A relaxed-certificate facial reduction algorithm based on subspace intersection

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Abstract

A "facial reduction"-like regularization algorithm is established for general conic optimization problems by relaxing requirements on the reduction certificates. This yields a rapid subspace reduction algorithm challenged only by representational issues of the regularized cone. A condition for practical usage is analyzed and shown to always be satisfied for single second-order cone optimization problems. Should the condition fail on some other class of instances, only partial regularization is achieved based on the success of the individual subspace intersection.

Keywords: Facial reduction; subspace intersection; conic optimization; second-order cones.

1. Introduction

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, consider the following primal-dual pair of conic optimization problems over the non-empty, closed, convex cone $\mathcal{K} \subseteq \mathbb{R}^n$ and its dual cone $\mathcal{K}^* \subseteq \mathbb{R}^n$:

$$\begin{array}{rll} \theta_P = \text{infimum} & c^T x & \\ \text{subject to} & Ax = b, & \qquad (\mathrm{P}) \\ & x \in \mathcal{K}, & \end{array}$$

$$\begin{array}{rll} \theta_D = \text{supremum} & b^T y & \\ \text{subject to} & c - A^T y = s, & \qquad (\mathrm{D}) \\ & s \in \mathcal{K}^*,\ y \in \mathbb{R}^m, & \end{array}$$

where $\theta_P, \theta_D \in \mathbb{R} \cup \{\pm\infty\}$ are the (possibly unattained) optimal values of (P) and (D), respectively. In contrast to linear optimization, valid reformulation of (P) may change the feasible set and optimal value of (D), and vice versa. This manifests itself in lacks of strong duality for feasible instances (i.e., $\theta_P > \theta_D$), in lacks of dual improving rays for infeasible instances (i.e., Farkas-type certificates), and in facial reduction algorithms able to amend such anomalies by reformulating the considered problem and thereby obtain a regularized form.

These facial reduction algorithms progress, iteratively, by reducing the conic domain $\mathcal{K}$ (resp. $\mathcal{K}^*$) to a face of itself containing the entire feasible set. Facial reduction certificates as used in [8, 10, 2, 5] often justify these steps, but alternatives can be used. A remark in [2] notes that by restriction of these certificates, a regularized form can be achieved in a single (computationally difficult) iteration. In contrast, this paper relax the certificates to achieve a regularized form in a linear number of easily computable iterations. Specifically, these relaxed certificates justify subspace intersections in general, paying no attention to facial properties, and have both pros and cons.

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2. Preliminaries

A *face* $\mathcal{F}$ of a set S, denoted $\mathcal{F} \trianglelefteq S$, is a subset for which any line segment in S, with a midpoint in $\mathcal{F}$, has both endpoints in $\mathcal{F}$ [9]. This generalizes optimal faces and extreme points/rays known from linear optimization. A non-empty subset of S—such as a face or intersection—is called *proper* if different from S.

The image of a set under a function, i.e., $f(S) := \{f(x) : x \in S\}$, is used implicitly. Hence, a subset $\mathcal{C} \subseteq \mathbb{R}^n$ is a *cone* if $\lambda\mathcal{C} = \mathcal{C}$ for any $\lambda > 0$, and the *dual cone* of $\mathcal{C}$ is denoted and defined by $\mathcal{C}^* := \{y \in \mathbb{R}^n : y^T\mathcal{C} \subseteq \mathbb{R}_+\}$. This paper is limited to non-empty, closed, convex cones, whereby $\mathcal{C}$ equals $(\mathcal{C}^*)^*$ and contains the origin [9].

A *subspace intersection* of $\mathcal{C}$ is the intersection of a linear subspace and $\mathcal{C}$. Let $z^\perp := \{x \in \mathbb{R}^n : z^T x = 0\}$. For $z \in \mathcal{C}^*$, the subspace intersection $\mathcal{C} \cap z^\perp$ contains the origin and is a face of $\mathcal{C}$ as it maximizes $-z^T x$ over $x \in \mathcal{C}$ [9]. Hence, if $z \in \mathcal{C}^* \setminus \mathcal{C}^\perp$, then $\mathcal{C} \cap z^\perp$ is a proper face of $\mathcal{C}$ by exclusion of the orthogonal complement.

