Strong Duality and Dual Pricing Properties in Semi-infinite Linear Programming–A Non-Fourier-Motzkin Elimination Approach

Qinghong Zhang

Department of Mathematics and Computer Science
Northern Michigan University
Marquette, MI 49855
qzhang@nmu.edu

Abstract

The Fourier-Motzkin elimination method has been recently extended to linear inequality systems that have infinitely many inequalities. It has been used in the study of linear semi-infinite programming by Basu, Martin, and Ryan. Following the idea of the conjecture for semi-infinite programming in a paper by Kortanek and Zhang recently published in Optimization, which states "all the duality results proved by applying FM (the Fourier-Motzkin elimination method) first can also be obtained by working with the problem directly", in this paper without using the Fourier-Motzkin elimination, we reproduce all the results presented in a recent paper by Basu, Martin, and Ryan on the strong duality and dual pricing properties in semi-infinite programming in which the main mechanism is the Fourier-Motzkin elimination. We also present some new results regarding the strong duality and dual pricing properties, which are the main topics in Basu-Martin-Ryan's paper.

Keywords. Semi-infinite programming; Strong duality property; Dual pricing property.

1 Introduction

The Basu-Martin-Ryan paper [2] provides a different approach to study semi-infinite programming problems by applying the Fourier-Motzkin (FM) elimination of variables method to linear inequality systems that have an infinite number of inequalities. Kortanek and Zhang [6] prove all the theorems that lead to the full eleven possible duality state classification theory by working with the reduced form of the primal-dual pair of semi-infinite programming problems obtained by applying the Fourier-Motzkin elimination. Although the semi-infinite classification theory was established more than four decades ago, see Ben-Israel, et al. [4] and Kortanek [5], for the first time it was established by the Fourier-Motzkin elimination method. Despite these positive aspects of the Fourier-Motzkin elimination to semi-infinite programming, Remark 5 in Kortanek and Zhang [6] states

"All duality results available in the literature are valid for (P) and (DP) because they are simply a primal-dual pair of linear SIP. Of course, the proof of these results could be simplified by working on the pair (P) and (DP) due to the simplicity of (P) and (DP). In this sense, every duality result in the literature can be obtained using FM method. On the other hand, every feasible solution of (P) or (DP) can be converted to a solution of $(SILP)$ or $(FDSILP)$ after a finite number of operations, we conjecture that all the duality results proved by applying FM first can also be obtained by working with the problem directly."

In this remark, $(SILP)$ and $(FDSILP)$ refer to the primal-dual pair of a semi-infinite programming problem. (P) and (D) is an equivalent primal-dual pair of $(SILP)$ and $(FDSILP)$ after the Fourier-Motzkin elimination is applied to the linear constraint system of $(SILP)$. Following the statement in this remark, in this paper, we show that there is no need to use the Fourier-Motzkin elimination method in the study of strong duality and dual pricing properties. We prove, without using the Fourier-Motzkin elimination method, all the results presented in Basu et al. [3] in which the main mechanism is the Fourier-Motzkin elimination method. We correct an error in Basu et al. [3]. We also present some new results regarding the strong duality and dual pricing properties. Specifically, we propose a new subspace under which the strong duality property holds. We give a necessary and sufficient condition for the dual pricing property to hold under this subspace, which is further used to examine the examples presented in Basu et al. [3].

Two properties, the *strong duality property* and *dual pricing property*, have been proposed in Basu et al. [3] for a semi-infinite programming problem of the following form under the assumption that the problem has a finite optimal value.

$$\begin{array}{lll} (SILP) & \inf & c^T x \\ & \text{subject to} & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i, \ i \in I, \end{array} \tag{1}$$

where $c \in \mathbb{R}^n$ (n-dimensional Euclidean space) is a given vector, $x = (x_1, x_2, \ldots, x_n)^T$, $c^T x$ is the inner product of c and x, I is the index set, $b_i \in \mathbb{R}$ (the set of real numbers) and $a_{ik} \in \mathbb{R}$ for $i \in I$ and $k = 1, 2, \ldots, n$. Let $\mathbb{R}^I$ be the vector space of all real-valued functions defined on I. Then $\{a_{ik} \mid i \in I\}$, $k = 1, 2, \ldots, n$, and $\{b_i \mid i \in I\}$ can be viewed as elements in $\mathbb{R}^I$. We use a^k, $k = 1, 2, \ldots, n$, and b to represent these elements in $\mathbb{R}^I$. In other words, $a^k : I \to \mathbb{R}$ with $a^k(i) = a_{ik}$, $k = 1, 2, \ldots, n$, and $b : I \to \mathbb{R}$ with $b(i) = b_i$ for $i \in I$. If $r \in \mathbb{R}^I$, we use $r \succeq 0$ to represent $r(i) \geq 0$ for all $i \in I$. Suppose that Y is a subspace of $\mathbb{R}^I$ that contains a^k, $k = 1, 2, \ldots, n$, and b with its algebraic dual denoted by Y'. While the choice of Y does not change the feasibility and optimality of $(SILP)$, it may result in different conclusions regarding the strong duality and dual pricing property. These two properties are based on the following dual problem of $(SILP)$.

$$
\begin{aligned}
(DSILP(Y)) \qquad & \sup \quad \psi(b) \\
& \text{subject to} \quad \psi(a^k) = c_k, \ \text{for } k = 1, 2, \ldots, n, \\
& \psi \in Y', \ \text{and} \ \psi \succeq_{Y'_+} 0,
\end{aligned} \tag{2}
$$

where $\succeq_{Y'_+}$ is the partial order induced by the cone Y'_+, the dual cone of the cone $Y \cap \mathbb{R}^I_+$, which is $Y'_+ = \{\psi \in Y' \mid \psi(y) \geq 0 \text{ for all } y \in Y \cap \mathbb{R}^I_+\}$. Here $\mathbb{R}^I_+$ is the cone of all the real-valued functions from I to $\mathbb{R}$ with nonnegative values. The subspace Y of $\mathbb{R}^I$ is called the *constraint space* of $(SILP)$ following Anderson and Nash [1] and Basu et al. [3].

The following weak duality theorem is well-known.

Theorem 1 *If x is a feasible solution of $(SILP)$ and ψ is a feasible solution of $(DSILP(Y))$, then $c^T x \geq \psi(b)$.*

The *strong duality* and *dual pricing* properties are defined for an instance of semi-infinite programming problems. In this paper, we always assume that a^k, $k = 1, 2, \ldots, n$, and c are fixed. Since b can be different, to make it clear about what the right hand side of the constraint system of (1) is, we use $(SILP(b))$ and $(DSILP(Y, b))$ to represent (1) and (2). Now let's review the definitions of the *strong duality property* and *dual pricing property*. They are defined for the primal-dual pair $(SILP(b))$ and $(DSILP(Y, b))$ with respect to a given constraint space Y of $\mathbb{R}^I$ under the assumption that the optimal value of $(SILP(b))$ is finite.

Definition 1 *The primal-dual pair $(SILP(b))$ and $(DSILP(Y, b))$ satisfies the* strong duality property *if there exists a dual optimal solution ψ^* such that $\psi^*(b)$ equals the optimal value of $(SILP(b))$.*

Because the *dual pricing property* involves the sensitivity analysis of the right hand side of the constraint of $(SILP(b))$, following the notation in Basu et al. [3] we use $OV(b)$ to represent the optimal value of $(SILP(b))$.

Definition 2 *The primal-dual pair* $(SILP(b))$ *and* $(DSILP(Y,b))$ *satisfies the* dual pricing property *if for every perturbation* $d \in Y$ *such that* $(SILP(b+d))$ *is feasible, there exists an optimal solution* ψ^* *of the dual problem* $(DSILP(Y,b))$ *and an* $\hat{\epsilon} > 0$ *such that* $OV(b+\epsilon d) = \psi^*(b+\epsilon d) = OV(b) + \epsilon\psi^*(d)$ *for all* $\epsilon \in [0,\hat{\epsilon}]$.

Even though the optimal solutions of the dual problems are required to be the same for all $\epsilon \in [0,\hat{\epsilon}]$ in Definition 2, the following lemma shows that we can choose different optimal solution for different $\epsilon \in [0,\hat{\epsilon}]$.

Lemma 1 *If for every perturbation* $d \in Y$ *such that* $(SILP(b+d))$ *is feasible, there exists an* $\hat{\epsilon} > 0$ *such that* $OV(b+\epsilon d)$ *is finite for each* $\epsilon \in [0,\hat{\epsilon}]$ *and for each* $\epsilon \in [0,\hat{\epsilon}]$ *there exists an optimal solution* ψ^*_ϵ *of the dual problem* $(DSILP(Y, b+\epsilon d))$ *such that* $OV(b+\epsilon d) = \psi^*_\epsilon(b+\epsilon d) = OV(b) + \epsilon\psi^*_\epsilon(d)$*, then the dual pricing property holds for the primal-dual pair.*

Proof: First, we know that $\psi^*_\epsilon(b) = OV(b)$ for all $\epsilon \in [0,\hat{\epsilon}]$ due to the assumption that $\psi^*_\epsilon(b+\epsilon d) = \psi^*_\epsilon(b) + \epsilon\psi^*_\epsilon(d) = OV(b) + \epsilon\psi^*_\epsilon(d)$.

