A decomposition approach for single allocation hub location problems with multiple capacity levels

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Abstract. In this paper we consider an extended version of the classical capacitated single allocation hub location problem in which the size of the hubs must be chosen from a finite and discrete set of allowable capacities. We develop a Lagrangian relaxation approach that exploits the problem structure and decomposes the problem into a set of smaller subproblems that can be solved efficiently. Upper bounds are derived by Lagrangean heuristics followed by a local search method. Moreover, we propose some reduction tests that allow us to decrease the size of the problem. Our computational experiments on some challenging benchmark instances from literature show the advantage of the decomposition approach over commercial solvers.

Keywords: Hub location, Capacity decisions, Lagrangian relaxation

1 Introduction

Given a complete graph $G = (N, A)$, where N represents the origins, destinations and possible hub locations, and A is the edge set. Hub location problems consider the location of hubs and the allocation of origin-destination nodes to hub nodes in order to route the flow w_{ij} from each origin $i \in N$ to each destination $j \in N$. Hub nodes are used to sort, consolidate, and redistribute flows and their main purpose is to realize economies of scale: while the construction and operation of hubs and the resulting detours lead to extra costs, the bundling of flows decreases costs. The economies of scale are usually modelled as being proportional to the transport volume, defined by multiplication with a discount factor $\alpha \in [0, 1]$.

Depending on the way in which non-hub nodes may be assigned to hub nodes, hub location problems can be classified as either multiple allocation [7] or single allocation [6, 15, 16, 13] hub location problems. In multiple allocation problems, the flow of the same non-hub node can be routed through different hubs, while in single allocation problems, each non-hub node is assigned to exactly one hub. In addition, each of these problems can be classified as capacitated or uncapacitated depending on various types of capacity restrictions. In particular, there can be

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limitations on the total flow routed on a hub-hub link [12] or on the volume of flow into the hub nodes [8]. For recent overviews on hub location problems we refer the reader to [1, 3]. Hub location problems have important applications including, among others, telecommunication systems [11], airline services [10], postal delivery services [6], and public transportation [14].

Due to the importance of capacity restrictions in real-world hub location problems, many papers can be found in the literature that address this type of problems in both multiple and single allocation cases [2, 8]. In what follows, we concentrate on different variants of the *capacitated single allocation hub location problem* (CSAHLP). The classical mixed integer linear programming (MILP) formulation for the CSAHLP was proposed by Campbell [2]. It allows a limit on the incoming flow at the hubs coming from both non-hub and hub nodes and defines set-up costs for establishing each of the hubs. Motivated by a postal delivery application, Ernst and Krishnamoorthy [8] studied a variant of the CSAHLP with capacity constraints on the incoming flow at the hubs coming only from non-hub nodes. They proposed an MILP formulation, two heuristics for obtaining upper bounds, and a branch and bound method. Labbé et al. [12] study a CSAHLP where for each hub there is a limit on the total flow traversing it. They studied some polyhedral properties of the problem and propose a branch-and-cut method. Costa et al. [9] proposed a bi-criteria approach to deal with the CSAHLP where the second objective function either minimizes the time that hubs take for processing the flow or minimizes the maximum service time at the hubs. Contreras et al. [4] present a branch-and-price approach for the CSAHLP where lower bounds are obtained using Lagrangian relaxation.

As an extension of the above models, in this paper, we consider a CSAHLP where the choice of capacity levels is explicitly included in the model. This problem was introduced by Correia et al. [5] and is called *capacitated single allocation hub location problem with multiple capacity levels* (CSAHLPM). In CSAHLPM the capacity restrictions are applied only on incoming flow from origins and each capacity level available incurs a specific set-up cost. All aforementioned CSAHLPs consider a discrete set of potential hub locations, with each hub location having an exogenously defined maximum capacity, while in CSAHLPM individual capacity levels can be installed for each hub location. Accordingly, not only have the hub nodes to be chosen but also the capacity level at which each of them will operate. In [5], the authors propose some MILP formulations for the problem, compare them in terms of the linear programming relaxation, and use state-of-the art optimization software to solve the problem.

