Exact SDP Relaxations with Truncated Moment Matrix for Binary Polynomial Optimization Problems

Shinsaku Sakaue* Akiko Takeda* Sunyoung Kim† Naoki Ito*

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Abstract

For binary polynomial optimization problems (POPs) of degree d with n variables, we prove that the $\lceil (n+d-1)/2 \rceil$th semidefinite (SDP) relaxation in Lasserre's hierarchy of the SDP relaxations provides the exact optimal value. If binary POPs involve only even-degree monomials, we show that it can be further reduced to $\lceil (n+d-2)/2 \rceil$. This bound on the relaxation order coincides with the conjecture by Laurent in 2003, which was recently proved by Fawzi, Saunderson and Parrilo, on binary quadratic optimization problems where $d = 2$. We also numerically confirm that the bound is tight. More precisely, we present instances of binary POPs that require solving at least the $\lceil (n+d-1)/2 \rceil$th SDP relaxation for general binary POPs and the $\lceil (n+d-2)/2 \rceil$th SDP relaxation for even-degree binary POPs to obtain the exact optimal values.

Keywords. Binary polynomial optimization problems, the hierarchy of the SDP relaxations, the bound for the exact SDP relaxation, even-degree binary polynomial optimization problems, chordal graph.

AMS Classification. 90C22, 90C25, 90C26.

1 Introduction

We are concerned with binary polynomial optimization problems (POPs):

$$(1) \qquad \begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \{-1, 1\}^n, \end{array}$$

where f is a polynomial of degree d and n is the number of variables. Binary POPs of form (1) arise in various applications. In particular, many applications in combinatorial optimization can be cast as binary quadratic optimization problems (QOPs), for instance, the maximum cut problem [6], the maximum stable set problems [9], and the quadratic assignment problems [14], which are special cases of (1). Moreover, binary POPs represent important classes of applications such as the maximum satisfiability problems [1] and the maximum weighted independent set problem [8].

*Department of Mathematical Informatics, The University of Tokyo, Tokyo 113-8656, Japan ({shinsaku_sakaue,takeda,naoki_ito}@mist.i.u-tokyo.ac.jp). The work of the second author was supported by Grant-in-Aid for Scientific Research (C), 15K00031.

†Department of Mathematics, Ewha W. University, 52 Ewhayeodaegil, Sudaemoon-gu, Seoul 120-750 Korea (skim@ewha.ac.kr). The research of the third author was supported by NRF 2014-R1A2A1A11049618.

Lasserre's hierarchy of semidefinite (SDP) relaxations [12] is one of the most extensively studied solution methods for general POPs including binary POPs (1). In his hierarchy of SDP relaxations parametrized by a parameter called the relaxation order ω, a sequence of increasingly large SDP relaxations should be solved for a tight optimal value. The sequence of SDP relaxations to be solved for the optimal value of general POPs is not finite. Unlike general POPs, a binary POP has a finite number of SDP relaxations to be solved in the hierarchy for the exact optimal value. Lasserre [11] showed that binary POPs were equivalently transformed into the SDP with $2^n - 1$ variables, or nth SDP relaxation where $\omega = n$ in the hierarchy. The result in [11] holds for arbitrary binary constrained POPs. For binary QOPs of form (1), Laurent in [13] conjectured that solving the SDP relaxation with the relaxation order $\omega = \lceil n/2 \rceil$ in the hierarchy provided the exact optimal value. Recently, Fawzi, Saunderson and Parrilo [4] proved the conjecture by applying their results on optimization problems on a finite abelian group to the binary QOPs. More recently, Kurpisz, Leppänen and Mastrolilli [10] presented the hardest binary problem where $n - 1$ relaxation order in Lasserre's hierarchy of SDP relaxations is not tight. However, any bound smaller than n on the relaxation order for the exact optimal value of binary POP (1) such as $\lceil n/2 \rceil$ for binary QOPs has not been proposed nor proved to the best of our knowledge.

We prove that the relaxation order ω to obtain the exact optimal value of (1) is $\lceil (n + d - 1)/2 \rceil$, in particular, $\lceil (n + d - 2)/2 \rceil$ if all monomials of f are even degree. For the binary QOPs where $d = 2$ and the degree of all monomials of f is even, our result coincides with $\lceil n/2 \rceil$ shown in [4]. However, the relaxation order $\lceil (n + d - 1)/2 \rceil$ presented in this paper includes more general binary POPs of degree greater than or equal to 3. We note that for the degree of (1), $d \leq n$ always holds since $x_i^2 = 1$ $(i = 1, \ldots, n)$. Thus, the relaxation order for the exact optimal value of (1) presented in this paper $\lceil (n + d - 1)/2 \rceil$ is bounded by n in [11]. For the binary POP shown to have an integrality gap at $\omega = n - 1$ in [10] by Kurpisz *et al.*, the degree of the binary POP was n as indicated in Theorem 3 of [10]. Since $\lceil (n + d - 1)/2 \rceil = n$ if $d = n$, their result on the binary POP [10] can be derived by our result in this paper.

For the proof of the relaxation order for the exact optimal value of binary POPs, we use Theorem 1D of [4] together with the concept of chordal graphs, especially, a chordal cover constructed from the cliques of the Cayley graph, and a Fourier support. We note that the Cayley graph is obtained from the monomials of binary POPs (1). To apply Theorem 1D of [4] to binary POPs, we need to find a chordal cover of the Cayley graph and then a Fourier support of the chordal cover, which leads to the bound on the relaxation order of Lasserre's hierarchy.

The main idea of finding a chordal cover of the Cayley graph corresponding to binary POP (1) in this paper is to *paste cliques* together along the complete graphs, utilizing the result from the graph theory that a graph is chordal if and only if it can be constructed recursively by pasting the complete subgraphs starting from complete graphs. Then, a chordal cover from the pasted cliques is found to construct a Fourier support for binary POP (1). As a result, we obtain the truncated moment matrix that corresponds to the $\lceil (n + d - 1)/2 \rceil$th SDP relaxation in the hierarchy.

The theoretical bounds are confirmed by numerical experiments on random instances of binary POPs (1) and the binary POP with all-one coefficients of f in (1). We randomly generated 100 binary POPs with $n = 8$ and $d = 1, 2, \ldots, 8$ to see that $\omega = \lceil (n + d - 1)/2 \rceil$ is necessary to obtain the optimal values of the test problems. The numerical results on all 100 randomly generated test problems show the discrepancy from the theoretical bound. More precisely, the exact optimal values of the randomly generated test problems are obtained by the SDP relaxation with ω which is smaller than $\lceil (n+d-1)/2 \rceil$ for binary POPs and $\lceil (n+d-2)/2 \rceil$ for even-degree binary POPs. However, we provide a binary POP and an even-degree binary POP whose exact optimal values are obtained only at $\omega = \lceil (n+d-1)/2 \rceil$ and $\omega = \lceil (n+d-2)/2 \rceil$, respectively, not with any smaller relaxation order. The problems are the binary POPs in the form of (1) where the coefficients of f are all ones. Thus, we observe that the numerical result is consistent with the theoretical bound.

1.1 Notation and symbols

We list the notation used in this paper as follows:

- For a given finite set F, $\mathbb{R}^F$ means the space of real vectors indexed by the elements of F. Similarly, $\mathbb{R}^{F \times F}$ denotes the space of matrices whose rows and columns are indexed by the elements of F.
- $[n] := \{1, 2, \ldots, n\}$, $2^{[n]} := \{S : S \subseteq [n]\}$, $\mathcal{S}^d := \{S \in 2^{[n]} : |S| \leq d\}$.
- For given $S, T \in 2^{[n]}$, $S \mathit{\Delta} T := (S \backslash T) \cup (T \backslash S)$. Note that $2^{[n]}$ forms a group with this operation; the identity element is $\emptyset$ and $S^{-1} = S$ for all $S \in 2^{[n]}$.
- For $\mathcal{S}, \mathcal{T} \subseteq 2^{[n]}$, $\mathcal{S} \mathit{\Delta} \mathcal{T} := \{S \mathit{\Delta} T : \forall S \in \mathcal{S}, \forall T \in \mathcal{T}\}$.
- $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for a vector of variables x and a nonnegative vector α. $(x^\alpha)_{\alpha \in \mathcal{D}}$ is a vector of monomials x^α for all α in the set $\mathcal{D}$ of vectors. $(\ell_\delta)_{\delta \in \mathcal{D}}$ is a vector of variables ℓ_δ with index $\delta \in \mathcal{D}$.
- $\mathcal{G} = (V, E)$ is a graph with the vertex set V and the edge set E.
- If two graphs $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ are induced from the same graph, then $\mathcal{G}_1 \cup \mathcal{G}_2$ is a graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. Similarly, $\mathcal{G}_1 \cap \mathcal{G}_2$ denotes a graph with the vertex set $V_1 \cap V_2$ and the edge set $E_1 \cap E_2$.

Let $\alpha \in \mathbb{Z}^n$ be a nonnegative vector, then the polynomial f is expressed as

$$f(x) = \sum_{\|\alpha\|_1 \leq d} c_\alpha x^\alpha,$$

where c_α are coefficients and $\|\alpha\|_1 := \sum_{i=1}^n \alpha_i$. Since (1) requires $x \in \{-1, 1\}^n$, we can assume $\alpha \in \{0, 1\}^n$. Thus, $\|\alpha\|_1$ denotes the number of nonzero elements in α. Note that $d \leq n$ and that the set of monomials $\{x^\alpha : \alpha \in \{0,1\}^n, \|\alpha\|_1 \leq d\}$ forms a basis for real-valued polynomials of degree d on $\{-1, 1\}^n$. Hence, (1) can be expressed as

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \displaystyle\sum_{\alpha \in \{0,1\}^n,\ \|\alpha\|_1 \leq d} c_\alpha x^\alpha \\ \text{subject to} & x \in \{-1, 1\}^n. \end{array} \tag{2}$$

1.2 An illustrative example

The following exemplary problem is used throughout the paper to illustrate our discussion.

$$\begin{array}{ll} \underset{x}{\text{minimize}} & 1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 \\ \text{subject to} & x \in \{-1, 1\}^3. \end{array} \tag{3}$$

Since $n = 3$ and $d = 2$ in (3), we notice that $[n] = \{1, 2, 3\}$,

$$2^{[3]} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},$$

and

$$\mathcal{S}^2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

We can also apply our discussion to a binary POP of degree higher than 2, but it is too complicated to display the Cayley graph of the binary POP, and a chordal cover of the Cayley graph, thus, (3) is used for illustration.

