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A characterization of Nash equilibrium for the games with random payoffs

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Abstract We consider a two player random bimatrix game where each player is interested in the payoffs which can be obtained with certain confidence. The players' payoff functions in such game theoretic problems are defined using chance constraints. We consider the case where the entries of each player's random payoff matrix jointly follow a multivariate elliptically symmetric distribution. We show an equivalence between a Nash equilibrium problem and a global maximization of a certain mathematical program. The case where the entries of the payoff matrices are independent normal/Cauchy random variables are also considered. The case of independent normally distributed random payoffs can be viewed as a special case of a multivariate elliptically symmetric distributed random payoffs. For the Cauchy distribution case, we show that a Nash equilibrium problem is equivalent to a global maximization of a certain quadratic program. Our theoretical results are illustrated by considering a random bimatrix game between two manufacturing firms acting on the same market.

Keywords Chance-constrained games · Nash equilibrium · Elliptically symmetric distribution · Cauchy distribution · Mathematical program · Quadratic program.

Mathematics Subject Classification (2000) 91A10, 90C15, 90C20, 90C26.

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1 Introduction

It is well known that there exists a mixed strategy saddle point equilibrium for a two player zero-sum matrix game (see [24]). Nash [17] showed the existence of a mixed strategy equilibrium for a finite strategic game. Such equilibrium is called a Nash equilibrium. For a two-player case, the game can be described using two payoff matrices (one for each player), and it is called a bimatrix game. Mangasarian and Stone [15] showed a one-to-one correspondence between a Nash equilibrium of a bimatrix game and a global maximum point of a certain quadratic program.

In [17, 24], the players' payoffs are known real values. However, in some cases the players' payoffs may be within certain ranges. Such games have been studied using fuzzy theory [7, 10, 12, 14]. In many practical situations the players' payoffs are better modeled using random variables due to the presence of various uncertain parameters. The wholesale electricity markets are the good examples [8, 16, 25, 26]. One way to handle such games is by taking the expectation of the random payoffs [25, 26]. Some recent papers on the games with random payoffs using expected payoff criterion include [9, 13, 19, 27].

The expected payoff criterion is not suitable when the random payoff has large variance. In this case, players will be more interested in payoffs which can be obtained with certain confidence. Such situations are better modeled using chance constraints. The payoff criterion based on chance constraint programming [4, 18] has received some attention in electricity market [8, 16]. These games are called chance-constrained games (CCGs). In [16], the randomness in payoffs is due to the installation of wind generators in the electricity market. They considered the case where the amount of wind through different wind generators are independent normal random variables. In [8], the payoffs are random due to consumers' random demand which is assumed to be normally distributed. Recently, Singh et al. [21, 22] proposed some contributions to develop the theory of CCGs. In [21], the authors showed the existence of a mixed strategy Nash equilibrium of a CCG when each player's payoff vector follows certain probability distributions, namely normal, Cauchy and elliptically symmetric distribution. In [22], they considered a CCG where the distribution of the payoff vector of each player is not completely known and belongs to a certain distributional uncertainty set. There is a scarce literature on zero sum CCGs available [2, 3, 5, 6, 23].

In this paper, we develop new techniques to compute the Nash equilibria of a two player CCG corresponding to different probability distributions. We consider the case where the entries of the payoff matrix of each player follow a multivariate elliptically symmetric distribution as well as the case where the entries are independent normal/Cauchy random variables. For each case we show an equivalence between a Nash equilibrium of a CCG and a global maximum of a certain mathematical program. Further, we show that a uniformly distributed strategy pair is a Nash equilibrium if the entries of the payoff matrices are independent and identically distributed normal random variables. When the entries of the payoff matrices are independent and identically Cauchy distributed random variables, all the strategy pairs are Nash equilibrium. To illustrate our theoretical results, we consider an example of a random bimatrix game between two manufacturing firms competing on the same market. Both firms compete for the customers by using different marketing strategies. A similar example in fuzzy setting has been studied in [12].

The rest of the paper is organized as follows. Section 2 contains the definition of a CCG. Section 3 presents a mathematical programming formulation for a CCG. Section 4 shows the numerical results for various instances of random bimatrix game between two manufacturing firms. We conclude the paper in Section 5.

2 The Model

We consider a two player bimatrix game where the payoff matrix of each player is random. Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ be the sets of actions of player 1 and player 2 respectively. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be the $m \times n$ random payoff matrices of player 1 and player 2 respectively. If player 1 chooses an action i and player 2 chooses an action j simultaneously, the payoff of player 1 is given by a random variable a_{ij}, and the payoff of player 2 is given by a random variable b_{ij}. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then, for each $i \in I$, $j \in J$, $a_{ij} : \Omega \to \mathbb{R}$ and $b_{ij} : \Omega \to \mathbb{R}$. The sets I and J are also called the sets of pure strategies of player 1 and player 2 respectively. Let $X = \{x \in \mathbb{R}^m | \sum_{i \in I} x_i = 1, x_i \geq 0, \ \forall\, i \in I\}$ and $Y = \{y \in \mathbb{R}^n | \sum_{j \in J} y_j = 1, y_j \geq 0, \ \forall\, j \in J\}$ be the sets of mixed strategies of player 1 and player 2 respectively. Then, for each $(x, y) \in X \times Y$ the payoff of player 1 $\big($resp. player 2$\big)$ given by $x^T A y$ $\big($resp. $x^T B y$ $\big)$ is also a random variable; T denotes a transposition. We consider the case where each player is interested in payoffs which can be obtained with at least a given level of confidence. The confidence level of each player is given a priori. We assume that the confidence level of one player is known to another player. Let $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, 1]$ be the confidence levels of player 1 and player 2 respectively; α_1 and α_2 also represent the risk levels. Let $\alpha = (\alpha_1, \alpha_2)$ be a confidence (risk) level vector. For a strategy pair (x, y) and a given α, the payoff of player 1 is given by

$$u_1^{\alpha_1}(x, y) = \sup\big\{\gamma_1 \mid P(x^T A y \geq \gamma_1) \geq \alpha_1\big\}, \tag{1}$$

and the payoff of player 2 is given by

$$u_2^{\alpha_2}(x, y) = \sup\big\{\gamma_2 \mid P(x^T B y \geq \gamma_2) \geq \alpha_2\big\}. \tag{2}$$

