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Abstract: In this paper we obtained approach for the optimal switching control problem with unknown switching points, which it is described in reference [1, 2]. In the references [1], the authors are studied Decomposition of Linear-Quadratic optimal Control problems for Two- Steps Systems. In [1], the authors assumed the switching point t1 is xed in the interval for state equation and boundary of the integral of minimization functional and it is given algorithm for solving Linear-Quadratic optimal Control problem. But in presented paper author assumed more general case, in the case of switching point is unknown and by using transformation, the main problem is reduced to the problem with known interval and unknown the boundary of the integral in minimization functional is reduced to the known one, which is dened in [1, 2]. It is given illustrated example at the end of the paper.Then by using Gradient Projection Method Algorithm, the problem is solved numerically by authors.

Dear Editor

We hereby submit the paper titled "Numerical Solution of Linear-Quadratic Optimal Control Problems for Switching System". In this paper I report our findings on the numerical results for optimal switching control system.In this article we tried solve the problem which is described in the reference [1,2] but in the case of the nonfixed switching points (in the reference [1,2], the authors solved problem with the fixed switching point case , but we think in the case of switching points nonfixed is very interesting and applicable to the engineering and finance) As discussed in the article, the presented method has very significant advantages over previous methods, such as simplicity, controlled particle size, and better morphology. I believe that this method has scientific and potential technical and engineering merit and may find industrial applications. I hope that the content and style meet your evaluation criteria.

Best Regards.
Shahlar Meherrem, Deniz Gucoglu and Samir Guliyev

Numerical Solution of Linear-Quadratic Optimal Control Problems for Switching System

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1

Abstract

In this paper we obtained an approach to the optimal switching control problem with unknown switching points which it is described in reference [1, 2]. In reference [1], the authors studied the Decomposition of Linear-Quadratic Optimal Control Problems for Two-Steps Systems. In [1], the authors assumed the switching point t_1 is fixed in the interval for state equation and boundary of the integral of the minimization functional an algorithm is given for solving the Linear-Quadratic Optimal Control Problem. But in present paper we assume a more general case, in the case in which the switching point is unknown and, by using transformation, the main problem is reduced to a problem with a known interval and unknown boundary of the integral in the minimization functional is reduced to the known one, which is defined in [1, 2]. This is illustrated by an example at the end of the paper. Then by using the Gradient Projection Method Algorithm, the problem is solved numerically by the authors.

Keywords: Optimal control, switching system, numerical solution, finite approximation.

1 Introduction

Recently, many articles have been published which dedicated to investigating the switching optimal control problems in a particular kind of hybrid system. Examples of switching system can be find in the area of engineering, chemical processes, automotive systems and military

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cervices. The published results in the literature can be classified into two categories, i.e., theoretical [1, 2, 4, 5, 6, 7, 8, 9, 14, 15, 16, 17] and practical [3, 10, 11, 12, 13, 18, 19, 20]. The very earliest result which proves a maximum principle for a hybrid system for autonomous switching system is described in [13]. More theoretical results of the maximum principle are obtained by Picolli in the [14] and Sussman in [15] which are respectivelly called the hybrid maximum principle and maximum principle for the hybrid system in the case in which the minimization functional is non-smooth. In the reference [16, 17] there the switching system is investigated by using a dynamical programming approach to derive Hamilton-Jacobi-Belmann equations. But there are some practical results for the switching optimal control problem which have significant applications to real-world problems. In [21] conceptual algorithms are given for general hybrid optimal control problems. In [18], for a class of discrete-time hybrid system an algorithm by using the constrained differential programming approach is given by the authors. An application to power train control can be found in [22]. Some heuristically oriented methods have been reported in [23], which is obtained algorithm prunes the search trees in discrete-time LQR control of a switched linear system. An efficient algorithm, called the time-optimal switching (TOS) algorithm, is proposed for the time-optimal switching control of nonlinear systems with a single control output by Kaya and Noakes in [19]. In [19], firstly, switching control is found using the STC (switching time computation) method to get from an initial point to a target point with a given number of switchings. Then by means of constrained optimization techniques, the cost being considered as a summation of the arc times, a minimum-time switching control solution is obtained. The rest of this paper is organized as follows: The problem formulation is given in section 2, the transformation for the given problem and related theorems are described in section 3, The Gradient Projection Method Algorithm for this problem is given in section 4, the numerical result to the example is given in section 5, section 6 presents the conclusion of the paper.