3. Subspace reduction certificates

Validity of $z^T x = 0$ in (P), for some $z \in \mathbb{R}^n$, justifies reformulation from cone $\mathcal{K}$ to cone $\mathcal{K} \cap z^\perp$. A certificate for proper intersections of this kind may hence be defined as solutions $z \in \mathbb{R}^n \setminus \mathcal{K}^\perp$, of the auxiliary problem

$$\omega^T A = z^T \text{ and } \omega^T b = 0, \text{ for some } \omega \in \mathbb{R}^m \qquad (1)$$

which justifies $z^T x = 0$ in (P). These certificates are called *subspace reduction certificates* and equal facial reduction certificates if $z \in \mathcal{K}^* \setminus \mathcal{K}^\perp$. Subspace reduction certificates for (D) are similarly solutions $z \in \mathbb{R}^n \setminus (\mathcal{K}^*)^\perp$, of

$$z^T c = 0 \text{ and } z^T A^T = 0 \qquad (2)$$

which justifies $z^T s = 0$ in (D). These certificates equal facial reduction certificates for (D) if $z \in \mathcal{K} \setminus (\mathcal{K}^*)^\perp$.

4. Subspace reduction algorithms

Subspace reduction algorithms progress, iteratively, by reducing the conic domain $\mathcal{K}$ (resp. $\mathcal{K}^*$) to a subspace intersection of itself containing the entire feasible set. If we justify each step with a subspace reduction certificate, and iterate until the corresponding auxiliary problem (1) or (2) becomes infeasible, the considered problem is regularized in $\mathcal{O}(n)$ iterations. This is formalized for Algorithm 1 in the following proposition.