1.3 Organization

This paper is organized as follows: In Section 2, we introduce basic concepts necessary for the subsequent discussion such as the group indicator vectors, the moment polytope, the truncated moment matrix, the Cayley graph, and the Fourier support. We use (3) to explain the concepts. Section 3 includes the main results of this paper. We describe pasting the cliques to construct a choral cover and a Fourier support, and how to obtain the truncated moment matrix. In Section 4, we focus on even-degree binary POPs. The method to obtain the reduced bound $\lceil (n + d - 2)/2 \rceil$ is discussed. In addition, we describe the method to extract the optimal solution of even-degree binary POPs from the truncated moment matrix. In Section 5, we test randomly generated binary POPs and the binary POP with all-one coefficients to confirm the theoretical bound obtained in Sections 3 and 4. Finally, we conclude in Section 6.

2 Preliminaries

We introduce the basics for the subsequent discussion and the results in [4] for $x \in \{-1, 1\}^n$.

2.1 The group of indicator vectors

For a given $S \in 2^{[n]}$, let $\delta_S \in \{0, 1\}^n$ be the indicator vector of S. We define $\mathcal{D} := \{\delta_S : S \in 2^{[n]}\}$. For a given set $\mathcal{S} \subseteq 2^{[n]}$, $\mathcal{D}_{\mathcal{S}} := \{\delta_S : S \in \mathcal{S}\}$. We often use $\delta \in \mathcal{D}$ to express an element of $\mathcal{D}$, abbreviating its index. The product of $\delta_S, \delta_T \in \mathcal{D}$ is defined as

$$\delta_S \delta_T := \delta_{S \triangle T}, \quad S, T \in 2^{[n]}.$$

For example, if $\delta_{\{1,2\}} = (1, 1, 0)$ and $\delta_{\{2,3\}} = (0, 1, 1)$, then $\delta_{\{1,2\}}\delta_{\{2,3\}} = \delta_{\{1,3\}} = (1, 0, 1)$. Note that the aforementioned operation returns the exclusive disjunction for each entry. For notational simplicity, we frequently identify an indicator vector δ with its numeric string, for instance, $\delta_{\{1,2\}} = 110$.

Note that $\mathcal{D}$ forms a group with the multiplication; the identity element is $\delta_\emptyset$ and $\delta^{-1} = \delta$. We also remark that $\{x^\delta : \delta \in \mathcal{D}\}$ is the set of all monomials on $\{-1, 1\}^n$ and $\{x^\delta : \delta \in \mathcal{D}_{\mathcal{S}^d}\}$ is the set of monomials on $\{-1, 1\}^n$ whose degree is less than or equal to d.

2.2 Moment polytopes and moment matrices

Linearizing problem (2), we obtain

$$(4) \qquad \underset{\ell \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^d}}}{\text{minimize}} \quad \sum_{\delta \in \mathcal{D}_{\mathcal{S}^d}} c_\delta \ell_\delta$$

$$\text{subject to} \quad (\ell_\delta)_{\delta \in \mathcal{D}_{\mathcal{S}^d}} \in \text{conv}\left\{ (x^\delta)_{\delta \in \mathcal{D}_{\mathcal{S}^d}} \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^d}} : x \in \{-1, 1\}^n \right\},$$

where ℓ_δ ($\forall \delta \in \mathcal{D}_{\mathcal{S}^d}$) are decision variables. The feasible set can be characterized by the *moment polytope* defined as follows.

Definition 2.1. *The moment polytope $\mathcal{M}(\mathcal{S})$ for given $\mathcal{S} \subseteq 2^{[n]}$ is defined as the convex hull of the vectors $(x^\delta)_{\delta \in \mathcal{D}_\mathcal{S}}$ for $x \in \{-1,1\}^n$, i.e.,*

$$\mathcal{M}(\mathcal{S}) := conv\left\{(x^\delta)_{\delta \in \mathcal{D}_\mathcal{S}} \in \mathbb{R}^{\mathcal{D}_\mathcal{S}} : x \in \{-1,1\}^n\right\}.$$

Note that problem (4) can be expressed with the moment polytope:

$$\begin{aligned} &\underset{\ell \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^d}}}{\text{minimize}} \quad \sum_{\delta \in \mathcal{D}_{\mathcal{S}^d}} c_\delta \ell_\delta \\ &\text{subject to} \quad (\ell_\delta)_{\delta \in \mathcal{D}_{\mathcal{S}^d}} \in \mathcal{M}(\mathcal{S}^d). \end{aligned} \tag{5}$$

Definition 2.2. *For a given vector $\ell \in \mathbb{R}^\mathcal{D}$, the moment matrix $M(\ell) \in \mathbb{R}^{\mathcal{D}\times\mathcal{D}}$ is defined as a matrix whose $(\delta, \widehat{\delta})$th element is given by $\ell_{\delta\widehat{\delta}}$ for all $\delta, \widehat{\delta} \in \mathcal{D}$.*

Example 2.3. *For $n = 3$, the set of indicator vectors is*

$$\mathcal{D} = \{000, 100, 010, 001, 110, 101, 011, 111\}.$$

With a given vector $\ell = (\ell_{000}\ \ell_{100}\ \ell_{010}\ \ell_{001}\ \ell_{110}\ \ell_{101}\ \ell_{011}\ \ell_{111})^\top \in \mathbb{R}^\mathcal{D}$, the moment matrix $M(\ell) \in \mathbb{R}^{\mathcal{D}\times\mathcal{D}}$ is defined as follows:

$$M(\ell) := \begin{bmatrix} \ell_{000} & \ell_{100} & \ell_{010} & \ell_{001} & \ell_{110} & \ell_{101} & \ell_{011} & \ell_{111} \\ \ell_{100} & \ell_{000} & \ell_{110} & \ell_{101} & \ell_{010} & \ell_{001} & \ell_{111} & \ell_{011} \\ \ell_{010} & \ell_{110} & \ell_{000} & \ell_{011} & \ell_{100} & \ell_{111} & \ell_{001} & \ell_{101} \\ \ell_{001} & \ell_{101} & \ell_{011} & \ell_{000} & \ell_{111} & \ell_{100} & \ell_{010} & \ell_{110} \\ \ell_{110} & \ell_{010} & \ell_{100} & \ell_{111} & \ell_{000} & \ell_{011} & \ell_{101} & \ell_{001} \\ \ell_{101} & \ell_{001} & \ell_{111} & \ell_{100} & \ell_{011} & \ell_{000} & \ell_{110} & \ell_{010} \\ \ell_{011} & \ell_{111} & \ell_{001} & \ell_{010} & \ell_{101} & \ell_{110} & \ell_{000} & \ell_{100} \\ \ell_{111} & \ell_{011} & \ell_{101} & \ell_{110} & \ell_{001} & \ell_{010} & \ell_{100} & \ell_{000} \end{bmatrix}.$$

Definition 2.4. *Let $\mathcal{T} \subseteq 2^{[n]}$. For a given vector $\ell \in \mathbb{R}^{\mathcal{D}_{\mathcal{T}\Delta\mathcal{T}}}$, the truncated moment matrix $M_\mathcal{T}(\ell) \in \mathbb{R}^{\mathcal{D}_\mathcal{T}\times\mathcal{D}_\mathcal{T}}$ is defined as a matrix whose $(\delta, \widehat{\delta})$th element is given by $\ell_{\delta\widehat{\delta}}$ for all $\delta, \widehat{\delta} \in \mathcal{D}_\mathcal{T}$.*

Example 2.5. *If $\mathcal{T} \subseteq 2^{[3]}$ is given by $\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$, then $\mathcal{T}\Delta\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Hence $\mathcal{D}_\mathcal{T} = \{000, 100, 010, 001, 110, 101, 011\}$ and $\mathcal{D}_{\mathcal{T}\Delta\mathcal{T}} = \{000, 100, 010, 001, 110, 101, 011, 111\}$. Thus for a given vector $\ell = (\ell_{000}\ \ell_{100}\ \ell_{010}\ \ell_{001}\ \ell_{110}\ \ell_{101}\ \ell_{011}\ \ell_{111})^\top \in \mathbb{R}^{\mathcal{D}_{\mathcal{T}\Delta\mathcal{T}}}$, the truncated moment matrix $M_\mathcal{T}(\ell) \in \mathbb{R}^{\mathcal{D}_\mathcal{T}\times\mathcal{D}_\mathcal{T}}$ is defined as follows:*

$$M_\mathcal{T}(\ell) := \begin{bmatrix} \ell_{000} & \ell_{100} & \ell_{010} & \ell_{001} & \ell_{110} & \ell_{101} & \ell_{011} \\ \ell_{100} & \ell_{000} & \ell_{110} & \ell_{101} & \ell_{010} & \ell_{001} & \ell_{111} \\ \ell_{010} & \ell_{110} & \ell_{000} & \ell_{011} & \ell_{100} & \ell_{111} & \ell_{001} \\ \ell_{001} & \ell_{101} & \ell_{011} & \ell_{000} & \ell_{111} & \ell_{100} & \ell_{010} \\ \ell_{110} & \ell_{010} & \ell_{100} & \ell_{111} & \ell_{000} & \ell_{011} & \ell_{101} \\ \ell_{101} & \ell_{001} & \ell_{111} & \ell_{100} & \ell_{011} & \ell_{000} & \ell_{110} \\ \ell_{011} & \ell_{111} & \ell_{001} & \ell_{010} & \ell_{101} & \ell_{110} & \ell_{000} \end{bmatrix}.$$

Notice that for the given moment matrix $M(\ell)$, the truncated moment matrix $M_\mathcal{T}(\ell)$ is the submatrix of $M(\ell)$ such that the rows and columns are obtained from $M(\ell)$ according to $\mathcal{D}_\mathcal{T}$.

The moment polytope is characterized by the moment matrix in the following proposition. For the proof, we refer the reader to [4].

Proposition 2.6. *The moment polytope with respect to $\mathcal{S} \subseteq 2^{[n]}$ can be expressed as follows:*

$$\mathcal{M}(\mathcal{S}) = \{\ell \in \mathbb{R}^{\mathcal{D}_\mathcal{S}} : \exists y \in \mathbb{R}^{\mathcal{D}} \text{ s.t. } y_\delta = \ell_\delta \text{ for all } \delta \in \mathcal{D}_\mathcal{S},\ y_{\delta_\emptyset} = 1,\ M(y) \succeq 0\}.$$

2.3 Expressing the moment polytope with the truncated moment matrix

In Proposition 2.6, the moment polytope has been represented with the moment matrix. To describe how the moment polytope can be expressed by the truncated moment matrix, we briefly introduce the chordal graph, the Cayley graph and a Fourier support. We note that the *chordal completion theorem* (see, e.g., [2, §12.3]) played an important role in the proof of Theorem 1D of [4], which is the main result for proving the conjecture in [13].