2 Problem Formulation

In the paper [1], the following minimizing optimal control problem is studied:
Problem I: Minimizing the functional

$$\begin{aligned} J(u,t_1) &= \frac{1}{2}\langle C_1x_1(t_1)-C_2x_2(t_1), F\left(C_1x_1(t_1)-C_2x_2(t_1)\right)\rangle \\ &+ \sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\left(\langle x_j(t), W_j(t)x_j(t)\rangle + \langle u_j(t), R_j(t)u_j(t)\rangle\right)dt \end{aligned} \tag{2.1}$$

where, $u=(u_1,u_2)$, with respect to the trajectories of the system

$$\dot{x}_j(t) = A_j(t)x_j(t) + B_j(t)u_j(t), \quad t_{j-1}\le t\le t_j,\ j=1,2 \tag{2.2}$$

with the following boundaries: $x_1(0) = x^0,\ x_2(T) = x^T$.

Here, $0 = t_0 < t_1 < t_2 = T$, the values t_0, t_2 are fixed, t_1 is not fixed, $x_j(t) \in X_j$, $u_j(t) \in U_j$, $A_j(t)$, $W_j \in L(X_j)$, $B_j(t) \in L(U_j,\ X_j)$, $R_j(t) \in L(U_j)$ for all $t \in [t_{j-1}, t_j]$, j=1, 2; $C_1 \in L(X_1, Y)$, $C_2 \in L(X_2, Y)$, F$\in L(Y)$, X_j, U_j, Y are real finite dimensional Euclidean spaces, the operators F, $W_j(t) \geq 0, R_j(t) > 0$ for all $t \in [t_{j-1}, t_j]$; $x^0 \in X_1$, $x^T \in X_2$ are given and symmetric, the operators F, C_1, C_2 are independent of t, but the other operators depend continually on t in the corresponding segment $[t_{j-1}, t_j]$, j=1, 2, $< .,. >$ means an inner product in appropriate spaces.

Note 1. In the reference [1], it is assumed the intermediate point t_1 is fixed. For this, the minimization functional has the form

$$\begin{aligned} J(u) &= \frac{1}{2}\langle C_1x_1(t_1) - C_2x_2(t_1), F\,(C_1x_1(t_1) - C_2x_2(t_1))\rangle \\ &+ \sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}(\langle x_j(t), W_j(t)x_j(t)\rangle + \langle u_j(t), R_j(t)u_j(t)\rangle)dt \end{aligned} \tag{2.3}$$

i,e., in the paper [1, 2], minimization functional is not depent on the switching point t_1 because t_1 is fixed. For this reason in the papers [1, 2], the minimizing functional is written in the form $J(u)$. But in the present paper, we consider a more general case. It is assumed that the point t_1 is unknown, the minimizing functional has the same form as in (2.1), i.e., $J(u, t_1)$. Let us make following substitution $u(t) = (u_1(t), u_2(t))$ and $x(t) = (x_1(t), x_2(t))$.

Definition I: The triple $w = (t_1, u(t), x(t))$ is called admissible, if it satisfies all constraints of *Problem I* (about the constraints see, [1])

Definition II: The triple $w^0 = (t_1, u(t), x(t))$ is called optimal control, if $J(w^0) \leq J(w)$ for all admissible process w.

3 Transformation

Let us take the following transformation. Assume a new parameter x_{n+1} such us satisfies following differential equation with initial condition in $[t_0, t_2]$,
$\frac{dx_{n+1}(t)}{dt} = 0$ with initial condition $x_{n+1}(0) = t_1$. This means x_{n+1} is constant in $[t_0, t_2]$.
Next, a new independent time variable τ is introduced as:

$$t = \begin{cases} t_0 + (x_{n+1} - t_0)\tau, & 0 \leq \tau < 1 \\ x_{n+1} + (t_2 - x_{n+1})(\tau - 1), & 1 \leq \tau \leq 2 \end{cases} \tag{3.4}$$

Then we can write that

$$dt = \begin{cases} (x_{n+1} - t_0)d\tau, & 0 \le \tau < 1 \\ (t_2 - x_{n+1})d\tau, & 1 \le \tau \le 2 \end{cases} \tag{3.5}$$

Clearly, (3.4) is linear mapping with $t : \tau \to [t_0, t_1]$ when $\tau \in [0, 1]$ and $t : \tau \to [t_1, t_2]$ when $\tau \in [1, 2]$. In fact, $\tau = 0$ corresponds to $t = t_0$, $\tau = 1$ corresponds to $t = t_1$, and $\tau = 2$ to $t = t_2$. By using the relation (3.4) it is easy to introduce inverse mapping $\tau = \frac{t-t_0}{x_{n+1}-t_0}$, for $0 \le \tau \le 1$ and $\tau = \frac{t-x_{n+1}}{t_2-x_{n+1}}$, for $1 \le \tau \le 2$. By introducing x_{n+1} and τ, and substitutions $y_i(\tau) = x_i(t(\tau))$, $v_i(\tau) = u_i(t(\tau)), i = 1, 2$ and using relation (3.5) the main problem is transcribed into the following equivalent form.