Algorithm 1: Regularizing (P) by use of subspace reduction certificates.

```
1 Let k ← 1
2 repeat
3 |  Compute z_k to solve the auxiliary problem (1).
4 |  Let K ← K ∩ z_k^⊥ and k ← k + 1.
5 until the auxiliary problem is infeasible;
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Proposition 1. *Algorithm 1 regularize the considered problem* (P) *no less than a facial reduction algorithm driven by facial reduction certificates and uses* $\mathcal{O}(n)$ *iterations.*

Proof. Termination: $\mathcal{K}^\perp \supseteq \operatorname{span}(z_1, \dots, z_{k-1})$ holds before each intersection on line 4. Hence, since $z_k \notin \mathcal{K}^\perp$ by (1), the span grows with each repeat until (1) is infeasible; e.g., at $\mathcal{K}^\perp = \mathbb{R}^n$. *Regularization*: Each subspace reduction is a valid reformulation of (P) and at termination there are no facial reduction certificates by infeasibility of (1). ☐

Algorithm 1, and the corresponding algorithm for (D), can be implemented to rapidly reveal the full list of subspace reduction certificates $z_1, \dots, z_k$ needed to regularize the considered problem. To see this, note first that the intersection on line 4 need not be computed inside the loop. In particular, all it takes to continue the loop is a simple update to the orthogonal complement used in the auxiliary problem on line 3; $\mathcal{K}^\perp \leftarrow (\mathcal{K} \cap z_k^\perp)^\perp = \operatorname{span}(\mathcal{K}^\perp, z_k)$.

The auxiliary problems (1) and (2) themselves can also be solved efficiently. Ignoring the domain of z, the former problem simply asks for vectors orthogonal to b, while the latter asks for vectors that expose row dependencies in the matrix $\begin{pmatrix} c, & A^T \end{pmatrix}$ as found, e.g, by Gaussian elimination. In both cases an equation of the form $z^T \tilde{x} = 0$, where $\tilde{x} \in \tilde{\mathcal{K}}$, is constructed by linear combination of rows from the equation system of the considered problem. Now, if $z^T \tilde{x} = 0$ was added explicitly to the constraint system, any one of the rows used to construct this equation would naturally become redundant. But if $z \in \tilde{\mathcal{K}}^\perp$, then $z^T \tilde{x} = 0$ is already in the constraint system because the domain of $\tilde{x}$ satisfy $\tilde{\mathcal{K}} = \tilde{\mathcal{K}} \cap z^\perp$. Hence, in case $z \in \tilde{\mathcal{K}}^\perp$, any one of the weighted rows can be removed. Doing so, at least locally within the auxiliary problem solver, allows one to find new vectors and resolve until the problem exhibits no more of the sought vectors, or $z \notin \tilde{\mathcal{K}}^\perp$ as required. This is realized for (1) in Algorithm 2.

Proposition 2. *The full list of subspace reduction certificates* $z_1, \dots, z_k$*, needed to regularize* (P)*, can be computed in* $\mathcal{O}(m)$ *basic operations.*

Proof. Termination: Let Algorithm 1 call Algorithm 2 on line 3 and update only the orthogonal complement of $\mathcal{K}$ within the loop as explained. As one row of $Ax = b$ can be removed after each pass through Algorithm 2, the number of passes taken at each call is bound by m. Remembering removals between calls, however, m also bounds the total number of passes taken. This bound is tight given that Algorithm 1 iterates until line 2 of Algorithm 2 is triggered. *Regularization*: By Proposition 1. ☐

Having avoided intersections, the resulting regularized cone is returned as $\tilde{\mathcal{K}} \cap z_1^\perp \cap \dots \cap z_k^\perp$. Formally, this corresponds to the intersection of $\tilde{\mathcal{K}}$ and the span of the feasible set of x (resp. s) as defined by the equation system of considered problem. This is shown for (P).

Proposition 3. *The list of subspace reduction certificates from Proposition 2 satisfy*

$$\mathcal{K} \cap z_1^\perp \cap \dots \cap z_k^\perp = \mathcal{K} \cap \operatorname{span}\{x \in \mathbb{R}^n : Ax = b\}.$$