We assume that the probability distribution of the payoff matrix of one player is known to another player. Then, for a given $\alpha \in [0, 1]^2$ the payoffs of one player defined above are known to another player. Therefore, a CCG is a non-cooperative game with complete information. For a given α, the set of best response strategies of player 1 against a fixed strategy y of player 2 is given by

$$BR^{\alpha_1}(y) = \left\{\bar{x} \in X \mid u_1^{\alpha_1}(\bar{x}, y) \geq u_1^{\alpha_1}(x, y),\ \forall\, x \in X\right\},$$

and the set of best response strategies of player 2 against a fixed strategy x of player 1 is given by

$$BR^{\alpha_2}(x) = \left\{\bar{y} \in Y \mid u_2^{\alpha_2}(x, \bar{y}) \geq u_2^{\alpha_2}(x, y),\ \forall\, y \in Y\right\}.$$

Definition 1 (Nash equilibrium) For a given $\alpha \in [0,1]^2$, a strategy pair (x^*, y^*) is said to be a Nash equilibrium of a CCG if the following inequalities hold:

$$u_1^{\alpha_1}(x^*, y^*) \geq u_1^{\alpha_1}(x, y^*),\ \forall\, x \in X,$$
$$u_2^{\alpha_2}(x^*, y^*) \geq u_2^{\alpha_2}(x^*, y),\ \forall\, y \in Y.$$

Notice that a strategy pair (x^*, y^*) is a Nash equilibrium if and only if $x^* \in BR^{\alpha_1}(y^*)$ and $y^* \in BR^{\alpha_2}(x^*)$.

3 Mathematical programming formulation

We consider the case where the entries of the payoff matrix A (resp. B) jointly follow a multivariate elliptically symmetric distribution. The distributions belonging to the class of multivariate elliptically symmetric distributions generalize the multivariate normal distribution [11]. Some famous multivariate distributions like normal, Cauchy, t, Laplace, and logistic distributions belong to the family of elliptically symmetric distributions. In this case, we show the equivalence between the Nash equilibrium problem of a chance-constrained game and the global maximization of a certain mathematical program. We also consider the case where the entries of the payoff matrices A and B are independent normal/Cauchy random variables. For the case of Cauchy distribution, we show that the Nash equilibrium problem of a CCG is equivalent to a global maximization of a certain quadratic program.

3.1 Payoffs following multivariate elliptical distribution

We represent the entries of payoff matrix A (resp. B) by an $mn \times 1$ vector $a = (a_1, a_2, \ldots, a_m)^T$ (resp. $b = (b_1, b_2, \ldots, b_m)^T$), where $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ (resp. $b_i = (b_{i1}, b_{i2}, \ldots, b_{in})$) for all $i \in I$. We assume that the vector a (resp. b) follows a multivariate elliptically symmetric distribution with location vector μ_1 (resp. μ_2) and scale matrix Σ_1 (resp. Σ_2) which is a positive definite matrix. Denote, $\mu_1 = (\mu_{1,1}, \mu_{1,2}, \ldots, \mu_{1,m})^T$, where $\mu_{1,i} = (\mu_{1,i1}, \mu_{1,i2}, \ldots, \mu_{1,in})$ for all $i \in I$. The vector μ_2 is defined similarly. For a strategy pair (x, y), define a vector $\eta(x, y) = (\eta_1, \eta_2, \ldots, \eta_m)^T$, where $\eta_i = (\eta_{i1}, \eta_{i2}, \ldots, \eta_{in})$, $i \in I$, such that

$\eta_{ij} = x_i y_j$ for all $i \in I$, $j \in J$. Using these notations, we can write the random payoff $x^T Ay$ (resp. $x^T By$) as $\big(\eta(x,y)\big)^T a$ $\big(\text{resp. } \big(\eta(x,y)\big)^T b\big)$. A linear combination of the components of a multivariate elliptically symmetric random vector follows a univariate elliptically symmetric distribution. Therefore, for a strategy pair (x,y), $x^T Ay$ (resp. $x^T By$) follows a univariate elliptically symmetric distribution with location parameter $\mu_1^T \eta(x,y)$ $\big(\text{resp. } \mu_2^T \eta(x,y)\big)$ and scale parameter $\big(\eta(x,y)\big)^T \Sigma_1 \eta(x,y)$ $\big(\text{resp. } \big(\eta(x,y)\big)^T \Sigma_2 \eta(x,y)\big)$. Since Σ_1 and Σ_2 are positive definite matrices, we can write $\sqrt{\big(\eta(x,y)\big)^T \Sigma_1 \eta(x,y)} = ||\Sigma_1^{1/2}\eta(x,y)||$ and $\sqrt{\big(\eta(x,y)\big)^T \Sigma_2 \eta(x,y)} = ||\Sigma_2^{1/2}\eta(x,y)||$, where $||\cdot||$ is the Euclidean norm. The matrices $\Sigma_1^{1/2}$ and $\Sigma_2^{1/2}$ are the unique positive definite square roots of the matrices Σ_1 and Σ_2 respectively. Then, $Z_1^S = \frac{x^T Ay - \mu_1^T \eta(x,y)}{||\Sigma_1^{1/2}\eta(x,y)||}$ and $Z_2^S = \frac{x^T By - \mu_2^T \eta(x,y)}{||\Sigma_2^{1/2}\eta(x,y)||}$ follow a univariate spherically symmetric distribution. Let $F^{-1}_{Z_1^S}(\cdot)$ and $F^{-1}_{Z_2^S}(\cdot)$ be the quantile functions of a univariate spherically symmetric distribution. From (1), for a strategy pair (x,y) and a confidence level α_1, the payoff of player 1 is given by

$$\begin{aligned}
u_1^{\alpha_1}(x,y) &= \sup\{\gamma_1 \mid P(x^T Ay \geq \gamma_1) \geq \alpha_1\} \\
&= \sup\left\{\gamma_1 \;\middle|\; P\left(\frac{x^T Ay - \mu_1^T \eta(x,y)}{||\Sigma_1^{1/2}\eta(x,y)||} \leq \frac{\gamma_1 - \mu_1^T \eta(x,y)}{||\Sigma_1^{1/2}\eta(x,y)||}\right) \leq 1 - \alpha_1\right\} \\
&= \sup\left\{\gamma_1 \mid \gamma_1 \leq \mu_1^T \eta(x,y) + ||\Sigma_1^{1/2}\eta(x,y)|| F^{-1}_{Z_1^S}(1-\alpha_1)\right\}.
\end{aligned}$$