The graph $\mathcal{G} = (V, E)$ is *chordal* if any cycle of length at least four has a chord. A *chordal cover* of $\mathcal{G}$ is a graph $\mathcal{G}' = (V, E')$ where $E \subset E'$ and $\mathcal{G}'$ is chordal. A subset $\mathcal{C} \subseteq V$ is a *clique* in $\mathcal{G}$ if $\{i, j\} \in E$ for all $i, j \in \mathcal{C},\ i \neq j$. The clique $\mathcal{C}$ is called *maximal* if it is not a strict subset of another clique $\mathcal{C}'$ of $\mathcal{G}$.

Definition 2.7. *For a given set $\mathcal{S} \subseteq 2^{[n]}$, the Cayley graph Cay($\mathcal{S}$) is the graph such that the vertex set is $2^{[n]}$ and two distinct vertices $S, T \in 2^{[n]}$ are connected by an edge if and only if $S \Delta T \in \mathcal{S}$.*

Definition 2.8. *Let Γ be a graph with vertices $2^{[n]}$. We say $\mathcal{T} \subseteq 2^{[n]}$ is a Fourier support of Γ if for any maximal clique $\mathcal{C} \subseteq 2^{[n]}$ of Γ, there exists a node $S_\mathcal{C} \in 2^{[n]}$ such that $S_\mathcal{C} \Delta \mathcal{C} \subseteq \mathcal{T}$.*

Example 2.9. *For $n = 3$ and $\mathcal{S}^2 = \{S \in 2^{[3]} : |S| \leq 2\}$, Figure 1 displays the Cayley graph Cay($\mathcal{S}^2$), the chordal cover Γ, and the Fourier support $\mathcal{T}$ of Γ. Cay($\mathcal{S}^2$) consists of all eight vertices of $2^{[3]}$ and edges shown by solid lines. The graph obtained by adding three dashed lines to Cay($\mathcal{S}^2$) is a chordal cover Γ of Cay($\mathcal{S}^2$). The vertex set $\mathcal{T} = \{T \in 2^{[3]} : |T| \leq 2\}$ is a Fourier support of Γ. Indeed, the chordal graph Γ has two maximal cliques $\mathcal{C}_0 = \{S \in 2^{[3]} : |S| \leq 2\}$ and $\mathcal{C}_1 = \{S \in 2^{[3]} : |S| \geq 1\}$, for which $S_{\mathcal{C}_0} = \emptyset$ satisfies $S_{\mathcal{C}_0} \Delta \mathcal{C}_0 \subseteq \mathcal{T}$ and $S_{\mathcal{C}_1} = \{1, 2, 3\}$ does $S_{\mathcal{C}_1} \Delta \mathcal{C}_1 \subseteq \mathcal{T}$.*

We are now in the position to describe Theorem 1D of [4] that gives the expression of the moment polytope with an appropriately truncated moment matrix. More precisely, the feasible set of the problems (5) can be expressed with a smaller moment matrix. For the proof, see [4].

Theorem 2.10. *[4] Let $\mathcal{S}$ be a given subset of $2^{[n]}$. If Cay($\mathcal{S}$) has a chordal cover Γ with Fourier support $\mathcal{T} \subseteq 2^{[n]}$, then the moment polytope can be expressed as follows:*

$$\mathcal{M}(\mathcal{S}) = \left\{ \ell \in \mathbb{R}^{\mathcal{D}_\mathcal{S}} \ :\ \exists y \in \mathbb{R}^{\mathcal{D}_{\mathcal{T}\Delta\mathcal{T}}} \text{ s.t. } \begin{array}{l} y_\delta = \ell_\delta \text{ for all } \delta \in \mathcal{D}_\mathcal{S}, \\ y_{\delta_\emptyset} = 1, \text{ and } M_\mathcal{T}(y) \succeq 0 \end{array} \right\}.$$

It is not hard to confirm $\mathcal{S} \subseteq \mathcal{T}\Delta\mathcal{T}$ to ensure that ℓ_δ ($\delta \in \mathcal{S}$) is well defined by y_δ ($\delta \in \mathcal{D}_{\mathcal{T}\Delta\mathcal{T}}$), although the proof in [4] is in a more general setting. Since $\mathcal{T}\Delta\mathcal{T}$ always contains $\emptyset$, we now show that arbitrary nonempty $S \in \mathcal{S}$ is in $\mathcal{T}\Delta\mathcal{T}$. If $S \in \mathcal{S}$, then $\emptyset \Delta S = S \in \mathcal{S}$, thus Cay($\mathcal{S}$) has the edge $(\emptyset, S)$, which means that the chordal cover Γ also has the edge $(\emptyset, S)$. Since any edge is included in some maximal clique, the edge $(\emptyset, S)$

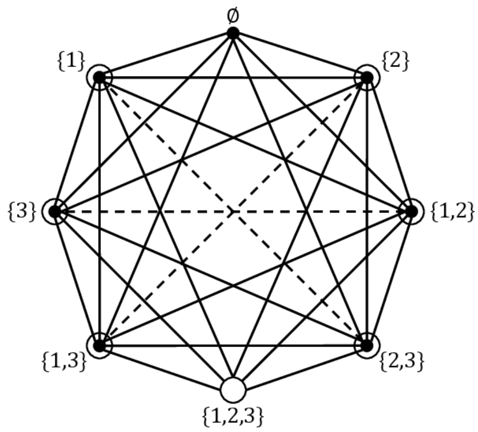

Figure 1: For the case $n = 3$ and $\mathcal{S}^2$, the solid lines represent the edges of the Cayley graph $\mathrm{Cay}(\mathcal{S}^2)$, and its chordal cover Γ is obtained by adding the dashed lines to the edge set. Note that Γ does not contain the edge between $\emptyset$ and $\{1, 2, 3\}$. The vertex set $\mathcal{T} = \{T \in 2^{[3]} : |T| \leq 2\}$ is the Fourier support (indicated by filled circles) of Γ since, for all maximal cliques, $\mathcal{C}_0 = \{S \in 2^{[3]} : |S| \leq 2\}$ and $\mathcal{C}_1 = \{S \in 2^{[3]} : |S| \geq 1\}$ of Γ, we have $\emptyset \Delta \mathcal{C}_0 \subseteq \mathcal{T}$ and $\{1, 2, 3\} \Delta \mathcal{C}_1 \subseteq \mathcal{T}$.

is included in some maximal clique $\mathcal{C}$ of Γ. Therefore, by the definition of the Fourier support $\mathcal{T}$, there exists a node $S_{\mathcal{C}}$ such that $S_{\mathcal{C}} \Delta \emptyset \in \mathcal{T}$ and $S_{\mathcal{C}} \Delta S \in \mathcal{T}$. Hence,

$$S = \emptyset \Delta S = (S_{\mathcal{C}} \Delta \emptyset) \Delta (S_{\mathcal{C}} \Delta S) \in \mathcal{T} \Delta \mathcal{T},$$

consequently, $\mathcal{S} \subseteq \mathcal{T} \Delta \mathcal{T}$ holds.

3 Truncation of the moment matrix for binary POPs

Recall that $\mathcal{M}(\mathcal{S}^d)$ in Definition 2.1 provides the feasible set of the problem (5). In what follows, we focus on finding an appropriate Fourier support $\mathcal{T}$ of a chordal graph Γ that covers $\mathrm{Cay}(\mathcal{S}^d)$. If such $\mathcal{T}$ is obtained, then we can characterize the moment polytope $\mathcal{M}(\mathcal{S}^d)$ with a truncated moment matrix $M_{\mathcal{T}}(y)$ using Theorem 2.10.

3.1 Constructing a chordal cover

If $\mathcal{G}$ is a graph with subgraphs $\mathcal{G}_1, \mathcal{G}_2$ such that $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, we say that $\mathcal{G}$ arises from $\mathcal{G}_1$ and $\mathcal{G}_2$ by *pasting* these graphs together along $\mathcal{H} := \mathcal{G}_1 \cap \mathcal{G}_2$. We have the following proposition for the construction of a chordal graph (see [3, Proposition 5.5.1.] for the proof).

Proposition 3.1. *A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.*

For example, in Figure 1, the chordal cover Γ is obtained by pasting two complete graphs $\mathcal{C}_0 = \{S \in 2^{[3]} : |S| \leq 2\}$ and $\mathcal{C}_1 = \{S \in 2^{[3]} : |S| \geq 1\}$ along the complete graph $\mathcal{C}_0 \cap \mathcal{C}_1$ whose vertex set is $\{S \in 2^{[3]} : 1 \leq |S| \leq 2\}$.

For notational convenience, let

$$\mathcal{T}_k := \{S \in 2^{[n]} : |S| = k\}, \quad k = 0, 1, \ldots, n. \tag{6}$$

Note that $2^{[n]} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$. We define $n-d+1$ cliques whose vertex sets are given by

$$\mathcal{C}_k := \mathcal{T}_k \cup \mathcal{T}_{k+1} \cup \cdots \cup \mathcal{T}_{k+d}, \quad k = 0, 1, \dots, n-d. \tag{7}$$

For simplicity, we also use $\mathcal{C}_k$ to express the clique itself whose vertex set is given by (7).

We now describe how to construct a chordal cover Γ of $\mathrm{Cay}(\mathcal{S}^d)$. Let Γ be a graph obtained by pasting $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ along the complete graph $\mathcal{C}_k \cap \mathcal{C}_{k+1}$ for $k = 0, 1, \dots, n-d-1$. Observe that the vertex set of Γ is $2^{[n]}$ and that $S, T \in 2^{[n]}$ are connected in Γ if and only if $||S| - |T|| \leq d$ holds. Then, Γ is chordal by Proposition 3.1 since Γ is obtained by pasting complete graphs $\mathcal{C}_k$ $(k = 0, 1, \dots, n-d)$ along the complete graph $\mathcal{C}_k \cap \mathcal{C}_{k+1}$. Moreover, Γ covers $\mathrm{Cay}(\mathcal{S}^d)$ since all connected vertices $S, T \in 2^{[n]}$ in $\mathrm{Cay}(\mathcal{S}^d)$ satisfy $|S \varDelta T| \leq d$, which implies $||S| - |T|| \leq d$.

