^{d-1} - 1)/2)\right)$ which completes the proof since $f_{d-1}(\mathcal{A}_\infty(d, 1)) = f_0(Z_\infty(d, 1))$ and $\bar{f}(d, m) - \bar{f}(d-1, m) = 2\binom{m-1}{d}$. In other words, the proof is based on the inductive build-up of $Z_\infty(d, 1)$ starting with the $(3^{d-1} - 3)/2$ generators with zero as first coordinate, and noticing that these $(3^{d-1} - 3)/2$ generators belong to a lower dimensional space. $\qquad\square$

3.2 A family of lattice polytopes arising as matroid polytopes

We consider a family of lattice polytopes introduced in [10] and defined as $M(d, r, s) = \text{conv}(W_d^s S_r^{s2^d})$ where W is the $\{0, 1\}$-valued $d \times s2^d$ matrix whose $s2^d$ columns consist of s copies of the 2^d elements of $\{0, 1\}^d$, and S is the (set of indicators of bases of the) uniform matroid $U_{s2^d}^r$ of rank r and order $s2^d$; that is, S consists of all vectors in $\{0, 1\}^{s2^d}$ with exactly r ones. We recall in Observation 3.6 some examples of $M(d, r, s)$ noted in [10].

Observation 3.6.

(i) $f_0(M(d, r, s \geq r)) = 2^d$ as $M(d, r, s \geq r)$ is the $\{0, \ldots, s\}^d$-cube,

(ii) $f_0(M(d, 2, 1)) = d2^{d-1}$ as $M(d, 2, 1)$ is the truncated $\{0, 1, 2\}^d$-cube,

(iii) $f_0(M(d, s+1, s \geq 2)) = d2^d$ as $M(d, s+1, s \geq 2)$ is the truncated $\{0, \ldots, s\}^d$-cube.

To illustrate the introduced family of matroid polytopes, Tables 5 and 6 provide the number of vertices of $M(d, r, s)$ followed by its diameter $\delta(M(d, r, s))$ and the k of this lattice (d, k)-polytope which were computed for small d, r and s. For instance, the entry $48(9, 5)$ for $(d, r, s) = (3, 5, 2)$ in Table 5 indicates that $M(3, 5, 2)$ has 48 vertices, diameter 9, and is a lattice $(3, 5)$-polytope. Note that $M(3, 5, 2)$ is homothetic to the *truncated cuboctahedron* represented in Figure 3, and thus a translation of $Z_1(3, 2)$.

$M(3,r,s)$	$r-s$ 1	2	3	4	5	6
1	12 (3,2)	24 (5,3)	14 (4,4)	24 (5,4)	12 (3,4)	8 (3,4)
2	24 (6,3)	12 (3,4)	48 (9,5)	24 (5,6)	48 (9,7)	14 (4,8)
3	24 (6,4)	24 (6,5)	12 (3,6)	48 (9,7)	48 (9,8)	24 (5,9)
4	24 (6,5)	24 (6,6)	24 (6,7)	12 (3,8)	48 (9,9)	48 (9,10)
5	24 (6,6)	24 (6,7)	24 (6,8)	24 (6,9)	12 (3,10)	48 (9,11)

Table 5: Number of vertices (diameter, integer range) of $M(3,r,s)$

$M(4,r,s)$	$r-s$ 1	2	3	4	$\ldots$	9
1	32 (4,2)	96(7,3)	88(6,4)	208 (8,5)	$\ldots$	256(10,8)
2	64 (8,3)	32 (4,4)	192 (12,5)	96 (7,6)	$\ldots$	672 (17,11)

Table 6: Number of vertices (diameter, integer range) of $M(4,r,s)$

Theorem 3.7 summarizes Theorems 3.3 and 3.5, and item (iii) of Observation 3.6.

Theorem 3.7. *The following inequalities hold for $d \geq 3$:*

$$\max\{d2^d, f_0(Z_\infty^+(d,1))\} \leq m(d,1) \leq f_0(Z_\infty(d,1)) \leq 2 \sum_{i=0}^{i=d-1} \binom{(3^d-3)/2}{i} - 2\binom{(3^{d-1}-3)/2}{d-1}$$

As noted in [10], $m(2,1) = 8$ as $f_0(M(2,3,2)) = f_0(Z_\infty(2,1)) = 8$. Finding (d,r,s) such that $M(d,r,s)$ has more that $\max\{d2^d, f_0(Z_\infty^+(d,1))\}$ vertices would strengthen the lower bound for $m(d,1)$ as reported in Observation 3.8.

Observation 3.8.

(i) $48 \leq m(3,1) \leq 96$ as $f_0(M(3,5,2)) = 48$ and $f_0(Z_\infty(3,1)) = 96$,

(ii) $672 \leq m(4,1) \leq 5\,376$ as $f_0(M(4,11,2)) = 672$ and $f_0(Z_\infty(4,1)) = 5\,376$,

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