Proof. In case $\mathcal{K} = \mathbb{R}^n$, the algorithm shows equivalence between $Ax = b$ for $x \in \mathbb{R}^n$ and the reduced problem $x \in z_1^\perp \cap \dots \cap z_k^\perp$ having no ($m = 0$) or one inhomogeneous equation ($m = 1$ and $b \neq 0$). The claim hence follows by taking the span of the two equivalent feasible sets, which for the reduced problem simplifies to $z_1^\perp \cap \dots \cap z_k^\perp$. For other cones, $\mathcal{K} = \mathcal{C}$, the exact same vectors are visited although $z_j \in \mathcal{C}^\perp$ are filtered out by line 5 of Algorithm 2. Hence $\mathcal{C} \cap z_1^\perp \cap \dots \cap z_k^\perp = \mathcal{C} \cap \operatorname{span}\{x \in \mathbb{R}^n : Ax = b\}$ holds for the unfiltered vectors $z_1, \dots, z_k$ (by case $\mathcal{K} = \mathbb{R}^n$), where intersections involving $z_j \in \mathcal{C}^\perp$ are redundant from the left side and can be filtered out to show the claim. ☐

Algorithm 2: Solving the auxiliary problem (1) given m rows in the equation system $Ax = b$.

1 **if** $m = 0$ or $[m = 1$ and $b \neq 0]$ **then**
2   **stop**. Trivially infeasible.
3 **else**
4   Let $z^T \leftarrow \omega^T A$ satisfy the equations of (1), with row weights $\omega_{3:m} = 0$ and

$$(\omega_1, \omega_2) = \begin{cases} (1, 0) & \text{if } b_1 = 0, \\ (0, 1) & \text{if } b_2 = 0, \\ (b_1^{-1}, -b_2^{-1}) & \text{otherwise.} \end{cases}$$

5   **if** $z \in \mathcal{K}^\perp$ **then**
6   Solve instead the auxiliary problem (1) after having removed the redundant i'th row in $Ax = b$ for any one $\omega_i \neq 0$.

It remains only to discuss when regularized cones of the computed form $\tilde{\mathcal{K}} \cap z_1^\perp \cap \ldots \cap z_k^\perp$ are useful in practice.

5. Representational issues

To solve the primal-dual pair (P) and (D) efficiently, a certain amount of information is needed about the cones $\mathcal{K}$ and $\mathcal{K}^*$. Good barrier functions are for example needed to deploy a primal-dual interior-point method [7]. Hence, when applying valid subspace intersections to the cones of a problem, it is important to ensure that optimization can also take place over the reduced cones.

The following definition materializes this property by providing a sufficient condition for intersections $\mathcal{C} \cap z^\perp$ to be representable within the same class of cones as $\mathcal{C}$. This class, formally denoted Ω, is arbitrary but could be the symmetric cones over which optimization is efficient.

Definition 1. *Suppose $\mathcal{C}$ is a cone of class Ω and $z \in \mathbb{R}^n$. The subspace intersection $\mathcal{C} \cap z^\perp$ is called Ω-representable in* (P) *if it is possible to satisfy*

$$\mathcal{C} \cap z^\perp = H\hat{\mathcal{K}}, \tag{3}$$

and Ω-representable in (D) *if it is possible to satisfy*

$$s \in \mathcal{C} \cap z^\perp \quad \Leftrightarrow \quad H^T s \in \hat{\mathcal{K}}, \tag{4}$$

for some cone $\hat{\mathcal{K}}$ of class Ω and some matrix $H \in \mathbb{R}^{n \times r}$ with column dimension $1 \le r \le n$.

These conditions, (3) and (4), allow subspace intersections without leaving the chosen class of cones as claimed.

Proposition 4. *An Ω-representable subspace intersection in either* (P) *or* (D) *leads to the reduced primal-dual pair:*

$$\hat{\theta}_P = \inf_{\hat{x}} \{(H^T c)^T \hat{x} : (AH)\hat{x} = b,\, \hat{x} \in \hat{\mathcal{K}}\}, \tag{$\hat{P}$}$$

$$\hat{\theta}_D = \sup_{\hat{s},y} \{b^T y : (H^T c) - (AH)^T y = \hat{s},\, \hat{s} \in \hat{\mathcal{K}}^*\}. \tag{$\hat{D}$}$$

If the subspace intersection is a valid reformulation of the considered problem—(P) *or* (D)*—its corresponding reduced form—*($\hat{P}$) *or* ($\hat{D}$)*—has equal optimal value and equivalent feasible set. Specifically, for* (P) *and* ($\hat{P}$)*, the postsolve is $x = H\hat{x}$, and for* (D) *and* ($\hat{D}$)*, it is $s = c - A^T y$.*

Proof. ($\hat{P}$) and ($\hat{D}$) forms a primal-dual pair. Hence, the claims follow since (3) leads from (P) to ($\hat{P}$) as seen by

$$\begin{aligned} \theta_P &= \inf_x \{c^T x : Ax = b,\, x \in \mathcal{K} \cap z^\perp\}, \\ &= \inf_x \{c^T x : Ax = b,\, x \in H\hat{\mathcal{K}}\}, \\ &= \inf_{\hat{x}} \{c^T (H\hat{x}) : A(H\hat{x}) = b,\, \hat{x} \in \hat{\mathcal{K}}\}, \end{aligned}$$

