Coordinate friendly structures, algorithms and applications*

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This paper focuses on *coordinate update methods*, which are useful for solving problems involving large or high-dimensional datasets. They decompose a problem into simple subproblems, where each updates one, or a small block of, variables while fixing others. These methods can deal with linear and nonlinear mappings, smooth and nonsmooth functions, as well as convex and nonconvex problems. In addition, they are easy to parallelize.

The great performance of coordinate update methods depends on solving simple subproblems. To derive simple subproblems for several new classes of applications, this paper systematically studies *coordinate friendly* operators that perform low-cost coordinate updates.

Based on the discovered coordinate friendly operators, as well as operator splitting techniques, we obtain new coordinate update algorithms for a variety of problems in machine learning, image processing, as well as sub-areas of optimization. Several problems are treated with coordinate update for the first time in history. The obtained algorithms are scalable to large instances through parallel and even asynchronous computing. We present numerical examples to illustrate how effective these algorithms are.

Keywords and phrases: coordinate update, fixed point, operator splitting, primal-dual splitting, parallel, asynchronous.

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1. Introduction

This paper studies *coordinate update methods*, which reduce a large problem to smaller subproblems and are useful for solving large-sized problems. These methods handle both linear and nonlinear maps, smooth and nonsmooth functions, and convex and nonconvex problems. The common special examples of these methods are the Jacobian and Gauss-Seidel algorithms for solving a linear system of equations, and they are also commonly used for solving differential equations (e.g., *domain decomposition*) and optimization problems (e.g., *coordinate descent*).

After coordinate update methods were initially introduced in each topic area, their evolution had been slow until recently, when data-driven applications (e.g., in signal processing, image processing, and statistical and machine learning) impose strong demand for scalable numerical solutions; consequently, numerical methods of *small footprints*, including coordinate update methods, become increasingly popular. These methods are generally applicable to many problems involving large or high-dimensional datasets.

Coordinate update methods generate simple subproblems that update one variable, or a small block of variables, while fixing others. The variables can be updated in the *cyclic*, *random*, or *greedy* orders, which can be selected to adapt to the problem. The subproblems that perform coordinate updates also have different forms. Coordinate updates can be applied either sequentially on a single thread or concurrently on multiple threads, or even in an asynchronous parallel fashion. They have been demonstrated to give rise to very powerful and scalable algorithms.

Clearly, the strong performance of coordinate update methods relies on solving *simple* subproblems. The cost of each subproblem must be proportional to how many coordinates it updates. When there are totally m coordinates, the cost of updating one coordinate should not exceed the average per-coordinate cost of the full update (made to all the coordinates at once). Otherwise, coordinate update is not *computationally worthy*. For example, let $f : \mathbb{R}^m \to \mathbb{R}$ be a C^2 function, and consider the Newton update $x^{k+1} \leftarrow x^k - \left(\nabla^2 f(x^k)\right)^{-1} \nabla f(x^k)$. Since updating each x_i (keeping others fixed) still requires forming the Hessian matrix $\nabla^2 f(x)$ (at least $O(m^2)$ operations) and factorizing it ($O(m^3)$ operations), there is little to save in computation compared to updating all the components of x at once; hence, the Netwon's method is generally not amenable to coordinate update.

The recent coordinate-update literature has introduced new algorithms. However, they are primarily applied to a few, albeit important, classes of problems that arise in machine learning. For many complicated problems,

it remains open whether simple subproblems can be obtained. We provide positive answers to several new classes of applications and introduce their coordinate update algorithms. Therefore, the focus of this paper is to build a set of tools for deriving simple subproblems and extending coordinate updates to new territories of applications.

We will frame each application into an equivalent fixed-point problem

$$x = \mathcal{T}x \tag{1}$$

by specifying the operator $\mathcal{T} : \mathbb{H} \to \mathbb{H}$, where $x = (x_1, \ldots, x_m) \in \mathbb{H}$, and $\mathbb{H} = \mathbb{H}_1 \times \cdots \times \mathbb{H}_m$ is a Hilbert space. In many cases, the operator $\mathcal{T}$ itself represents an iteration:

$$x^{k+1} = \mathcal{T}x^k \tag{2}$$

such that the limit of the sequence $\{x^k\}$ exists and is a fixed point of $\mathcal{T}$, which is also a solution to the application or from which a solution to the application can be obtained. We call the scheme (2) a *full update*, as opposed to updating one x_i at a time. The scheme (2) has a number of interesting special cases including methods of gradient descent, gradient projection, proximal gradient, operator splitting, and many others.

We study the structures of $\mathcal{T}$ that make the following coordinate update algorithm *computationally worthy*

$$x_i^{k+1} = x_i^k - \eta_k (x^k - \mathcal{T}x^k)_i, \tag{3}$$

where η_k is a step size and $i \in [m] := \{1, \ldots, m\}$ is arbitrary. Specifically, the cost of performing (3) is roughly $\frac{1}{m}$, or lower, of that of performing (2). We call such $\mathcal{T}$ a **Coordinate Friendly (CF)** operator, which we will formally define.

This paper will explore a variety of CF operators. Single CF operators include linear maps, projections to certain simple sets, proximal maps and gradients of (nearly) separable functions, as well as gradients of sparsely supported functions. There are many more composite CF operators, which are built from single CF and non-CF operators under a set of rules. The fact that some of these operators are CF is not obvious.

These CF operators let us derive powerful coordinate update algorithms for a variety of applications including, but not limited to, linear and second-order cone programming, variational image processing, support vector machine, empirical risk minimization, portfolio optimization, distributed computing, and nonnegative matrix factorization. For each application, we present

an algorithm in the form of (2) so that its coordinate update (3) is efficient. In this way we obtain new coordinate update algorithms for these applications, some of which are treated with coordinate update for the first time.

The developed coordinate update algorithms are easy to parallelize. In addition, the work in this paper gives rise to parallel and asynchronous extensions to existing algorithms including the Alternating Direction Method of Multipliers (ADMM), primal-dual splitting algorithms, and others.

The paper is organized as follows. §1.1 reviews the existing frameworks of coordinate update algorithms. §2 defines the CF operator and discusses different classes of CF operators. §3 introduces a set of rules to obtain composite CF operators and applies the results to operator splitting methods. §4 is dedicated to primal-dual splitting methods with CF operators, where existing ones are reviewed and a new one is introduced. Applying the results of previous sections, §5 obtains novel coordinate update algorithms for a variety of applications, some of which have been tested with their numerical results presented in §6.

Throughout this paper, all functions f, g, h are proper closed convex and can take the extended value ∞, and all sets X, Y, Z are nonempty closed convex. The indicator function $\iota_X(x)$ returns 0 if $x \in X$, and ∞ elsewhere. For a positive integer m, we let $[m] := \{1, \dots, m\}$.

1.1. Coordinate Update Algorithmic Frameworks

This subsection reviews the *sequential* and *parallel* algorithmic frameworks for coordinate updates, as well as the relevant literature.

The general framework of coordinate update is

1. set $k \leftarrow 0$ and initialize $x^0 \in \mathbb{H} = \mathbb{H}_1 \times \cdots \times \mathbb{H}_m$
2. while *not converged* do
3. select an index $i_k \in [m]$;
4. update x_i^{k+1} for $i = i_k$ while keeping $x_i^{k+1} = x_i^k$, $\forall\, i \neq i_k$;
5. $k \leftarrow k + 1$;

Next we review the index rules and the methods to update x_i.

1.1.1. Sequential Update. In this framework, there is a sequence of coordinate indices $i_1, i_2, \dots$ chosen according to one of the following rules: cyclic, cyclic permutation, random, and greedy rules. At iteration k, only the i_kth coordinate is updated:

$$\begin{cases} x_i^{k+1} = x_i^k - \eta_k (x^k - \mathcal{T} x^k)_i, & i = i_k, \\ x_i^{k+1} = x_i^k, & \text{for all } i \neq i_k. \end{cases}$$

Sequential updates have been applied to many problems such as the Gauss-Seidel iteration for solving a linear system of equations, alternating projection [75, 4] for finding a point in the intersection of two sets, ADMM [31, 30] for solving monotropic programs, and Douglas-Rachford Splitting (DRS) [26] for finding a zero to the sum of two operators.