Suppose that $\epsilon_1 \in [0,\hat{\epsilon}]$, $\epsilon_2 \in [0,\hat{\epsilon}]$, and $\epsilon_1 \neq \epsilon_2$. Then $\psi^*_{\epsilon_1}(b) = \psi^*_{\epsilon_2}(b) = OV(b)$. Noticing that $\psi^*_{\epsilon_1}$ is an optimal solution of

$$\begin{array}{lll} (DSILP(Y, b+\epsilon_1 d)) & \sup & \psi(b+\epsilon_1 d) \\ & \text{subject to} & \psi(a^k) = c_k, \text{ for } k = 1,2,\ldots,n, \\ & \psi \in Y', \text{ and } \psi \succeq_{Y'_+} 0, & \end{array} \tag{3}$$

and $\psi^*_{\epsilon_2}$ is an optimal solution of

$$\begin{array}{lll} (DSILP(Y, b+\epsilon_2 d)) & \sup & \psi(b+\epsilon_2 d) \\ & \text{subject to} & \psi(a^k) = c_k, \text{ for } k = 1,2,\ldots,n, \\ & \psi \in Y', \text{ and } \psi \succeq_{Y'_+} 0, & \end{array} \tag{4}$$

we obtain that $\psi^*_{\epsilon_1}$ is a feasible solution of $(DSILP(Y, b+\epsilon_2 d))$. So $\psi^*_{\epsilon_1}(b+\epsilon_2 d) \leq \psi^*_{\epsilon_2}(b+\epsilon_2 d)$, which gives that $\psi^*_{\epsilon_1}(d) \leq \psi^*_{\epsilon_2}(d)$.

Similarly, $\psi^*_{\epsilon_2}$ is a feasible solution of $(DSILP(Y, b+\epsilon_1 d))$. So $\psi^*_{\epsilon_1}(b+\epsilon_1 d) \geq \psi^*_{\epsilon_2}(b+\epsilon_1 d)$, which gives that $\psi^*_{\epsilon_1}(d) \geq \psi^*_{\epsilon_2}(d)$. Hence, we get $\psi^*_{\epsilon_1}(d) = \psi^*_{\epsilon_2}(d)$.

Now, in Definition 2 taking $\psi^* = \psi^*_{\hat{\epsilon}}$, we can easily see that ψ^* is an optimal solution to the dual problem $(DSILP(Y,b))$ which satisfies all the requirements in Definition 2. □

Remark 1 *To prove if the dual pricing property holds for the primal-dual pair* $(SILP(b))$ *and* $(DSILP(Y,b))$*, we need to show*

- *For any $d \in Y$ such that $(SILP(b+d))$ is feasible, there exists an $\hat{\epsilon} > 0$ such that $OV(b+\epsilon d)$ is finite for every $\epsilon \in [0, \hat{\epsilon}]$.*
- *For any $\epsilon \in [0, \hat{\epsilon}]$, there is an optimal solution ψ^*_ϵ to the dual problem $(DSILP(Y, b+\epsilon d))$, such that $OV(b+\epsilon d) = \psi^*_\epsilon(b+\epsilon d) = OV(b) + \epsilon \psi^*_\epsilon(d)$.*

The next lemma is very important in this paper. It basically says that there is no improving ray if a semi-infinite programming problem is finite. It is well-known in the literature. We include a proof for completeness.

Lemma 2 *If $(SILP(b))$ is feasible and has a finite optimal value, then for any $x = (x_1, x_2, \ldots, x_n)^T$ with $\sum_{k=1}^n x_k a^k \succeq 0$, $c^T x \geq 0$. Hence, $OV(\theta) = 0$, where θ represents the zero function from I to $\mathbb{R}$, that is $\theta(i) = 0$ for all $i \in I$.*

Proof: Let $y = (y_1, y_2, \ldots, y_n)^T$ be a feasible solution of $(SILP(b))$. It is easy to verify that $\alpha x + y$ with $\alpha > 0$ is also a feasible solution of $(SILP(b))$. The value of objective function of $(SILP(b))$ at $\alpha x + y$ is $\alpha c^T x + c^T y$. If $c^T x < 0$, then we obtain that $OV(b) = -\infty$, which contradicts the assumption that $(SILP(b))$ is bounded. □

2 The smallest constraint space

Let $U = span(a^1, a^2, \ldots, a^n, b)$. It is the smallest vector space that makes the formulation of $(DSILP(Y, b))$ meaningful.

Theorem 2 *Consider $(SILP(b))$ that is feasible and has a finite optimal value. Then the dual problem $(DSILP(U, b))$ is solvable and the strong duality property holds for $(SILP(b))$ and $(DSILP(U, b))$. Moreover, $(DSILP(U, b))$ has a unique optimal solution.*

Proof: Because $U = span(a^1, a^2, \ldots, a^n, b)$, to define a linear function on U, we only need to define the values of the function at $a^1, a^2, \ldots, a^n$, and b. We let $\psi^* : U \to \mathbb{R}$ be defined by $\psi^*(a^k) = c_k$ and $\psi^*(b) = OV(b)$. We show that such ψ^* is well defined.

In the case that $a^1, a^2, \ldots, a^n, b$ are linearly independent, then $a^1, a^2, \ldots, a^n, b$ form a basis of U. Therefore, ψ^* is well defined.

If $a^1, a^2, \ldots, a^n, b$ are linearly dependent, then either some a^p is a linear combination of a^k for $k = 1, 2, \ldots, n$ and $p \neq k$, or b is a linear combination of a^k for $k = 1, 2, \ldots, n$.

Without loss of generality, we may assume that $a^1 = x_2 a^2 + x_3 a^3 + \cdots + x_n a^n$. We conclude that $c_1 = x_2 c_2 + x_3 c_3 + \cdots + x_n c_n$. Otherwise, we may assume that $c_1 < x_2 c_2 + x_3 c_3 + \cdots + x_n c_n$ or $c_1 > x_2 c_2 + x_3 c_3 + \cdots + x_n c_n$. Because $a^1 - x_2 a^2 - x_3 a^3 - \cdots - x_n a^n = 0$, by lemma 2 we obtain that $(SILP(b))$ is not bounded, which is a contradiction.

Now if b is a linear combination of a^k for $k = 1, 2, \ldots, n$. We may assume that $b = x_1a^1 + x_2a^2 + \cdots + x_na^n$. We claim that $OV(b) = x_1c_1 + x_2c_2 + \cdots + x_nc_n$. Otherwise, $OV(b) < x_1c_1 + x_2c_2 + \cdots + x_nc_n$ because of the feasibility of $x = (x_1, x_2, \ldots, x_n)^T$ of $(SILP(b))$. This implies that there exists a feasible solution $y = (y_1, y_2, \ldots, y_n)^T$ of $(SILP(b))$ such that $c^Tx > c^Ty$. Because y is a feasible solution of $(SILP(b))$, we know $y_1a^1 + y_2a^2 + \cdots + y_na^n \succeq b$. Therefore, we have $\sum_{k=1}^n (y_k - x_k)a^k \succeq 0$ and $\sum_{k=1}^n (y_k - x_k)c_k < 0$, indicating that $(SILP(b))$ is not bounded by Lemma 2.

Combining all we have proved, we know that ψ^* is well-defined.

Now we prove that $\psi^* \in U'_+$. We let $u = \sum_{k=1}^n x_ka^k + \alpha b \succeq 0$. If $\alpha = 0$, then by Lemma 2 we know that $\psi^*(u) = c^Tx = \sum_{k=1}^n c_kx_k \geq 0$.

If $\alpha < 0$, then we know that $\sum_{k=1}^n \frac{x_k}{-\alpha}a^k \succeq b$. Therefore, $\frac{1}{-\alpha}x$ is a feasible solution of $(SILP(b))$. Hence, $c^T(\frac{x}{-\alpha}) \geq OV(b)$, which shows that $\psi^*(u) = c^Tx + \alpha OV(b) \geq 0$.

If $\alpha > 0$, then without loss of generality we may assume $\alpha = 1$. Because $(SILP(b))$ is finite, for any $\delta > 0$ there exists a feasible solution y of $(SILP(b))$ such that $c^Ty \leq OV(b) + \delta$. Because $\sum_{k=1}^n x_ka^k + b \succeq 0$ and $\sum_{k=1}^n y_ka^k \succeq b$, we know $\sum_{k=1}^n (x_k + y_k)a^k \succeq 0$. By Lemma 2, we know that $c^Tx + c^Ty = c^T(x + y) \geq 0$. Therefore, $c^Tx + OV(b) + \delta \geq c^Tx + c^Ty \geq 0$, which gives $c^Tx + OV(b) + \delta \geq 0$ for all $\delta > 0$. Letting $\delta \to 0$, we obtain that $\psi^*(u) = c^Tx + OV(b) \geq 0$.

By combining the proofs for $\alpha = 0$, $\alpha < 0$, and $\alpha > 0$, we know that $\psi^* \in U'_+$, which together with the definition of $\psi^*(a^k) = c_k,\ k = 1, 2, \ldots, n$, and $\psi^*(b) = OV(b)$ shows that the result of the theorem is true. □

Next we show that the dual pricing property holds with respect to U.

Theorem 3 *The dual pricing property holds for the primal-dual pair* $(SILP(b))$ *and* $(DSILP(U, b))$.