In this paper we consider a general form of the CSAHLPM where distances between possible hub locations are not necessarily Euclidean distances in the plane. Starting from a natural quadratic binary program, we first provide a reformulation of the problem which shifts the quadratic term from the objective function to a set of constraints. This allow us to deal with an even more general form of the problem where transportation costs do not need to be linear anymore [19, 17]. We develop a Lagrangian relaxation scheme of a path-based MILP formulation by relaxing the assignment constraints and also constraints that link the assignment variables with the path variables. The Lagrangian function exploits the problem structure and decomposes the problem into a set of smaller subproblems that can be solved efficiently. Some of the latter can be reduced to continuous knapsack problems that can be solved quickly. Since the proposed Lagrangian relaxation does not have the so-called integrality property, the obtained bound will be stronger than the one given by the continuous relaxation of the MILP. To calculate feasible solutions we propose a two-phase heuristic where a greedy algorithm is used to construct an initial solution in the first phase and a local search scheme tries to improve the initial solution in the second phase. Finally, we present some reduction tests that allow us to decrease the size of the problem without affecting the set of optimal solutions and, accordingly, to obtain tighter bounds with less computational effort.

2 Problem formulations

Let a directed graph $G = (N, A)$ be given, where the set N contains the nodes, representing the origins, destinations and possible hub locations, and A is the edge set. Let w_{ij} be the amount of flow to be transported from node i to node j. We denote by $O_i = \sum_{j \in N} w_{ij}$ and $D_i = \sum_{j \in N} w_{ji}$ the total outgoing flow from node i and the total incoming flow to node i, respectively. For each $k \in N$, we consider $Q_k = \{1, 2, \ldots, s_k\}$ as a set of different capacity levels available for a potential hub to be installed at node k. For each $k \in N$ and each $\ell \in Q_k$, let $f_{k\ell}$ and $\Gamma_{k\ell}$ represent the fixed set-up cost and the capacity of hub k associated with capacity level ℓ. The capacity of a hub represents an upper bound on the total incoming flow that can be processed in the hub. Thus, it refers only to the sum of the flow generated at the nodes that are assigned to the hub and not taking into account the inter-hub flow. The cost per unit of flow for each path $i{-}j{-}k{-}m$ from an origin node i to a destination node j which passes hubs k and m respectively, is $\chi d_{ik} + \alpha d_{kl} + \delta d_{lj}$, where χ, α, and δ are the nonnegative collection, transfer and distribution costs respectively, and d_{ij} represents the distance between nodes i and j. Note that we do not require that the distances satisfy the triangle inequality. The CSAHLPM now consists in selecting a subset of nodes as hubs with specific capacity levels and assigning the remaining nodes to these hubs such that each non-hub node is assigned to exactly one hub node without exceeding its chosen capacity, with the minimum overall cost.

2.1 Quadratic binary formulation

In order to model the problem as an integer quadratic program, we define binary variables x_{ik} indicating whether a source/sink i is allocated to a hub k. In particular, the variables x_{kk} are used to indicate whether k becomes a hub. Moreover, for each $k \in N$ and $\ell \in Q_k$, we define a binary variable $t_{k\ell}$ indicating whether node k receives a hub with capacity level ℓ. For ease of presentation, we set $c_{ik} := d_{ik}\,(\chi\, O_i + \delta\, D_i)$. The CSAHLPM can then be formulated as follows:

$$\text{P:} \quad \min \quad \sum_i \sum_k c_{ik} x_{ik} + \sum_i \sum_j \sum_k \sum_m \alpha d_{km} w_{ij}\, x_{ik} x_{jm} + \sum_k \sum_{\ell \in Q_k} f_{k\ell} t_{k\ell}$$

$$\text{s.t.} \quad \textstyle\sum_k x_{ik} = 1 \quad (i \in N) \tag{1}$$

$$x_{ik} \le x_{kk} \quad (i, k \in N) \tag{2}$$

$$\textstyle\sum_i O_i x_{ik} \le \sum_{\ell \in Q_k} t_{k\ell} \Gamma_{k\ell} \quad (k \in N) \tag{3}$$