In optimization, *coordinate descent* algorithms, at each iteration, minimize the function $f(x_1, \ldots, x_m)$ by fixing all but one variable x_i. Let

$$x_{i-} := (x_1, \ldots, x_{i-1}), \quad x_{i+} = (x_{i+1}, \ldots, x_m)$$

collect all but the ith coordinate of x. Coordinate descent solves one of the following subproblems:

$$(\mathcal{T}x^k)_i = \arg\min_{x_i} f(x^k_{i-}, x_i, x^k_{i+}), \tag{4a}$$

$$(\mathcal{T}x^k)_i = \arg\min_{x_i} f(x^k_{i-}, x_i, x^k_{i+}) + \frac{1}{2\eta_k}\|x_i - x^k_i\|^2, \tag{4b}$$

$$(\mathcal{T}x^k)_i = \arg\min_{x_i} \langle \nabla_i f(x^k), x_i\rangle + \frac{1}{2\eta_k}\|x_i - x^k_i\|^2, \tag{4c}$$

$$(\mathcal{T}x^k)_i = \arg\min_{x_i} \langle \nabla_i f^{\text{diff}}(x^k), x_i\rangle + f^{\text{prox}}_i(x_i) + \frac{1}{2\eta_k}\|x_i - x^k_i\|^2, \tag{4d}$$

which are called *direct* update, *proximal* update, *gradient* update, and *prox-gradient* update, respectively. The last update applies to the function

$$f(x) = f^{\text{diff}}(x) + \sum_{i=1}^{m} f^{\text{prox}}_i(x_i),$$

where f^{diff} is differentiable and each f^{prox}_i is proximable (its proximal map takes $O\big(\dim(x_i)\,\text{polylog}(\dim(x_i))\big)$ operations to compute).

Sequential-update literature. Coordinate descent algorithms date back to the 1950s [35], when the *cyclic* index rule was used. Its convergence has been established under a variety of cases, for both convex and nonconvex objective functions; see [77, 85, 57, 33, 45, 70, 32, 72, 58, 8, 36, 78]. Proximal updates are studied in [32, 1] and developed into prox-gradient updates in [74, 73, 13] and mixed updates in [81].

The *random* index rule first appeared in [48] and then [61, 44]. Recently, [82, 80] compared the convergence speeds of cyclic and stochastic update-orders. The gradient update has been relaxed to stochastic gradient update for large-scale problems in [21, 83].

The *greedy* index rule leads to fewer iterations but is often impractical since it requires a lot of effort to calculate scores for all the coordinates. However, there are cases where calculating the scores is inexpensive [11, 41, 79] and the save in the total number of iterations significantly outweighs the extra calculation [74, 25, 55, 49].

A simple example. We present the coordinate update algorithms under different index rules for solving a simple least squares problem:

$$\underset{x}{\text{minimize}}\ f(x) := \frac{1}{2}\|Ax - b\|^2,$$

where $A \in \mathbb{R}^{p\times m}$ and $b \in \mathbb{R}^p$ are Gaussian random. Our goal is to numerically demonstrate the advantages of coordinate updates over the full update of gradient descent:

$$x^{k+1} = x^k - \eta_k A^\top (Ax^k - b).$$

The four tested index rules are: cyclic, cyclic permutation, random, and greedy under the Gauss-Southwell[1] rule. Note that because this example is very special, the comparisons of different index rules are far from conclusive.

In the full update, the step size η_k is set to the theoretical upper bound $\frac{2}{\|A\|_2^2}$, where $\|A\|_2$ denotes the matrix operator norm and equals the largest singular value of A. For each coordinate update to x_i, the step size η_k is set to $\frac{1}{(A^\top A)_{ii}}$. All of the full and coordinate updates have the same *per-epoch* complexity, so we plot the objective errors in Figure 1.

1.1.2. Parallel Update. As one of their main advantages, coordinate update algorithms are easy to parallelize. In this subsection, we discuss both synchronous (sync) and asynchronous (async) parallel updates.

Sync-parallel (Jacobi) update specifies a sequence of index subsets $\mathbb{I}_1, \mathbb{I}_2, \ldots \subseteq [m]$, and at each iteration k, the coordinates in $\mathbb{I}_k$ are updated in parallel by multiple agents:

$$\begin{cases} x_i^{k+1} = x_i^k - \eta_k (x^k - \mathcal{T} x^k)_i, & i \in \mathbb{I}_k, \\ x_i^{k+1} = x_i^k, & i \notin \mathbb{I}_k. \end{cases}$$

Synchronization across all agents ensures that all x_i in $\mathbb{I}_k$ are updated and also written to the memory before the next iteration starts. Note that, if

[1] it selects $i_k = \arg\max_i \|\nabla_i f(x^k)\|$.

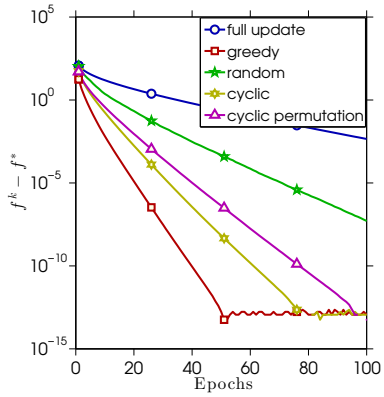


Figure 1: Gradient descent: the coordinate updates are faster than the full update since the former can take larger steps at each step.

$\mathbb{I}_k = [m]$ for all k, then all the coordinates are updated and, thus, each iteration reduces to the full update: $x^{k+1} = x^k - \eta_k(x^k - \mathcal{T}x^k)$.

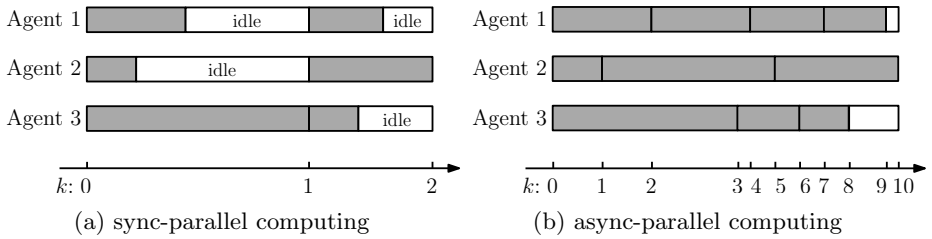
Async-parallel update. In this setting, a set of agents still perform parallel updates, but synchronization is eliminated or weakened. Hence, each agent continuously applies (5), which reads x from and writes x_i back to the shared memory (or through communicating with other agents without shared memory):

$$\begin{cases} x_i^{k+1} = x_i^k - \eta_k \left((\mathcal{I} - \mathcal{T})x^{k-d_k}\right)_i, & i = i_k, \\ x_i^{k+1} = x_i^k, & \text{for all } i \neq i_k. \end{cases} \tag{5}$$

Unlike before, k increases whenever any agent completes an update.

The lack of synchronization often results in computation with out-of-date information. During the computation of the kth update, other agents make d_k updates to x in the shared memory; when the kth update is written, its input is already d_k iterations out of date. This number is referred to as the asynchronous delay. In (5), the agent reads x^{k-d_k} and commits the update to $x_{i_k}^k$. Here we have assumed *consistent* reading, i.e., x^{k-d_k} lying in the set $\{x^j\}_{j=1}^k$. This requires implementing a memory lock. Removing the lock can lead to *inconsistent* reading, which still has convergence guarantees; see [54, Section 1.2] for more details.