Then,

$$u_1^{\alpha_1}(x,y) = \mu_1^T \eta(x,y) + ||\Sigma_1^{1/2}\eta(x,y)|| F^{-1}_{Z_1^S}(1-\alpha_1). \tag{3}$$

Similarly, from (2), for a strategy pair (x,y) and a confidence level α_2, the payoff of player 2 is given by

$$u_2^{\alpha_2}(x,y) = \mu_2^T \eta(x,y) + ||\Sigma_2^{1/2}\eta(x,y)|| F^{-1}_{Z_2^S}(1-\alpha_2). \tag{4}$$

Singh et al. [21] showed that there exists a mixed strategy Nash equilibrium for all $\alpha \in (0.5,1]^2$. If the random payoff matrices have strictly positive density function, a mixed strategy Nash equilibrium exists for all $\alpha \in [0.5,1]^2$. At $\alpha = (1,1)$, $F^{-1}_{Z_1^S}(1-\alpha_1) = F^{-1}_{Z_2^S}(1-\alpha_2) = -\infty$. Then, $u_1^{\alpha_1}(x,y) = u_2^{\alpha_2}(x,y) = -\infty$ for all $(x,y) \in X \times Y$. In this case, each strategy pair (x,y) would be a Nash equilibrium because there is no incentive for any player to deviate unilaterally. Therefore, we consider the case where $\alpha \in (0.5,1)^2$ so that the payoff functions defined by (3) and (4) have finite values.

Lemma 1 *For every $y \in X$, $u_1^{\alpha_1}(\cdot,y)$ defined by* (3) *is a concave function of x for all $\alpha_1 \in (0.5,1)$, and for every $x \in X$, $u_2^{\alpha_2}(x,\cdot)$ defined by* (4) *is a concave function of y for all $\alpha_2 \in (0.5,1)$.*

Proof The proof follows from Lemma 3.5 of [21].

Remark 1 Lemma 1 holds for all $\alpha_1 \in [0.5, 1)$ and $\alpha_2 \in [0.5, 1)$ if the payoff matrices (A, B) have strictly positive density functions [21].

3.1.1 Best response convex programs

From Lemma 1, for a fixed strategy y of player 2 and $\alpha_1 \in (0.5, 1)$, a best response strategy of player 1 can be obtained by solving the following convex quadratic program:

$$
\begin{aligned}
\text{[P1]} \quad \min_{x} \quad & -\mu_1^T \eta(x,y) - ||\Sigma_1^{1/2}\eta(x,y)||F^{-1}_{Z_1^S}(1-\alpha_1) \\
\text{s.t.} \quad & \\
& \sum_{i\in I} x_i = 1, \\
& x_i \geq 0, \ i \in I.
\end{aligned}
$$

Let $X^+ = \{x \in \mathbb{R}^m \mid x_i \geq 0, \forall\, i \in I\}$. Then, the Lagrangian dual problem of [P1] is

$$
\max_{\lambda_1 \in \mathbb{R}} \min_{x \in X^+} \left[-\mu_1^T \eta(x,y) - ||\Sigma_1^{1/2}\eta(x,y)||F^{-1}_{Z_1^S}(1-\alpha_1) + \lambda_1 \left(1 - \sum_{i\in I} x_i \right) \right].
$$

For a fixed $\lambda_1 \in \mathbb{R}$, we have

$$
\begin{aligned}
& \min_{x \in X^+} \left[-\mu_1^T \eta(x,y) - ||\Sigma_1^{1/2}\eta(x,y)||F^{-1}_{Z_1^S}(1-\alpha_1) + \lambda_1 \left(1 - \sum_{i\in I} x_i \right) \right] \\
&= \min_{x\in X^+} \max_{\substack{v_1 \in \mathbb{R}^{mn\times 1} \\ ||v_1|| \leq 1}} \left[\sum_{i\in I} x_i \left(-\sum_{j\in J} \mu_{1,ij} y_j - \sum_{j\in J} y_j \big(\Sigma_1^{1/2} v_1\big)_{ij} F^{-1}_{Z_1^S}(1-\alpha_1) - \lambda_1 \right) + \lambda_1 \right] \\
&= \max_{\substack{v_1 \in \mathbb{R}^{mn\times 1} \\ ||v_1|| \leq 1}} \min_{x\in X^+} \left[\sum_{i\in I} x_i \left(-\sum_{j\in J} \mu_{1,ij} y_j - \sum_{j\in J} y_j \big(\Sigma_1^{1/2} v_1\big)_{ij} F^{-1}_{Z_1^S}(1-\alpha_1) - \lambda_1 \right) + \lambda_1 \right],
\end{aligned}
$$

where $\big(\Sigma_1^{1/2} v_1\big)_{ij}$ is an $(ij)^{\text{th}}$ element of the vector $\Sigma_1^{1/2} v_1$. The first equality is obtained by using Cauchy-Schwartz inequality. The second equality follows from Corollary 37.3.2 of [20]. The minimum in the second equality is unbounded unless

$$
\lambda_1 \leq -\sum_{j\in J} \mu_{1,ij} y_j - \sum_{j\in J} y_j \big(\Sigma_1^{1/2} v_1\big)_{ij} F^{-1}_{Z_1^S}(1-\alpha_1), \ \forall\, i \in I.
$$

Hence, the Lagrangian dual problem of [P1] is

$$
\begin{aligned}
&[\text{D1}] \quad \max_{\lambda_1, v_1} \lambda_1 \\
&\quad \text{s.t.} \\
&\quad\quad \lambda_1 \leq -\sum_{j \in J} \mu_{1,ij} y_j - \sum_{j \in J} y_j \big(\Sigma_1^{1/2} v_1\big)_{ij} F^{-1}_{Z_1^S}(1-\alpha_1), \ \forall\, i \in I, \\
&\quad\quad ||v_1|| \leq 1.
\end{aligned}
$$