Proof: Let $d \in U$ be any perturbation and assume that $(SILP(b + d))$ is feasible. To prove the dual pricing property holds for $(SILP(b))$ and $(DSILP(U, b))$, by Lemma 1 we need to show (i) there exists an $\hat{\epsilon} > 0$ such that $(SILP(b + \epsilon d))$ is finite for any $\epsilon \in [0, \hat{\epsilon}]$. (ii) For each $\epsilon \in [0, \hat{\epsilon}]$, there is an optimal solution ψ^*_ϵ of the dual problem $(DSILP(U, b + \epsilon d))$ such that $\psi^*_\epsilon(b + \epsilon d) = OV(b + \epsilon d) = OV(b) + \epsilon\psi^*_\epsilon(d)$.

Because $d \in U$, we can assume $d = x_1a^1 + x_2a^2 + \cdots + x_na^n + \alpha b$. If $\alpha \geq -1$, then it is straightforward to show that for any feasible solution $y = (y_1, y_2, \ldots, y_n)^T$ of $(SILP(b + \epsilon d))$ with $0 \leq \epsilon \leq 1$, $\frac{1}{1+\epsilon\alpha}(y - \epsilon x)$ is a feasible solution of $(SILP(b))$. Therefore, $\frac{1}{1+\epsilon\alpha}c^T(y - \epsilon x) \geq OV(b)$, which shows that $c^Ty \geq (1 + \epsilon\alpha)OV(b) + \epsilon c^Tx$. Because y is any feasible solution of $(SILP(b + \epsilon d))$, we have $OV(b + \epsilon d) \geq (1 + \epsilon\alpha)OV(b) + \epsilon c^Tx$. Now let $z = (z_1, z_2, \ldots, z_n)^T$ be any feasible solution of $(SILP(b))$, then it is easy to verify that $(1 + \epsilon\alpha)z + \epsilon x$ is a feasible solution of $(SILP(b + \epsilon d))$. Hence, we have $(1 + \epsilon\alpha)c^Tz + \epsilon c^Tx \geq OV(b + \epsilon d)$. Because z is any feasible solution of $(SILP(b))$, we know $(1+\epsilon\alpha)OV(b)+\epsilon c^T x \geq OV(b+\epsilon d)$, which together with $(1+\epsilon\alpha)OV(b)+\epsilon c^T x \leq OV(b+\epsilon d)$ gives $OV(b+\epsilon d) = (1+\epsilon\alpha)OV(b)+\epsilon c^T x$. If $\alpha < -1$, then for $0 < \epsilon \leq \frac{1}{-\alpha}$, we know that $(1+\epsilon\alpha) \geq 0$, and hence, we can easily verify that $OV(b+\epsilon d) = (1+\epsilon\alpha)OV(b)+\epsilon c^T x$. In either case, there is $\hat{\epsilon} > 0$ such that for any $\epsilon \in [0, \hat{\epsilon}]$, $OV(b+\epsilon d)$ is finite. Hence, (i) is true.

Now let's prove (ii). We first consider the case that $\alpha \geq -1$. We let ψ^* be the unique optimal solution for the dual problem $(DSILP(U,b))$ defined in Theorem 2, that is $\psi^* \in U'_+$, $\psi^*(a^k) = c_k$, and $\psi^*(b) = OV(b)$. Similarly, we let ϕ^* be the unique optimal solution for the dual problem $(DSILP(U^*, b+d))$, that is $\phi^* \in (U^*)'_+$, $\phi^*(a^k) = c_k$, and $\phi^*(b+d) = OV(b+d)$, where $U^* = span(a^1, a^2, \ldots, a^n, b+d)$.

Since $d \in U$, it is obvious that $U^* \subseteq U$. If $U^* = U$, then ψ^* and ϕ^* are feasible to both $(DSILP(U,b))$ and $(DSILP(U^*,b))$. Because of the optimality of ψ^* and ϕ^*, we know

$$\psi^*(b) \geq \phi^*(b) \tag{5}$$
$$\phi^*(b+d) \geq \psi^*(b+d). \tag{6}$$

Noticing that $\psi^*(a^k) = \phi^*(a^k) = c_k$ and by virtue of (6), we obtain that $(1+\alpha)\phi^*(b) \geq (1+\alpha)\psi^*(b)$. If $\alpha > -1$, then we get that $\phi^*(b) \geq \psi^*(b)$, which together with (5) shows that $\phi^*(b) = \psi^*(b)$. Therefore, $\phi^* = \psi^*$, which implies

$$OV(b+d) = \phi^*(b+d) = \psi^*(b+d) = \psi^*(b) + \psi^*(d) = OV(b) + \psi^*(d). \tag{7}$$

If $\alpha = -1$, then $U^* = U$ indicates that $b \in span(a^1, a^2, \ldots, a^n)$. In this case, both ψ^* and ϕ^* are determined by the values at a^k, $k = 1, 2, \ldots, n$. Because the values of ψ^* and ϕ^* at a^k, $k = 1, 2, \ldots, n$, are equal, we know that $\psi^* = \phi^*$. Hence, $OV(b+d) = \phi^*(b+d) = \phi^*(b) + \phi^*(d) = \psi^*(b) + \phi^*(d) = OV(b) + \phi^*(d)$.

Now, consider the case that $U^* \neq U$. In this case, we know that $b \notin span(a^1, a^2, \ldots, a^n)$ and $b+d \in span(a^1, a^2, \ldots, a^n)$. Therefore, $OV(b+d) = \phi^*(b+d) = \psi^*(b+d) = \psi^*(b) + \psi^*(d) = OV(b) + \psi^*(d)$.

Combining all of the above, we know that if $d = x_1 a^1 + x_2 a^2 + \cdots + x_n a^n + \alpha b$ with $\alpha \geq -1$, then there is an optimal solution ψ^* to the dual problem $(DSILP(U,b))$, such that $OV(b+d) = \psi^*(b+d) = OV(b) + \psi^*(d)$.

Now if $d = x_1 a^1 + x_2 a^2 + \cdots + x_n a^n + \alpha b$ with $\alpha < -1$, then we can let $0 < \hat{\epsilon} \leq \frac{1}{-\alpha}$. Then for each $\epsilon \in [0, \hat{\epsilon}]$, we know that $\epsilon d = \epsilon x_1 a^1 + \epsilon x_2 a^2 + \cdots + \epsilon x_n a^n + \epsilon\alpha b$ with $\epsilon\alpha \geq -1$. By replacing d in the previous discussion with ϵd, we obtain that there is an optimal solution ψ^*_ϵ to $(DSILP(U,b))$ such that $OV(b+\epsilon d) = \psi^*(b+\epsilon d) = OV(b) + \epsilon\psi^*(d)$.

Combining all these cases, we obtain that for $d = x_1 a^1 + x_2 a^2 + \cdots + x_n a^n + \alpha b$, $\epsilon\alpha \geq -1$ for all $0 \leq \epsilon \leq \hat{\epsilon} = \min\{1, \frac{1}{|\alpha|}\}$. Hence, $OV(b+\epsilon d) = \psi^*(b+\epsilon d) = OV(b) + \epsilon\psi^*(d)$ for $\epsilon \in [0, \hat{\epsilon}]$.

Therefore, the dual pricing property holds for the primal-dual pair $(SILP(b))$ and $(DSILP(U, b))$ by Lemma 1. □

Remark 2 *Basu et al. [3] prove Theorems 2 and 3 by applying the Fourier-Motzkin elimination to the constraint system of* $(SILP(b))$ *first. Here we provide a direct proof by constructing the linear function on* U *that satisfies the desired requirements. Basu et al. [3] also state a stronger version of the dual pricing property. By the stronger dual pricing property, it means that for any perturbation* $d \in U$ *with* $(SILP(b+d))$ *finite, there exists an optimal dual solution* ψ^* *such that* $OV(b+d) = OV(b) + \psi^*(d)$*. This actually is not correct based on Theorem 3 proved above. A counter example can be easily constructed by giving* $d = x_1a^1 + x_2a^2 + \cdots + x_na^n + \alpha b$ *with* $\alpha < -1$*.*

3 Larger Constraint Spaces

We first prove a lemma, which states that $OV(r) : \mathbb{R}^I \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is sublinear.

Lemma 3 *(i)* $OV(r+s) \le OV(r) + OV(s)$ *for* $r \in \mathbb{R}^I$ *and* $s \in \mathbb{R}^I$*.*
(ii) $OV(\alpha r) = \alpha OV(r)$ *for any* $r \in \mathbb{R}^I$ *and* $\alpha \ge 0$*.*

Proof: (i) If one of $(SILP(r))$ and $(SILP(s))$ is inconsistent, for example, $(SILP(r))$ is inconsistent, then $OV(r) = +\infty$ and the inequality is trivially true.

If one of $OV(r)$ and $OV(s)$ equals $-\infty$, for example, $OV(r) = -\infty$, then there exists a sequence $\{x^m\}_{m=1}^{+\infty} \subset \mathbb{R}^n$ such that x^m is a feasible solution of $(SILP(r))$ for each $m = 1, 2, \ldots$ and the objective value of $(SILP(r))$ at x^m approaches $-\infty$ as $m \to +\infty$. Let y be any feasible solution of $(SILP(s))$. Then $x^m + y$ is a feasible solution of $(SILP(r+s))$ for each m, and the value of the objective function of $(SILP(r+s))$ at $x^m + y$ is the sum of the value of the objective function of $(SILP(r))$ at x^m and the value of the objective function of $(SILP(s))$ at y, showing that the values of the objective function of $(SILP(r+s))$ at $x^m + y$ approaches $-\infty$. Therefore, $OV(r+s) = -\infty$ and the inequality holds.