Synchronization across all agents means that all agents will wait for the last (slowest) agent to complete. Async-parallel updates eliminate such idle time, spread out memory access and communication, and thus often run



(a) sync-parallel computing (b) async-parallel computing

Figure 2: Sync-parallel computing (left) versus async-parallel computing (right). On the left, all the agents must wait at idle (white boxes) until the slowest agent has finished.

much faster. However, async-parallel is more difficult to analyze because of the asynchronous delay.

Parallel-update literature. Async-parallel methods can be traced back to [17] for systems of linear equations. For function minimization, [12] introduced an async-parallel gradient projection method. Convergence rates are obtained in [69]. Recently, [14, 62] developed parallel randomized methods.

For fixed-point problems, async-parallel methods date back to [3] in 1978. In the pre-2010 methods [2, 10, 6, 27] and the review [29], each agent updates its own subset of coordinates. Convergence is established under the *P-contraction* condition and its variants [10]. Papers [6, 7] show convergence for async-parallel iterations with simultaneous reading and writing to the same set of components. Unbounded but stochastic delays are considered in [67].

Recently, random coordinate selection appeared in [19] for fixed-point problems. The works [47, 59, 43, 42, 37] introduced async-parallel stochastic methods for function minimization. For fixed-point problems, [54] introduced async-parallel stochastic methods, as well as several applications.

1.2. Contributions of This Paper

The paper systematically discusses the CF properties found in both single and composite operators underlying many interesting applications. We introduce approaches to recognize CF operators and develop coordinate-update algorithms based on them. We provide a variety of applications to illustrate our approaches. In particular, we obtain new coordinate-update algorithms for image deblurring, portfolio optimization, second-order cone programming, as well as matrix decomposition. Our analysis also provides

guidance to the implementation of coordinate-update algorithms by specifying how to compute certain operators and maintain certain quantities in memory. We also provide numerical results to illustrate the efficiency of the proposed coordinate update algorithms.

This paper does *not* focus on the convergence perspective of coordinate update algorithms, though a convergence proof is provided in the appendix for a new primal-dual coordinate update algorithm. In general, in fixed-point algorithms, the iterate convergence is ensured by the monotonic decrease of the distance between the iterates and the solution set, while in minimization problems, the objective value convergence is ensured by the monotonic decrease of a certain energy function. The reader is referred to the existing literature for details.

The structural properties of operators discussed in this paper are irrelevant to the convergence-related properties such as nonexpansiveness (for an operator) or convexity (for a set or function). Hence, the algorithms developed can be still applied to nonconvex problems.

2. Coordinate Friendly Operators

2.1. Notation

For convenience, we do not distinguish *a coordinate* from *a block of coordinates* throughout this paper. We assume our variable x consists of m coordinates:

$$x = (x_1, \ldots, x_m) \in \mathbb{H} := \mathbb{H}_1 \times \cdots \times \mathbb{H}_m \quad \text{and} \quad x_i \in \mathbb{H}_i,\ i = 1, \ldots, m.$$

For simplicity, we assume that $\mathbb{H}_1, \ldots, \mathbb{H}_m$ are finite-dimensional real Hilbert spaces, though most results hold for general Hilbert spaces. A function maps from $\mathbb{H}$ to $\mathbb{R}$, the set of real numbers, and an operator maps from $\mathbb{H}$ to $\mathbb{G}$, where the definition of $\mathbb{G}$ depends on the context.

Our discussion often involves two points $x, x^+ \in \mathbb{H}$ that *differ over one coordinate*: there exists an index $i \in [m]$ and a point $\delta \in \mathbb{H}$ supported on $\mathbb{H}_i$, such that

$$x^+ = x + \delta. \tag{6}$$

Note that $x_j^+ = x_j$ for all $j \neq i$. Hence, $x^+ = (x_1, \ldots, x_i + \delta_i, \ldots, x_m)$.

Definition 1 (number of operations)**.** *We let* $\mathfrak{M}[a \mapsto b]$ *denote* the number of basic operations *that it takes to compute the quantity* b *from the input* a*.*

For example, $\mathfrak{M}\left[x \mapsto (\mathcal{T}x)_i\right]$ denotes the number of operations to compute the ith component of $\mathcal{T}x$ given x. We explore the possibility to compute $(\mathcal{T}x)_i$ with much fewer operations than what is needed to first compute $\mathcal{T}x$ and then take its ith component.

2.2. Single Coordinate Friendly Operators

This subsection studies a few classes of CF operators and then formally defines the CF operator. We motivate the first class through an example.

In the example below, we let $A_{i,:}$ and $A_{:,j}$ be the ith row and jth column of a matrix A, respectively. Let $A^\top$ be the transpose of A and $A_{i,:}^\top$ be $(A^\top)_{i,:}$, i.e., the ith row of the transpose of A.

Example 1 (least squares I)**.** *Consider the least squares problem*

$$\underset{x}{\text{minimize}}\, f(x) := \frac{1}{2}\|Ax-b\|^2, \tag{7}$$

where $A \in \mathbb{R}^{p\times m}$ and $b \in \mathbb{R}^p$. In this example, assume that $m = \Theta(p)$, namely, m and p are of the same order. We compare the full update of gradient descent to its coordinate update.[2] *The full update is referred to as the iteration $x^{k+1} = \mathcal{T}x^k$ where $\mathcal{T}$ is given by*

$$\mathcal{T}x := x - \eta\nabla f(x) = x - \eta A^\top Ax + \eta A^\top b. \tag{8}$$

Assuming that $A^\top A$ and $A^\top b$ are already computed, we have $\mathfrak{M}\left[x \mapsto \mathcal{T}x\right] = O(m^2)$. The coordinate update at the kth iteration performs

$$x_{i_k}^{k+1} = (\mathcal{T}x^k)_{i_k} = x_{i_k}^k - \eta\nabla_{i_k} f(x^k),$$

and $x_j^{k+1} = x_j^k, \forall j \neq i_k$, where i_k is some selected coordinate.

Since for all i, $\nabla_i f(x^k) = \left(A^\top(Ax-b)\right)_i = (A^\top A)_{i,:}\cdot x - (A^\top b)_i$, we have $\mathfrak{M}\left[x \mapsto (\mathcal{T}x)_i\right] = O(m)$ and thus $\mathfrak{M}\left[x \mapsto (\mathcal{T}x)_i\right] = O(\frac{1}{m}\mathfrak{M}\left[x \mapsto \mathcal{T}x\right])$. Therefore, the coordinate gradient descent is computationally worthy.

The operator $\mathcal{T}$ in the above example is a special *Type-I CF* operator.

Definition 2 (Type-I CF)**.** *For an operator $\mathcal{T}: \mathbb{H} \to \mathbb{H}$, let $\mathfrak{M}\left[x \mapsto (\mathcal{T}x)_i\right]$ be the number of operations for computing the ith coordinate of $\mathcal{T}x$ given x*

[2] Although gradient descent is seldom used to solve least squares, it often appears as a part in first-order algorithms for problems involving a least squares term.

and $\mathfrak{M}[x \mapsto \mathcal{T}x]$ *the number of operations for computing* $\mathcal{T}x$ *given* x*. We say* $\mathcal{T}$ *is* Type-I CF *(denoted as* $\mathcal{F}_1$*) if for any* $x \in \mathbb{H}$ *and* $i \in [m]$*, it holds*

$$\mathfrak{M}[x \mapsto (\mathcal{T}x)_i] = O\left(\frac{1}{m}\mathfrak{M}[x \mapsto \mathcal{T}x]\right).$$

Example 2 (least squares II)**.** *We can implement the coordinate update in Example 1 in a different manner by maintaining the result* $\mathcal{T}x^k$ *in the memory. This approach works when* $m = \Theta(p)$ *or* $p \gg m$*. The full update (8) is unchanged. At each coordinate update, from the maintained quantity* $\mathcal{T}x^k$*, we immediately obtain* $x_{i_k}^{k+1} = (\mathcal{T}x^k)_{i_k}$*. But we need to update* $\mathcal{T}x^k$ *to* $\mathcal{T}x^{k+1}$*. Since* x^{k+1} *and* x^k *differ only over the coordinate* i_k*, this update can be computed as*

$$\mathcal{T}x^{k+1} = \mathcal{T}x^k + x^{k+1} - x^k - \eta(x_{i_k}^{k+1} - x_{i_k}^k)(A^\top A)_{:,i_k},$$

which is a scalar-vector multiplication followed by vector addition, taking only $O(m)$ *operations. Computing* $\mathcal{T}x^{k+1}$ *from scratch involves a matrix-vector multiplication, taking* $O(\mathfrak{M}[x \mapsto \mathcal{T}(x)]) = O(m^2)$ *operations. Therefore,*

$$\mathfrak{M}\left[\{x^k, \mathcal{T}x^k, x^{k+1}\} \mapsto \mathcal{T}x^{k+1}\right] = O\left(\frac{1}{m}\mathfrak{M}\left[x^{k+1} \mapsto \mathcal{T}x^{k+1}\right]\right).$$

The operator $\mathcal{T}$ in the above example is a special *Type-II CF* operator.