Problem II:

$$subsystem(1) : \begin{cases} \frac{dy_1(\tau)}{d\tau} &= (x_{n+1} - t_0)\left(A_1(\tau)y_1(\tau) + B_1(\tau)v_1(\tau)\right) \\ \frac{dx_{n+1}}{d\tau} &= 0 \\ x_{n+1}(0) &= t_1 \end{cases} \tag{3.6}$$

in the interval $\tau \in [0, 1)$ and

$$subsystem(2) : \begin{cases} \frac{dy_2(\tau)}{d\tau} &= (t_2 - x_{n+1})\left(A_2(\tau)y_2(\tau) + B_2(\tau)v_2(\tau)\right) \\ \frac{dx_{n+1}}{d\tau} &= 0 \\ x_{n+1}(0) &= t_1 \end{cases} \tag{3.7}$$

in the interval $\tau \in [1, 2]$ and minimizing functional takes the form

$$\begin{aligned} \tilde{J}(v, x_{n+1}) &= \frac{1}{2}\langle C_1y_1(1) - C_2y_2(1), F(C_1y_1(1)) - C_2y_2(1))\rangle \\ &+ \int_0^1 (x_{n+1} - t_0)(\langle y_1(\tau), W_1(\tau)y_1(\tau)\rangle + \langle v_1(\tau), R_1(\tau)v_1(\tau)\rangle)d\tau \\ &+ \int_1^2 (t_2 - x_{n+1})(\langle y_2(\tau), W_2(t)y_2(\tau)\rangle + \langle v_2(\tau), R_2(\tau)v_2(\tau)\rangle)d\tau \end{aligned} \tag{3.8}$$

After this transformation, we reduce *Problem I*, to *Problem II*. In *Problem II*, the state trajectory is $y(\tau) = (y_1(\tau), y_2(\tau))$ and the control is
$v(\tau) = (v_1(\tau), v_2(\tau), x_{n+1}), 0 \le \tau \le 2$
Since x_{n+1} is an unknown constant (parameter) in the interval $[0, 2]$ (see, (3.6) and (3.7)), after the transformation, the dimensional of *Problem II* will be same as the dimensional of *Problem I*.

Theorem 3.1 *There is a one-to-one correspondence between the admissible process* $(t_1, x(t), u(t))$ *for Problem I and the admissible process* $(y(\tau), v(\tau))$ *for Problem II.*

Proof : In fact by using transformation from the admissible process $(t_1, x(t), u(t))$, we obtained admissible process $(y(\tau), v(\tau))$. Let us prove the inverse opinion. If $(y(\tau), v(\tau)$ is an admissible process (where $v(\tau) = (v_1(\tau), v_2(\tau))$ in the problem (3.6)-(3.7), then by using relation (3.4) we can say, if we take $\tau = 0$ then $t = t_0, \tau = 1$ then $t = x_{n+1}$ (in fact $x_{n+1}(0) = t_1$), and for $\tau = 2$ then $t = t_2$. It means we obtained intervals $[t_0, t_1]$ and interval $[t_1, t_2]$. From the relation (3.4), we can $\tau = \frac{t-t_0}{x_{n+1}-t_0},\ 0 \leq \tau \leq 1$ and $\tau = \frac{t-x_{n+1}}{t_2-x_{n+1}},\ 1 \leq \tau \leq 2$. Then, if we denote, $x_1(t) = y_1(\tau(t))$ and $x_2(t) = y_2(\tau(t))$ then we obtain, $\dot{x}_1 = \dot{y}_1(\tau(t))\frac{1}{x_{n+1}-t_0}$ and $\dot{x}_2 = \dot{y}_2(\tau(t))\frac{1}{t_2-x_{n+1}}$. If we consider this in the equations (3.6) and (3.7), we can come to the point that, $(t_1, x(t), u(t))$ is the admissible process for the equations (2.1) and (2.2).