Now considering that both $OV(r)$ and $OV(s)$ are finite. We assume that $\{x^m\}_{m=1}^{+\infty}$ and $\{y^m\}_{m=1}^{+\infty}$ are sequences of feasible solutions of $(SILP(r))$ and $(SILP(s))$, respectively, with the value of the objective function of $(SILP(r))$ at x^m approaching $OV(r)$ and the value of the objective function of $(SILP(s))$ at y^m approaching $OV(s)$. We easily see that $x^m + y^m$ is a sequence of feasible solutions of $(SILP(r+s))$ with the value of the objective function of $(SILP(r+s))$ at $x^m + y^m$ approaching $OV(r) + OV(s)$. Hence, $OV(r+s) \le OV(r) + OV(s)$.

(ii) If $\alpha = 0$, then by Lemma 1 we know that $0 = OV(\theta) = OV(0y) = 0OV(y)$. If $\alpha > 0$, noticing that if x is a feasible solution of $(SILP(r))$, then αx is a feasible solution of $(SILP(\alpha r))$, we obtain that $OV(\alpha r) = \alpha OV(r)$. □

Let W be a subset of $\mathbb{R}^I$ defined by $W = \{r \in \mathbb{R}^I \mid -\infty < OV(r) < \infty \text{ and } -\infty < OV(-r) < \infty\}$. Since $OV(\theta) = 0$, we know that W is not empty. Here again θ represents the zero function from I to $\mathbb{R}$. The next lemma shows that W is actually a linear subspace of $\mathbb{R}^I$.

Lemma 4 *W is a subspace of $\mathbb{R}^I$. If $W \neq \{\theta\}$, then W contains a^k, $k = 1, 2, \ldots, n$.*

Proof: If W contains only one element θ, then it is obvious that W is a subspace of $\mathbb{R}^I$.

To prove that W is a subspace, we need to show that W is closed under scalar multiplication and addition. Since for each $y \in W$ both $OV(r)$ and $OV(-r)$ are finite, we easily see that W is closed under scalar multiplication because $OV(\alpha r) = \alpha OV(r)$ for any $\alpha \geq 0$.

Now let $r \in W$ and $s \in W$. By Lemma 3, $OV(r + s) \leq OV(r) + OV(s)$, we know that $OV(r + s) < +\infty$. We now show that $OV(r + s) > -\infty$. This can be done by $OV(r) = OV(r + s + (-s)) \leq OV(r + s) + OV(-s)$, which gives $OV(r) - OV(-s) \leq OV(r + s)$. Similarly, we can show that $OV(-r - s)$ is finite. Therefore, W is closed under addition.

Suppose that W contains more than one element. To prove $a^k \in W$ for $k = 1, 2, \ldots, n$, we need to show that for each $k = 1, 2, \ldots, n$, $OV(\alpha a^k)$ is finite for any $\alpha \in \mathbb{R}$. Suppose that $x = (x_1, x_2, \ldots, x_n)^T$ is a feasible solution of $(SILP(\alpha a^k))$. Then it is easy to see that $(x_1, x_2, \ldots, x_{k-1}, x_k - \alpha, x_{k+1}, \ldots, x_n)^T$ is feasible to $(SILP(\theta))$. Because W contains more than one element, there is $b \in \mathbb{R}^I$ such that $OV(b)$ and $OV(-b)$ are finite. Therefore, by Lemma 2, the value of the objective function of $(SILP(\theta))$ for any feasible solution must be bigger than or equal to 0. Hence, $c^T x - \alpha c_k \geq 0$. This shows that for any feasible solution of $(SILP(\alpha a^k))$, the value of the objective function at this feasible solution is at least αc_k, which is attained by setting $x_i = 0$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, n$ and $x_k = \alpha$. Hence $OV(\alpha a^k) = \alpha c_k$, which gives that for each $k = 1, 2, \ldots, n$, $a^k \in W$. □

Remark 3 *In Basu et al. [3], a set $\hat{Y}$ consisting of all $r \in \mathbb{R}^I$ with $-\infty < OV(r) < +\infty$ is defined. A sufficient condition under which this set is a subspace is given. Then all the results proved in Basu et al. [3] are given under this sufficient condition. The advantage of the set W defined above in this paper is that we can still study the strong duality property and the dual pricing property even though the sufficient condition presented in Basu et al. [3] is not satisfied and $\hat{Y}$ is not a subspace. We believe W is the largest subspace in $\mathbb{R}^I$ that we can use to study the strong duality property and dual pricing property without using topological properties or other structures or properties of the specific underlined constraint spaces.*

Theorem 4 *Let $b \in W$. Then the strong duality property holds for the primal-dual pair $(SILP(b))$ and $(DSILP(W,b))$.*

Proof: Since $b \in W$, we know that $OV(b)$ and $OV(-b)$ are finite. We define a linear function on the space $U = span(a^1, a^2, \ldots, a^n, b)$ by $\psi^*(a^k) = c_k$ for $k = 1, 2, \ldots, n$, and $\psi^*(b) = OV(b)$. As we proved in Theorem 2, this ψ^* is well-defined and is in U'_+. It is easy to see that ψ^* is a feasible solution of $(DSILP(U, r))$ for any $r \in U$. By the weak duality result, we obtain $\psi^*(r) \leq OV(r)$ for $r \in U$. Let $\phi(r) = OV(r)$ for all $r \in W$. Then by Lemma 3 we know that ϕ is sublinear defined on W. Noticing that $\psi^*(r) \leq \phi(r)$ on U, by the Hahn-Banach Theorem, we know that ψ^* can be extended to W with $\psi^*(r) \leq OV(r) = \phi(r)$ for all $r \in W$. We now prove that this ψ^* defined on W is an optimal solution for $(DSILP(W, b))$. Since $\psi^*(a^k) = c_k$ for $k = 1, 2, \ldots, n$, and $\psi^*(b) = OV(b)$, we only need to show that $\psi^* \in W'_+$. We let $r \in W$ with $r \succeq 0$. Then $0a^1 + 0a^2 + \cdots + 0a^n \succeq -r$. This shows that 0 is a feasible solution of $(SILP(-r))$. Hence, $OV(-r) \leq 0$. But we want $\psi^*(r) \geq 0$, which is equivalent to $\psi^*(-r) = -\psi^*(r) \leq 0$. This is true because $\psi^*(-r) \leq OV(-r) \leq 0$. Therefore, ψ^* is an optimal solution of $(DSILP(W, b))$ with $\psi^*(b) = OV(b)$, which shows the strong duality property holds. □

Now, let's turn to the discussion of the dual pricing property. Let $b \in W$ and $d \in W$. If $d \in U = span(a^1, a^2, \ldots, a^n, b)$, then the dual pricing property holds because for the constraint space U, we can define a linear function ψ^* by $\psi^*(a^k) = c_k$ and $\psi^*(b) = OV(b)$ and this ψ^* can be extended to W by using the Hahn-Banach Theorem.

If $d \notin U$, then we let $V = span(a^1, a^2, \ldots, a^n, b, d)$, which is the same as $span(a^1, a^2, \ldots, a^n, b, b + \epsilon d)$ for any $\epsilon > 0$. We define a linear function ψ^* for each $\epsilon > 0$ by

$$\psi^*(a^k) = c_k, \ \psi^*(b) = OV(b), \ \text{and } \psi^*(d) = \frac{OV(b + \epsilon d) - OV(b)}{\epsilon}. \tag{8}$$

Since $d \notin U$, we know that ψ^* is well-defined on V and $\psi^*(b + \epsilon d) = OV(b + \epsilon d)$. Now we would like to find under what condition $\psi^* \in V'_+$. For this purpose, we let $e = x_1 a^1 + x_2 a^2 + \cdots + x_n a^n - \alpha b - \beta(b + \epsilon d) \in V$ and $e \succeq 0$.

Case 1: If $\alpha = \beta = 0$, then $\psi^*(e) \geq 0$. Otherwise, by Lemma 2 it contradicts the assumption that $OV(b)$ is finite.

Case 2: If $\alpha < 0$, without loss of generality we assume that $\alpha = -1$. Since $OV(b)$ is finite, we know that for any small $\delta > 0$ there is $y \in \mathbb{R}^n$ such that

$$\sum_{k=1}^{n} y_k a^k \succeq b \tag{9}$$

$$c^T y \leq OV(b) + \delta. \tag{10}$$

By adding (9) to $e=\sum_{k=1}^n x_k a^k + b - \beta(b+\epsilon d) \succeq 0$, we obtain that

$$\sum_{k=1}^{n} (x_k + y_k) a^k - \beta(b+\epsilon d) \succeq 0. \tag{11}$$

If $\beta > 0$, then $\frac{x+y}{\beta}$ is a feasible solution of $(SILP(b+\epsilon d))$. So we know that $c^T(\frac{x+y}{\beta}) \geq OV(b+\epsilon d)$. Therefore, $c^T x + OV(b) + \delta - \beta OV(b+\epsilon d) \geq c^T x + c^T y - \beta OV(b+\epsilon d) \geq 0$. We can let $\delta \to 0$ and obtain $\psi^*(\sum_{i=1}^n x_i a^i + b - \beta(b+\epsilon d)) \geq 0$, that is $\psi^*(e) \geq 0$.