Definition 3 (Type-II CF)**.** *An operator* $\mathcal{T}$ *is called* Type-II CF *(denoted as* $\mathcal{F}_2$*) if, for any* i, x *and* $x^+ := \big(x_1, \ldots, (\mathcal{T}x)_i, \ldots, x_m\big)$*, the following holds*

$$\mathfrak{M}\left[\{x, \mathcal{T}x, x^+\} \mapsto \mathcal{T}x^+\right] = O\left(\frac{1}{m}\mathfrak{M}\left[x^+ \mapsto \mathcal{T}x^+\right]\right). \tag{9}$$

The next example illustrates an efficient coordinate update by maintaining certain quantity other than $\mathcal{T}x$.

Example 3 (least squares III)**.** *For the case* $p \ll m$*, we should avoid pre-computing the relative large matrix* $A^\top A$*, and it is cheaper to compute* $A^\top(Ax)$ *than* $(A^\top A)x$*. Therefore, we change the implementations of both the full and coordinate updates in Example 1. In particular, the full update*

$$x^{k+1} = \mathcal{T}x^k = x^k - \eta\nabla f(x^k) = x^k - \eta A^\top(Ax^k - b),$$

pre-multiplies x^k *by* A *and then* $A^\top$*. Hence,* $\mathfrak{M}\left[x^k \mapsto \mathcal{T}(x^k)\right] = O(mp)$*.*

We change the coordinate update to maintain the intermediate quantity Ax^k*. In the first step, the coordinate update computes*

$$(\mathcal{T}x^k)_{i_k} = x^k_{i_k} - \eta(A^\top(Ax^k) - A^\top b)_{i_k},$$

by pre-multiplying Ax^k *by* $A^\top_{i_k,:}$*. Then, the second step updates* Ax^k *to* Ax^{k+1} *by adding* $(x^{k+1}_{i_k} - x^k_{i_k})A_{:,i_k}$ *to* Ax^k*. Both steps take* $O(p)$ *operations, so*

$$\mathfrak{M}\left[\{x^k, Ax^k\} \mapsto \{x^{k+1}, Ax^{k+1}\}\right] = O(p) = O\left(\frac{1}{m}\mathfrak{M}\left[x^k \mapsto \mathcal{T}x^k\right]\right).$$

Combining Type-I and Type-II CF operators with the last example, we arrive at the following CF definition.

Definition 4 (CF operator)**.** *We say that an operator* $\mathcal{T} : \mathbb{H} \to \mathbb{H}$ *is* CF *if, for any* i, x *and* $x^+ := \big(x_1, \ldots, (\mathcal{T}x)_i, \ldots, x_m\big)$*, the following holds*

$$\mathfrak{M}\left[\{x, \mathcal{M}(x)\} \mapsto \{x^+, \mathcal{M}(x^+)\}\right] = O\left(\frac{1}{m}\mathfrak{M}\left[x \mapsto \mathcal{T}x\right]\right), \tag{10}$$

where $\mathcal{M}(x)$ *is some quantity maintained in the memory to facilitate each coordinate update and refreshed to* $\mathcal{M}(x^+)$*.* $\mathcal{M}(x)$ *can be empty, i.e., except* x*, no other varying quantity is maintained.*

The left-hand side of (10) measures the cost of performing one coordinate update (including the cost of updating $\mathcal{M}(x)$ to $\mathcal{M}(x^+)$) while the right-hand side measures the average per-coordinate cost of updating all the coordinates together. When (10) holds, $\mathcal{T}$ is amenable to coordinate updates.

By definition, a Type-I CF operator $\mathcal{T}$ is CF without maintaining any quantity, i.e., $\mathcal{M}(x) = \emptyset$.

A Type-II CF operator $\mathcal{T}$ satisfies (10) with $\mathcal{M}(x) = \mathcal{T}x$, so it is also CF. Indeed, given any x and i, we can compute x^+ by immediately letting $x^+_i = (\mathcal{T}x)_i$ (at $O(1)$ cost) and keeping $x^+_j = x_j$, $\forall j \neq i$; then, by (9), we update $\mathcal{T}x$ to $\mathcal{T}x^+$ at a low cost. Formally, letting $\mathcal{M}(x) = \mathcal{T}x$,

$$\begin{aligned}
&\mathfrak{M}\left[\{x, \mathcal{M}(x)\} \mapsto \{x^+, \mathcal{M}(x^+)\}\right] \\
\leq &\mathfrak{M}\left[\{x, \mathcal{T}x\} \mapsto x^+\right] + \mathfrak{M}\left[\{x, \mathcal{T}x, x^+\} \mapsto \mathcal{T}x^+\right] \\
\overset{(9)}{=} &O(1) + O\left(\frac{1}{m}\mathfrak{M}\left[x^+ \mapsto \mathcal{T}x^+\right]\right) \\
= &O\left(\frac{1}{m}\mathfrak{M}\left[x \mapsto \mathcal{T}x\right]\right).
\end{aligned}$$

In general, the set of CF operators is much larger than the union of Type-I and Type-II CF operators.

Another important subclass of CF operators are operators $\mathcal{T} : \mathbb{H} \to \mathbb{H}$ where $(\mathcal{T}x)_i$ only depends on one, or a few, entries among $x_1, \ldots, x_m$. Based on how many input coordinates they depend on, we partition them into three subclasses.

Definition 5 (separable operator)**.** *Consider* $\mathfrak{T} := \{\mathcal{T} \mid \mathcal{T} : \mathbb{H} \to \mathbb{H}\}$. *We have the partition* $\mathfrak{T} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, *where*

- separable operator: $\mathcal{T} \in \mathcal{C}_1$ *if, for any index* i, *there exists* $\mathcal{T}_i : \mathbb{H}_i \to \mathbb{H}_i$ *such that* $(\mathcal{T}x)_i = \mathcal{T}_i x_i$, *that is,* $(\mathcal{T}x)_i$ *only depends on* x_i.
- nearly-separable operator: $\mathcal{T} \in \mathcal{C}_2$ *if, for any index* i, *there exists* $\mathcal{T}_i$ *and index set* $\mathbb{I}_i$ *such that* $(\mathcal{T}x)_i = \mathcal{T}_i(\{x_j\}_{j\in\mathbb{I}_i})$ *with* $|\mathbb{I}_i| \ll m$, *that is, each* $(\mathcal{T}x)_i$ *depends on a few coordinates of* x.
- non-separable operator: $\mathcal{C}_3 := \mathfrak{T} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$. *If* $\mathcal{T} \in \mathcal{C}_3$, *there exists some* i *such that* $(\mathcal{T}x)_i$ *depends on many coordinates of* x.

Throughout the paper, we assume the coordinate update of a (nearly-) separable operator costs roughly the same for all coordinates. Under this assumption, separable operators are both Type-I CF and Type-II CF, and nearly-separable operators are Type-I CF.[3]

2.3. Examples of CF Operators

In this subsection, we give examples of CF operators arising in different areas including linear algebra, optimization, and machine learning.