$$\textstyle\sum_{\ell \in Q_k} t_{k\ell} = x_{kk} \quad (k \in N) \tag{4}$$

$$\textstyle\sum_k x_{kk} \ge p \tag{5}$$

$$x_{ik} \in \{0, 1\} \quad (i, k \in N) \tag{6}$$

$$t_{\ell k} \in \{0, 1\} \quad (k \in N, \ell \in Q_k) \tag{7}$$

where the objective function measures the total transport costs consisting of the collection and distribution costs of nonhub-hub and hub-nonhub connections, the hub-hub transfer costs, as well as the set-up costs of the hubs. Constraints (1) force every node to be allocated to precisely one hub node. Constraints (2) state that i can only be allocated to k if k is chosen as a hub. Constraints (3) are capacity constraints and ensure that the overall incoming flow of nodes assigned to a hub does not exceed its capacity. Constraints (4) assure that if a hub is installed at a node then exactly one capacity level is chosen.

Finally, Constraint (5) sets a lower bound p on the number of chosen hubs. We use this constraint only to strengthen the lower bounds given by this model, choosing a value for p that is a lower bound on the number of hubs already due to the remaining constraints, so that Constraint (5) is redundant. For details about the computation of p we refer the reader to [5].

Now, let us consider a reformulation of the above model in which the quadratic term from the objective function is shifted to the set of constraints. To this end, we define a new continuous variable z_{km} that models the traffic on the hub-hub connection $(k, m) \in A$. This allows us to rewrite the above model as:

$$\text{P1:} \quad \min \quad \textstyle\sum_i \sum_k c_{ik}\, x_{ik} + \sum_k \sum_m \alpha\, d_{km}\, z_{km} + \sum_k \sum_{\ell \in Q_k} f_{k\ell} t_{k\ell}$$

$$\text{s.t.} \quad \textstyle\sum_i \sum_j w_{ij}\, x_{ik} x_{jm} \le z_{km} \quad (k, m \in N) \tag{8}$$

$$(1) - (7)$$

where the traffic variable z_{km} for $(k, m) \in A$ is determined by Constraints (8). Note that since all data are non-negative, there exists an optimal solution for problem P1 where all constraints (8) are tight. Therefore, the problems P and P1 are equivalent.

2.2 Linearization

All formulations proposed in [5] for the CSAHLPM are, in fact, based on the classical path based formulation of [2, 18] and the flow based formulation of [8]. Moreover, in [5] the authors assume that the distances between nodes satisfy the triangle inequality. However, if the distances considered are, for example, road distances which do not necessarily satisfy the triangle inequality, then the flow based formulations cannot be applied.

To linearize Problem P1, we follow the path based formulation of [18] for uncapacitated hub location problems and define a set of binary variables y_{ikjm} for $i, k, j, m \in N$ to indicate whether the flow from node i to node j travels via hubs located at nodes k and m or not. The resulting formulation is as follows:

$$\begin{aligned}
\text{ILP1:}\quad \min\quad & \textstyle\sum_i \sum_k c_{ik} x_{ik} + \sum_k \sum_m \alpha\, d_{km}\, z_{km} + \sum_k \sum_{\ell \in Q_k} f_{k\ell} t_{k\ell} \\
\text{s.t.}\quad & \textstyle\sum_i \sum_j w_{ij}\, y_{ikjm} \le z_{km} \quad (k, m \in N) && (9)\\
& \textstyle\sum_k y_{ikjm} = x_{jm} \quad (i, j, m \in N) && (10)\\
& \textstyle\sum_m y_{ikjm} = x_{ik} \quad (i, j, k \in N) && (11)\\
& 0 \le y_{ikjm} \le 1 \quad (i, k, j, m \in N) && (12)\\
& (1) - (7).
\end{aligned}$$

This problem is equivalent to Problem P. However, if the integrality restrictions on variables x or t are relaxed, it is no longer equivalent to P, but only provides a lower bound on its objective function value.