Theorem 3.2 *The corresponding mapping between the admissible processes* $(t_1, x(t), u(t))$ *and* $(y(t), v(t))$ *for the equations* (2.2), (3.6) *and* (3.7) *preserve the value of the cost functionals* (2.1) *and* (3.8).

Proof : In fact, assume process $(t_1^0, x^0(t), u^0(t))$ is the optimal control for *Problem I.* Let us take process $(y^0(\tau), v^0(\tau))$, which is obtained from the optimal process $(t_1^0, x^0(t), u^0(t))$ above mentioned transformation. Assume that, $(y^0(\tau), v^0(\tau))$, is not the optimal process and there exists another optimal process $(\tilde{y}(\tau), \tilde{v}(\tau))$ with $\tilde{J}(\tilde{y}(\tau), \tilde{v}(\tau)) \leq J(y^0(\tau), v^0(\tau))$. Take the corresponding admissible process, which is obtained inverse transformation from the process $(\tilde{x}_{n+1}, \tilde{y}(\tau), \tilde{v}(\tau))$ and denote it by $(t_1, u(t), x(t))$. Then it is clear that,
$J(t_1, u(t), x(t)) = \tilde{J}(\tilde{y}(\tau), \tilde{v}(\tau)) \leq \tilde{J}(y^0(\tau), v^0(\tau) = \tilde{J}(t_1^0, x^0(t), u^0(t))$.
But this is a contradiction of the optimality of the process $(t_1^0, x^0(t), u^0(t))$. in Definition II.
The inverse opinion can be proved in the same way.
By using the last two theorems, it is possible to prove the following Corollary.

Corollary : If the process $(t_1^0, x^0(t), u^0(t))$ gives minimum for *Problem I*, then the process $(y^0(\tau), v^0(\tau))$, which is obtained after transformation, gives minimum value for *Problem II*, and vice versa.

4 Gradient Projection Method Algorithm

We have three optimized arguments:
The first one is the scaler argument $t_1 \in [t_0, t_f]$, the second one is a first control function $v_1(t)$ for $t \in [t_0, t_{mid}]$ and the last one is a second control function $v_2(t)$, for $t \in [t_{mid}, t_f]$.

That is $x = (t_1, v_1(t), v_2(t))$ with the cost function $J(t_1, v_1(t), v_2(t))$ and with the only constraint placed on $t_1 : t_0 \leq t_1 \leq t_f$.
In the present form, the above admissible process arguments represent an infinite-dimensional optimization problem. By applying the "parametrization technique", we can reduce the initial infinite-dimensional optimization problem to a finite-dimensional optimization problem. The usefulness of this procedure is that for the solution of a finite-dimensional optimization problem there exists a sufficiently powerful arsenal of methods and algorithms.
To convert the problem into an finite-dimensional optimization problem we apply the following parametrization technique:
Let's partition the sections $[t_0, t_{mid}]$ and $[t_{mid}, t_f]$ into finite number of sub-segments:

$[t_0, t_{mid}] = \bigcup_{i=1}^{N}[a_i, b_i)$ and $[t_{mid}, t_f] = \bigcup_{j=1}^{M}[c_j, d_j)$

Instead of the functions $v_1(t)$ and $v_2(t)$ we consider their piecewise constant approximations:

$v_1(t) = u_1^i =$ constant, if $t \in [a_i, b_i)$, $i = 1, 2, ..., N$;

$v_2(t) = u_2^j =$ constant, if $t \in [c_j, d_j)$, $j = 1, 2, ..., M$;

Thus, instead of the admissible process arguments we obtain a finite-dimensional optimization problem:

t_1, u_1^i, u_2^i with the cost function: $J(t_1; u_1^1, u_1^2, ..., u_1^N; u_2^1, u_2^2, ..., u_2^M)$.

To solve the above finite-dimensional optimization problem we propose to use first-order optimization techniques, i.e. the gradient-based methods, e.g. gradient projection procedure. Here are the steps of this procedure:

1) As an initial guess we choose some values for the optimized arguments of the cost function : $x^0 = (t_1^0, u_1^{1^0}, u_1^{2^0}, ..., u_1^{N^0}; u_2^{1^0}, u_2^{2^0}, ..., u_2^{M^0})$ such that the constraint is satisfied.

2) Then the considered procedure is an ordinary gradient method:

$$x^{k+1} = x^k - \alpha_k . \nabla f(x_k) \tag{4.9}$$

where $\nabla f(x_k)$ is the gradient of the cost functional at the point x_k ; α_k is the step in the direction of the anti-gradient.