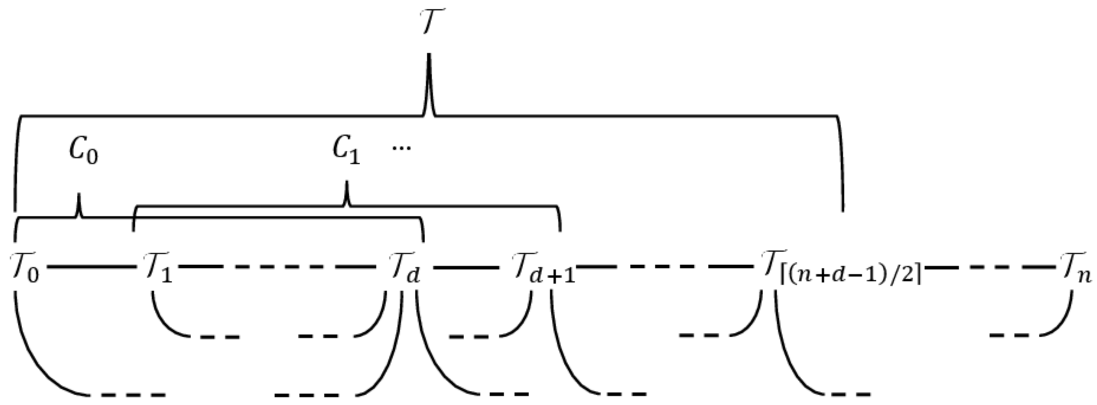

Figure 2: A sketch of the Cayley graph $\mathrm{Cay}(\mathcal{S}^d)$, the chordal cover Γ and the Fourier support $\mathcal{T}$. In $\mathrm{Cay}(\mathcal{S}^d)$, there exists a pair of connected nodes $S \in \mathcal{T}_i$ and $T \in \mathcal{T}_j$ if and only if $|i-j| \leq d$. The chordal cover Γ is obtained by pasting the cliques $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ along $\mathcal{C}_k \cap \mathcal{C}_{k+1}$ for $k = 0, 1, \dots, n-d-1$ and the Fourier support of Γ is $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{\lceil (n+d-1)/2 \rceil}$, which is shown in the next subsection.

3.2 A Fourier support of the chordal cover

Let $\mathcal{T} \subseteq 2^{[n]}$ be a vertex set defined as

$$\mathcal{T} := \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{\lceil \frac{n+d-1}{2} \rceil}. \tag{8}$$

We show that $\mathcal{T}$ is a Fourier support of Γ defined in Section 3.1.

We first examine whether the cliques $\mathcal{C}_k$ $(k = 0, 1, \dots, n-d)$ are maximal in Γ. To show this, we prove that for all $S \notin \mathcal{C}_k$ there exists $T \in \mathcal{C}_k$ such that S and T are disjoint in Γ. Note that all vertices $S \notin \mathcal{C}_k$ satisfy either $|S| < k$ or $|S| > k+d$. If $|S| < k$, for a vertex $T \in \mathcal{T}_{k+d} \subset \mathcal{C}_k$, we have

$$|S \varDelta T| \geq ||S| - |T|| > d.$$

Thus S and T are disjoint. Similarly, if $|S| > k+d$, there is a vertex $T \in \mathcal{T}_k \subset \mathcal{C}_k$ that satisfies $|S \varDelta T| > d$. Since the aforementioned discussion holds for $k = 0, 1, \dots, n-d$, the cliques $\mathcal{C}_k$ $(k = 0, 1, \dots, n-d)$ are maximal in Γ.

We next prove that $\mathcal{T}$ defined in (8) is a Fourier support of Γ, i.e., for all maximal cliques $\mathcal{C}_k$ $(k = 0, 1, \ldots, n-d)$, there exists $S_{\mathcal{C}_k} \in 2^{[n]}$ such that $S_{\mathcal{C}_k} \Delta \mathcal{C}_k \subseteq \mathcal{T}$. Let us choose $S_{\mathcal{C}_k}$ for $\mathcal{C}_k$ $(k = 0, 1, \ldots, n-d)$ as follows:

If $k \le \lceil \frac{n+d-1}{2} \rceil - d$, then $\mathcal{C}_k \subseteq \mathcal{T}$ and thus $S_{\mathcal{C}_k} = \emptyset$.

If $k \ge \lceil \frac{n+d-1}{2} \rceil - d + 1$, then

$$n - k \le n + d - 1 - \left\lceil \frac{n+d-1}{2} \right\rceil \le \left\lceil \frac{n+d-1}{2} \right\rceil .$$

Hence,

$$[n] \Delta \mathcal{C}_k = \mathcal{T}_{n-(k+d)} \cup \mathcal{T}_{n-(k+d)+1} \cup \cdots \cup \mathcal{T}_{n-k} \subseteq \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{\lceil \frac{n+d-1}{2} \rceil} = \mathcal{T}.$$

Thus, $S_{\mathcal{C}_k} = [n]$.

As a consequence, we obtain the following result by Theorem 2.10.

Theorem 3.2. *Let $\mathcal{T}$ be the subset of $2^{[n]}$ defined by (8). Then the moment polytope $\mathcal{M}(\mathcal{S}^d)$ can be expressed as follows:*

$$\mathcal{M}(\mathcal{S}^d) = \left\{ \ell \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^d}} \ : \ \exists y \in \mathbb{R}^{\mathcal{D}_{\mathcal{T}\Delta\mathcal{T}}} \ s.t. \ \begin{array}{l} y_\delta = \ell_\delta \ \textit{for all} \ \delta \in \mathcal{D}_{\mathcal{S}^d}, \\ y_{\delta_\emptyset} = 1, \ \textit{and} \ M_{\mathcal{T}}(y) \succeq 0 \end{array} \right\}.$$

For binary POP (1), Theorem 3.2 provides the exact SDP relaxation with the truncated moment matrix indexed by $\delta_S \in \mathcal{D}$ such that $|S| \le \lceil \frac{n+d-1}{2} \rceil$.

3.3 Illustrating the truncation for example (3)

For the problem (3) in Section 1.2, linearizing the moment polytope leads to the following problem with variables $\ell \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^2}}$:

$$\begin{array}{ll} \underset{\ell \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^2}}}{\text{minimize}} & \ell_{000} + \ell_{100} + \ell_{010} + \ell_{001} + \ell_{110} + \ell_{101} + \ell_{011} \\ \text{subject to} & \begin{pmatrix} \ell_{000} \\ \ell_{100} \\ \ell_{010} \\ \ell_{001} \\ \ell_{110} \\ \ell_{101} \\ \ell_{011} \end{pmatrix} \in \left\{ \begin{array}{l} \ell \ : \ \exists y_{111} \ \ \text{s.t.} \ \ \ell_{000} = 1 \ \ \text{and} \\ \begin{bmatrix} 1 & \ell_{100} & \ell_{010} & \ell_{001} & \ell_{110} & \ell_{101} & \ell_{011} & y_{111} \\ \ell_{100} & 1 & \ell_{110} & \ell_{101} & \ell_{010} & \ell_{001} & y_{111} & \ell_{011} \\ \ell_{010} & \ell_{110} & 1 & \ell_{011} & \ell_{100} & y_{111} & \ell_{001} & \ell_{101} \\ \ell_{001} & \ell_{101} & \ell_{011} & 1 & y_{111} & \ell_{100} & \ell_{010} & \ell_{110} \\ \ell_{110} & \ell_{010} & \ell_{100} & y_{111} & 1 & \ell_{011} & \ell_{101} & \ell_{001} \\ \ell_{101} & \ell_{001} & y_{111} & \ell_{100} & \ell_{011} & 1 & \ell_{110} & \ell_{010} \\ \ell_{011} & y_{111} & \ell_{001} & \ell_{010} & \ell_{101} & \ell_{110} & 1 & \ell_{100} \\ y_{111} & \ell_{011} & \ell_{101} & \ell_{110} & \ell_{001} & \ell_{010} & \ell_{100} & 1 \end{bmatrix} \succeq 0 \end{array} \right\}. \end{array}$$

The Cayley graph, the chordal cover and the Fourier support are as shown in Example 2.9. Since $\lceil \frac{n+d-1}{2} \rceil = 2$ for $n = 3$ and $d = 2$, Theorem 3.2 indicates that the moment matrix can be truncated by $\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ and the above problem can be equivalently rewritten as

$$\begin{aligned}(9)\qquad &\underset{\ell\in\mathbb{R}^{\mathcal{DS}^2}}{\text{minimize}} \quad \ell_{000}+\ell_{100}+\ell_{010}+\ell_{001}+\ell_{110}+\ell_{101}+\ell_{011}\\ &\text{subject to} \quad \begin{pmatrix}\ell_{000}\\ \ell_{100}\\ \ell_{010}\\ \ell_{001}\\ \ell_{110}\\ \ell_{101}\\ \ell_{011}\end{pmatrix} \in \left\{ \begin{array}{l} \ell \ :\ \exists y_{111} \quad \text{s.t.} \quad \ell_{000}=1 \quad \text{and} \\ \begin{bmatrix} 1 & \ell_{100} & \ell_{010} & \ell_{001} & \ell_{110} & \ell_{101} & \ell_{011}\\ \ell_{100} & 1 & \ell_{110} & \ell_{101} & \ell_{010} & \ell_{001} & y_{111}\\ \ell_{010} & \ell_{110} & 1 & \ell_{011} & \ell_{100} & y_{111} & \ell_{001}\\ \ell_{001} & \ell_{101} & \ell_{011} & 1 & y_{111} & \ell_{100} & \ell_{010}\\ \ell_{110} & \ell_{010} & \ell_{100} & y_{111} & 1 & \ell_{011} & \ell_{101}\\ \ell_{101} & \ell_{001} & y_{111} & \ell_{100} & \ell_{011} & 1 & \ell_{110}\\ \ell_{011} & y_{111} & \ell_{001} & \ell_{010} & \ell_{101} & \ell_{110} & 1 \end{bmatrix} \succeq 0 \end{array} \right\}.\end{aligned}$$

Note that this problem has the positive semidefinite matrix whose size is one less than the original one.

4 Further truncating for even-degree binary POPs

In this section, we assume that only even-degree monomials appear in f of problem (1). Under the assumption, we prove that the truncated moment matrix $M_{\mathcal{T}}(y)$ shown in Theorem 3.2 can be further reduced.

Let the degree of f be $2\widehat{d}$ throughout this section. We note that the discussion in this section is a generalization of [4, §4.1], where $\widehat{d}$ is restricted to 1.