3 Solution method

3.1 Lagrangian relaxation

Due to the large number of variables and constraints, solving the linear relaxation of ILP1 requires considerable running time as the size of the instances increases. To overcome this problem, we develop a Lagrangian relaxation approach based on relaxing Constraints (1), (10), and (11) of the ILP1 formulation. Using Lagrangian multipliers π, λ, and μ, respectively, we obtain the following Lagrangian function:

$$\begin{aligned}
\mathrm{L}(\pi, \lambda, \mu):\quad \min\quad & \textstyle\sum_i \pi_i + \sum_i \sum_k \bar{c}_{ik} x_{ik} + \sum_k \sum_{\ell \in Q_k} f_{k\ell} t_{k\ell} \\
& \textstyle + \sum_k \sum_m \alpha\, d_{km}\, z_{km} + \sum_i \sum_j \sum_k \sum_m (\lambda_{ijm} + \mu_{jik}) y_{ikjm} \\
\text{s.t.}\quad & (2) - (7), (9), (12)
\end{aligned}$$

where

$$\bar{c}_{ik} := c_{ik} - \pi_i - \textstyle\sum_j (\lambda_{jik} + \mu_{jik}) \quad (i, k \in N)\ .$$

The best lower bound is then obtained by solving the Lagrangian dual problem given as $\max_{\pi,\lambda,\mu} \mathrm{L}(\pi, \lambda, \mu)$.

Considering the independence between the two groups of variables (x, t) and (y, z) in $\mathrm{L}(\pi, \lambda, \mu)$, we can first decompose the latter into two subproblems $\mathrm{L}_{xt}(\pi, \lambda, \mu)$ and $\mathrm{L}_{yz}(\lambda, \mu)$:

$$\begin{aligned} \mathrm{L}_{xt}(\pi,\lambda,\mu): \quad & \min \quad \textstyle\sum_i \sum_k \bar{c}_{ik} x_{ik} + \sum_k \sum_{\ell \in Q_k} f_{k\ell} t_{k\ell} \\ & \text{s.t.} \quad (2), (3), (4), (5), (6) \end{aligned} \tag{13}$$

$$\begin{aligned} \mathrm{L}_{yz}(\lambda,\mu): \quad & \min \quad \textstyle\sum_k \sum_m d_{km}\, z_{km} + \sum_i \sum_j \sum_k \sum_m (\lambda_{ijm} + \mu_{jik}) y_{ikjm} \\ & \text{s.t.} \quad (9), (12). \end{aligned} \tag{14}$$

Solving $L_{xt}(\pi, \lambda, \mu)$: To solve Subproblem (13), let us suppose that node k receives a hub with capacity level ℓ, that is, $t_{k\ell} = 1$. Then the remaining nodes that will be assigned to hub k can be found by solving the following binary knapsack problem:

$$\begin{aligned} \xi_{k\ell} = \min \quad & \textstyle\sum_{i \neq k} \bar{c}_{ik} x_{ik} \\ \text{s.t.} \quad & \textstyle\sum_{i \neq k} O_i x_{ik} \leq \Gamma_{k\ell} - O_i \\ & x_{ik} \in \{0, 1\} \quad (i \in N,\ i \neq k). \end{aligned}$$

Now suppose that a hub is located at $k \in N$, such that $x_{kk} = 1$. Then the following problem just selects the best capacity level of the hub from Q_k:

$$\begin{aligned} \varepsilon_k = \min \quad & \textstyle\sum_{\ell \in Q_k} (f_{k\ell} + \xi_{k\ell}) t_{k\ell} \\ \text{s.t.} \quad & \textstyle\sum_{\ell \in Q_k} t_{k\ell} = 1 \\ & t_{k\ell} \in \{0, 1\} \quad (\ell \in Q_k). \end{aligned}$$

Finally, solving the following problem gives the optimal value of $\mathrm{L}_{xt}(\pi, \lambda, \mu)$:

$$\begin{aligned} \min \quad & \textstyle\sum_k (\varepsilon_k + \bar{c}_{kk}) x_{kk} \\ \text{s.t.} \quad & \textstyle\sum_k x_{kk} \geq p \\ & x_{kk} \in \{0, 1\} \quad (k \in N). \end{aligned}$$

This problem can be solved easily by sorting and choosing at least the p hubs with smallest objective coefficient $\varepsilon_k + \bar{c}_{kk}$.