and (4) leads from (D) to ($\hat{D}$) as seen by

$$\begin{aligned} \theta_D &= \sup_{s,y} \{b^T y : c - A^T y = s,\, s \in \mathcal{K}^* \cap z^\perp\}, \\ &= \sup_{s,y} \{b^T y : c - A^T y = s,\, H^T s \in \hat{\mathcal{K}}^*\}, \\ &= \sup_{\hat{s},y} \{b^T y : H^T (c - A^T y) = \hat{s},\, \hat{s} \in \hat{\mathcal{K}}^*\}. \end{aligned}$$

□

Having established an effective way to handle subspace intersections satisfying Definition 1, the question is when these conditions can be fulfilled. To address the general case of non-empty, closed, convex cones, it is first analyzed when intersections can be split into simpler parts.

Proposition 5. *Suppose $z = \sum_{j=1}^k z_j \in \mathbb{R}^n$. Then a cone $\mathcal{C}$ can be intersected by each addend independently,*

$$\mathcal{C} \cap z^\perp = \mathcal{C} \cap z_1^\perp \cap \ldots \cap z_k^\perp,$$

if and only if one of the following conditions hold:

1. *The addends are either all in $\mathcal{C}^*$ or all in $-\mathcal{C}^*$.*
2. *$z_j \in \mathcal{C}^\perp$ holds for all but one addend.*

Proof. The intersection $\mathcal{C} \cap z^\perp$ is always a superset,

$$\begin{aligned} \mathcal{C} \cap z^\perp &= \{x \in \mathcal{C} : x^T z = 0\}, \\ &\supseteq \{x \in \mathcal{C} : x^T z_j = 0 \text{ for } j = 1, \ldots, k\}, \\ &= \mathcal{C} \cap z_1^\perp \cap \ldots \cap z_k^\perp, \end{aligned}$$

with equality if and only if $x^T z = 0$ implies $x^T z_j = 0$ for all $j \in \{1, \ldots, k\}$ over the domain $x \in \mathcal{C}$. Given $x^T z = \sum_{j=1}^k x^T z_j = 0$, this requires all terms $x^T z_j$ to have the same sign (statement 1), or that only a single term $x^T z_j$ can take nonzero values (statement 2). □

The possibilities left open by Proposition 5 are limited and implies the following unfortunate corollary which itself is a generalization of [11, eq. (7.2.13)].

Corollary 1. *Suppose $z = (z^1, \ldots, z^k) \in \mathbb{R}^n$. Then a cone $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ can be intersected by each Cartesian factor independently,*

$$\mathcal{C} \cap z^\perp = \left(\mathcal{C}_1 \cap (z^1)^\perp\right) \times \cdots \times \left(\mathcal{C}_k \cap (z^k)^\perp\right),$$

if and only if $z \in \mathcal{C}^*$, $z \in -\mathcal{C}^*$, or $z^j \in \mathcal{K}_j^\perp$ holds for all but one Cartesian factor.

Proof. Define $z_j^T = (0, \ldots, 0, (z^j)^T, 0, \ldots, 0)$, nonzero only on the support of K_j, and use Proposition 5. □

The consequences of Corollary 1 are a major setback for general purpose usage of subspace reduction algorithms. To see why, suppose all subspace intersections of two cones, $\mathcal{C}_1$ and $\mathcal{C}_2$, are shown Ω-representable for the considered problem. Even then, subspace intersections of their Cartesian product $\mathcal{C}_1 \times \mathcal{C}_2$ are not readily available, and may perhaps not even be Ω-representable, if falling outside the conditions of Corollary 1. Whether this can be clarified for any particular cone class is left open. Instead, it is now shown that subspace reduction algorithms succeed when the conic domain is an intersection of $\{0\}$-sets and one second-order cone. Specifically, in this case, the condition $z^j \in \mathcal{K}_j^\perp$ for all but one Cartesian factor (in Corollary 1) trivially holds. Moreover, all subspace intersections of a single second-order cone are Ω-representable in both (P) and (D) as now established.

6. All second-order cone intersections

This section offers a unified treatment of all second-order cone subspace intersections generalizing previous work on facial reductions, e.g., [3, 6]. The two cones addressed here are the quadratic cone $\mathcal{Q}^n$ and rotated quadratic cone $\mathcal{Q}_r^n$ (see, e.g., [1]), defined by and closely related as

$$\begin{aligned} \mathcal{Q}^n &= \left\{ x \in \mathbb{R}^n : x_1^2 \geq \textstyle\sum_{j=2}^n x_j^2, \text{ and } x_1 \geq 0 \right\}, \\ &= W \mathcal{Q}_r^n, \text{ for } W = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & \mathrm{I} \end{pmatrix}. \end{aligned}$$