If $\beta = 0$, then (11) becomes $\sum_{k=1}^n (x_k + y_k) a^k \succeq 0$. Since $OV(b)$ is finite, we know that $0 \leq c^T(x+y) = c^T x + c^T y$. Therefore, $OV(b) + \delta + c^T x \geq c^T y + c^T x \geq 0$. Letting $\delta \to 0$, we get $\psi^*(e) = c^T x + OV(b) \geq 0$.

If $\beta < 0$, then based on the assumption that $OV(b)$ and $OV(d)$ are finite, we know that $OV(b+\epsilon d)$ is also finite, Hence, for any small positive $\tau > 0$, there is a feasible solution $z = (z_1, z_2, \ldots, z_n)^T$ of $(SILP(-\beta(b+\epsilon d))$ such that

$$\sum_{k=1}^{n} z_k a^k \succeq -\beta(b+\epsilon d) \tag{12}$$

$$c^T z \leq OV(-\beta(b+\epsilon d)) + \tau. \tag{13}$$

Adding $\sum_{k=1}^n z_k a^k \succeq -\beta(b+\epsilon d)$ to (11), we have $\sum_{k=1}^n (x_k+y_k+z_k) a^k \succeq 0$. By Lemma 2 and noticing that $OV(b)$ is finite, we know that $\psi^*(\sum_{k=1}^n (x_k+y_k+z_k) a^k) = c^T(x+y+z)$ must be nonnegative. Therefore, $c^T x + OV(b) + \delta + OV(-\beta(b+\epsilon d)) + \tau \geq c^T(x+y+z) \geq 0$. Letting $\delta \to 0$ and $\tau \to 0$, we obtain that $\psi^*(\sum_{k=1}^n x_k a^k + b - \beta(b+\epsilon d)) = c^T \alpha + OV(b) - \beta OV(b+\epsilon d)) \geq 0$.

Case 3: If $\beta < 0$, then by following the same argument as in Case 2, we can prove that $\psi^*(e) \geq 0$.

Now it comes to the situations that $\alpha \geq 0$ and $\beta \geq 0$, and both of them cannot be zero at the same time.

Case 4: If $\beta = 0$ and $\alpha > 0$, or $\beta > 0$ and $\alpha = 0$, then by the proof of the previous theorem for U, we know that $\psi^*(e) \geq 0$.

Case 5: If $\alpha > 0$ and $\beta > 0$, then this is the only case that we need to impose some conditions such that $\psi^*(e) \geq 0$ is guaranteed. We have the following result.

Theorem 5 *ψ^* defined in (8) is in V'_+ if and only if $OV(\alpha b + \beta(b+\epsilon d)) = \alpha OV(b) + \beta OV(b+\epsilon d)$ for all $\alpha \geq 0$ and $\beta \geq 0$.*

Proof: Suppose that $\psi^* \in V'_+$. If $\alpha = 0$ or $\beta = 0$, then $OV(\alpha b + \beta(b+\epsilon d)) = \alpha OV(b) + \beta OV(b+\epsilon d)$ is trivially true. So we let $\alpha > 0$ and $\beta > 0$. Since $OV(b)$ and $OV(d)$ are finite, we know that $OV(\alpha b + \beta(b+\epsilon d))$ is also finite. Hence, for any small

$\sigma > 0$, there is a feasible solution $x = (x_1, x_2, \ldots, x_n)^T$ of $(SILP(\alpha b + \beta(b + \epsilon d)))$ such that

$$\sum_{k=1}^{n} x_k a^k \succeq \alpha b + \beta(b + \epsilon d) \tag{14}$$

$$c^T x \leq OV(\alpha b + \beta(b + \epsilon d)) + \sigma. \tag{15}$$

Because $\psi^* \in V'_+$, we know that $\psi^*(\sum_{k=1}^{n} x_k a^k - \alpha b - \beta(b + \epsilon d)) \geq 0$, which shows $c^T x - \alpha OV(b) - \beta OV(b + \epsilon d)) \geq 0$. Therefore, we obtain $\alpha OV(b) + \beta OV(b + \epsilon d) \leq c^T x \leq OV(\alpha b + \beta(b + \epsilon d)) + \sigma$. However, we know that $OV(\alpha b + \beta(b + \epsilon d) \leq \alpha OV(b) + \beta OV(b + \epsilon d))$ by Lemma 3, which further gives $\alpha OV(b) + \beta OV(b + \epsilon d) \leq c^T x \leq OV(\alpha b + \beta(b + \epsilon d)) + \sigma \leq \alpha OV(b) + \beta OV(b + \epsilon d) + \sigma$. By letting $\sigma \to 0$, we therefore prove that $OV(\alpha b + \beta(b + \epsilon d) = \alpha OV(b) + \beta OV(b + \epsilon d))$.

Now, let $e = x_1 a^1 + x_2 a^2 + \cdots + x_n a^n - \alpha b - \beta(b + \epsilon d)$ and $e \succeq 0$ be an element in V. If one of α and β is negative, by the discussion before we know that $\psi^*(e) \geq 0$. So we let $\alpha \geq 0$ and $\beta \geq 0$. The assumption that $e \succeq 0$ implies that x is a feasible solution of $(SILP(\alpha b + \beta(b + \epsilon d)))$. Hence, $c^T x \geq OV(\alpha b + \beta(b + \epsilon d))$, which implies that $c^T x - OV(\alpha b + \beta(b + \epsilon d)) \geq 0$. If we have that $OV(\alpha b + \beta(b + \epsilon d) = \alpha OV(b) + \beta OV(b + \epsilon d))$, then $\psi^*(e) = c^T x - \alpha OV(b) - \beta OV(b + \epsilon d) = c^T x - OV(\alpha b + \beta(b + \epsilon d)) \geq 0$. Therefore, $\psi^*(e) \geq 0$. This completes the proof. □

Now let's state and prove a necessary and sufficient condition under which the dual pricing property holds for the constraint space W.

Theorem 6 *Let $b \in W$. Then the dual pricing property holds for the primal-dual pair $(SILP(b))$ and $(DSILP(W, b))$ if and only if for any $d \in W$ there exists $\hat{\epsilon} > 0$ such that $OV(\alpha b + \beta(b + \epsilon d)) = \alpha OV(b) + \beta OV(b + \epsilon d)$ for any $\alpha \geq 0$, $\beta \geq 0$, and $\epsilon \in [0, \hat{\epsilon}]$.*

Proof: Suppose that the dual pricing property holds for the primal-dual pair $(SILP(b))$ and $(DSILP(W, b))$. Then for any $d \in W$, there exists an $\hat{\epsilon} > 0$ such that there is a feasible solution ψ^* of $(DSILP(W, b))$ with $\psi^*(b + \epsilon d) = OV(b + \epsilon d) = OV(b) + \epsilon \psi^*(d)$ for any $\epsilon \in [0, \hat{\epsilon}]$. Let $V = span(a^1, a^2, \ldots, a^n, b, d)$ and define $\psi^*|_V \in V'$ by $\psi^*|_V(v) = \psi^*(v)$ for all $v \in V$. Because $\psi^* \in W'_+$ and $V \subseteq W$, we know that $\psi^*|_V \in V'_+$. Moreover $\psi^*|_V(a^k) = c_k$ for $k = 1, 2, \ldots, n$, $\psi^*|_V(b) = OV(b)$, and $\psi^*|_V(b + \epsilon d) = OV(b + \epsilon d)$, by applying Theorem 5 to $\psi^*|_V$, we obtain that $OV(\alpha b + \beta(b + \epsilon d)) = \alpha OV(b) + \beta OV(b + \epsilon d)$ for any $\alpha \geq 0$, $\beta \geq 0$, and $\epsilon \in [0, \hat{\epsilon}]$.

Now suppose that for any $d \in W$, there exists an $\hat{\epsilon} > 0$ such that $OV(\alpha b + \beta(b + \epsilon d)) = \alpha OV(b) + \beta OV(b + \epsilon d)$ for all $\alpha \geq 0$, $\beta \geq 0$, and $\epsilon \in [0, \hat{\epsilon}]$. Then for any $\epsilon \in [0, \hat{\epsilon}]$, we define a linear function on $V = span(a^1, a^2, \ldots, a^n, b, b + \epsilon d)$ by $\psi^*(a^k) = c_k$, $\psi^*(b) = OV(b)$, and $\psi^*(b + \epsilon d) = OV(b + \epsilon d)$. Since $OV(\alpha b + \beta(b + \epsilon d)) = \alpha OV(b) + \beta OV(b + \epsilon d)$ for all $\alpha \geq 0$, $\beta \geq 0$, and $\epsilon \in [0, \hat{\epsilon}]$, we know that $\psi^* \in V'_+$ by Theorem 5. By applying the weak duality theorem to $(SILP(v))$ and $(DSILP(V, v))$, we know that $\psi^*(v) \leq OV(v)$ for all $v \in V$. Since ψ^* is a linear function on V and $OV(w)$ is sublinear on W, by the Hahn-Banach Theorem, we conclude that ψ^* can be extended to W with the property that $\psi^*(w) \leq OV(w)$ for all $w \in W$. This inequality further gives that the extension of ψ^* to W also belongs to W'_+. Therefore by Lemma 1, we know that the dual pricing property holds. □

The following corollary provides a necessary and sufficient condition for the stronger version of the dual pricing property to hold for the constraint space W.