Solving $L_{yz}(\lambda, \mu)$: Subproblem (14) can be further decomposed into n^2 subproblems, one for each pair (k, m), as follows:

$$\begin{aligned} \max \quad & -\alpha d_{km}\, z_{km} - \textstyle\sum_i \sum_j (\lambda_{ijm} + \mu_{jik}) y_{ikjm} \\ \text{s.t.} \quad & \textstyle\sum_i \sum_j w_{ij}\, y_{ikjm} \leq z_{km} \\ & 0 \leq y_{ikjm} \leq 1 \quad (i, j \in N) \end{aligned}$$

where we may assume $\lambda_{ijm} + \mu_{jik} < 0$ for all $i, j \in N$. This problem is a special knapsack problem where the capacity of the knapsack is part of the decision making process with per unit cost αd_{km}. An optimal solution for this problem can be found by adding only those items to the knapsack whose profit per unit of weight exceeds the cost of one unit of capacity. More precisely, for each item $i, j \in N$, if

$$-(\lambda_{ijm} + \mu_{jik}) y_{ikjm} / w_{ij} > \alpha d_{km}$$

we add this item to the knapsack and set $y_{ikjm} = 1$, otherwise we set $y_{ikjm} = 0$.

3.2 Primal heuristics

To obtain a valid upper bound for our given formulation, we aim at creating a feasible solution in a two step process, starting from the current solution of the Lagrangian relaxation procedure. In the first step, we try to derive a feasible solution and, if we found one, we reassign hubs with a local-search algorithm to improve the solution.

Let (x, t, z) be the solution of the Lagrangian relaxation in the current iteration. As this solution does not need to be feasible, we first try to find valid variable assignment. Firstly, we will only consider the current solution if the opened capacity is large enough to meet the demand. The procedure to heuristically generate a feasible solution $(\bar{x}, \bar{t}, \bar{z})$ is as follows: whenever $x_{kk} = 1$ in the Lagrangian relaxation, node k will be a hub in the heuristic solution, i.e., $\bar{x}_{kk} = 1$, and we set $\bar{t}_{k\ell} = t_{k\ell}$ for all $\ell \in Q_k$. If a node i is assigned to more than one hub, we only assign it to the hub with the least cost. The remaining unassigned nodes will now be sorted according to their weights O_i and we consider the free capacity γ_k in each hub given by

$$\gamma_k := \textstyle\sum_{\ell \in Q_k} \mathit{\Gamma}_{k\ell} \bar{t}_{k\ell} - \sum_{i=1}^{n} O_i \bar{x}_{ik} \ .$$

Now we iteratively assign each remaining node to a hub, starting with the one with the highest O_i. The node with the highest weight will be assigned to the hub with the lowest cost and so on. If we cannot assign every node to a hub in this way, we stop our heuristic.

In the second step we try to improve the upper bound found in the first step by means of a local search algorithm. For that, we only consider reassigning nodes if the calculated upper bound from the first step is not considerably worse than the best known upper bound. The local search phase then tries to find possible shift or swap moves that improve the current solution. While shifting tries to assign nodes to different hubs with enough free capacity, swapping exchanges two assigned hubs. Note that when we reassign a node we only need to calculate the estimated cost $\mathit{\Delta}$ of the new feasible solution in terms of the old upper bound, corrected by the influence of the new assignment. For example in the shifting step, when node i is reassigned from hub k to hub m, the value of $\mathit{\Delta}$ is computed as:

$$\begin{aligned}\mathit{\Delta} = {} & \bar{z}_{km} + \bar{z}_{mk} + c_{im} - c_{ik} \\ & + \sum_{j \neq m} \alpha (d_{jm} \bar{z}_{jm} + d_{mj} \bar{z}_{mj}) + \sum_{j \neq k} \alpha (d_{jk} \bar{z}_{jk} + d_{kj} \bar{z}_{kj}).\end{aligned}$$

If $\Delta < 0$, then the so found upper bound is smaller than the original one and we update the solution accordingly.