Similarly, for a fixed $x \in X$ and $\alpha_2 \in (0.5, 1)$, a best response strategy of player 2 can be obtained by solving the following convex quadratic program:

$$
\begin{aligned}
&[\text{P2}] \quad \min_{y} \quad -\mu_2^T \eta(x,y) - ||\Sigma_2^{1/2} \eta(x,y)||\ F^{-1}_{Z_2^S}(1-\alpha_2) \\
&\quad \text{s.t.} \\
&\quad\quad \sum_{j \in J} y_j = 1, \\
&\quad\quad y_j \geq 0,\ j \in J.
\end{aligned}
$$

From the similar arguments used above, the dual of [P2] is

$$
\begin{aligned}
&[\text{D2}] \quad \max_{\lambda_2, v_2} \lambda_2 \\
&\quad \text{s.t.} \\
&\quad\quad \lambda_2 \leq -\sum_{i \in I} \mu_{2,ij} x_i - \sum_{i \in I} x_i \big(\Sigma_2^{1/2} v_2\big)_{ij} F^{-1}_{Z_2^S}(1-\alpha_2), \ \forall\, j \in J, \\
&\quad\quad ||v_2|| \leq 1.
\end{aligned}
$$

3.1.2 Mathematical program

We denote the decision variables and the objective function of the mathematical program [MP] by $\zeta = (\lambda_1, \lambda_2, v_1, v_2, x, y)$ and $\psi(\cdot)$ respectively. By using the best response convex programs [P1], [D1], [P2], [D2] we have the following characterization.

Theorem 1 *Consider a random bimatrix game (A, B), where all the entries of matrix A (resp. B) jointly follow a multivariate elliptically symmetric distribution with location vector μ_1 (resp. μ_2) and scale matrix Σ_1 (resp. Σ_2). Let Σ_1 and Σ_2 be positive definite matrices. Then, for an $\alpha \in (0.5, 1)^2$*

1. *If (x^*, y^*) is a Nash equilibrium of a CCG, there exists a vector $\zeta^* = (\lambda_1^*, \lambda_2^*, v_1^*, v_2^*, x^*, y^*)$ such that it is a global maximum of the following mathematical program [MP]*

$$
\text{[MP]} \quad \max_{\zeta} \Bigg[\Big(\lambda_1 + \mu_1^T \eta(x,y) + ||\Sigma_1^{1/2}\eta(x,y)||F^{-1}_{Z_1^S}(1-\alpha_1)\Big) + \Big(\lambda_2 + \mu_2^T \eta(x,y) + ||\Sigma_2^{1/2}\eta(x,y)||F^{-1}_{Z_2^S}(1-\alpha_2)\Big) \Bigg]
$$

s.t.

$$
\lambda_1 \leq -\sum_{j\in J} \mu_{1,ij} y_j - \sum_{j\in J} y_j \big(\Sigma_1^{1/2} v_1\big)_{ij} F^{-1}_{Z_1^S}(1-\alpha_1), \ \forall \ i \in I, \tag{5}
$$

$$
\lambda_2 \leq -\sum_{i\in I} \mu_{2,ij} x_i - \sum_{i\in I} x_i \big(\Sigma_2^{1/2} v_2\big)_{ij} F^{-1}_{Z_2^S}(1-\alpha_2), \ \forall \ j \in J, \tag{6}
$$

$$
||v_1|| \leq 1, \tag{7}
$$

$$
||v_2|| \leq 1, \tag{8}
$$

$$
\sum_{i\in I} x_i = 1, \tag{9}
$$

$$
\sum_{j\in J} y_j = 1, \tag{10}
$$

$$
x_i \geq 0, \ \forall \ i \in I, \tag{11}
$$

$$
y_j \geq 0, \ \forall \ j \in J, \tag{12}
$$

with objective function value $\psi(\zeta^*) = 0$.

2. *If* $\zeta^* = (\lambda_1^*, \lambda_2^*, v_1^*, v_2^*, x^*, y^*)$ *is a global maximum of the mathematical program [MP],* (x^*, y^*) *is a Nash equilibrium of a CCG.*

Proof Let (x^*, y^*) be a Nash equilibrium of a CCG. The constraints (9)-(12) of [MP] are satisfied by (x^*, y^*) because x^* and y^* are the mixed strategies. From the definition of a Nash equilibrium, x^* is an optimal solution of [P1] for fixed y^*, and y^* is an optimal solution of [P2] for fixed x^*. The convex quadratic program [P1] $\big($resp. [P2]$\big)$ satisfies all the conditions of strong duality Theorem 6.2.4 of [1]. Hence, there exists an optimal solution (v_1^*, λ_1^*) $\big($resp. $(v_2^*, \lambda_2^*)\big)$ of the dual program [D1] $\big($resp. [D2]$\big)$ such that the objective function values of [P1] and [D1] $\big($resp. [P2] and [D2]$\big)$ are the same. That is, the constraints (5) and (7) $\big($resp. (6) and (8) $\big)$ of [MP] are satisfied at $(v_1^*, \lambda_1^*, y^*)$ (resp. $(v_2^*, \lambda_2^*, x^*)$), and

$$
\lambda_1^* = -\mu_1^T \eta(x^*, y^*) - ||\Sigma_1^{1/2}\eta(x^*, y^*)|| \ F^{-1}_{Z_1^S}(1-\alpha_1),
$$

$$
\lambda_2^* = -\mu_2^T \eta(x^*, y^*) - ||\Sigma_2^{1/2}\eta(x^*, y^*)|| \ F^{-1}_{Z_2^S}(1-\alpha_2).
$$

Therefore, ζ^* is a feasible point of [MP] such that $\psi(\zeta^*) = 0$. Next, we show that ζ^* is a global maximum of [MP]. Let ζ be a feasible point of [MP]. Multiply constraint (5) by x_i for each $i \in I$, and sum over all $i \in I$. Then, by using Cauchy-Schwartz inequality and constraints (7), (9), (11), we have

$$\lambda_1 \leq -\mu_1^T \eta(x, y) - ||\Sigma_1^{1/2} \eta(x, y)|| \; F_{Z_1^S}^{-1}(1 - \alpha_1). \tag{13}$$

Similarly,

$$\lambda_2 \leq -\mu_2^T \eta(x, y) - ||\Sigma_2^{1/2} \eta(x, y)|| \; F_{Z_2^S}^{-1}(1 - \alpha_2). \tag{14}$$

From (13) and (14), $\psi(\zeta) \leq 0$ for all feasible points ζ of [MP]. Therefore, ζ^* is a global maximum of [MP].