As in Section 2.2, problem (1) is linearized in the form of (5) with an appropriate set $\mathcal{S} \subseteq 2^{[n]}$. Since f has only even-degree monomials, we define $\mathcal{S}$ as $\mathcal{S}^{2\widehat{d}}_{\text{even}}$:

$$\mathcal{S}^{2\widehat{d}}_{\text{even}} := \{S \in 2^{[n]} :\ |S| = 0, 2, \ldots, 2\widehat{d}\}.$$

Observe that any two vertices $S, T \in 2^{[n]}$ are connected in $\text{Cay}(\mathcal{S}^{2\widehat{d}}_{\text{even}})$ if and only if $|S\varDelta T|$ is even and $|S\varDelta T| \leq 2\widehat{d}$ holds. Thus $\text{Cay}(\mathcal{S}^{2\widehat{d}}_{\text{even}})$ has two connected components

$$\mathcal{T}_{\text{even}} := \mathcal{T}_0 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{2\lfloor n/2\rfloor} \quad \text{and} \quad \mathcal{T}_{\text{odd}} := \mathcal{T}_1 \cup \mathcal{T}_3 \cup \cdots \cup \mathcal{T}_{2\lceil n/2\rceil-1},$$

where $\mathcal{T}_k$ $(k = 0, 1, \ldots, n)$ are defined by (6). More precisely, the Cayley graph $\text{Cay}(\mathcal{S}^{2\widehat{d}}_{\text{even}})$ has an edge between S and T if and only if

(i) $S, T \in \mathcal{T}_{\text{even}}$ and $|S\varDelta T| \leq 2\widehat{d}$, or,

(ii) $S, T \in \mathcal{T}_{\text{odd}}$ and $|S\varDelta T| \leq 2\widehat{d}$.

We define a map $\phi : 2^{[n]} \to 2^{[n]}$ by $\phi(S) := \{1\}\varDelta S$ for the succeeding discussion. Note that ϕ is an automorphism of $\text{Cay}(\mathcal{S}^{2\widehat{d}}_{\text{even}})$ that exchanges $\mathcal{T}_{\text{even}}$ and $\mathcal{T}_{\text{odd}}$. In addition, $\phi(S)\varDelta\phi(T) = S\varDelta T$ holds for all $S, T \in 2^{[n]}$.

Example 4.1. *We show an example of the Cayley graph $Cay(\mathcal{S}^{2\widehat{d}}_{even})$ and the map ϕ for an even-degree binary POP with $n = 4$ and $\widehat{d} = 1$, a modified problem of* (3) *where all linear terms are multiplied with an additional variable x_4. The resulting problem is*

$$\begin{aligned}&\underset{x}{\text{minimize}} \quad 1 + x_1x_4 + x_2x_4 + x_3x_4 + x_1x_2 + x_1x_3 + x_2x_3\\ &\text{subject to} \quad x \in \{-1, 1\}^4.\end{aligned}$$

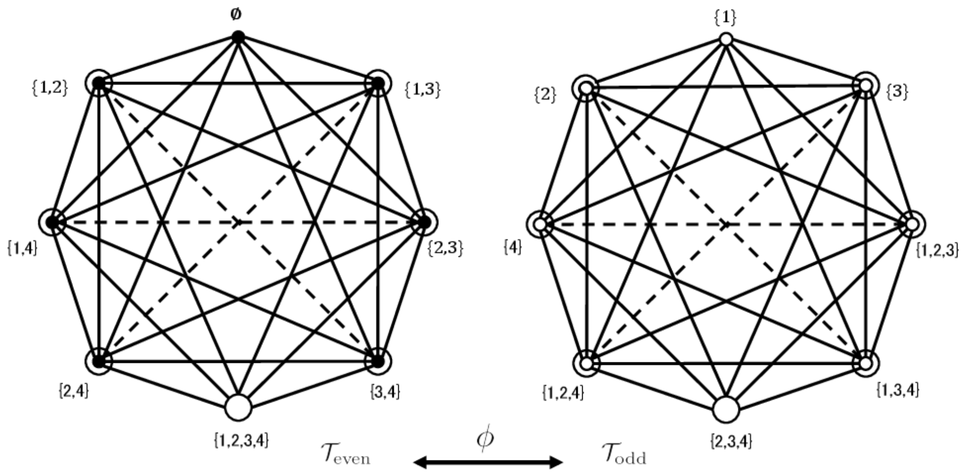

Figure 3: An example of the Cayley graph for the case of $n = 4$ and $\mathcal{S}^2_{\rm even} = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$. Observe that $\phi(S) = \{1\}\Delta S$ is the automorphism of the Cayley graph that exchanges $\mathcal{T}_{\rm even}$ and $\mathcal{T}_{\rm odd}$. The graph generated by adding dash lines to $\mathrm{Cay}(\mathcal{S}^2_{\rm even})$ is a chordal cover Γ of $\mathrm{Cay}(\mathcal{S}^2_{\rm even})$ and the Fourier support $\mathcal{T}$ is $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_2$ (indicated by filled circles) of Γ.

In Section 4.5*, we see that the solution for original problem* (3) *can be obtained by solving the above problem.*

Figure 3 *shows the Cayley graph Cay*$(\mathcal{S}^2_{even})$ *for the case of* $n = 4$ *and* $\mathcal{S}^2_{even} = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$*. Cay*$(\mathcal{S}^2_{even})$ *consists of two connected components,* $\mathcal{T}_{even}$ *and* $\mathcal{T}_{odd}$*. The vertex set of Cay*$(\mathcal{S}^2_{even})$ *is* $2^{[4]}$ *and edges are shown by solid lines. Notice that* $\phi(S) = \{1\}\Delta S$ *is the automorphism of the Cayley graph that exchanges* $\mathcal{T}_{even}$ *and* $\mathcal{T}_{odd}$*.*

Next, as in Section 3, we find a Fourier support $\mathcal{T}$ of a chordal graph Γ that covers $\mathrm{Cay}(\mathcal{S}^{2\widehat{d}}_{\rm even})$. For that, we check whether $\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil$ of a given even-degree POP is even or odd. We first discuss the case where $\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil$ is even in detail. Then, we briefly mention the required modification for the case where $\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil$ is odd. Let us first assume $\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil$ is even.

4.1 Constructing a chordal cover

For the construction of a chordal cover for $\mathrm{Cay}(\mathcal{S}^{2\widehat{d}}_{\rm even})$, we define $\left\lfloor \frac{n}{2} \right\rfloor - \widehat{d} + 1$ cliques whose vertex sets are given by

$$\mathcal{C}_k := \mathcal{T}_k \cup \mathcal{T}_{k+2} \cup \cdots \cup \mathcal{T}_{k+2\widehat{d}}, \quad k = 0, 2, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor - 2\widehat{d}. \tag{10}$$

As previously, we abuse the notation $\mathcal{C}_k$ to express the clique itself whose vertex set is given by (10). Note that the cliques $\mathcal{C}_k$ are indexed by even integer k. With the cliques, we construct a chordal cover Γ of $\mathrm{Cay}(\mathcal{S}^{2\widehat{d}}_{\rm even})$ as follows:

(i) Paste $\mathcal{C}_k$ and $\mathcal{C}_{k+2}$ along the complete graph $\mathcal{C}_k \cap \mathcal{C}_{k+2}$ for $k = 0, 2, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor - 2\widehat{d} - 2$. We denote the resulting connected component by $\Gamma_{\rm even}$.

(ii) Paste $\phi(\mathcal{C}_k)$ and $\phi(\mathcal{C}_{k+2})$ along the complete graph $\phi(\mathcal{C}_k) \cap \phi(\mathcal{C}_{k+2})$ for $k = 0, 2, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor - 2\widehat{d} - 2$. The obtained connected component is denoted by Γ_{odd}.

Now, define $\Gamma := \Gamma_{\text{even}} \cup \Gamma_{\text{odd}}$, which is chordal by Proposition 3.1. Observe that Γ_{even} and Γ_{odd} satisfy the followings:

(iii) $S, T \in \mathcal{T}_{\text{even}}$, which is the vertex set of Γ_{even}, are connected in Γ_{even} if and only if $||S| - |T|| \le 2\widehat{d}$ holds.

(iv) $S, T \in \mathcal{T}_{\text{odd}}$, which is the vertex set of Γ_{odd}, are connected in Γ_{odd} if and only if $||\phi(S)| - |\phi(T)|| \le 2\widehat{d}$ holds.

Thus, $\Gamma = \Gamma_{\text{even}} \cup \Gamma_{\text{odd}}$ covers $\text{Cay}(\mathcal{S}^{2\widehat{d}}_{\text{even}})$.

4.2 A Fourier support of the chordal cover

Let $\mathcal{T} \subseteq 2^{[n]}$ be a vertex set defined as

(11) $$\mathcal{T} := \mathcal{T}_0 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil}.$$

We see that $\mathcal{T}$ is a Fourier support of Γ.

First, check that the cliques $\mathcal{C}_k$ and $\phi(\mathcal{C}_k)$ for $k = 0, 2, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor - 2\widehat{d}$ are maximal in Γ. To show this, we observe that for all $S \notin \mathcal{C}_k$, there exists $T \in \mathcal{C}_k$ such that S and T are disjoint in Γ. If $S \notin \mathcal{C}_k$, then S satisfies

(a) $S \in \mathcal{T}_{\text{odd}}$ or,

(b) $S \in \mathcal{T}_{\text{even}}$ and $|S| < k$ or,

(c) $S \in \mathcal{T}_{\text{even}}$ and $|S| > k + 2\widehat{d}$.

If $S \in \mathcal{T}_{\text{odd}}$, then all vertices in $\mathcal{C}_k$ are not connected to S. If $S \in \mathcal{T}_{\text{even}}$ and $|S| < k$, then all vertices in $\mathcal{T}_{k+2\widehat{d}} \subseteq \mathcal{C}_k$ are not connected to S. If $S \in \mathcal{T}_{\text{even}}$ and $|S| > k + 2\widehat{d}$, then all vertices in $\mathcal{T}_k \subseteq \mathcal{C}_k$ are not connected to S. Therefore, for any $S \notin \mathcal{C}_k$, there exists $T \in \mathcal{C}_k$ such that S and T are disjoint in Γ, which implies $\mathcal{C}_k$ ($k = 0, 2, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor - 2\widehat{d}$) are maximal in Γ. Since ϕ is an automorphism of Γ, the cliques $\phi(\mathcal{C}_k)$ are also maximal in Γ.