This relation means that subspace intersections of one leads to the subspace intersections of the other.

Proposition 6. *The representational conditions for $\mathcal{Q}^n$ from Definition 1, implies the conditions for $\mathcal{Q}_r^n$:*

1. $\mathcal{Q}_r^n \cap z^\perp = W(\mathcal{Q}^n \cap (Wz)^\perp)$*;*
2. $s \in \mathcal{Q}_r^n \cap z^\perp \quad \Leftrightarrow \quad Ws \in \mathcal{Q}^n \cap (Wz)^\perp$*.*

Proof. The matrix W is orthogonal. Hence, $f(x) = Wx$ is injective such that $f(X \cap Y) = f(X) \cap f(Y)$ for all sets X and Y. Moreover, $W\mathcal{Q}_r^n = \mathcal{Q}^n$ and $Wz^\perp = (W^{-T}z)^\perp = (Wz)^\perp$. Finally, $WW = \mathrm{I}$ by symmetry. $\square$

In the following derivation of subspace intersections for the quadratic cone $\mathcal{Q}^n$, many results follow by definition. Note that x_1 in this definition is sometimes called the radius entry, and $\hbar = x_{2:n}$ the hyperball entries, since the quadratic cone correspond to an $(n-1)$-dimensional hyperball with radius x_1 centered around the origin. First, the elimination of zero-valued entries is formalized.

Proposition 7. *Consider $(x_1, \hbar) \in \mathcal{Q}^n$.*

1. *Given $\hbar_i = 0$, the membership is equivalent to*

$$\begin{pmatrix} x_1 \\ \hbar_{1:(i-1)} \\ \hbar_{(i+1):n} \end{pmatrix} \in \mathcal{Q}^{n-1}.$$

2. *Given $x_1 = 0$, the membership is equivalent to*

$$\hbar \in \{0\}^{n-1}.$$

Next, the elimination of scaled duplicates is formalized by describing an aggregation of squares in the definition of $\mathcal{Q}^n$. In particular, scaled duplicates within the hyperball entries are eliminated by aggregating a sum of two squares, while scaled radius-hyperball duplicates are eliminated by aggregating a difference of two squares.

Proposition 8. *Consider $(x_1, \hbar) \in \mathcal{Q}^n$.*

1. *Given $\hbar_i = \alpha \hbar_j$, where $j < i$ is assumed without loss of generality, the membership is equivalent to*

$$\begin{pmatrix} x_1 \\ \hbar_{1:(j-1)} \\ \sqrt{1+\alpha^2}\hbar_j \\ \hbar_{(j+1):(i-1)} \\ \hbar_{(i+1):n} \end{pmatrix} \in \mathcal{Q}^{n-1}.$$

2. *Let $\hbar_i = \alpha x_1$. If $\alpha^2 \geq 1$, the aggregation yields an empty radius entry allowing use of Proposition 7-(2).*
   (a) *Given $\alpha^2 < 1$, the membership is equivalent to*

$$\begin{pmatrix} \sqrt{1-\alpha^2}x_1 \\ \hbar_{1:(i-1)} \\ \hbar_{(i+1):n} \end{pmatrix} \in \mathcal{Q}^{n-1}.$$

   (b) *Given $\alpha^2 = 1$, the membership is equivalent to*

$$\begin{pmatrix} \hbar_{1:(i-1)} \\ \hbar_{(i+1):n} \end{pmatrix} \in \{0\}^{n-1} \quad \textit{and} \quad x_1 \geq 0.$$

   (c) *Given $\alpha^2 > 1$, the membership is equivalent to*

$$\begin{pmatrix} \hbar_{1:(i-1)} \\ \sqrt{\alpha^2-1}\, x_1 \\ \hbar_{(i+1):n} \end{pmatrix} \in \{0\}^{n-1},$$

   where $x_1 \geq 0$ is redundant.

The applicability of these eliminations are greatly widened by the fact that the hyperball entries can be modified by orthogonal transformations [11].

Proposition 9. *The hyperball entries are invariant to orthogonal transformations. That is, given $H = H^{-T}$, then*

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & H \end{smallmatrix} \right) \mathcal{Q}^n = \mathcal{Q}^n.$$

Proof. $\sum_{j=2}^n x_j^2 = x_{2:n}^T x_{2:n} = (Hx_{2:n})^T (Hx_{2:n})$. $\square$

The Householder matrix defines a particularly useful orthogonal transformation matrix able to rotate any nonzero vector to the main axis $\mathrm{e}_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$. This follows from the results of [4].

Proposition 10. *The Householder matrix solving $H\lambda = \|\lambda\|_2 \mathrm{e}_1$ for nonzero $\lambda \in \mathbb{R}^n$ is given by*

$$\begin{aligned} H &= \left(\lambda/\|\lambda\|_2 ,\ V \right) = \begin{pmatrix} \lambda^T/\|\lambda\|_2 \\ V^T \end{pmatrix} = \mathrm{I} - 2uu^T, \\ u &= \begin{cases} \frac{\lambda/\|\lambda\|_2 - \mathrm{e}_1}{\|\lambda/\|\lambda\|_2 - \mathrm{e}_1\|_2} & \textit{if } \lambda/\|\lambda\|_2 \neq \mathrm{e}_1, \\ 0 & \textit{otherwise}, \end{cases} \end{aligned}$$