Let ζ^* be a global maximum of [MP]. It follows from the first part of the theorem that the objective function value at a global maximum point is zero. Then, $\psi(\zeta^*) = 0$. Since, ζ^* is a feasible point of [MP], ζ^* will satisfy (13) and (14). Hence, each term of the objective function is non-positive at ζ^*. This implies that (13) and (14) are equalities at ζ^*. For a given ζ^* multiply the constraint (5) by x_i for each $i \in I$, and sum over all $i \in I$. Then, by using Cauchy-Schwartz inequality, we have

$$\lambda_1^* \leq -\mu_1^T \eta(x, y^*) - ||\Sigma_1^{1/2} \eta(x, y^*)|| \; F_{Z_1^S}^{-1}(1 - \alpha_1), \; \forall \; x \in X.$$

Using the fact that (13) is an equality at ζ^*, we have

$$u_1^{\alpha_1}(x^*, y^*) \geq u_1^{\alpha_1}(x, y^*), \; \forall \; x \in X. \tag{15}$$

Similarly,

$$u_2^{\alpha_2}(x^*, y^*) \geq u_2^{\alpha_2}(x^*, y), \; \forall \; y \in Y. \tag{16}$$

From (15) and (16), (x^*, y^*) is a Nash equilibrium of a CCG. □

Remark 2 Theorem 1 holds for all $\alpha_1 \in [0.5, 1)$ and $\alpha_2 \in [0.5, 1)$, if the payoff matrices (A, B) have strictly positive density functions [21].

3.1.3 Special case

We consider the case where the entries of the payoff matrices A and B are independent normal random variables. For each $i \in I$, $j \in J$, let a_{ij} $\big($resp. $b_{ij}\big)$ follows a normal distribution with mean $\mu_{1,ij}$ $\big($resp. $\mu_{2,ij}\big)$ and variance $\sigma_{1,ij}^2$ $\big($resp. $\sigma_{2,ij}^2\big)$. It is well known that if a multivariate normal random vector is uncorrelated, i.e., the covariance matrix is a diagonal matrix, then, its components are independent normal random variables. A multivariate normal distribution belongs to the family of elliptically symmetric distributions. Therefore, the equivalent mathematical program for the case of independent normal random payoffs can be obtained from [MP] where the mean vector and covariance matrix of player 1 (resp. Player 2) are μ_1 (resp. μ_2) and $\Sigma_1 = \text{diag}(\sigma_{1,11}^2, \sigma_{1,12}^2, \ldots, \sigma_{1,mn}^2)$ $\big($resp. $\Sigma_2 = \text{diag}(\sigma_{2,11}^2, \sigma_{2,12}^2, \ldots, \sigma_{2,mn}^2)\big)$ respectively; $\text{diag}(\cdot)$ denotes a diagonal matrix. Further if the entries of A (resp. B) are also identically distributed with mean μ (resp. $\bar{\mu}$) and variance σ^2 (resp. $\bar{\sigma}^2$), we show that a uniformly distributed strategy pair is a Nash equilibrium. The result is summarized in Theorem 2.

Theorem 2 *Consider a random bimatrix game (A, B), where all the entries of matrix A are independent and identically distributed (i.i.d.) normal random variables with mean μ and variance σ^2, and all the entries of matrix B are i.i.d. normal random variables with mean $\bar{\mu}$ and variance $\bar{\sigma}^2$. The strategy pair (x^*, y^*), where*

$$x_i^* = \frac{1}{m},\ \forall\ i \in I, \quad y_j^* = \frac{1}{n}, \forall\ j \in J, \tag{17}$$

is a Nash equilibrium of a CCG for all $\alpha \in [0.5, 1)^2$.

Proof The proof is given in Appendix A. □

3.2 Payoffs following Cauchy distribution

In general, the components of an uncorrelated random vector do not need to be independent. It holds only for the case of a multivariate normal distribution. Therefore, we discuss the case of independent Cauchy random payoffs separately. We assume that all the entries of the payoff matrix A $\big($resp. $B\big)$ are independent Cauchy random variables. For each $i \in I$, $j \in J$, let a_{ij} $\big($resp. $b_{ij}\big)$ follows a Cauchy distribution with location and scale parameters $\mu_{1,ij}$ $\big($resp. $\mu_{2,ij}\big)$ and $\sigma_{1,ij}$ $\big($resp. $\sigma_{2,ij}\big)$ respectively. Therefore, for a strategy pair (x, y), $x^T Ay$ $\big($resp. $x^T By$ $\big)$ follows a Cauchy distribution with location parameter $\mu_1(x, y) = \sum_{i\in I, j\in J} x_i y_j \mu_{1,ij}$ $\big($resp. $\mu_2(x, y) = \sum_{i\in I, j\in J} x_i y_j \mu_{2,ij}\big)$ and scale parameter $\sigma_1(x, y) = \sum_{i\in I, j\in J} x_i y_j \sigma_{1,ij}$ $\big($resp. $\sigma_2(x, y) = \sum_{i\in I, j\in J} x_i y_j \sigma_{2,ij}$ $\big)$. Then, $Z_1^C = \frac{x^T Ay - \mu_1(x,y)}{\sigma_1(x,y)}$ and $Z_2^C = \frac{x^T By - \mu_2(x,y)}{\sigma_2(x,y)}$ follow a standard Cauchy distribution. Let $F^{-1}_{Z_1^C}(\cdot)$ and $F^{-1}_{Z_2^C}(\cdot)$ be the quantile functions of a standard Cauchy distribution. Similar to the previous case, for a strategy pair (x, y) and a confidence level α_1, the payoff of player 1 is given by

$$\begin{aligned}
u_1^{\alpha_1}(x, y) &= \sup\{\gamma_1 \mid P(x^T Ay \geq \gamma_1) \geq \alpha_1\} \\
&= \sup\left\{\gamma_1 \,\middle|\, P\left(\frac{x^T Ay - \mu_1(x, y)}{\sigma_1(x, y)} \leq \frac{\gamma_1 - \mu_1(x, y)}{\sigma_1(x, y)}\right) \leq 1 - \alpha_1\right\} \\
&= \sup\Big\{\gamma_1 \mid \gamma_1 \leq \mu_1(x, y) + \sigma_1(x, y) F^{-1}_{Z_1^C}(1 - \alpha_1)\Big\}.
\end{aligned}$$