We then prove that $\mathcal{T}$ defined in (11) is a Fourier support of Γ. To show this, we examine that for all $\mathcal{C}_k$ ($k = 0, 2, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor - 2\widehat{d}$), there exists $S_{\mathcal{C}_k} \in 2^{[n]}$ such that $S_{\mathcal{C}_k} \Delta \mathcal{C}_k \subseteq \mathcal{T}$. This is sufficient because, for the cliques $\phi(\mathcal{C}_k)$, we have $\phi(S_{\mathcal{C}_k}) \Delta \phi(\mathcal{C}_k) = S_{\mathcal{C}_k} \Delta \mathcal{C}_k \subseteq \mathcal{T}$. The following gives an appropriate choice of $S_{\mathcal{C}_k}$:

- If $k \le \left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil - 2\widehat{d}$, then $\mathcal{C}_k \subseteq \mathcal{T}$ and thus $S_{\mathcal{C}_k} = \emptyset$.
- If $k \ge \left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil - 2\widehat{d} + 2$ and n is even, we have

$$n - k \le n + 2\widehat{d} - 2 - \left\lceil \frac{n + 2\widehat{d} - 2}{2} \right\rceil \le \left\lceil \frac{n + 2\widehat{d} - 2}{2} \right\rceil.$$

  Thus,

$$[n] \Delta \mathcal{C}_k = \mathcal{T}_{n-(k+2\widehat{d})} \cup \mathcal{T}_{n-(k+2\widehat{d})+2} \cup \cdots \cup \mathcal{T}_{n-k} \subseteq \mathcal{T}_0 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil} = \mathcal{T}.$$

  Thus, $S_{\mathcal{C}_k} = [n]$.

- If $k \geq \left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil - 2\widehat{d}+2$ and n is odd, we have $\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil + \widehat{d} - 1 = n - \left\lceil \frac{n}{2} \right\rceil + \widehat{d}$ and thus

$$n - k + 1 \leq n + 2\widehat{d} - 1 - \left\lceil \frac{n + 2\widehat{d} - 2}{2} \right\rceil = n - \left\lceil \frac{n}{2} \right\rceil + \widehat{d} = \left\lceil \frac{n + 2\widehat{d} - 2}{2} \right\rceil.$$

Hence,

$$\begin{aligned}
\phi([n]) \mathit{\Delta} \mathcal{C}_k &= [n] \mathit{\Delta} \phi(\mathcal{C}_k) \\
&\subseteq [n] \mathit{\Delta} (\mathcal{T}_{k-1} \cup \mathcal{T}_{k+1} \cup \cdots \cup \mathcal{T}_{k+2\widehat{d}+1}) \\
&\subseteq \mathcal{T}_{n-k-2\widehat{d}-1} \cup \mathcal{T}_{n-k-2\widehat{d}+1} \cup \cdots \cup \mathcal{T}_{n-k+1} \\
&\subseteq \mathcal{T}_0 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{\lceil (n+2\widehat{d}-2)/2 \rceil} \\
&= \mathcal{T}.
\end{aligned}$$

As a result, $S_{\mathcal{C}_k} = \phi([n]) = \{2, 3, \ldots, n\}$.

Consequently, $\mathcal{T}$ is a Fourier support of Γ that covers the Cayley graph $\mathrm{Cay}(\mathcal{S}^{2\widehat{d}}_{\mathrm{even}})$.

Example 4.2. *As in Figure 3, if $n = 4$ and $\widehat{d} = 1$, the graph obtained by adding dash lines to Cay($\mathcal{S}^2_{even}$) is a chordal cover Γ of Cay($\mathcal{S}^2_{even}$). The Fourier support $\mathcal{T}$ is $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_2$ (indicated by filled circles). Note that there are four maximal cliques for Γ; from the left component in Figure 3, $\mathcal{C}_0 := \mathcal{T}_0 \cup \mathcal{T}_2$ (induced by small filled circles), $\mathcal{C}_2 := \mathcal{T}_2 \cup \mathcal{T}_4$ (by large open circles), and from the right component, $\phi(\mathcal{C}_0) := \{1\} \mathit{\Delta} \mathcal{C}_0$ (by small open circles), $\phi(\mathcal{C}_2) := \{1\} \mathit{\Delta} \mathcal{C}_2$ (by large open circles). We can make sure that $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_2$ is a Fourier support from the following*

- $\emptyset \mathit{\Delta} \mathcal{C}_0 \subseteq \mathcal{T}$,
- $[4] \mathit{\Delta} \mathcal{C}_2 \subseteq \mathcal{T}$,
- $\phi(\emptyset) \mathit{\Delta} \phi(\mathcal{C}_0) = \emptyset \mathit{\Delta} \mathcal{C}_0 \subseteq \mathcal{T}$ *and*
- $\phi([4]) \mathit{\Delta} \phi(\mathcal{C}_2) = [4] \mathit{\Delta} \mathcal{C}_2 \subseteq \mathcal{T}$.

4.3 A Fourier support for the odd case

In the case where $\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil$ is odd, we define the cliques

$$\mathcal{C}_k := \mathcal{T}_k \cup \mathcal{T}_{k+2} \cup \cdots \cup \mathcal{T}_{k+2\widehat{d}}, \quad k = 1, 3, \ldots, 2 \left\lceil \frac{n}{2} \right\rceil - 2\widehat{d} - 1.$$

Note that $\mathcal{C}_k$ are indexed by odd integers. Then we construct Γ_{even} and Γ_{odd} as follows:

- Construct Γ_{odd} by pasting $\mathcal{C}_k$ and $\mathcal{C}_{k+2}$ along $\mathcal{C}_k \cap \mathcal{C}_{k+2}$ for $k = 1, 3, \ldots, 2 \left\lceil \frac{n}{2} \right\rceil - 2\widehat{d} - 3$.
- Construct Γ_{even} by pasting $\phi(\mathcal{C}_k)$ and $\phi(\mathcal{C}_{k+2})$ along $\phi(\mathcal{C}_k) \cap \phi(\mathcal{C}_{k+2})$ for $k = 1, 3, \ldots, 2 \left\lceil \frac{n}{2} \right\rceil - 2\widehat{d} - 3$.

Thus we obtain the chordal cover $\Gamma := \Gamma_{\mathrm{odd}} \cup \Gamma_{\mathrm{even}}$ for $\mathrm{Cay}(\mathcal{S}^{2\widehat{d}}_{\mathrm{even}})$. The maximal cliques of Γ are $\mathcal{C}_k$ for $k = 1, 3, \ldots, 2 \left\lceil \frac{n}{2} \right\rceil - 2\widehat{d} - 1$ together with the corresponding $\phi(\mathcal{C}_k)$.

The Fourier support of Γ is given by

$$\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_3 \cup \cdots \cup \mathcal{T}_{\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil}.$$

We check this by choosing $S_{\mathcal{C}_k}$ satisfying $S_{\mathcal{C}_k} \mathit{\Delta} \mathcal{C}_k \subseteq \mathcal{T}$ for each maximal clique $\mathcal{C}_k$ as follows:

- If $k \le \left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil - 2\widehat{d}$, then $S_{\mathcal{C}_k} = \emptyset$.
- If $k \ge \left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil - 2\widehat{d} + 2$ and n is even, then $S_{\mathcal{C}_k} = [n]$.
- If $k \ge \left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil - 2\widehat{d} + 2$ and n is odd, then $S_{\mathcal{C}_k} = \phi([n])$.

Thus, our desired result for the case with odd $\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil$ follows. By the discussion up to this point and Theorem 2.10, we present the following result:

Theorem 4.3. *Let $\mathcal{T}$ be the subset of $2^{[n]}$ defined as*

$$\mathcal{T} := \begin{cases} \mathcal{T}_0 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil} & \text{if } \left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil \text{ is even} \\ \mathcal{T}_1 \cup \mathcal{T}_3 \cup \cdots \cup \mathcal{T}_{\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil} & \text{if } \left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil \text{ is odd} \end{cases}.$$

Then, the moment polytope $\mathcal{M}(\mathcal{S}^{2\widehat{d}}_{even})$ can be expressed as

$$\mathcal{M}(\mathcal{S}^{2\widehat{d}}_{even}) = \left\{ \ell \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^{2\widehat{d}}_{even}}} \ : \ \exists y \in \mathbb{R}^{\mathcal{D}_{\mathcal{T}\Delta\mathcal{T}}} \ s.t. \ \begin{array}{l} y_\delta = \ell_\delta \text{ for all } \delta \in \mathcal{D}_{\mathcal{S}^{2\widehat{d}}_{even}}, \\ y_{\delta_\emptyset} = 1, \text{ and } M_{\mathcal{T}}(y) \succeq 0 \end{array} \right\}.$$

Thus, if the objective function f consists of only even monomials, problem (1) can be equivalently transformed into a smaller size SDP whose moment matrix is truncated with $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil}$ or $\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_3 \cup \cdots \cup \mathcal{T}_{\left\lceil \frac{n+2\widehat{d}-2}{2} \right\rceil}$.

Note that for the binary QOPs where $\widehat{d} = 1$ and all monomials have even degrees, the above $\mathcal{T}$ becomes $\mathcal{T}_0 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{\lceil \frac{n}{2} \rceil}$ or $\mathcal{T}_1 \cup \mathcal{T}_3 \cup \cdots \cup \mathcal{T}_{\lceil \frac{n}{2} \rceil}$. The result coincides with Laurent conjecture in [13] that solving the SDP relaxation with the relaxation order $\omega = \lceil n/2 \rceil$ in the hierarchy gives the exact optimal value, which is recently shown in [4].

4.4 Computing the optimal solution for the original problem

By solving the SDP relaxation with the truncated moment matrix, the optimal solution of linearized decision variables $\ell^* \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^{2\widehat{d}}_{\text{even}}}}$ can be obtained. However, the variables $\ell_{\delta_{\{1\}}}, \ell_{\delta_{\{2\}}}, \dots, \ell_{\delta_{\{n\}}}$ corresponding to $x_1, x_2, \dots, x_n$ are not included in $\ell \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^{2\widehat{d}}_{\text{even}}}}$, and thus it is not trivial to extract the optimal solution $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ from the truncated moment matrix for the original problem.