3.3 Reduction test

The size of ILP1 may be reduced by eliminating the hub variables which do not appear in any optimal solution of a given instance. For this, we use information obtained from the Lagrangean function at a given iteration for any given Lagrangian multipliers π, λ and μ. Our reduction test is based on testing if a hub k with a specific level ℓ is going to be excluded in the optimal solution of a given instance. The main idea of fixing a variable to zero is to check if including this variable in the solution will lead to a lower bound greater than the best upper bound found so far. If so, the variable cannot belong to an optimal solution. We consider variables $t_{k\ell}$, $k \in N, \ell \in Q_k$, that are not already fixed and are equal to zero in the current iteration. We thus impose an additional constraint $t_{k\ell} = 1$ to the current Lagrangian function. Let $LB^{k\ell}$ represent the new value of the Lagrangian function. If $LB^{k\ell} \geq UB$, then we fix $t_{k\ell} = 0$ and add it to the list of fixed variables.

The main question arising here is how to compute $LB^{k\ell}$ without resolving the Lagrangian function. To answer this question, we distinguish between the following two situations: (i) If $x_{kk} = 1$, then there exists a level $\ell' \in Q_k$ such that $t_{k\ell'} = 1$. Therefore, to open hub k with different capacity level ℓ, we need to exclude hub k with capacity level ℓ'. Hence we have:

$$LB^{k\ell} = LB - (f_{k\ell'} + \xi_{k\ell'}) + (f_{k\ell} + \xi_{k\ell}).$$

(ii) If $x_{kk} = 0$, we consider two different scenarios. If the number of open hubs agrees with the lower bound p and all values $\varepsilon_m + \bar{c}_{mm}$ of open hubs are non-positive, or the number of open hubs is strictly grater than p, we do not need to close any hubs in order to open hub k, which means:

$$LB^{k\ell} = LB + (\bar{c}_{kk} + f_{k\ell} + \xi_{k\ell}) \tag{15}$$

Otherwise, if for some open hubs the values $\varepsilon_m + \bar{c}_{mm}$ are positive, we close the most expensive one and subtract $\varepsilon_{m^*} + \bar{c}_{m^*m^*}$ from $LB^{k\ell}$, i.e,

$$LB^{k\ell} = LB + (\bar{c}_{kk} + f_{k\ell} + \xi_{k\ell}) - (\varepsilon_{m^*} + \bar{c}_{m^*m^*})$$

where $m^* = \text{argmax}\{\varepsilon_m + \bar{c}_{mm} \mid x_{mm} = 1\}$ is the most expensive open hub. Note that if for every k and all $\ell \in Q_k$ we have $t_{k\ell} = 0$, we can fix all x_{kk} and x_{ik} to zero as well.

In our computational experiments we performed this reduction test in any iteration of the subgradient method. When the set of potential hubs or the set of capacity levels for a given hub was reduced, we updated Subproblems (13) and (14) accordingly.

4 Extension to link capacities

The reformulation of Problem P as Problem P1 allows to extend the CSAHLPM in order to include link capacities. In particular, consider an application in telecommunication networks containing set-up cost s_{ij} for installing needed capacities on the connection $(i,j) \in A$; see [19]. The capacity is provided by the installation of an integer number of links of a fixed capacity q. The resulting problem is as follows:

$$\begin{aligned} \text{MILP1:} \quad \min \quad & \textstyle\sum_i \sum_k c_{ik}\, x_{ik} + \sum_k \sum_m s_{km}\, z_{km} + \sum_{k,\ell \in Q_k} f_{k\ell} t_{k\ell} \\ \text{s.t.} \quad & \textstyle\sum_i \sum_j w_{ij}\, y_{ikjm} \leq q\, z_{km} \quad (k,m \in N) \qquad (16) \\ & (1)-(7),(9)-(12) \\ & z_{ij} \geq 0, \text{integer} \quad (i,j) \in A \end{aligned}$$

where z_{ij} indicates the number of installed links on $(i,j) \in A$. Constraints (16) relate the number of installed links on each hub-hub connection to the total flow passing the link and state that the flow on each hub-hub connection cannot exceed the capacity of the link. Moreover, since the total incoming and outgoing flow for each node i is given, the cost factors c_{ik} can be precomputed. Note that if we divide all w_{ij} by q and set $s_{km} = \alpha d_{km}$, and if we relax the integrality of the z variables, Problem MILP1 agrees with Problem ILP1.

Such stepwise cost functions make the problem much harder to solve. They have been considered also in [17] for the uncapacitated single allocation hub location problem arising in transportations where the stepwise function results from the integrality of the number of vehicles on hub-hub-connections.

Note that our decomposition approach is still valid for problem MILP1 with a simple modification in Subproblem (14) to include the integrality of z variables. An efficient solution method can be found in [17].

5 Computational experiments

In this section we present our computational experiments on the lower bound and upper bound computations for the CSAHLPM. We apply our decomposition approach to both problems ILP1 and MILP1, and compare the results with those obtained from the linear relaxation of these models solved by Cplex 12.6. To compute the optimal (or near-optimal) Lagrangian multipliers, we use a subgradient optimization method with a maximum number of 2000 iterations. We implemented the algorithms in C++ and performed all experiments on an Intel Xeon processor running at 2.5 GHz.

For our numerical tests, we considered the Australian Post data set (AP) from the OR library[1], which is used frequently for different hub location problems. We followed the pattern proposed in [5] to generate hub levels: for the highest hub level, the capacity and the set-up cost are equal to the values that are included

[1] http://people.brunel.ac.uk/~mastjjb/jeb/orlib/phubinfo.html

Table 1: Computational results for the original CSAHLPM.

instance			Lagrangian relaxation				cplex	
type	level	Ub	gap(%)	time (s)	ch(%)	chl(%)	gap(%)	time(s)
$\rho = 1.1$:								
25LL	3	216557.6	0.1	18.0	4.0	66.7	0.1	15.9
	5	202773.6	8.3	14.6	0.0	5.6	7.3	6.0
25LT	3	278062.7	1.0	16.8	8.0	56.0	1.1	11.5
	5	260850.8	2.2	17.1	4.0	42.4	3.5	11.7
50LL	3	217183.0	1.1	405.2	6.0	59.3	1.0	993.2
	5	200016.4	1.5	408.5	0.0	45.6	2.1	815.9
50LT	3	290376.8	1.9	390.4	18.0	54.0	2.2	1837.3
	5	272592.7	1.1	384.4	24.0	58.0	1.9	1651.4
$\rho = 1.2$:								
25LL	3	227274.4	0.1	16.4	64.0	86.7	4.8	12.1
	5	211538.1	0.3	17.3	0.0	69.6	11.2	4.9
25LT	3	289881.1	0.6	16.7	16.0	58.7	0.9	11.7
	5	282810.3	0.8	16.9	8.0	56.8	2.3	14.6
50LL	3	229339.1	1.4	407.2	4.0	41.3	6.3	1414.6
	5	219429.2	1.0	406.1	2.0	46.8	10.7	1205.2
50LT	3	303998.6	1.5	393.7	12.0	43.3	1.3	1715.3
	5	291289.2	0.2	372.0	56.0	85.2	1.5	1522.3

in the AP data set. Additional levels are produced recursively, starting from the second highest level, according to the formulae

$$\Gamma_{k\ell} = 0.7 \cdot \Gamma_{k\,\ell+1} \quad \text{and} \quad f_{k\ell} = \rho \cdot 0.7 \cdot f_{k\,\ell+1} \ ,$$

where $\rho = 1.1$ or 1.2 is a factor to model economies of scale. We consider instances with either three or five levels, with the highest capacity level equal to the loos (L) capacity for the potential hub in the corresponding instance in the AP data set. For the highest set-up cost level we use both the tight (T) and the loose (L) set-up costs from the AP data set. For the data used in MILP1 we followed [16]: for all $i, j \in N$, we set $s_{ij} = \alpha\, q\, d_{ij}$, where $q = \sum_{i,j} w_{ij}/p^2$.