3) If after completing the next iteration of (4.9) we trespass the allowable boundaries for the argument x_1^{k+1}, which in our case is t_1^{k+1}, we put it back into $[t_0, t_f]$ according to the following formula:

$$t_1^{k+1} = \begin{cases} 0, \ if \ t_1^{k+1} < 0 \\ 2, \ if \ t_1^{k+1} > 2 \end{cases} \tag{4.10}$$

4) We repeat steps 2-3 for new $k := k + 1$ until some exit criterion is satisfied. Possible exit criterions:

- $\|\nabla f(x_k)\| \leq \epsilon_1$
- $|f(x^{k+1}) - f(x^k)| < \epsilon_2$
- $|x^{k+1} - x^k| < \epsilon_3$

5 An Example

This example is taken from the reference [2], but in this case the switching point t_1 is considered to be unfixed. We will try reduce the unknown switching case to the known switching case, after which can be used the procedure for Gradient Projection Method Algorithm in section 4. Consider the following problem of minimizing the functional,

$$\begin{aligned} J(x, u_1, u_2, t_1) &= \frac{1}{2}[(x_{11}(t_1) + x_{21}(t_1))^2 + \int_0^{t_1} (x_{11}^2(t) + 2x_{11}(t)x_{12}(t) \\ &+ 3x_{12}^2(t) + u_1^2(t))dt + \int_{t_1}^2 (x_{21}^2(t) + 8x_{22}^2(t) + u_2^2(t))dt] \end{aligned} \tag{5.11}$$

with respect to the trajectories of the systems

$$subsystem(1) : \begin{cases} \dot{x}_{11}(t) - x_{11}(t) &= 0 \\ x_{12}(t) + u_1(t) &= 0 \\ x_{11}(0) &= -1 \quad \text{for } t \in [0, t_1) \end{cases} \tag{5.12}$$

$$subsystem(2) : \begin{cases} \dot{x}_{21}(t) &= 0 \\ x_{22}(t) - u_2(t) &= 0 \\ x_{21}(2) &= 1 \quad \text{for } t \in [t_1, 2] \end{cases} \tag{5.13}$$

We will use the transformation (3.4) which reduces the problem (5.12)-(5.14) to the new problem without an unknown switching point.

For this aim, take the new variable $\dot{x}_{n+1}(t) = 0$, $x_{n+1}(0) = t_1$. From this differential equation, it is clear $x_{n+1} = t_1$ is an unknown constant in $[0,2]$. Take also, $y_{i,j}(\tau) = x_{i,j}(t(\tau)),\ v_i(\tau) = u_i(t(\tau))$ where $i,j = 1,2$. Let us use also interval transformation in (3.4) with $t_0 = 0$ and $t_2 = 2$. Then we can come to the point at which, if $\tau = 0$ then $t = 0$, if $\tau = 1$ then $t = x_{n+1} = t_1$, and, if $\tau = 2$ then $t = 2$. If we use all these transformation, then the minimizing functional and state equations will take the following form:

$$\begin{aligned} J(v) &= \frac{1}{2}[(y_{11}(1) + y_{21}(1))^2 + t_1 \int_0^1 (y_{11}^2(\tau) + 2y_{11}(\tau)y_{21}(\tau) + 3y_{12}^2 + v_1^2(\tau))d\tau \\ &+ (2 - t_1) \int_1^2 (y_{21}^2(\tau) + 8y_{22}^2(\tau) + v_2^2(\tau))d\tau] \end{aligned} \tag{5.14}$$

where, $v = (v_1, v_2)$, and state equations takes the form

$$subsystem(1) : \begin{cases} \dot{y}_{11}(t) - t_1 y_{11}(t) &= 0 \\ y_{12}(t) + v_1(t) &= 0 \\ y_{11}(0) &= -1 \quad \text{for } \tau \in [0,1) \end{cases} \tag{5.15}$$

$$subsystem(2) : \begin{cases} \dot{y}_{21}(t) &= 0 \\ y_{22}(t) - v_2(t) &= 0 \\ y_{21}(2) &= 1 \quad \text{for } \tau \in [1,2] \end{cases} \tag{5.16}$$

If we solve the subsystem(1) with respect to the states $y_{11}(t)$ and $y_{12}(t)$ and subsystem(2) with respect to the states $y_{21}(t)$ and $y_{22}(t)$, then putting these in (5.14), the minimizing functional assumes following form:

$$\begin{aligned} J(t_1, v_1, v_2) &= \frac{1}{2}[(1 - \exp(t_1))^2 + t_1 \int_0^1 (\exp(2t_1\tau) + 2\exp(t_1\tau)v_1(\tau) + 4v_1^2(\tau))d\tau \\ &+ (2 - t_1) \int_1^2 (1 + 9v_2^2(\tau))d\tau] \end{aligned} \tag{5.17}$$