For even-degree binary POPs, we describe a method to extract the optimal solution x^* from $\ell^* \in \mathbb{R}^{\mathcal{D}_{\mathcal{S}^{2\widehat{d}}_{\text{even}}}}$. The map $\phi(S) := \{1\}\Delta S$ is applied to $\ell^*_{\delta_S}$, $\forall S \in \{\emptyset, \{1,2\}, \{1,3\}, \dots, \{1,n\}\}$, which leads to the optimal solution $\ell^*_{\delta_S}$, $\forall S \in \{\{1\}, \{2\}, \{3\}, \dots, \{n\}\}$. First, define a vector $\widehat{\ell} \in \{-1,1\}^{n-1}$ using the elements of ℓ^* as follows:

$$\widehat{\ell} := (\ell^*_{\delta_{\{1,2\}}}, \ell^*_{\delta_{\{1,3\}}}, \dots, \ell^*_{\delta_{\{1,n\}}}).$$

Since $\ell^*_{\delta_{\{1,i\}}}$ is the linearization of $x_1^* x_i^*$, we have $x_1^* \ell^*_{\delta_{\{1,i\}}} = {x_1^*}^2 x_i^* = x_i^*$ for $i = 2, 3, \dots, n$. As a result, we have two candidates for the optimal solution x^* corresponding to the two cases $x_1^* = \pm 1$ as follows:

$$x^* = (x_1^*, x_2^*, \dots, x_n^*) = (x_1^*, x_1^* \widehat{\ell}) = \begin{cases} (1, \widehat{\ell}) & \text{if } x_1^* = 1 \\ (-1, -\widehat{\ell}) & \text{if } x_1^* = -1 \end{cases}.$$

Note that f is an even function, and thus, both of these two solutions give the optimal value, i.e., the optimal value is $f((1, \widehat{\ell})) = f(-(1, \widehat{\ell}))$.

4.5 Formulating general binary POPs as even-degree binary POPs

The objective function f of general binary POP (1) can be written as the sum of the monomials of odd and even degree. Let $\widehat{c}_\alpha$ denote the coefficients of the monomials of even degree and $\widetilde{c}_\alpha$ the coefficients of the monomials of odd degree. Then,

$$f(x) = \sum_{||\alpha||_1 \le d} c_\alpha x^\alpha = \sum_{||\alpha||_1 \le d} \widehat{c}_\alpha x^\alpha + \sum_{||\alpha||_1 \le d} \widetilde{c}_\alpha x^\alpha.$$

By introducing the additional variable $x_{n+1} \in \{-1, 1\}$, binary POP (1) can be expressed as the following *even-degree* binary POP on $\{-1, 1\}^{n+1}$, to which Theorem 4.3 can be applied:

$$\begin{aligned} \text{(12)} \qquad & \underset{x}{\text{minimize}} \quad \widetilde{f}(x) := \sum_{||\alpha||_1 \le d} \widehat{c}_\alpha x^\alpha + \sum_{||\alpha||_1 \le d} \widetilde{c}_\alpha x^\alpha x_{n+1} \\ & \text{subject to} \quad x \in \{-1, 1\}^{n+1}. \end{aligned}$$

Let the degree of $\widetilde{f}$ be $\widetilde{d}$. Obviously, $\widetilde{d} = d$ if d is even, and $\widetilde{d} = d+1$ if d is odd. Therefore, $d \le \widetilde{d}$.

If problem (12) has an optimal solution $x^* \in \{-1, 1\}^{n+1}$ with $x^*_{n+1} = 1$, then an optimal solution for the original problem (1) is obtained as $(x^*_1, x^*_2, \dots, x^*_n)$. We note that problem (12) always has an optimal solution with $x^*_{n+1} = 1$. More precisely, if problem (12) has an optimal solution x^* such that $x^*_{n+1} = -1$, then $-x^*$ is also an optimal solution since $\widetilde{f}$ is an even function. Therefore, any general binary POP (1) with the constraint $x \in \{-1, 1\}^n$ can be formulated as an even-degree binary POP with the constraint $x \in \{-1, 1\}^{n+1}$.

However, we mention that applying Theorem 4.3 to problem (12) results in a larger truncated moment matrix than the one obtained by applying Theorem 3.2 to problem (1). To see this, we define $\mathcal{T}^n_k := \{S \in 2^{[n]} : |S| = k\}$. Observe that the size of $[n]$ is included in the definition of $\mathcal{T}^n_k$. If Theorem 3.2 is applied to problem (1), the moment matrix is truncated by the following Fourier support:

$$\mathcal{T}^n := \mathcal{T}^n_0 \cup \mathcal{T}^n_1 \cup \dots \cup \mathcal{T}^n_{\lceil \frac{n+d-1}{2} \rceil}.$$

On the other hand, if Theorem 4.3 is applied to problem (12), the moment matrix is truncated by the Fourier support defined as follows:

$$\widetilde{\mathcal{T}}^{n+1} := \begin{cases} \mathcal{T}^{n+1}_0 \cup \mathcal{T}^{n+1}_2 \cup \dots \cup \mathcal{T}^{n+1}_{\lceil \frac{(n+1)+\widetilde{d}-2}{2} \rceil} & \text{if } \left\lceil \frac{(n+1)+\widetilde{d}-2}{2} \right\rceil \text{ is even} \\ \mathcal{T}^{n+1}_1 \cup \mathcal{T}^{n+1}_3 \cup \dots \cup \mathcal{T}^{n+1}_{\lceil \frac{(n+1)+\widetilde{d}-2}{2} \rceil} & \text{if } \left\lceil \frac{(n+1)+\widetilde{d}-2}{2} \right\rceil \text{ is odd} \end{cases}.$$

We now prove $|\mathcal{T}^n| \le |\widetilde{\mathcal{T}}^{n+1}|$. Let $\omega := \lceil (n+d-1)/2 \rceil$ and $\widetilde{\omega} := \left\lceil (n+\widetilde{d}-1)/2 \right\rceil$. Notice that $\omega \le \widetilde{\omega}$ by $d \le \widetilde{d}$, and thus

$$\text{(13)} \qquad |\mathcal{T}^n| = |\mathcal{T}^n_0| + |\mathcal{T}^n_1| + \dots + |\mathcal{T}^n_\omega| = \sum_{i=0}^{\omega} \binom{n}{i} \le \sum_{i=0}^{\widetilde{\omega}} \binom{n}{i}.$$

We show that the right hand side of (13) coincides with $|\widetilde{\mathcal{T}}^{n+1}|$ by considering the two cases where $\widetilde{\omega}$ is even and where $\widetilde{\omega}$ is odd. If $\widetilde{\omega}$ is even, then

$$\sum_{i=0}^{\widetilde{\omega}} \binom{n}{i} = \binom{n}{0} + \sum_{i=1}^{\widetilde{\omega}/2} \left\{ \binom{n}{2i-1} + \binom{n}{2i} \right\} = \binom{n+1}{0} + \sum_{i=1}^{\widetilde{\omega}/2} \binom{n+1}{2i} = |\widetilde{\mathcal{T}}^{n+1}|.$$

Similarly, if $\widetilde{\omega}$ is odd, then

$$\sum_{i=0}^{\widetilde{\omega}} \binom{n}{i} = \sum_{i=0}^{\lfloor \widetilde{\omega}/2 \rfloor} \left\{ \binom{n}{2i} + \binom{n}{2i+1} \right\} = \sum_{i=0}^{\lfloor \widetilde{\omega}/2 \rfloor} \binom{n+1}{2i+1} = |\widetilde{\mathcal{T}}^{n+1}|.$$

Consequently, the desired result $|\mathcal{T}^n| \le |\widetilde{\mathcal{T}}^{n+1}|$ follows.

Example 4.4. *In Section* 3.3*, we have seen that the truncated moment matrix of the problem* (3) *is* $\mathcal{T}^3 := \mathcal{T}_0^3 \cup \mathcal{T}_1^3 \cup \mathcal{T}_2^3$ *where the size of the truncated moment matrix is* $|\mathcal{T}^3| = 7$.

We can reformulate problem (3) *as an even-degree binary POP in Example* 4.1 *using an additional variable* x_4*. Then the Fourier support is* $\widetilde{\mathcal{T}}^4 := \mathcal{T}_0^4 \cup \mathcal{T}_2^4 = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ *which gives a truncated moment matrix with the size* $|\widetilde{\mathcal{T}}^4| = 7$*. Consequently, problem* (3) *can be equivalently rewritten with the truncated moment matrix by* $\widetilde{\mathcal{T}}^4$ *as*

$$\begin{array}{ll} \underset{\ell \in \mathbb{R}^{\mathcal{S}^2_{even}}}{\text{minimize}} & \ell_{0000} + \ell_{1001} + \ell_{0101} + \ell_{0011} + \ell_{1100} + \ell_{1010} + \ell_{0110} \\ \text{subject to} & \begin{pmatrix} \ell_{0000} \\ \ell_{1100} \\ \ell_{1010} \\ \ell_{1001} \\ \ell_{0110} \\ \ell_{0101} \\ \ell_{0011} \end{pmatrix} \in \left\{ \begin{array}{l} \ell \ : \ \exists y_{1111} \quad s.t. \quad \ell_{0000} = 1 \quad and \\ \begin{bmatrix} 1 & \ell_{1100} & \ell_{1010} & \ell_{1001} & \ell_{0110} & \ell_{0101} & \ell_{0011} \\ \ell_{1100} & 1 & \ell_{0110} & \ell_{0101} & \ell_{1010} & \ell_{1001} & y_{1111} \\ \ell_{1010} & \ell_{0110} & 1 & \ell_{0011} & \ell_{1100} & y_{1111} & \ell_{1001} \\ \ell_{1001} & \ell_{0101} & \ell_{0011} & 1 & y_{1111} & \ell_{1100} & \ell_{1010} \\ \ell_{0110} & \ell_{1010} & \ell_{1100} & y_{1111} & 1 & \ell_{0011} & \ell_{0101} \\ \ell_{0101} & \ell_{1001} & y_{1111} & \ell_{1100} & \ell_{0011} & 1 & \ell_{0110} \\ \ell_{0011} & y_{1111} & \ell_{1001} & \ell_{1010} & \ell_{0101} & \ell_{0110} & 1 \end{bmatrix} \succeq 0 \end{array} \right\}. \end{array}$$

As shown in Section 4.4*, we can retrieve an optimal solution* $(x_1^*, x_2^*, x_3^*, x_4^*)$ *from* $(\ell_{1100}^*, \ell_{1010}^*, \ell_{1001}^*)$*. We note that solving the above modified problem does not have any computational advantage over solving problem* (9) *generated by* $\mathcal{T}^3 := \mathcal{T}_0^3 \cup \mathcal{T}_1^3 \cup \mathcal{T}_2^3$ *since the sizes of the moment matrices of the two problems are the same.*

5 Numerical experiments

We performed numerical experiments to see whether the bounds shown theoretically in Sections 3 and 4 could be obtained numerically. We tested our theoretical bound with 100 randomly generated binary POP (1) with $n = 8$ and the binary POPs with all-one coefficients of f for $d = 1, 2, \ldots, 8$. The numerical results on these problems are presented in Section 5.1. In Section 5.2, we show the numerical results for even-degree binary POPs.