Then,

$$u_1^{\alpha_1}(x, y) = \sum_{i\in I, j\in J} x_i y_j \left(\mu_{1,ij} + \sigma_{1,ij} F^{-1}_{Z_1^C}(1 - \alpha_1)\right). \tag{18}$$

Similarly, for a strategy pair (x, y) and a confidence level α_2, the payoff of player 2 is given by

$$u_2^{\alpha_2}(x, y) = \sum_{i\in I, j\in J} x_i y_j \left(\mu_{2,ij} + \sigma_{2,ij} F^{-1}_{Z_2^C}(1 - \alpha_2)\right). \tag{19}$$

The quantile function of a standard Cauchy distribution is not finite at 0 and 1. Therefore, we consider the case of $\alpha \in (0,1)^2$ so that the payoff functions defined by (18) and (19) have finite values.

3.2.1 Equivalent bimatrix game

Define a matrix $\tilde{A}(\alpha_1) = \left[\tilde{a}_{ij}(\alpha_1)\right]$, where

$$\tilde{a}_{ij}(\alpha_1) = \mu_{1,ij} + \sigma_{1,ij} F^{-1}_{Z_1^C}(1-\alpha_1),$$

and a matrix $\tilde{B}(\alpha_2) = \left[\tilde{b}_{ij}(\alpha_2)\right]$, where

$$\tilde{b}_{ij}(\alpha_2) = \mu_{2,ij} + \sigma_{2,ij} F^{-1}_{Z_2^C}(1-\alpha_2).$$

Then, we can write (18) and (19) as

$$u_1^{\alpha_1}(x,y) = x^T \tilde{A}(\alpha_1) y,$$
$$u_2^{\alpha_2}(x,y) = x^T \tilde{B}(\alpha_2) y.$$

Therefore, for a given $\alpha \in (0,1)^2$, a CCG is equivalent to a deterministic bimatrix game $\big(\tilde{A}(\alpha_1), \tilde{B}(\alpha_2)\big)$. Hence, the existence of a Nash equilibrium in this case follows from [17].

Remark 3 For the case of i.i.d. Cauchy random variables each strategy pair (x,y) is a Nash equilibrium because the players' payoff functions (18) and (19) are constant.

3.2.2 Quadratic program

We have the following characterization for a CCG corresponding to Cauchy distribution which follows from [15].

Theorem 3 *Consider a random bimatrix game (A,B), where all the entries of A are independent Cauchy random variables, and all the entries of B are also independent Cauchy random variables. For all $i \in I$, $j \in J$, the location and scale parameters of a_{ij} $\big($resp. $b_{ij}\big)$ are $\mu_{1,ij}$ $\big($resp. $\mu_{2,ij}\big)$ and $\sigma_{1,ij}$ $\big($resp. $\sigma_{2,ij}\big)$ respectively. Then, for an $\alpha \in (0,1)^2$*

1. *If (x^*, y^*) is a Nash equilibrium of a CCG, there exists a vector $\zeta^* = (\lambda_1^*, \lambda_2^*, x^*, y^*)$ such that it is a global maximum of the following quadratic program*

$$\text{[QP]} \quad \max_{\zeta} \left[\left(x^T \tilde{A}(\alpha_1) y - \lambda_1 \right) + \left(x^T \tilde{B}(\alpha_2) y - \lambda_2 \right) \right]$$

s.t.

$$\tilde{A}(\alpha_1) y \leq \lambda_1 \mathbf{1}_m,$$

$$\tilde{B}^T(\alpha_2)x \leq \lambda_2 \mathbf{1}_n,$$
$$\sum_{i\in I} x_i = 1,$$
$$\sum_{j\in J} y_j = 1,$$
$$x_i \geq 0,\ \forall\ i \in I,$$
$$y_j \geq 0,\ \forall\ j \in J,$$

with objective function value $\psi(\zeta^*) = 0$.

2. *If* $\zeta^* = (\lambda_1^*, \lambda_2^*, x^*, y^*)$ *is a global maximum of the quadratic program [QP],* (x^*, y^*) *is a Nash equilibrium of a CCG.*

Proof For a given $\alpha \in (0,1)^2$, a CCG corresponding to Cauchy distribution is equivalent to the bimatrix game $\left(\tilde{A}(\alpha_1), \tilde{B}(\alpha_2)\right)$ defined in Section 3.2.1. Then, the proof follows from [15]. □

4 Competition between two manufacturing firms

We consider two manufacturing firms which produce similar products and compete on the same market. They plan to release a new product with similar features. To attract more customers, each firm uses different marketing strategies, e.g., TV and web advertisements, paid media advertisements, gift coupons, cash back, special offers, etc. We assume that each firm has finite number of marketing strategies. In this case, a mixed strategy represents the percentage allocation of the total marketing budget among different marketing strategies. The profit of each firm increases with the number of customers. The customers' demand can be random which also depends upon the marketing strategies of the firms. Therefore, the payoff of each firm corresponding to different pair of marketing strategies of both the firms is better modeled by a random variable. Hence, the competition between the firms can be modeled as a random bimatrix game. We assume that both the firms are interested in the payoffs that can be obtained with at least a given probability level. We study this game theoretic situation using CCG framework developed in this paper. We consider a general case where there is a correlation among different marketing strategies. We assume that the entries of the payoff matrix jointly follow a multivariate normal distribution. For illustration purpose, we consider the example given below.