Example 4 ((block) diagonal matrix)**.** *Consider the diagonal matrix*

$$A = \begin{bmatrix} a_{1,1} & & 0 \\ & \ddots & \\ 0 & & a_{m,m} \end{bmatrix} \in \mathbb{R}^{m\times m}.$$

Clearly $\mathcal{T} : x \mapsto Ax$ *is separable.*

[3]Not all nearly-separable operators are Type-II CF. Indeed, consider a sparse matrix $A \in \mathbb{R}^{m\times m}$ whose non-zero entries are only located in the last column. Let $\mathcal{T}x = Ax$ and $x^+ = x+\delta_m$. As x^+ and x differ over the last entry, $\mathcal{T}x^+ = \mathcal{T}x+(x_m^+ - x_m)A_{:,m}$ takes m operations. Therefore, we have $\mathfrak{M}[\{x, \mathcal{T}x, x^+\} \mapsto \mathcal{T}x^+] = O(m)$. Since $\mathcal{T}x^+ = x_m^+ A_{:,m}$ takes m operations, we also have $\mathfrak{M}[x^+ \mapsto \mathcal{T}x^+] = O(m)$. Therefore, (9) is violated, and there is no benefit from maintaining $\mathcal{T}x$.

Example 5 (gradient and proximal maps of a separable function)**.** *Consider a* separable *function*

$$f(x) = \sum_{i=1}^{m} f_i(x_i).$$

Then, both ∇f *and* $\mathbf{prox}_{\gamma f}$ *are separable, in particular,*

$$(\nabla f(x))_i = \nabla f_i(x_i) \quad \textit{and} \quad (\mathbf{prox}_{\gamma f}(x))_i = \mathbf{prox}_{\gamma f_i}(x_i).$$

Here, $\mathbf{prox}_{\gamma f}(x)$ *(*$\gamma > 0$*) is the* proximal operator *that we define in Definition 10 in Appendix A.*

Example 6 (projection to box constraints)**.** *Consider the "box" set* $B := \{x : a_i \leq x_i \leq b_i,\ i \in [m]\} \subset \mathbb{R}^m$*. Then, the projection operator* $\mathbf{proj}_B$ *is separable. Indeed,*

$$\left(\mathbf{proj}_B(x)\right)_i = \max(b_i, \min(a_i, x_i)).$$

Example 7 (sparse matrices)**.** *If every row of the matrix* $A \in \mathbb{R}^{m \times m}$ *is sparse,* $\mathcal{T} : x \mapsto Ax$ *is nearly-separable.*

Examples of sparse matrices arise from various finite difference schemes for differential equations, problems defined on sparse graphs. When most pairs of a set of random variables are conditionally independent, their inverse covariance matrix is sparse.

Example 8 (sum of sparsely supported functions)**.** *Let* E *be a class of index sets and every* $e \in E$ *be a small subset of* $[m]$*,* $|e| \ll m$*. In addition* $\#\{e : i \in e\} \ll \#\{e\}$ *for all* $i \in [m]$*. Let* $x_e := (x_i)_{i \in e}$*, and*

$$f(x) = \sum_{e \in E} f_e(x_e).$$

The gradient map ∇f *is nearly-separable.*

An application of this example arises in wireless communication over a graph of m *nodes. Let each* x_i *be the spectrum assignment to node* i*, each* e *be a neighborhood of nodes, and each* f_e *be a utility function. The input of* f_e *is* x_e *since the utility depends on the spectra assignments in the neighborhood.*

In machine learning, if each observation only involves a few features, then each function of the optimization objective will depend on a small number of components of x*. This is the case when graphical models are used [64, 9].*

Example 9 (squared hinge loss function)**.** *Consider for $a, x \in \mathbb{R}^m$,*

$$f(x) := \frac{1}{2}\big(\max(0, 1 - \beta a^\top x)\big)^2,$$

which is known as the squared hinge loss function. Consider the operator

$$\mathcal{T}x := \nabla f(x) = -\beta \max(0, 1 - \beta a^\top x)a. \tag{11}$$

Let us maintain $\mathcal{M}(x) = a^\top x$. For arbitrary x and i, let

$$x_i^+ := (\mathcal{T}x)_i = -\beta \max(0, 1 - \beta a^\top x)a_i$$

and $x_j^+ := x_j, \forall j \neq i$. Then, computing x_i^+ from x and $a^\top x$ takes $O(1)$ (as $a^\top x$ is maintained), and computing $a^\top x^+$ from $x_i^+ - x_i$ and $a^\top x$ costs $O(1)$. Formally, we have

$$\begin{aligned}
&\mathfrak{M}\left[\{x, a^\top x\} \mapsto \{x^+, a^\top x^+\}\right] \\
\leq &\mathfrak{M}\left[\{x, a^\top x\} \mapsto x^+\right] + \mathfrak{M}\left[\{a^\top x, x_i^+ - x_i\} \mapsto a^\top x^+\right] \\
= &O(1) + O(1) = O(1).
\end{aligned}$$

On the other hand, $\mathfrak{M}[x \mapsto \mathcal{T}x] = O(m)$. Therefore, (10) holds, and $\mathcal{T}$ defined in (11) is CF.

3. Composite Coordinate Friendly Operators

Compositions of two or more operators arise in algorithms for problems that have composite functions, as well as algorithms that are derived from operator splitting methods. To update the variable x^k to x^{k+1}, two or more operators are sequentially applied, and therefore the structures of all operators determine whether the update is CF. This is where CF structures become less trivial but more interesting. This section studies composite CF operators. The exposition leads to the recovery of existing algorithms, as well as powerful new algorithms.

3.1. Combinations of Operators

We start by an example with numerous applications. It is a generalization of Example 9.

Example 10 (scalar map pre-composing affine function)**.** *Let $a_j \in \mathbb{R}^m, b_j \in \mathbb{R}$, and $\phi_j : \mathbb{R} \to \mathbb{R}$ be differentiable functions, $j \in [p]$. Let*

$$f(x) = \sum_{j=1}^{p} \phi_j(a_j^\top x + b_j).$$

Assume that evaluating ϕ_j' costs $O(1)$ for each j. Then, ∇f is CF. Indeed, let

$$\mathcal{T}_1 y := A^\top y, \quad \mathcal{T}_2 y := [\phi_1'(y_1); \ldots; \phi_p'(y_p)], \quad \mathcal{T}_3 x := Ax + b,$$

where $A = [a_1^\top; a_2^\top; \ldots; a_p^\top] \in \mathbb{R}^{p \times m}$ and $b = [b_1; b_2; \ldots; b_p] \in \mathbb{R}^{p \times 1}$. Then we have $\nabla f(x) = \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_3 x$. For any x and $i \in [m]$, let $x_i^+ = \nabla_i f(x)$ and $x_j^+ = x_j, \forall j \neq i$, and let $\mathcal{M}(x) := \mathcal{T}_3 x$. We can first compute $\mathcal{T}_2 \circ \mathcal{T}_3 x$ from $\mathcal{T}_3 x$ for $O(p)$ operations, then compute $\nabla_i f(x)$ and thus x^+ from $\{x, \mathcal{T}_2 \circ \mathcal{T}_3 x\}$ for $O(p)$ operations, and finally update the maintained $\mathcal{T}_3 x$ to $\mathcal{T}_3 x^+$ from $\{x, x^+, \mathcal{T}_3 x\}$ for another $O(p)$ operations. Formally,

$$\begin{aligned}
&\mathfrak{M}\left[\{x, \mathcal{T}_3 x\} \mapsto \{x^+, \mathcal{T}_3 x^+\}\right] \\
\leq &\mathfrak{M}\left[\mathcal{T}_3 x \mapsto \mathcal{T}_2 \circ \mathcal{T}_3 x\right] + \mathfrak{M}\left[\{x, \mathcal{T}_2 \circ \mathcal{T}_3 x\} \mapsto x^+\right] + \mathfrak{M}\left[\{x, \mathcal{T}_3 x, x^+\} \mapsto \{\mathcal{T}_3 x^+\}\right] \\
= &O(p) + O(p) + O(p) = O(p).
\end{aligned}$$

Since $\mathfrak{M}[x \mapsto \nabla f(x)] = O(pm)$, therefore $\nabla f = \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_3$ is CF.