5.1 General binary POPs

In the case of general binary POP (1), the relaxation order ω means that the moment matrix is truncated by $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_\omega$.

5.1.1 Random instances

We generated 100 test problems randomly by choosing the coefficient c_α ($\alpha \in \{0,1\}^8$) of binary POP (2) such that

$$c_\alpha = 10 \times u_\alpha,$$

where u_α is sampled from the standard normal distribution. We increased the relaxation order from $\lceil d/2 \rceil$ to see what relaxation order is needed to obtain the optimal values of the test problems.

In Table 1, we observe that the optimal values of all randomly generated test problems could be obtained with the relaxation order ω, which is smaller than the theoretical bound $\bar{\omega} := \lceil \frac{n+d-1}{2} \rceil$. The underlined numbers in Table 1 mean that all test problems with specified d could be exactly solved with the relaxation order ω in the corresponding row in the first column. For instance, when $d = 2$, the optimal values of all test problems could be obtained with $\omega = 2$.

Table 1: The number of random instances solved.

ω	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
1	<u>100</u>	15						
2	100	<u>100</u>	<u>100</u>	<u>100</u>				
3	100	100	100	100	<u>100</u>	<u>100</u>		
4	100	100	100	100	100	100	<u>100</u>	99
5	100	100	100	100	100	100	100	<u>100</u>
$\bar{\omega}$	4	5	5	6	6	7	7	8

5.1.2 Binary POPs with all-one coefficients

For binary POP (2) where $c_\alpha = 1$ for all $\alpha \in \{0,1\}^8$, the degree of f was varied as $d = 1, 2, \ldots, 8$ for numerical experiments. The optimal values of the problems are known as shown in Table 2. The last row shows the theoretical bound $\bar{\omega} := \lceil \frac{n+d-1}{2} \rceil$. As previously, the underlined number for each d in Table 2 denotes that the exact optimal value of the test problem was obtained with the corresponding order ω in the first column, which is the minimum order for the exactness.

Notice that the exact optimal values of the problem of $d = 2, 8$ were obtained at the theoretical bound $\bar{\omega}$, while the exact optimal values of the other problems were obtained with ω smaller than the theoretical bound.

Table 2: The objective values obtained for the all-ones instances using $\omega = 1$ to 8.

ω	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
1	<u>−7.00</u>	−3.50						
2	−7.00	−3.50	<u>−35.0</u>	−7.67				
3	−7.00	−3.50	−35.0	−7.67	<u>−21.0</u>	−8.19		
4	−7.00	−3.50	−35.0	−6.18	−21.0	<u>−5.00</u>	−2.88	−2.74
5	−7.00	<u>−3.00</u>	−35.0	<u>−5.00</u>	−21.0	−5.00	−1.05	−4.24e-1
6	−7.00	−3.00	−35.0	−5.00	−21.0	−5.00	<u>−1.00</u>	−5.38e-2
7	−7.00	−3.00	−35.0	−5.00	−21.0	−5.00	−1.00	−3.94e-3
8	−7.00	−3.00	−35.0	−5.00	−21.0	−5.00	−1.00	<u>−1.28e-12</u>
OptV.	−7	−3	−35	−5	−21	−5	−1	0
$\bar{\omega}$	4	5	5	6	6	7	7	8

5.2 Even-degree binary POPs

In the case of even-degree binary POPs, the relaxation order ω indicates that the moment matrix is truncated by

$$\mathcal{T} = \begin{cases} \mathcal{T}_0 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_\omega & \text{if } \omega \text{ is even} \\ \mathcal{T}_1 \cup \mathcal{T}_3 \cup \cdots \cup \mathcal{T}_\omega & \text{if } \omega \text{ is odd} \end{cases}.$$

5.2.1 Random instances

As in Section 5.1.1, we generated 100 test problems randomly by choosing the coefficient c_α ($\alpha \in \{0,1\}^8$) of even-degree binary POP (2) such that

$$c_\alpha = 10 \times u_\alpha,$$

where u_α is sampled from the standard normal distribution, and solved them via the SDP relaxations with the truncated moment matrices.

The underline in Table 3 stands for the minimum order ω, with which the exact optimal values of all test problems with specified d could be obtained. We observe that the optimal values of randomly generated test problems could be obtained with the relaxation order smaller than theoretical bound $\bar{\omega} := \lceil \frac{n+d-2}{2} \rceil$.

Table 3: The number of random instances solved

ω	$d = 2$	$d = 4$	$d = 6$	$d = 8$
1	0			
2	$\underline{100}$	$\underline{100}$		
3	100	100	$\underline{100}$	
4	100	100	100	$\underline{100}$
5	100	100	100	100
$\bar{\omega}$	4	5	6	7

5.2.2 Binary POPs with all-one coefficients

We consider even-degree binary POP (2) where $c_\alpha = 1$ for all $\alpha \in \{0,1\}^8$ such that $||\alpha||_1$ is even. Even-degree binary POP (2) of degree $d = 2, 4, 6, 8$ with all-one coefficients were tested. The optimal values of the problems are known as in Table 4.

The meaning of the underline in Table 4 is the same as the previous tables. We observe that the exact optimal value of the problem of $d = 8$ is obtained at the theoretical bound $\bar{\omega} := \lceil \frac{n+d-2}{2} \rceil$, while the exact optimal values of the problems of $d = 2, 4, 6$ were obtained with ω smaller than the theoretical bound.

Table 4: The objective values obtained for the all-ones instances using $\omega = 1$ to 7.

ω	$d = 2$	$d = 4$	$d = 6$	$d = 8$
1	$\underline{-3.00}$			
2	-3.00	-5.17		
3	-3.00	-5.17	-3.63	
4	-3.00	$\underline{-5.00}$	-1.36	-8.66e-1
5	-3.00	-5.00	$\underline{-1.00}$	-1.11e-1
6	-3.00	-5.00	-1.00	-7.94e-3
7	-3.00	-5.00	-1.00	$\underline{-1.75\text{e-}14}$
Opt.V.	-3	-5	-1	0
$\bar{\omega}$	4	5	6	7

6 Concluding remarks

We have shown that the exact optimal values of binary POPs are obtained by solving the $\lceil (n + d - 1)/2 \rceil$th SDP relaxation of Lasserre's hierarchy of SDP relaxations, and the $\lceil (n + d - 2)/2 \rceil$th SDP relaxation for even-degree binary POPs. The bound $\lceil (n + d - 2)/2 \rceil$ for even-degree binary POPs is the generalization of the result in [4] for binary QOPs. We have also discussed on the construction of the optimal solution from the truncated moment matrix for even-degree binary POPs. These bounds have been confirmed by the numerical results in Section 5.

The moment matrix of a given binary POP has been truncated using a chordal graph of the Cayley graph from the binary POP in this paper. In [7], the sparse SDP relaxation was proposed by exploiting the sparsity of a given POP. Specifically, the sizes of the moment matrices for the sparse SDP relaxation were reduced using the maximal cliques of an extended chordal graph obtained from the sparsity pattern graph of the given POP. It will be interesting to investigate the relationship between the two approaches.

Solving POPs by the hierarchy of SDP relaxations is a very challenging problem since it requires to solve increasingly large SDP relaxations for the desired accuracy. Moreover, the SDPs induced from POPs are frequently degenerate, causing a great deal of numerical difficulties for SDP solvers. It is well-known that the SDP solvers based on the interior-point methods [5, 15, 16] cannot handle the SDPs with more than several thousand variables, as a result, POPs with moderate size n and d cannot be solved by the solvers. From the computational viewpoints, it is essential to have the size of the SDPs as small as possible for accurate solutions of POPs. The bounds presented in this paper for binary POPs provide the important information on the largest size of the SDP to be solved for the exact optimal value of binary POPs in advance.

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References

[1] M. F. Anjos. Semidefinite optimization approaches for satisfiability and maximum-satisfiability problems. *J. Satisf. Boolean Model. Comput.*, 1:1–47, 2005.

[2] R. B. Bapat. *Graphs and matrices.* Springer, 2010.

[3] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics.* Springer, Heidelberg, fourth edition, 2010.

[4] H. Fawzi, J. Saunderson, and P. A. Parrilo. Sparse sum-of-squares certificates on finite abelian groups. *ArXiv:1503.01207*, 2015.

[5] K. Fujisawa, M. Fukuda, K. Kobayashi, M. Kojima, K. Nakata, M. Nakata, and M. Yamashita. SDPA (SemiDefinite Programming Algorithm) user's manual – Version 7.0.5. Technical report, Research Report B-448, Dept. of Mathematical and Computing Sciences, Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo 152-8552, Japan, 2008.

[6] M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. ACM*, 42(6):1115–1145, 1995.

[7] M. Kojima H. Waki, S. Kim and M. Muramatsu. Sums of squares and semidefinite programming relaxations for polynomial optimization problems with structured sparsity. *SIAM J. Optim.*, 17:218–242, 2006.

[8] S. He, Z. Li, and S. Zhang. Approximation algorithms for discrete polynomial optimization. *J. Oper. Res. Soc. China*, 1(1):3–36, 2013.

[9] A. Hertz. Polynomially solvable cases for the maximum stable set problem. *Discrete Appl. Math.*, 60:195–210, 1995.

[10] A. Kurpisz, S. Leppänen, and M. Mastrolilli. On the hardest problem formulations for the 0/1 Lasserre hierarchy. In *Automata, Languages, and Programming*, pages 872–885. Springer, 2015.

[11] J. B. Lasserre. An explicit exact SDP relaxation for nonlinear 0-1 programs. In *Integer Programming and Combinatorial Optimization*, pages 293–303. Springer, 2001.

[12] J. B. Lasserre. Global optimization with polynomials and the problems of moments. *SIAM J. Optim.*, 11:796–817, 2001.

[13] M. Laurent. Lower bound for the number of iterations in semidefinite hierarchies for the cut polytope. *Math. Oper. Res.*, 28(4):871–883, 2003.

[14] J. Povh and F. Rendl. Copositive and semidefinite relaxations of the quadratic assignment problem. *Discrete Optim.*, pages 231–224, 2009.

[15] J. F. Sturm. SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optim. Methods and Softw.*, 11&12:625–653, 1999.

[16] R. H. Tütüncü, K. C. Toh, and M. J. Todd. Solving semidefinite-quadratic-linear programs using SDPT3. *Math. Program.*, 95:189–217, 2003.