where H is symmetric and orthogonal by definition.

Finally, the necessary results are in place to characterize all subspace intersections of the quadratic cone.

Theorem 1. *Suppose $z = (z_1, \lambda^T)^T \in \mathbb{R}^n$ is nonzero. Further, when $\lambda \neq 0$, consider $\alpha = -z_1/\|\lambda\|_2$ and the submatrix V of H from Proposition 10 solving $H\lambda = \|\lambda\|_2 \mathrm{e}_1$. The conditions of Definition 1 is satisfied for* (P) *by:*

1. $\mathcal{Q}^n \cap z^\perp = \{0\}^n$, *for* $z_1^2 > \|\lambda\|_2^2$*;*
2. $\mathcal{Q}^n \cap z^\perp = \left(\begin{smallmatrix} 1 \\ \alpha\lambda/\|\lambda\|_2 \end{smallmatrix} \right) \mathbb{R}_+$, *for* $0 \neq z_1^2 = \|\lambda\|_2^2$*;*
3. $\mathcal{Q}^n \cap z^\perp = \begin{pmatrix} \frac{1}{\sqrt{1-\alpha^2}} & 0 \\ \frac{\alpha}{\sqrt{1-\alpha^2}}\lambda/\|\lambda\|_2 & V \end{pmatrix} \mathcal{Q}^{n-1}$, *for* $z_1^2 < \|\lambda\|_2^2$,

and for (D) *by:*

4. $x \in \mathcal{Q}^n \cap z^\perp \;\Leftrightarrow\; x \in \{0\}^n,$ *for* $z_1^2 > \|\lambda\|_2^2$*;*

5. $x \in \mathcal{Q}^n \cap z^\perp \;\Leftrightarrow\; \begin{pmatrix} 1 & 0 \\ 0 & V^T \\ -\alpha & \lambda^T/\|\lambda\|_2 \end{pmatrix} x \in \mathbb{R}_+ \times \{0\}^{n-1},$
for $0 \neq z_1^2 = \|\lambda\|_2^2$*;*

6. $x \in \mathcal{Q}^n \cap z^\perp \;\Leftrightarrow\; \begin{pmatrix} \sqrt{1-\alpha^2} & 0 \\ 0 & V^T \\ -\alpha & \lambda^T/\|\lambda\|_2^2 \end{pmatrix} x \in \mathcal{Q}^{n-1} \times \{0\},$
for $z_1^2 < \|\lambda\|_2^2$.

Proof. If $\lambda = 0$ (a subcase of statement 1), the claim follows from Proposition 7-(2). Otherwise $\lambda \neq 0$, and the subspace intersections of $\mathcal{Q}^n$ are characterized by

$$\begin{aligned} \mathcal{Q}^n \cap z^\perp &= W\big((W\mathcal{Q}^n) \cap (Wz)^\perp\big), \\ &= W\left(\mathcal{Q}^n \cap \left(\begin{smallmatrix} z_1 \\ \|\lambda\|_2 e_1 \end{smallmatrix}\right)^\perp\right), \end{aligned}$$

for the symmetric and orthogonal matrix $W = \left(\begin{smallmatrix} 1 & 0 \\ 0 & H \end{smallmatrix}\right)$. This follows firstly by arguments for the proof of Proposition 6, and secondly by Proposition 9 and the definition of H. The set $\mathcal{Q}^n \cap \left(\begin{smallmatrix} z_1 \\ \|\lambda\|_2 e_1 \end{smallmatrix}\right)^\perp = \{x \in \mathcal{Q}^n : z_1 x_1 + \|\lambda\|_2 x_2 = 0\}$ is characterized below, and the claims follow from left-multiplication by W. If $z_1 = 0$ (a subcase of statement 3), the claim hence follows from Proposition 7-(1) as

$$\begin{aligned} \mathcal{Q}^n \cap \left(\begin{smallmatrix} z_1 \\ |\lambda\|_2 e_1 \end{smallmatrix}\right)^\perp &= \{x \in \mathbb{R}^n : \left(\begin{smallmatrix} x_1 \\ x_{3:n} \end{smallmatrix}\right) \in \mathcal{Q}^{n-1},\ x_2 = 0\}, \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \mathrm{I} \end{smallmatrix}\right) \mathcal{Q}^{n-1}. \end{aligned}$$

Otherwise $z_1 \neq 0$, and Proposition 8-(2) is used to eliminate the dependency $x_2 = \alpha x_1$ where $\alpha = -z_1/\|\lambda\|_2$. If $\alpha^2 > 1$ (the last of statement 1), the claim follows by

$$\begin{aligned} &\mathcal{Q}^n \cap \left(\begin{smallmatrix} z_1 \\ \|\lambda\|_2 e_1 \end{smallmatrix}\right)^\perp \\ &\quad = \{x \in \mathbb{R}^n : \begin{pmatrix} \sqrt{\alpha^2-1}\, x_1 \\ x_{3:n} \end{pmatrix} \in \{0\}^{n-1},\ x_2 = \alpha x_1\}, \\ &\quad = \{0\}^n. \end{aligned}$$

If $\alpha^2 = 1$ (statement 2), the claim follows by

$$\begin{aligned} &\mathcal{Q}^n \cap \left(\begin{smallmatrix} z_1 \\ \|\lambda\|_2 e_1 \end{smallmatrix}\right)^\perp \\ &\quad = \{x \in \mathbb{R}^n : x_{3:n} \in \{0\}^{m-2},\ x_1 \geq 0,\ x_2 = \alpha x_1\}, \\ &\quad = \begin{pmatrix} 1 \\ \alpha \\ \{0\}^{m-2} \end{pmatrix} \mathbb{R}_+. \end{aligned}$$

If $\alpha^2 < 1$ (the last of statement 3), the claim follows by