Tables 1 and 2 present the results for problems ILP1 and MILP1, respectively. Each of these tables is divided into two parts where we separately report results for $\rho = 1.1$ or 1.2. In both tables, the first columns indicate the problem type ($|N|Lx$) with $|N| \in \{25, 50\}$ and $x \in \{L, T\}$. The next two columns give the number of levels (level) and the best upper bound (Ub) obtained with our primal heuristic. The next columns present the results of our Lagrangian relaxation and the results of Cplex. For each algorithm, gap(%), and time(s) represent the relative gap in percent and the total required time (in seconds), respectively. The formula we used to compute the relative gaps is $100 \times (Ub - Lb)/Ub$, where

Table 2: Computational results for the CSAHLPM with link capacities.

instance			Lagrangian relaxation				cplex	
type	level	Ub	gap (%)	time(s)	ch(%)	chl(%)	gap(%)	time(s)
$\rho = \mathbf{1.1}$:								
25LL	3	292025.9	3.4	62.2	0.0	14.7	25.9	11.4
	5	292025.9	3.5	70.1	0.0	13.6	35.6	4.7
25LT	3	353996.0	4.6	54.5	4.0	13.3	22.3	11.1
	5	362634.8	6.6	56.7	4.0	5.6	30.6	11.1
50LL	3	288545.2	3.4	1144.1	2.0	14.7	25.5	772.1
	5	284577.4	2.1	1167.4	2.0	18.4	31.2	751.6
50LT	3	339367.0	7.2	957.0	8.0	20.7	16.3	1152.4
	5	350838.8	9.8	883.3	4.0	10.0	23.8	1616.1
$\rho = \mathbf{1.2}$:								
25LL	3	292025.9	3.5	55.2	4.0	17.3	22.3	11.8
	5	292025.9	3.4	60.8	0.0	12.8	28.2	7.7
25LT	3	360757.7	6.0	58.5	4.0	6.7	20.3	11.4
	5	360757.7	5.9	60.8	4.0	7.2	23.4	14.4
50LL	3	292622.1	5.4	1169.5	2.0	4.7	23.1	1123.3
	5	289010.6	3.4	1193.7	2.0	13.2	26.1	804.2
50LT	3	358772.3	11.7	894.6	8.0	10.7	16.4	1462.5
	5	359860.4	12.1	1010.3	6.0	8.0	20.3	1427.7

Lb stands for the value of the lower bound. The columns under the headings ch(%) and chl(%) of the Lagrangian relaxation present, respectively, the percent of closed hubs and closed hub levels by our reduction tests.

As we can observe from Table 1, the Lagrangian relaxation almost always outperforms Cplex in terms of both the bound tightness and running times. More precisely, the duality gaps for instances with $\rho = 1.1$ and $\rho = 1.2$ are, on average, 2.1% and 0.7% for the Lagrangian relaxation, and 2.4% and 4.9% for Cplex. The results reported in columns 6 and 7 show the effectiveness of the proposed reduction tests in closing hub levels: on average, 49% and 61% of levels have been closed for instances with $\rho = 1.1$ and $\rho = 1.2$, respectively. This has a significant positive effect on the required computational times of our Lagrangian relaxation.

Table 2 reports the results for the CSAHLPM with link capacities (MILP1). As we can observe, in principle, the problem is much more difficult than ILP1. However, the Lagrangian relaxation outperforms Cplex significantly: the average duality gap is 5.7% for the Lagrangian relaxation, compared with an average duality gap of 24.5% for Cplex.

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