Example 1 We consider the case where each firm has three marketing strategies, i.e., $I = \{1,2,3\}$ and $J = \{1,2,3\}$. We assume that the entries of the random payoff matrix A (resp. B) jointly follow a multivariate normal distribution. The mean vector μ_1 (resp. μ_2) and the covariance matrix Σ_1 (resp. Σ_2) for the payoff matrix A (resp. B) are given below:

$$\mu_1 = (10, 9, 11, 8, 12, 10, 7, 8, 13),\ \mu_2 = (9, 7, 8, 9, 10, 10, 10, 9, 8).$$

$$\varSigma_1 = \begin{pmatrix} 6&4&3&3&2&3&4&2&4\\ 4&6&3&4&3&3&3&2&3\\ 3&3&8&4&2&3&3&2&4\\ 3&4&4&6&2&3&3&3&2\\ 2&3&2&2&6&2&4&3&3\\ 3&3&3&3&2&6&3&3&4\\ 4&3&3&3&4&3&8&4&3\\ 2&2&2&3&3&3&4&6&4\\ 4&3&4&2&3&4&3&4&8 \end{pmatrix}, \qquad \varSigma_2 = \begin{pmatrix} 6&3&3&3&3&2&4&3&2\\ 3&6&3&3&2&2&3&3&4\\ 3&3&6&3&3&3&4&3&4\\ 3&3&3&6&3&2&2&3&3\\ 3&2&3&3&6&4&2&2&3\\ 2&2&3&2&4&6&3&3&4\\ 4&3&4&2&2&3&6&3&2\\ 3&3&3&3&2&3&3&6&3\\ 2&4&4&3&3&4&2&3&6 \end{pmatrix}.$$

We compute the Nash equilibria of the CCG by solving the mathematical program [MP]. Our numerical experiments were carried out on an Intel(R) 32-bit core(TM) i3-3110M CPU @ 2.40GHz×4 and 3.8 GiB of RAM machine. We solve the equivalent minimization problem of mathematical program [MP] using **fmincon** in MATLAB optimization toolbox. We run the numerical experiments with an initial point $\zeta_0 = \big(-1, -2, 0, \frac{1}{2}, \frac{1}{2}, 2, 0, 3, 0, 0, 0, 0, 1, 3, \frac{1}{2}, \frac{1}{4}, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\big)$. Table 1 summarizes the Nash equilibria for various values of α. The exitflag

Table 1: Nash equilibria for various values of α

α		Nash Equilibrium		$\psi(\zeta^*)$	exitflag
α_1	α_2	x^*	y^*		
0.55	0.55	$\left(\frac{2046}{10000}, \frac{6164}{10000}, \frac{1790}{10000}\right)$	$\left(\frac{2143}{10000}, \frac{3055}{10000}, \frac{4802}{10000}\right)$	0	1
0.6	0.6	$\left(\frac{2078}{10000}, \frac{6340}{10000}, \frac{1582}{10000}\right)$	$\left(\frac{2121}{10000}, \frac{3071}{10000}, \frac{4808}{10000}\right)$	0	1
0.7	0.7	(1, 0, 0)	(1, 0, 0)	0	1
0.8	0.8	(1, 0, 0)	$\left(\frac{9804}{10000}, 0, \frac{196}{10000}\right)$	0	1

value 1 indicates that the point ζ^* is a local maximum. The objective function value $\psi(\zeta^*) = 0$ shows that ζ^* is a global maximum of the mathematical program [MP]. Therefore, from Theorem 1, the strategy part (x^*, y^*) of ζ^* given in column 3 and column 4 of Table 1 is a Nash equilibrium of the CCG.

To test our theoretical results for large game instances, we perform numerical experiments by considering different sizes of randomly generated instances. For each i, $i = 1, 2$, we take the mean vector $\mu_i = \textbf{randi}\,([m+n, m+n+2], mn, 1)$. It generates $mn \times 1$ integer vector within interval $[m+n, m+n+2]$. For each i, $i = 1, 2$, we take the covariance matrix $\varSigma_i = B + B^T + \theta \cdot I_{mn\times mn}$, where $B = \textbf{randi}(2, mn)$ is an $mn \times mn$ integer random matrix with entries not more than 2, and θ is sufficiently large so that $\varSigma_i$ is a positive definite matrix, and $I_{mn\times mn}$ is an $mn \times mn$ identity matrix. In our experiments, we take $\theta = m + n$. For these games, the mathematical program [MP] has $2 + 2mn + m + n$ variables and $4+2m + 2n$ constraints. We take $\alpha_1 = \alpha_2 = 0.6$. We run the numerical experiments with a randomly generated initial point $\zeta_0 = \mathbf{rand}(1, 2 + 2mn + m + n)$, where **rand** is a random number generator. Table 2 summarizes the numerical results for different sizes of the considered games. The columns 1-3 represent the size

Table 2: Average time for computing Nash equilibrium

Number of instances	Number of actions		Average order of magnitude k of $\psi(\zeta^*)$		Average time (s)	Average $\psi(\zeta^*)$	exitflag (all instances)
	m	n	$k = -2$	$k \leq -3$			
10	5	5	20%	80%	3.55	-1.4×10^{-3}	1
10	10	10	10%	90%	21.44	-5.1×10^{-3}	1
10	15	15	20%	80%	56.15	-4.6×10^{-3}	1
10	20	20	10%	90%	387.91	-2.6×10^{-3}	1

of the game problem. The order of magnitude for $\psi(\zeta^*)$ is k if its value is a constant multiple of 10^k. In 10%-20% of the instances $\psi(\zeta^*)$ is a constant multiple of 10^{-2} as given in column 4. In 80%-90% of the instances $\psi(\zeta^*)$ is a constant multiple of 10^{-k}, $k \leq -3$, as given in column 5. The average time to solve the mathematical program [MP] is given in column 6. The average value of $\psi(\zeta^*)$ is given in columns 7. The exitflag value 1 given in column 8 shows that every time we get a local maximum. In most of the time $\psi(\zeta^*)$ is zero or close to zero. This implies that we obtain a global maximum of the mathematical program [MP] or very close to it in most of the cases. Based on our computational experience, we can see that solving the mathematical program [MP] is not time consuming. Therefore, we can solve large instances in reasonable time.

5 Conclusions

We formulate a two player random bimatrix game as a CCG. We consider multivariate elliptically symmetric as well as independent normal/Cauchy distributed payoffs. For each case we show that a Nash equilibrium of a CCG can be obtained by computing a global maximum of a certain optimization problem. To illustrate our theoretical results, we consider a random bimatrix game between two manufacturing firms producing similar products. Both firms compete for the customers by using different marketing strategies. We take different sizes of random instances of the game. We use MATLAB to perform numerical experiments. Our approaches can be used for solving large instances as shown by the low computational effort in the considered numerical examples.