$$\begin{aligned} &\mathcal{Q}^n \cap \left(\begin{smallmatrix} z_1 \\ |\lambda\|_2 e_1 \end{smallmatrix}\right)^\perp \\ &\quad = \{x \in \mathbb{R}^n : \begin{pmatrix} \sqrt{1-\alpha^2}\, x_1 \\ x_{3:n} \end{pmatrix} \in \mathcal{Q}^{n-1},\ x_2 = \alpha x_1\}, \\ &\quad = \begin{pmatrix} \frac{1}{\sqrt{1-\alpha^2}} & 0 \\ \frac{\alpha}{\sqrt{1-\alpha^2}} & 0 \\ 0 & \mathrm{I} \end{pmatrix} \mathcal{Q}^{n-1}. \end{aligned}$$

The dual statements 4-6, are characterized from the above derivations using that $x \in \mathcal{Q}^n \cap z^\perp$ is equivalent to

$$\begin{aligned} PWx \in PW(\mathcal{Q}^n \cap z^\perp) &= P\big((W\mathcal{Q}^n) \cap (Wz)^\perp\big), \\ &= P\left(\mathcal{Q}^n \cap \left(\begin{smallmatrix} z_1 \\ \|\lambda\|_2 e_1 \end{smallmatrix}\right)^\perp\right), \end{aligned}$$

for full rank matrices P and $W = \left(\begin{smallmatrix} 1 & 0 \\ 0 & H \end{smallmatrix}\right)$ as above. The statements are obtained using $P = \mathrm{I}$ for statement 1, $P = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{I} \\ -\alpha & 1 & 0 \end{smallmatrix}\right)$ for statement 2, and $P = \begin{pmatrix} \sqrt{1-\alpha^2} & 0 & 0 \\ 0 & 0 & \mathrm{I} \\ -\alpha & 1 & 0 \end{pmatrix}$ for statement 3. □

6.1. Tricks for multiple intersections

Consider the subspace intersection

$$\mathcal{Q}^n \cap \operatorname{span}(z_1, z_2, \ldots)^\perp = \mathcal{Q}^n \cap z_1^\perp \cap \ldots \cap z_k^\perp.$$

As $z_j^\perp = (-z_1)^\perp$, one may sign-normalize and aggregate, by summation, the subset of so-called facially exposing vectors $z_j \in \pm(\mathcal{Q}^n)^*$ by Proposition 5. The intersection of $\mathcal{Q}^n$ with all purely hyperball-supported vectors can moreover be aggregated into one computational step.

Theorem 2. *Let* $\mathcal{Z} = \operatorname{span}(z_1, \ldots, z_k) \subseteq \mathbb{R}^n$ *for nonzero* $z_j = (0, \lambda_j^T)^T \in \mathbb{R}^n$*, and consider the QR-decomposition with pivoting* $(Q_1, Q_2)\left(\begin{smallmatrix} R_1 \\ 0 \end{smallmatrix}\right) = (\lambda_1, \ldots, \lambda_k)P$ *for full row rank* $R_1 \in \mathbb{R}^{r\times k}$*. The conditions of Definition 1 is satisfied for* (P) *by:*

1. $\mathcal{Q}^n \cap \mathcal{Z}^\perp = \left(\begin{smallmatrix} 1 & 0 \\ 0 & Q_2 \end{smallmatrix}\right) \mathcal{Q}^{n-r}$,

and for (D) *by:*

2. $x \in \mathcal{Q}^n \cap \mathcal{Z}^\perp \;\Leftrightarrow\; \begin{pmatrix} 1 & 0 \\ 0 & Q_2^T \\ 0 & Q_1^T \end{pmatrix} x \in \mathcal{Q}^{n-r} \times \{0\}^r.$

Proof. The subspace reduction of $\mathcal{Q}^n$ is characterized by

$$\mathcal{Q}^n \cap \mathcal{Z}^\perp \;=\; W^T\big(\mathcal{Q}^n \cap (W\mathcal{Z})^\perp\big),$$

for nonsymmetric but orthogonal $W = \begin{pmatrix} 1 & 0 \\ 0 & Q_2^T \\ 0 & Q_1^T \end{pmatrix}$, following arguments for the proof of Proposition 6. Moreover, in terms of the column space operator $\mathcal{C}(\cdot)$, then

$$W\mathcal{Z} = W\mathcal{C}\begin{pmatrix} 0, & \ldots, & 0 \\ \lambda_1, & \ldots, & \lambda_k \end{pmatrix} = W\mathcal{C}\left(\begin{smallmatrix} 0 \\ Q_1 \end{smallmatrix}\right) = \mathcal{C}\left(\begin{smallmatrix} 0 \\ \left(\begin{smallmatrix} 0 \\ \mathrm{I} \end{smallmatrix}\right) \end{smallmatrix}\right),$$

by definition, where $Q_1^T Q_1 = \mathrm{I} \in \mathbb{R}^{r\times r}$ is the identity matrix. Hence, by Proposition 7-(1),

$$\mathcal{Q}^n \cap (W\mathcal{Z})^\perp = \mathcal{Q}^{n-r} \times \{0\}^r,$$

showing statement 1 after left-multiplication by W^T. Statement 2 is shown from the above derivation using that $x \in \mathcal{Q}^n \cap \mathcal{Z}^\perp$ is equivalent to $Wx \in \mathcal{Q}^n \cap (W\mathcal{Z})^\perp$. □

7. Final comments

The subspace reduction algorithm shows a potential for rapid regularization. Nevertheless, it is likely doomed to partial regularization for most applications unless the conditions of Definition 1 can be weakened. This is due to the consequences of Corollary 1 where only a subset of intersections of a Cartesian product are shown readily available. It moreover remains unknown to what degree subspace intersections can be characterized for cones other than the second-order cones treated here.

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