If $p = m$, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ all map from $\mathbb{R}^m$ to $\mathbb{R}^m$. Then, it is easy to check that $\mathcal{T}_1$ is Type-I CF, $\mathcal{T}_2$ is separable, and $\mathcal{T}_3$ is Type-II CF. The last one is crucial since not maintaining $\mathcal{T}_3 x$ would disqualify $\mathcal{T}$ from CF. Indeed, to obtain $(\mathcal{T}x)_i$, we must multiply $A_i^\top$ to all the entries of $\mathcal{T}_2 \circ \mathcal{T}_3 x$, which in turn needs all the entries of $\mathcal{T}_3 x$, computing which from scratch would cost $O(pm)$.

There are general rules to preserve Type-I and Type-II CF. For example, $\mathcal{T}_1 \circ \mathcal{T}_2$ is still Type-I CF, and $\mathcal{T}_2 \circ \mathcal{T}_3$ is still CF, but there are counter examples where $\mathcal{T}_2 \circ \mathcal{T}_3$ can be neither Type-I nor Type-II CF. Such properties are important for developing efficient coordinate update algorithms for complicated problems; we will formalize them in the following.

The operators $\mathcal{T}_2$ and $\mathcal{T}_3$ in the above example are prototypes of *cheap* and *easy-to-maintain* operators from $\mathbb{H}$ to $\mathbb{G}$ that arise in operator compositions.

Definition 6 (cheap operator)**.** *For a composite operator $\mathcal{T} = \mathcal{T}_1 \circ \cdots \circ \mathcal{T}_p$, an operator $\mathcal{T}_i : \mathbb{H} \to \mathbb{G}$ is cheap if $\mathfrak{M}[x \mapsto \mathcal{T}_i x]$ is less than or equal to the number of remaining coordinate-update operations, in order of magnitude.*

Case	$\mathcal{T}_1 \in$	$\mathcal{T}_2 \in$	$(\mathcal{T}_1 \circ \mathcal{T}_2) \in$
1	$\mathcal{C}_1$ (separable)	$\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$	$\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, respectively
2	$\mathcal{C}_2$ (nearly-sep.)	$\mathcal{C}_1$, $\mathcal{C}_3$	$\mathcal{C}_2$, $\mathcal{C}_3$, resp.
3	$\mathcal{C}_2$	$\mathcal{C}_2$	$\mathcal{C}_2$ or $\mathcal{C}_3$, case by case
4	$\mathcal{C}_3$ (non-sep.)	$\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$	$\mathcal{C}_3$

Table 1: $\mathcal{T}_1 \circ \mathcal{T}_2$ inherits the weaker separability property from those of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Case	$\mathcal{T}_1 \in$	$\mathcal{T}_2 \in$	$(\mathcal{T}_1 \circ \mathcal{T}_2) \in$	Example
5	$\mathcal{C}_1 \cup \mathcal{C}_2$	$\mathcal{F}$, $\mathcal{F}_1$	$\mathcal{F}$, $\mathcal{F}_1$, resp.	Examples 11 and 13
6	$\mathcal{F}$, $\mathcal{F}_2$	$\mathcal{C}_1$	$\mathcal{F}$, $\mathcal{F}_2$, resp.	Example 10
7	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}$	Example 12
8	cheap	$\mathcal{F}_2$	$\mathcal{F}$	Example 13
9	$\mathcal{F}_1$	cheap	$\mathcal{F}_1$	Examples 10 and 13

Table 2: Summary of how $\mathcal{T}_1 \circ \mathcal{T}_2$ inherits CF properties from those of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Definition 7 (easy-to-maintain operator)**.** *For a composite operator* $\mathcal{T} = \mathcal{T}_1 \circ \cdots \circ \mathcal{T}_p$*, the operator* $\mathcal{T}_p : \mathbb{H} \to \mathbb{G}$ *is* easy-to-maintain*, if for any* x, i, x^+ *satisfying* (6)*,* $\mathfrak{M}[\{x, \mathcal{T}_p x, x^+\} \mapsto \mathcal{T}_p x^+]$ *is less than or equal to the number of remaining coordinate-update operations, in order of magnitude,* or *belongs to* $O(\frac{1}{\dim \mathbb{G}} \mathfrak{M}[x^+ \mapsto \mathcal{T} x^+])$.

The splitting schemes in §3.2 below will be based on $\mathcal{T}_1 + \mathcal{T}_2$ or $\mathcal{T}_1 \circ \mathcal{T}_2$, as well as a sequence of such combinations. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are both CF, $\mathcal{T}_1 + \mathcal{T}_2$ remains CF, but $\mathcal{T}_1 \circ \mathcal{T}_2$ is not necessarily so. This subsection discusses how $\mathcal{T}_1 \circ \mathcal{T}_2$ inherits the properties from $\mathcal{T}_1$ and $\mathcal{T}_2$. Our results are summarized in Tables 1 and 2 and explained in detail below.

The combination $\mathcal{T}_1 \circ \mathcal{T}_2$ generally inherits the *weaker* property from $\mathcal{T}_1$ and $\mathcal{T}_2$.

The separability ($\mathcal{C}_1$) property is preserved by composition. If $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are separable, then $\mathcal{T}_1 \circ \cdots \circ \mathcal{T}_n$ is separable. However, combining nearly-separable ($\mathcal{C}_2$) operators may not yield a nearly-separable operator since composition introduces more dependence among the input entries. Therefore, composition of nearly-separable operators can be either nearly-separable or non-separable.

Next, we discuss how $\mathcal{T}_1 \circ \mathcal{T}_2$ inherits the CF properties from $\mathcal{T}_1$ and $\mathcal{T}_2$. For simplicity, we only use matrix-vector multiplication as examples to illustrate the ideas; more interesting examples will be given later.

- If $\mathcal{T}_1$ is separable or nearly-separable ($\mathcal{C}_1 \cup \mathcal{C}_2$), then as long as $\mathcal{T}_2$ is CF ($\mathcal{F}$), $\mathcal{T}_1 \circ \mathcal{T}_2$ remains CF. In addition, if $\mathcal{T}_2$ is Type-I CF ($\mathcal{F}_1$), so is $\mathcal{T}_1 \circ \mathcal{T}_2$.

  Example 11. *Let $A \in \mathbb{R}^{m\times m}$ be* sparse *and $B \in \mathbb{R}^{m\times m}$* dense. *Then $\mathcal{T}_1 x = Ax$ is nearly-separable and $\mathcal{T}_2 x = Bx$ is Type-I CF*[4]*. For any i, let $\mathbb{I}_i$ index the set of nonzeros on the ith row of A. We first compute $(Bx)_{\mathbb{I}_i}$, which costs $O(|\mathbb{I}_i|m)$, and then $a_{i,\mathbb{I}_i}(Bx)_{\mathbb{I}_i}$, which costs $O(|\mathbb{I}_i|)$, where $a_{i,\mathbb{I}_i}$ is formed by the nonzero entries on the ith row of A. Assume $O(|\mathbb{I}_i|) = O(1)$, $\forall i$. We have, from the above discussion, that $\mathfrak{M}[x \mapsto (\mathcal{T}_1 \circ \mathcal{T}_2 x)_i] = O(m)$, while $\mathfrak{M}[x \mapsto \mathcal{T}_1 \circ \mathcal{T}_2 x] = O(m^2)$. Hence, $\mathcal{T}_1 \circ \mathcal{T}_2$ is Type-I CF.*

- Assume that $\mathcal{T}_2$ is separable ($\mathcal{C}_1$). It is easy to see that if $\mathcal{T}_1$ is CF ($\mathcal{F}$), then $\mathcal{T}_1 \circ \mathcal{T}_2$ remains CF. In addition if $\mathcal{T}_1$ is Type-II CF ($\mathcal{F}_2$), so is $\mathcal{T}_1 \circ \mathcal{T}_2$; see Example 10.
  Note that, if $\mathcal{T}_2$ is nearly-separable, we do not always have CF properties for $\mathcal{T}_1 \circ \mathcal{T}_2$. This is because $\mathcal{T}_2 x$ and $\mathcal{T}_2 x^+$ can be totally different (so updating $\mathcal{T}_2 x$ is expensive) even if x and x^+ only differ over one coordinate; see the footnote 3 on Page 13.
- Assume that $\mathcal{T}_1$ is Type-I CF ($\mathcal{F}_1$). If $\mathcal{T}_2$ is Type-II CF ($\mathcal{F}_2$), then $\mathcal{T}_1 \circ \mathcal{T}_2$ is CF ($\mathcal{F}$).