Corollary 1 *Let $b \in W$. Then for any $d \in W$ there is a dual optimal solution $\psi^* \in W'_+$ such that $\psi^*(b + d) = OV(b + d) = OV(b) + \psi^*(d)$ if and only if $OV(-b) = -OV(b)$.*

Proof: Let $\alpha \geq 0$, $\beta \geq 0$, and $\hat{\epsilon} = 1$. Then for $\epsilon \in [0, \hat{\epsilon}]$, we have $OV(\alpha b + \beta(b + \epsilon d)) \leq \alpha OV(b) + \beta OV(b + \epsilon d)$ and $OV(\beta(b + \epsilon d)) = OV(\alpha b + \beta(b + \epsilon d) + (-\alpha b)) \leq OV(\alpha b + \beta(b + \epsilon d)) + OV(-\alpha b)$. If $OV(-b) = -OV(b)$, then we have $OV(\alpha b + \beta(b + \epsilon d)) = \alpha OV(b) + \beta OV(b + \epsilon d)$. By Theorem 6, the dual pricing property holds for the primal-dual pair $(SILP(b)$ and $(DSILP(W, b))$. Hence, for $\epsilon = \hat{\epsilon} = 1$, there exists a dual feasible solution ψ^* such that $\psi^*(b + d) = OV(b + d) = OV(b) + \psi^*(d)$.

Suppose that for any $d \in W$ there is a dual optimal solution $\psi^* \in W'_+$ such that $\psi^*(b+d) = OV(b+d) = OV(b)+\psi^*(d)$. If we take $d = -2b$, we can see that $OV(-b) = -OV(b)$. □

Let $b \in W$. Then by Theorem 4, we know that the strong duality property holds for the pair $(SILP(b))$ and $(DSILP(W, b))$. This shows that there is a feasible solution ψ^* of $(DSILP(W, b))$, such that $\psi^*(b) = OV(b)$. Let $S = \{\psi^* \in W'_+ \mid \psi^* \text{ is a feasible solution of } (DSILP(W, b)) \text{ and } \psi^*(b) < OV(b)\}$. The next corollary gives a sufficient condition under which the dual pricing property holds.

Corollary 2 *If $\sup_{\psi \in S} \psi(b) < OV(b)$, then the dual pricing property holds for the primal-dual pair $(SILP(b))$ and $(DSILP(W, b))$.*

Proof: By Theorem 6, we only need to show that for any $d \in W$ there exists $\hat{\epsilon} > 0$ such that $OV(\alpha b + \beta(b + \epsilon d)) = \alpha OV(b) + \beta OV(b + \epsilon d)$ for any $\alpha \geq 0$, $\beta \geq 0$, and $\epsilon \in [0, \hat{\epsilon}]$. Since for $\beta = 0$ this is automatically true, we may assume $\beta = 1$ (otherwise, we can divide β on both sides).

Let $s = \sup_{\psi \in S} \psi(b)$ and $\hat{\epsilon} = \frac{OV(b)-s}{OV(d)+OV(-d)}$ if $OV(d)+OV(-d) \neq 0$. Otherwise, set $\hat{\epsilon} = 1$. Since we assume that $b \in W$ and $d \in W$, we know that $b+\epsilon d \in W$. By Theorem 4, there is a dual optimal solution ψ^* of $(DSILP(W, b+\epsilon d))$ such that $\psi^*(b+\epsilon d) = OV(b+\epsilon d)$.

We claim that $\psi^*(b) = OV(b)$ if $\epsilon < \hat{\epsilon}$. Suppose $\psi^*(b) \neq OV(b)$. Then by the feasibility of ψ^* of the dual problem $(DSILP(W, b + \epsilon d))$, we get that ψ^* is also a feasible solution of the dual problem $(DSILP(W, b))$. Therefore, $\psi^*(b) \leq OV(b)$, which together with $\psi^*(b) \neq OV(b)$ shows $\psi^*(b) < OV(b)$. Because

$$\begin{aligned} OV(b) - \epsilon OV(-d) &\leq OV(b + \epsilon d) \\ &= \psi^*(b + \epsilon d) \\ &= \psi^*(b) + \epsilon\psi^*(d) \\ &\leq s + \epsilon OV(d), \end{aligned}$$

where the last inequality is due to the fact that ψ^* is also feasible to $(DSILP(W, d))$, we have

$$OV(b) - s \leq \epsilon(OV(d) + OV(-d)),$$

which is impossible because $\epsilon < \hat{\epsilon}$. Therefore, $\psi^*(b) = OV(b)$. Now because ψ^* is a feasible solution of $(DSILP(W, \alpha b + (b + \epsilon d))$, we know that $\psi^*(\alpha b + (b + \epsilon d)) \leq OV(\alpha b + (b + \epsilon d))$. Hence, $OV(\alpha b + (b + \epsilon d)) = \psi^*(\alpha b + (b + \epsilon d)) = \alpha\psi^*(b) + \psi^*(b + \epsilon d) = \alpha OV(b) + OV(b + \epsilon d)$. By Theorem 6, we know that the dual pricing property holds for the primal-dual pair $(SILP(b))$ and $DSILP(W, b))$. □

In the related literature, the standard dual of a semi-infinite programming problem is different from the ones discussed in this paper. The standard dual of a semi-infinite programming problem is formulated as follows:

$$\begin{aligned} (FDSILP) \qquad & \sup \sum_{i \in I} b_i v_i \\ & \text{subject to} \quad \sum_{i \in I} a_{ik} v_i = c_k, \text{ for } k = 1, 2, \ldots, n, \\ & v \in \mathbb{R}_+^{(I)}, \end{aligned} \tag{16}$$

where $\mathbb{R}_+^{(I)}$ is the subset of generalized finite sequences defined on I with positive values, in other words, $v \in \mathbb{R}_+^{(I)}$ means that $v \succeq 0$ and $\{i \in I \mid v_i \equiv v(i) \neq 0\}$ is a finite set.

Optimal values for the problems $(SILP)$ and $(FDSILP)$ may not be the same in general. It is well-known that the asymptotic extension of $(FDSILP)$ can be used to fill out the duality gap and the dual optimal value is attained. The asymptotic extension is formed as follows:

$$\begin{aligned} (D_2(b)) \qquad v(D_2) = \sup \quad & \overline{\lim_{m \to +\infty}} \sum_{i \in I} v_i^m b_i \\ \text{subject to} \quad & \lim_{m \to +\infty} \sum_{i \in I} v_i^m a_{ik} = c_k, \ k = 1, 2, \ldots, n \\ & v^m \in \mathbb{R}_+^{(I)}. \end{aligned} \tag{17}$$

By a close look at the formulation of $(D_2(b))$, we find that a feasible solution of $(D_2(b))$, which is a sequence of elements v^m in $\mathbb{R}^{(I)}_+$, can be used to define a positive linear function on $U = span\{a^1, a^2, \ldots, a^n, b\}$. Also if $OV(b)$ is finite, then every positive linear function on U can be written in this form. We state these results as a proposition.

Proposition 1 *(i) A feasible solution $\{v^m\}_{m=1}^{+\infty}$ of $(D_2(b))$ defines a feasible solution of $(DSILP(U, b))$. Moreover, if $b \in W$, then ψ can be extended to W with the property that $\psi(w) \leq OV(w)$ for $w \in W$.*
(ii) If $OV(b)$ is finite, then every feasible solution of $(DSILP(U, b))$ can be written in this form.

Proof: (i) A linear function on U is determined by the values of the function at $a^1, a^2, \ldots, a^n$, and b.

Let's define a linear function $\psi : U \to \mathbb{R}$ by $\psi(a^k) = \lim\limits_{m\to+\infty} \sum\limits_{i\in I} v_i^m a_{ik} = c_k$, $k = 1, 2, \ldots, n$, and $\psi(b) = \overline{\lim\limits_{m\to+\infty}} \sum\limits_{i\in I} v_i^m b_i$. It is straightforward to prove that ψ is well defined by considering the cases that $a^1, a^2, \ldots, a^n$, and b are linearly independent or linearly dependent. The positivity of ψ is an immediate result of the fact that $v^m \in \mathbb{R}^{(I)}_+$. Therefore, ψ is a feasible solution of $(DSILP(U, b))$. By the weak duality theorem, we also know that $\psi(u) \leq OV(u)$ for all $u \in U$. Because $\phi : W \to \mathbb{R}$ by $\phi(w) \equiv OV(w)$ is sublinear, by the Hahn-Banach Theorem, ψ can be extended to W with the property $\psi(w) \leq \phi(w) = OV(w)$.
(ii) We first prove the theorem under the assumption that $b \in W$, that is, both $OV(b)$ and $OV(-b)$ are finite. Because there is no duality gap between $(SILP(b))$ and $(D_2(b))$ and $(D_2(b))$ is solvable, there is a sequence $\{v^m\}_{m=1}^{+\infty} \subset \mathbb{R}^{(I)}_+$ such that $\lim\limits_{m\to+\infty} \sum\limits_{i\in I} v_i^m a_{ik} = c_k$, $k = 1, 2, \ldots, n$, and $OV(b) = \overline{\lim\limits_{m\to+\infty}} \sum\limits_{i\in I} v_i^m b_i$. Since for any subsequence $\{v^{m_p}\}_{p=1}^{+\infty}$ of $\{v^m\}_{m=1}^{+\infty}$, we always have $\lim\limits_{p\to+\infty} \sum\limits_{i\in I} v_i^{m_p} a_{ik} = c_k$, $k = 1, 2, \ldots, n$, we may simply assume that $\lim\limits_{m\to+\infty} \sum\limits_{i\in I} v_i^m a_{ik} = c_k$, $k = 1, 2, \ldots, n$, and $OV(b) = \lim\limits_{m\to+\infty} \sum\limits_{i\in I} v_i^m b_i$ by relabeling if needed (because $OV(b) = \overline{\lim\limits_{m\to+\infty}} \sum\limits_{i\in I} v_i^m b_i$ implies that there is a subsequence $\{v^{m_p}\}_{p=1}^{+\infty}$ of $\{v^m\}_{m=1}^{+\infty}$ such that $OV(b) = \lim\limits_{p\to+\infty} \sum\limits_{i\in I} v_i^{m_p} b_i$).