Appendix A Proof of Theorem 2

Proof Fix $\alpha \in [0.5, 1)^2$. To show (x^*, y^*) defined by (17) is a Nash equilibrium, it is sufficient to show that there exists a vector $(\lambda_1^*, \lambda_2^*, v_1^*, v_2^*)$ which together with (x^*, y^*) is a feasible point of [MP] with objective function value zero (see Theorem 1). By using i.i.d. property, we have $\Sigma_1^{1/2} = \sigma I_{mn\times mn}$, $\Sigma_2^{1/2} = \bar{\sigma} I_{mn\times mn}$. Take,

$$\left.\begin{aligned} \lambda_1^* &= -\mu - \frac{\sigma\ F^{-1}_{Z_1^S}(1-\alpha_1)}{\sqrt{mn}}, \\ \lambda_2^* &= -\bar{\mu} - \frac{\bar{\sigma}\ F^{-1}_{Z_2^S}(1-\alpha_2)}{\sqrt{mn}}, \\ v_1^* &= \frac{1}{\sqrt{mn}}\mathbf{1}_{mn}, \\ v_2^* &= \frac{1}{\sqrt{mn}}\mathbf{1}_{mn}, \end{aligned}\right\}$$

where $\mathbf{1}_k$ denotes a $k \times 1$ vector of ones. It is easy to check that $\zeta^* = (\lambda_1^*, \lambda_2^*, v_1^*, v_2^*, x^*, y^*)$ is a feasible point of [MP] and $\psi(\zeta^*) = 0$. Hence, (x^*, y^*) defined by (17) is a Nash equilibrium of a CCG.

References

1. M.S. Bazaraa, H.D. Sherali, and C.M. Shetty. *Nonlinear Programming: Theory and Algorithms*. John Wiley and Sons, Inc., U.S.A, Third ed., 2006.
2. R. A. Blau. Random-payoff two person zero-sum games. *Operations Research*, 22(6):1243–1251, 1974.
3. R. G. Cassidy, C. A. Field, and M. J. L. Kirby. Solution of a satisficing model for random payoff games. *Management Science*, 19(3):266–271, 1972.
4. A. Charnes and W. W. Cooper. Deterministic equivalents for optimizing and satisficing under chance constraints. *Operations Research*, 11(1):18–39, 1963.
5. A. Charnes, M. J. L. Kirby, and W. M. Raike. Zero-zero chance-constrained games. *Theory of Probability and its Applications*, 13(4):628–646, 1968.
6. Jianqiang Cheng, Janny Leung, and Abdel Lisser. Random-payoff two-person zero-sum game with joint chance constraints. *European Journal of Operational Research*, 252(1):213–219, 2016.
7. W. D. Collins and C. Hu. Studying interval valued matrix games with fuzzy logic. *Soft Computing*, 12:147–155, 2008.

8. P. Couchman, B. Kouvaritakis, M. Cannon, and F. Prashad. Gaming strategy for electric power with random demand. *IEEE Transactions on Power Systems*, 20(3):1283–1292, 2005.
9. V. DeMiguel and H. Xu. A stochastic multiple leader Stackelberg model: analysis, computation, and application. *Operations Research*, 57(5):1220–1235, 2009.
10. Jiang-Xia Nan Deng-Feng Li and Mao-Jun Zhang. Interval programming models for matrix games with interval payoffs. *Optimization Methods and Software*, 27(1):1–16, 2012.
11. K.-T. Fang, S. Kotz, and K.-W. Ng. *Symmetric Multivariate and Related Distributions*. Chapman and Hall, London, New York, 1990.
12. Jinwu Gao. Uncertain bimatrix game with applications. *Fuzzy Optimization and Decision Making*, 12(1):65–78, 2013.
13. B. Jadamba and F. Raciti. Variational inequality approach to stochastic Nash equilibrium problems with an application to Cournot oligopoly. *Journal of Optimization Theory and Application*, 165(3):1050–1070, 2015.
14. Deng-Feng Li. Linear programming approach to solve interval-valued matrix games. *Journal of Omega*, 39(6):655–666, 2011.
15. O. L. Mangasarian and H. Stone. Two-person nonzero-sum games and quadratic programming. *Journal of Mathematical Analysis and Applications*, 9:348–355, 1964.
16. M. Mazadi, W. D. Rosehart, H. Zareipour, O. P. Malik, and M. Oloomi. Impact of wind integration on electricity markets: A chance-constrained Nash-Cournot model. *International Transactions on Electrical Energy Systems*, 23(1):83–96, 2013.
17. J. F. Nash. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36(1):48–49, 1950.
18. A. Prékopa. *Stochastic Programming*. Springer, Netherlands, 1995.
19. U. Ravat and U. V. Shanbhag. On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games. *SIAM Journal of Optimization*, 21(3):1168–1199, 2011.
20. R. T. Rockafellar. *Convex analysis*. Princeton University Press, 1970.
21. V. V. Singh, O. Jouini, and A. Lisser. Existence of Nash equilibrium for chance-constrained games. *Operations Research Letters*, 44:640–644, 2016.
22. Vikas Vikram Singh, Oualid Jouini, and Abdel Lisser. Distributionally robust chance-constrained games: existence and characterization of Nash equilibrium. *Optimization Letters*, 2017. DOI:10.1007/s11590-016-1077-6.
23. T. Song. *Systems and Management Science by Extremal Methods*, chapter On random payoff matrix games, pages 291–308. Springer Science + Business Media, LLC, 1992.
24. J. von Neumann. On the theory of games. *Math. Annalen*, 100(1):295–320, 1928.
25. J. Valenzuela and M. Mazumdar. Cournot prices considering generator availability and demand uncertainty. *IEEE Transactions on Power Systems*, 22(1):116–125, 2007.

26. D. De Wolf and Y. Smeers. A stochastic version of a Stackelberg-Nash-Cournot equilibrium model. *Management Science*, 43(2):190–197, 1997.
27. H. Xu and D. Zhang. Stochastic Nash equilibrium problems: sample average approximation and applications. *Computational Optimization and Applications*, 55(3):597–645, 2013.