  Example 12. *Let $A, B \in \mathbb{R}^{m\times m}$ be dense. Then $\mathcal{T}_1 x = Ax$ is Type-I CF and $\mathcal{T}_2 x = Bx$ Type-II CF (by maintaining Bx; see Example 2). For any x and i, let x^+ satisfy (6). Maintaining $\mathcal{T}_2 x$, we can compute $(\mathcal{T}_1 \circ \mathcal{T}_2 x)_j$ for $O(m)$ operations for any j and update $\mathcal{T}_2 x^+$ for $O(m)$ operations. On the other hand, computing $\mathcal{T}_1 \circ \mathcal{T}_2 x^+$ without maintaining $\mathcal{T}_2 x$ takes $O(m^2)$ operations.*

- Assume that one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is cheap. If $\mathcal{T}_2$ is cheap, then as long as $\mathcal{T}_1$ is Type-I CF ($\mathcal{F}_1$), $\mathcal{T}_1 \circ \mathcal{T}_2$ is Type-I CF. If $\mathcal{T}_1$ is cheap, then as long as $\mathcal{T}_2$ is Type-II CF ($\mathcal{F}_2$), $\mathcal{T}_1 \circ \mathcal{T}_2$ is CF ($\mathcal{F}$); see Example 13.

We will see more examples of the above cases in the rest of the paper.

3.2. Operator Splitting Schemes

We will apply our discussions above to operator splitting and obtain new algorithms. But first, we review several major operator splitting schemes and

[4]For this example, one can of course pre-compute AB and claim that $(\mathcal{T}_1 \circ \mathcal{T}_2)$ is Type-I CF. Our arguments keep A and B separate and only use the nearly-separability of $\mathcal{T}_1$ and Type-I CF property of $\mathcal{T}_2$, so our result holds for any such composition even when $\mathcal{T}_1$ and $\mathcal{T}_2$ are nonlinear.

discuss their CF properties. We will encounter important concepts such as *(maximum) monotonicity* and *cocoercivity*, which are given in Appendix A. For a monotone operator $\mathcal{A}$, the *resolvent operator* $\mathcal{J}_{\mathcal{A}}$ and the *reflective-resolvent operator* $\mathcal{R}_{\mathcal{A}}$ are also defined there, in (67) and (68), respectively.

Consider the following problem: given three operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$, possibly set-valued,

$$\text{find } x \in \mathbb{H} \qquad \text{such that} \qquad 0 \in \mathcal{A}x + \mathcal{B}x + \mathcal{C}x, \tag{12}$$

where "+" is the Minkowski sum. This is a high-level abstraction of many problems or their optimality conditions. The study began in the 1960s, followed by a large number of algorithms and applications over the last fifty years. Next, we review a few basic methods for solving (12).

When $\mathcal{A}, \mathcal{B}$ are maximally monotone (think it as the subdifferential ∂f of a proper convex function f) and $\mathcal{C}$ is β-cocoercive (think it as the gradient ∇f of a $1/\beta$-Lipschitz differentiable function f), a solution can be found by the iteration (2) with $\mathcal{T} = \mathcal{T}_{3\mathrm{S}}$, introduced recently in [24], where

$$\mathcal{T}_{3\mathrm{S}} := \mathcal{I} - \mathcal{J}_{\gamma\mathcal{B}} + \mathcal{J}_{\gamma\mathcal{A}} \circ (2\mathcal{J}_{\gamma\mathcal{B}} - \mathcal{I} - \gamma\mathcal{C} \circ \mathcal{J}_{\gamma\mathcal{B}}). \tag{13}$$

Indeed, by setting $\gamma \in (0, 2\beta)$, $\mathcal{T}_{3S}$ is $(\frac{2\beta}{4\beta-\gamma})$-averaged (think it as a property weaker than the Picard contraction; in particular, $\mathcal{T}$ may not have a fixed point). Following the standard convergence result (cf. textbook [5]), provided that $\mathcal{T}$ has a fixed point, the sequence from (2) converges to a fixed-point x^* of $\mathcal{T}$. Note that, instead of x^*, $\mathcal{J}_{\gamma\mathcal{B}}(x^*)$ is a solution to (12).

Following §3.1, $\mathcal{T}_{3\mathrm{S}}$ is CF if $\mathcal{J}_{\gamma\mathcal{A}}$ is separable ($\mathcal{C}_1$), $\mathcal{J}_{\gamma\mathcal{B}}$ is Type-II CF ($\mathcal{F}_2$), and $\mathcal{C}$ is Type-I CF ($\mathcal{F}_1$).

We give a few special cases of $\mathcal{T}_{3\mathrm{S}}$ below, which have much longer history. They all converge to a fixed point x^* whenever a solution exists and γ is properly chosen. If $\mathcal{B} \neq 0$, then $\mathcal{J}_{\gamma\mathcal{B}}(x^*)$, instead of x^*, is a solution to (12).

Forward-Backward Splitting (FBS): Letting $\mathcal{B} = 0$ yields $\mathcal{J}_{\gamma\mathcal{B}} = \mathcal{I}$. Then, $\mathcal{T}_{3\mathrm{S}}$ reduces to FBS [52]:

$$\mathcal{T}_{\mathrm{FBS}} := \mathcal{J}_{\gamma\mathcal{A}} \circ (\mathcal{I} - \gamma\mathcal{C}) \tag{14}$$

for solving the problem $0 \in \mathcal{A}x + \mathcal{C}x$.

Backward-Forward Splitting (BFS): Letting $\mathcal{A} = 0$ yields $\mathcal{J}_{\gamma\mathcal{A}} = \mathcal{I}$. Then, $\mathcal{T}_{3\mathrm{S}}$ reduces to BFS:

$$\mathcal{T}_{\mathrm{BFS}} := (\mathcal{I} - \gamma\mathcal{C}) \circ \mathcal{J}_{\gamma\mathcal{B}} \tag{15}$$

for solving the problem $0 \in \mathcal{B}x + \mathcal{C}x$. When $\mathcal{A} = \mathcal{B}$, $\mathcal{T}_{\text{FBS}}$ and $\mathcal{T}_{\text{BFS}}$ apply the same pair of operators in the opposite orders, and they solve the same problem. Iterations based on $\mathcal{T}_{\text{BFS}}$ are rarely used in the literature because they need an extra application of $\mathcal{J}_{\gamma B}$ to return the solution, so $\mathcal{T}_{\text{BFS}}$ is seemingly an unnecessary variant of $\mathcal{T}_{\text{FBS}}$. However, they become different for coordinate update; in particular, $\mathcal{T}_{\text{BFS}}$ is CF (but $\mathcal{T}_{\text{FBS}}$ is generally not) when $\mathcal{J}_{\gamma\mathcal{B}}$ is Type-II CF ($\mathcal{F}_2$) and $\mathcal{C}$ is Type-I CF ($\mathcal{F}_1$). Therefore, $\mathcal{T}_{\text{BFS}}$ is worth discussing alone.