Similarly, there is a sequence $\{u^m\}_{m=1}^{+\infty} \subset \mathbb{R}^{(I)}_+$ such that $\lim\limits_{m\to+\infty} \sum\limits_{i\in I} u_i^m a_{ik} = c_k$, $k = 1, 2, \ldots, n$, and $OV(-b) = \lim\limits_{m\to+\infty} \sum\limits_{i\in I} u_i^m (-b_i)$.

Now let ψ be any feasible solution of $(DSILP(U, b))$. Then ψ is also a feasible solution of $(DSILP(U, -b))$ because $(DSILP(U, b))$ and $(DSILP(U, -b))$ have the same constraints. Therefore, $\psi(b) \leq OV(b)$ and $\psi(-b) \leq OV(-b)$, which indicate that $-OV(-b) \le \psi(b) \le OV(b)$. If $OV(b) = -OV(-b)$, then $\{v^m\}_{m=1}^{+\infty}$ defines ψ. If $OV(b) \neq -OV(-b)$, then we let $\lambda = \frac{OV(b)-\psi(b)}{OV(b)+OV(-b)}$. So $\psi(b) = -\lambda OV(-b)+(1-\lambda)OV(b)$. Let $w^m = \lambda u^m + (1-\lambda)v^m$. Then we can verify that $w^m \in \mathbb{R}_+^{(I)}$ for $m = 1, 2, \dots$, $\lim_{m\to+\infty} \sum_{i\in I} w_i^m a_{ik} = c_k$, $k = 1, 2, \dots, n$, and $\psi(b) = \lim_{m\to+\infty} \sum_{i\in I} w_i^m b_i$. Hence $\{w^m\}_{m=1}^{+\infty}$ defines ψ.

Next, we prove the theorem for the case that $b \notin W$. In this case, since $OV(b)$ is finite, we can prove by contradiction that $(SILP(-b))$ is inconsistent, which shows that $(D_2(-b))$ must be unbounded. Therefore, for any feasible solution ψ of $(DSILP(U, b))$, we can find a sequence $\{u^m\}_{m=1}^{+\infty} \subset \mathbb{R}_+^{(I)}$ such that $\lim_{m\to+\infty} \sum_{i\in I} u_i^m a_{ik} = c_k$, $k = 1, 2, \dots, n$, and $\psi(b) \ge -\lim_{m\to+\infty} \sum_{i\in I} u_i^m(-b_i) = \lim_{m\to+\infty} \sum_{i\in I} u_i^m b_i$. The remaining of the proof is exactly the same as the one for the case that $b \in W$. We complete the proof. □

Remark 4 *In Basu et al. [3], two conditions called DP.1 and DP.2 are proposed under which the dual pricing property holds. In view of Proposition 1 and Lemma 4 in Basu et al. [2], the conditions DP.1 and DP.2 together is equivalent to* $\sup_{\psi\in S} \psi(b) < OV(b)$ *presented in Corollary 2 in this paper. Theorem 5.12 in Basu et al. [3] is the same as Corollary 2. But here again we do not use the Fourier-Motzkin elimination.*

Remark 5 *All theorems and lemmas proved in this paper for the constraint space W are also true for all the subspaces of W containing U. This can be seen by restricting the dual feasible solution, which is a function from W to $\mathbb{R}$, on the subspaces.*

4 Revisiting examples in Basu et al. [3]

Example 1 (Example 5.3 in Basu et al. [3])

$$\begin{array}{lll} (SILP1(b)) & \inf & x_1 \\ & \textit{subject to} & (1/i)x_1 + (1/i^2)x_2 \ge (1/i),\ i \in \mathbb{N} \equiv \{1, 2, 3, \dots\}. \end{array} \quad (18)$$

Obviously, $a^1 : \mathbb{N} \to \mathbb{R}$ and $a^1(i) = \frac{1}{i}$, $a^2 : \mathbb{N} \to \mathbb{R}$ and $a^2(i) = \frac{1}{i^2}$, and $b : \mathbb{N} \to \mathbb{R}$ and $b(i) = \frac{1}{i}$. Multiplying both sides of the constraints by i and letting $i \to +\infty$, we obtain that the optimal value of $(SILP1(b))$ is 1, that is, $OV(b) = 1$. Similarly, we know that $OV(-b) = -1$. Therefore, by Corollary 1 the stronger version of the dual pricing property holds for the primal-dual pair with respect to the constraint space W, which is defined to be all the functions $w : \mathbb{N} \to \mathbb{R}$ with $OV(w)$ and $OV(-w)$ finite.

In Basu et al. [3], it is shown that the dual pricing property does not hold with respect to the constraint space l_2. This does not contradict what we have just shown, that is, the *dual pricing property holds for the primal-dual pair with respect to the constraint space W. We notice that $l_2 \nsubseteq W$ because $p: \mathbb{N} \to \mathbb{R}$ by $p(i) = \frac{1}{i^{0.9}}$ is an element in l_2 but it is not in W.*

Example 2 (Example 5.10 in Basu et al. [3])

$$\begin{array}{llllll} (SILP2(b)) & \inf & x_1 & & & \\ & \textit{subject to} & x_1 & & & \geq -1 \\ & & & -x_2 & & \geq -1 \\ & & & & -x_3 & \geq -1 \\ & & x_1 & +x_2 & & \geq 0 \\ & & x_1 & -\frac{1}{i}x_2 & +\frac{1}{i^2}x_3 & \geq 0, \ i = 5, 6, \ldots. \end{array} \tag{19}$$

It is obvious that the index set $I = \mathbb{N}$, a^1 can be represented by the sequence $(1, 0, 0, 1, 1, 1, \ldots)$, a^2 can be represented by the sequence $(0, -1, 0, 1, -\frac{1}{5}, -\frac{1}{6}, \ldots)$, a^3 can be represented by the sequence $(0, 0, -1, 0, \frac{1}{5^2}, \frac{1}{6^2}, \ldots)$, and b can be represented by the sequence $(-1, -1, -1, 0, 0, 0 \ldots)$. To find $OV(b)$, we notice that x_1 must be nonnegative. Otherwise, in the last constraint if we let $i \to +\infty$, we will arrive at a contradiction. Therefore, $x_1 \geq 0$. However, $x_1 = x_2 = x_3 = 0$ is a feasible solution, which shows $OV(b) = 0$. Now let's find $OV(-b)$. The first constraint now becomes $x_1 \geq 1$, which indicates that $OV(-b) \geq 1$. However, $x_1 = 1$, $x_2 = x_3 = -1$ is a feasible solution of $(SILP2(b))$. Hence, $OV(-b) = 1$, which together with $OV(b) = 0$ shows that $b \in W$. Therefore, the strong duality property holds for the pair.

Let $d = (0, 0, 0, 1, 0, 0, \ldots)$. We know $d \in W$ because the first constraint gives a lower bound of x_1. Next we want to find $OV(\lambda b + \epsilon d)$ for $\lambda > 0$ and $\epsilon > 0$. It is easy to see that $x_1 = x_2 = x_3 = 0$ is not a feasible solution of $(SILP2(\lambda b + \epsilon d))$. We consider the last constraint $x_1 - \frac{1}{i}x_2 + \frac{1}{i^2}x_3 \geq 0, \ i = 5, 6, \ldots$. From this constraint, we know that (i) x_1 cannot be 0 or negative. So the first constraint $x_1 \geq -\lambda$ is redundant. (ii) Because $-x_3 \geq -\lambda$, x_3 in the last constraint can be replaced by λ. (iii) Since $x_1 + x_2 \geq \epsilon$ that is equivalent to $x_1 - \epsilon \geq -x_2$, $-x_2$ in the last constraint can be replaced by $x_1 - \epsilon$. Hence, we have $x_1 + \frac{1}{i}x_1 - \frac{1}{i}\epsilon + \frac{\lambda}{i^2} \geq 0$, which implies that $x_1 \geq \frac{1}{(1+i)\epsilon} - \frac{\lambda}{i(1+i)}$ for $i = 5, 6, \ldots$. Hence, $OV(\lambda b + \epsilon d) = \sup_{i \geq 5} \left(\frac{1}{(1+i)\epsilon} - \frac{\lambda}{i(1+i)} \right)$. If we set λ to be a positive integer and $\epsilon = \frac{1}{p}$ for $p \in \mathbb{N}$, then using basic Calculus we obtain that the function $f(i) \equiv \frac{1}{(1+i)\epsilon} - \frac{\lambda}{i(1+i)} = \frac{p+\lambda}{1+i} - \frac{\lambda}{i}$ is decreasing for $i \geq \frac{\sqrt{\lambda}}{\sqrt{p+\lambda} - \sqrt{\lambda}}$. Now let p be big enough such that $\frac{1}{\sqrt{p+1}-1} \leq 4$ and $\frac{\sqrt{2}}{\sqrt{p+2}-\sqrt{2}} \leq 4$. By setting $\lambda = 1$ and noticing $5 > 4 \geq \frac{1}{\sqrt{p+1}-1}$, we know $OV(b+\epsilon d) = \sup_{i \geq 5} \left(\frac{1}{(1+i)\epsilon} - \frac{1}{i(1+i)} \right) = \sup_{i \geq 5} \left(\frac{p}{(1+i)} - \frac{1}{i(1+i)} \right) = \frac{p}{(1+5)} - \frac{1}{5(1+5)} = \frac{1}{30}(5p - 1)$. Similarly, if we set $\lambda = 2$ and notice $\frac{\sqrt{2}}{\sqrt{p+2}-\sqrt{2}} \leq 4 < 5$, we obtain that $OV(2b+\epsilon d) = \sup_{i \geq 5} \left(\frac{1}{(1+i)\epsilon} - \frac{2}{i(1+i)} \right) = \sup_{i \geq 5} \left(\frac{p}{(1+i)} - \frac{2}{i(1+i)} \right) = \frac{p}{(1+5)} - \frac{2}{5(1+5)} = \frac{1}{30}(5p-$