To solve (5.17) using finite-optimization techniques first we transform the functional into a finite-dimensional problem as follows:

$$\begin{aligned} J(t_1, w_1, w_2) &= \frac{1}{2}[(1 - \exp(t_1))^2 + t_1 \sum_{i=1}^{N} \int_0^1 (\exp(2t_1\tau) + 2\exp(t_1\tau)w_1^i(\tau) + 4(w_1^i)^2(\tau))d\tau \\ &+ (2 - t_1) \sum_{j=1}^{M} \int_1^2 (1 + 9(w_2^j)^2(\tau))d\tau] \end{aligned} \tag{5.18}$$

where, $v_1(t) = w_1^i =$ constant, if $t \in [0,1)$; $v_2(t) = w_2^j =$ constant, if $t \in [1,2]$.

Then, by using the Gradient Projection Method we can obtain the following optimal control input and state variable histories numerically in figures. As a result, by applying the Gradient Algorithm for the initial nominal $t_1 = 1.0$, after 145 iterations we find that the optimal switching time $t_1^* = 0.0656$ and optimal cost $J^* = 0.9958$. The computation takes about 6.856 seconds of CPU time when using C♯ on an Intel(R)Core(TM)i7-3720QM 2.60 GHz PC with 8GB of RAM.

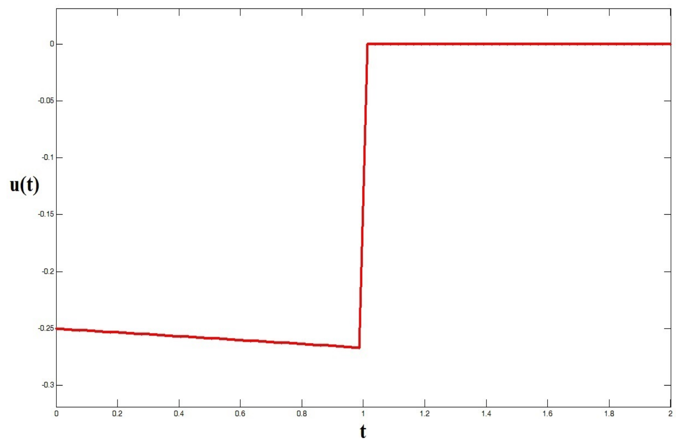

Figure 1: Optimal Control Input

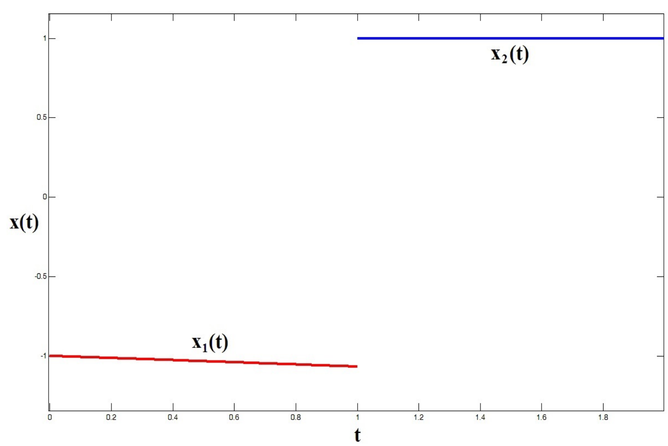

Figure 2: State Trajectories

6 Conclusion

If there are K numbers of switchings, then there is no difficulty in applying the previous method to the problems with several subsystems. If there exist non fixed switchings, $t_1, t_2, ..., t_K$ with $0 = t_0 < t_1 < t_2 <, ..., < t_K < T = 0$, then we can transcribe the problem into an equivalent problem by introducing K new state variables $x_{n+1}, x_{n+2}, ..., x_{n+K}$ which correspond to the switching instants $t_1, t_2, ..., t_K$ and satisfies,
$\frac{dx_{n+i}}{d\tau} = 0,\ x_{n+i}(0) = t_i,\ i = 1, 2, ..., K$
The new independent time variable τ has a linear relationships with t where $\tau = 0$ corresponds to $t = t_0,\ \tau = 1$ corresponds to $t = t_1 ... \tau = K + 1$ corresponds to $t = t_T$.

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