2). *Therefore, we have* $OV(2b+\epsilon d) = \frac{1}{30}(5p-2) \neq 0 + \frac{1}{30}(5p-1) = OV(b)+OV(b+\epsilon d)$. *Because* $\epsilon = \frac{1}{p}$ *can be as small as we want, by Theorem 6 we know that the dual pricing property does not hold for the primal-dual pair.*

Example 3 (Example 5.14 in Basu et al. [3])

$$(SILP3(b)) \qquad \inf \quad x_1$$
$$\text{subject to} \quad x_1 + \frac{1}{m+n}x_2 \geq -\frac{1}{n^2}, \ (m,n) \in I. \tag{20}$$

where $I = \{(m,n) \mid (m,n) \in \mathbb{N} \times \mathbb{N}\}$.

It is easy to see that $a^1(i) = 1$ *for all* $i \in I$, *and* $a^2(i) = \frac{1}{m+n}$ *for* $i = (m,n) \in I$, *and* $b(i) = -\frac{1}{n^2}$ *for* $i = (m,n) \in I$. *Obviously,* $x_1 = 0$, $x_2 = 0$ *is a feasible solution of* $(SILP3(b))$. *However, for any* $x_1 < 0$, *there is* $N \in \mathbb{N}$ *such that* $-\frac{1}{n^2} > x_1$ *for* $n \geq N$. *Hence, for any fixed* x_2 *and* $n \geq N$, *if we let* m *to be sufficiently big, then we know that* $x_1 + \frac{1}{m+n}x_2 < -\frac{1}{n^2}$ *indicating the constraint is violated. Hence,* $OV(b) = 0$. *To find* $OV(-b)$, *we notice that* $x_1 = 1$ *and* $x_2 = 0$ *is a feasible solution of* $(SILP3(-b))$ *and that for any* $x_1 < 1$ *and let* m *be big enough the constraint of* $(SILP3(-b))$ *is violated. Hence,* $OV(-b) = 1$. *Therefore,* $b \in W$.

To show that the dual pricing property does not hold for the primal-dual pair, by Theorem 6 we only need to prove that there exists $d \in W$ *such that for any* $\hat{\epsilon} > 0$, *there exists* $\epsilon \in [0, \hat{\epsilon}]$, $\alpha \geq 0$, *and* $\beta \geq 0$ *such that* $OV(\alpha b+\beta(b+\epsilon d)) \neq \alpha OV(b)+\beta OV(b+\epsilon d)$.

We let $d : I \to \mathbb{R}$ *be defined by* $d(i) = \frac{1}{n}$ *for* $i = (m,n)$. *By a similar argument, we obtain that* $OV(d) = 1$ *and* $OV(-d) = 0$. *Therefore,* $d \in W$. *We also have* $OV(\lambda b + \epsilon d) = \sup_{n\in\mathbb{N}} \left(-\frac{\lambda}{n^2} + \frac{\epsilon}{n}\right)$ *for any* $\lambda > 0$ *and* $\epsilon > 0$. *If we set* λ *to be a positive integer and* $\epsilon = \frac{1}{p}$ *with* $p \in \mathbb{N}$, *then it is easy to verify that* $\sup_{n\in\mathbb{N}} \left(-\frac{\lambda}{n^2} + \frac{\epsilon}{n}\right)$ *is attained at* $n = 2\lambda p$. *Therefore,* $OV(\lambda b + \frac{1}{p}d) = \frac{1}{4\lambda p}$ *for* $\lambda \in \mathbb{N}$. *Now we set* $\lambda = 1 + \alpha$. *Then* $OV((1+\alpha)b + \frac{1}{p}d) = \frac{1}{4p(1+\alpha)}$ *for* $\alpha \in \mathbb{N}$. *If we set* $\lambda = 1$, *then we have* $OV(b + \frac{1}{p}d) = \frac{1}{4p}$. *Therefore, we have* $OV(\alpha b + (b + \frac{1}{p}d) = OV((1+\alpha)b + \frac{1}{p}d) = \frac{1}{4p(1+\alpha)} \neq 0 + \frac{1}{4p} = OV(b) + OV(b + \frac{1}{p}d)$ *if* $\alpha > 0$. *Therefore, by Theorem 6 we know that the dual pricing property does not hold for the primal-dual pair.*

Example 4 (Example 5.16 in Basu et al. [3])

$$(SILP4(b)) \qquad \inf \quad x_1$$
$$\text{subject to} \quad x_1 + \frac{1}{i^2}x_2 \geq \frac{2}{i}, \ i \in \mathbb{N}. \tag{21}$$

In this example, we know that $a^1 : \mathbb{N} \to \mathbb{R}$ *by* $a^1(i) = 1$, $a^2 : \mathbb{N} \to \mathbb{R}$ *by* $a^2(i) = \frac{1}{i^2}$, *and* $b : \mathbb{N} \to \mathbb{R}$ *by* $b(i) = \frac{2}{i}$. *For any feasible solution* $(x_1, x_2)^T$, $x_1 \geq 0$. *Otherwise, by letting*

$i \to \infty$, the constraint is violated. We can also see that $(\frac{1}{i}, i)^T$ is a feasible solution of $(SILP4(b))$. Hence, we obtain that $OV(b) = 0$. Since for any given x_2, $-\frac{2}{i} - \frac{1}{i^2}x_2 \to 0$ as $i \to +\infty$, we obtain that $OV(-b) = 0$. Hence, $OV(b) = -OV(-b) = 0$, which by Corollary 1 indicates that for any $d \in W$ there exists a dual optimal solution ψ^* of $(DSILP(W, b))$ such that $\psi^*(b+d) = OV(b+d) = OV(b) + \psi^*(d)$, a stronger version of the dual pricing property holds. Since l_∞ consists of all bounded sequences, we know that for each $p \in l_\infty$, $OV(p)$ and $OV(-p)$ are both finite. Hence, $l_\infty \subseteq W$. Therefore, the dual pricing property holds for the primal-dual pair $(SILP4(b))$ and $(DSILP(l_\infty, b))$ too.

5 Conclusions

We have shown in this paper that the Fourier-Motzkin elimination is not necessary in the study of the strong duality and dual pricing properties in semi-infinite programming. All the results in Basu et al. [3], which do not specifically mention the Fourier-Motzkin elimination in their statements, have been proved directly by following a straightforward idea, that is, to construct the linear function that satisfies the desired requirements. We have proposed a constraint space W. Because W is a subspace, unlike the set $\hat{Y}$ defined in Basu et al. [3], no sufficient condition needs to be imposed on W to make it a subspace under which the strong duality and dual pricing properties can be discussed. We believe that W is the largest subspace of $\mathbb{R}^I$ that one can use to prove the strong duality and dual pricing properties without using topological or other structures of the underlined constraint spaces. We have proved that the strong duality property holds for the primal-dual pair with respect to the constraint space W. We have provided a necessary and sufficient condition for the dual pricing property to hold for the constraint space W. Examples in Basu et al. [3] have been revisited by applying the results proved in this paper to them.

References

[1] E.J. Anderson and P. Nash, *Linear Programming in Infinite-Dimensional Spaces: Theory and Applications.* Wiley, 1987.

[2] A. Basu, K. Martin and C. T. Ryan, *Projection: A unified approach to semi–infinite linear programs and duality in convex programming. Math. Oper. Res.* 40 (2015), 146-170.

[3] A. Basu, K. Martin and C. T. Ryan, *Strong duality and sensitivity analysis in semi-infinite linear programming. arXiv:1510.07531 [math.OC]* .

[4] A. Ben-Israel, A. Charnes and K. O. Kortanek, *Duality and asymptotic solvability over cones,* Bull. Amer. Math. Soc. 75 (1969), 318–324. Erratum, Bull. Amer. Math. Soc. 76 (1970), 426.

[5] K. O. Kortanek, *Classifying convex extremum problems over linear topologies having separation properties,* J. Math. Anal. Appl. 46 (1974), 725–755.

[6] K.O. Kortanek and Q. Zhang, *Extending the mixed algebraic-analysis Fourier-Motzkin elimination method for classifying linear semi-infinite programmes. Optimization*, http://dx.doi.org/10.1080/02331934.2015.1080254