Douglas-Rachford Splitting (DRS): Letting $\mathcal{C} = 0$, $\mathcal{T}_{\text{3S}}$ reduces to

$$\mathcal{T}_{\text{DRS}} := \mathcal{I} - \mathcal{J}_{\gamma\mathcal{B}} + \mathcal{J}_{\gamma\mathcal{A}} \circ (2\mathcal{J}_{\gamma\mathcal{B}} - \mathcal{I}) = \frac{1}{2}(\mathcal{I} + \mathcal{R}_{\gamma\mathcal{A}} \circ \mathcal{R}_{\gamma\mathcal{B}}) \tag{16}$$

introduced in [26] for solving the problem $0 \in \mathcal{A}x + \mathcal{B}x$. A more general splitting is the Relaxed Peaceman-Rachford Splitting (RPRS) with $\lambda \in [0, 1]$:

$$\mathcal{T}_{\text{RPRS}} = (1 - \lambda)\,\mathcal{I} + \lambda\,\mathcal{R}_{\gamma\mathcal{A}} \circ \mathcal{R}_{\gamma\mathcal{B}}, \tag{17}$$

which recovers $\mathcal{T}_{\text{DRS}}$ by setting $\lambda = \frac{1}{2}$ and Peaceman-Rachford Splitting (PRS) [53] by letting $\lambda = 1$.

Forward-Douglas-Rachford Splitting (FDRS): Let V be a linear subspace, and $\mathcal{N}_V$ and $\mathcal{P}_V$ be its normal cone and projection operator, respectively. The FDRS [15]

$$\mathcal{T}_{\text{FDRS}} = \mathcal{I} - \mathcal{P}_V + \mathcal{J}_{\gamma\mathcal{A}} \circ (2\mathcal{P}_V - \mathcal{I} - \gamma\mathcal{P}_V \circ \tilde{\mathcal{C}} \circ \mathcal{P}_V),$$

aims at finding a point x such that $0 \in \mathcal{A}\,x + \tilde{\mathcal{C}}\,x + \mathcal{N}_V\,x$. If an optimal x exists, we have $x \in V$ and $\mathcal{N}_V x$ is the orthogonal complement of V. Therefore, the problem is equivalent to finding x such that $0 \in \mathcal{A}\,x + \mathcal{P}_V \circ \tilde{\mathcal{C}} \circ \mathcal{P}_V\,x + \mathcal{N}_V\,x$. Thus, $\mathcal{T}_{\text{3S}}$ recovers $\mathcal{T}_{\text{FDRS}}$ by letting $\mathcal{B} = \mathcal{N}_V$ and $\mathcal{C} = \mathcal{P}_V \circ \tilde{\mathcal{C}} \circ \mathcal{P}_V$.

Forward-Backward-Forward Splitting (FBFS): Composing $\mathcal{T}_{\text{FBS}}$ with one more forward step gives $\mathcal{T}_{\text{FBFS}}$ introduced in [71]:

$$\mathcal{T}_{\text{FBFS}} = -\gamma\mathcal{C} + (\mathcal{I} - \gamma\mathcal{C})\mathcal{J}_{\gamma\mathcal{A}}(\mathcal{I} - \gamma\mathcal{C}). \tag{18}$$

$\mathcal{T}_{\text{FBFS}}$ is not a special case of $\mathcal{T}_{\text{3S}}$. At the expense of one more application of $(\mathcal{I} - \gamma\mathcal{C})$, $\mathcal{T}_{\text{FBFS}}$ relaxes the convergence condition of $\mathcal{T}_{\text{FBS}}$ from the cocoercivity of $\mathcal{C}$ to its monotonicity. (For example, a nonzero skew symmetric matrix is monotonic but not cocoercive.) From Table 2, we know that $\mathcal{T}_{\text{FBFS}}$ is CF if both $\mathcal{C}$ and $\mathcal{J}_{\gamma\mathcal{A}}$ are separable.

3.2.1. Examples in Optimization. Consider the optimization problem

$$\underset{x\in X}{\text{minimize}}\ f(x)+g(x), \tag{19}$$

where X is the feasible set and f and g are objective functions. We present examples of operator splitting methods discussed above.

Example 13 (proximal gradient method)**.** *Let $X=\mathbb{R}^m$, f be differentiable, and g be proximable in (19). Setting $\mathcal{A}=\partial g$ and $\mathcal{C}=\nabla f$ in (14) gives $\mathcal{J}_{\gamma\mathcal{A}}=\mathbf{prox}_{\gamma g}$ and reduces $x^{k+1}=\mathcal{T}_{\rm FBS}(x^k)$ to prox-gradient iteration:*

$$x^{k+1}=\mathbf{prox}_{\gamma g}(x^k-\gamma\nabla f(x^k)). \tag{20}$$

A special case of (20) with $g=\iota_X$ is the projected gradient iteration:

$$x^{k+1}=\mathcal{P}_X(x^k-\gamma\nabla f(x^k)). \tag{21}$$

If ∇f is CF and $\mathbf{prox}_{\gamma g}$ is (nearly-)separable (e.g., $g(x)=\|x\|_1$ or the indicator function of a box constraint) or if ∇f is Type-II CF and $\mathbf{prox}_{\gamma g}$ is cheap (e.g., $\nabla f(x)=Ax-b$ and $g=\|x\|_2$), then the FBS iteration (20) is CF. In the latter case, we can also apply the BFS iteration (15) (i.e, compute $\mathbf{prox}_{\gamma g}$ and then perform the gradient update), which is also CF.

Example 14 (ADMM)**.** *Setting $X=\mathbb{R}^m$ simplifies (19) to*

$$\underset{x,y}{\text{minimize}}\ f(x)+g(y),\quad \text{subject to } x-y=0. \tag{22}$$

The ADMM method iterates:

$$x^{k+1}=\mathbf{prox}_{\gamma f}(y^k-\gamma s^k), \tag{23a}$$

$$y^{k+1}=\mathbf{prox}_{\gamma g}(x^{k+1}+\gamma s^k), \tag{23b}$$

$$s^{k+1}=s^k+\frac{1}{\gamma}(x^{k+1}-y^{k+1}). \tag{23c}$$

(The iteration can be generalized to handle the constraint $Ax-By=b$.) The dual problem of (22) is $\min_s f^*(-s)+g^*(s)$, where f^* is the convex conjugate of f. Letting $\mathcal{A}=-\partial f^*(-\cdot)$ and $\mathcal{B}=\partial g^*$ in (16) recovers the iteration (23) through (see the derivation in Appendix B)

$$t^{k+1}=\mathcal{T}_{\rm DRS}(t^k)=t^k-\mathcal{J}_{\gamma\mathcal{B}}(t^k)+\mathcal{J}_{\gamma\mathcal{A}}\circ(2\mathcal{J}_{\gamma\mathcal{B}}-\mathcal{I})(t^k).$$

From the results in §3.1, a sufficient condition for the above iteration to be CF is that $\mathcal{J}_{\gamma\mathcal{A}}$ is (nearly-)separable and $\mathcal{J}_{\gamma\mathcal{B}}$ being CF.

The above abstract operators and their CF properties will be applied in §5 to give interesting algorithms for several applications.

4. Primal-dual Coordinate Friendly Operators

We study how to solve the problem

$$\underset{x\in\mathbb{H}}{\text{minimize}}\ f(x)+g(x)+h(Ax), \tag{24}$$

with primal-dual splitting algorithms, as well as their coordinate update versions. Here, f is differentiable and A is a "p-by-m" linear operator from $\mathbb{H}=\mathbb{H}_1\times\cdots\times\mathbb{H}_m$ to $\mathbb{G}=\mathbb{G}_1\times\cdots\times\mathbb{G}_p$. Problem (24) abstracts many applications in image processing and machine learning.

Example 15 (image deblurring/denoising)**.** *Let u^0 be an image, where $u_i^0\in[0,255]$, and B be the blurring linear operator. Let $\|\nabla u\|_1$ be the anisotropic*[5] *total variation of u (see (49) for definition). Suppose that b is a noisy observation of Bu^0. Then, we can try to recover u^0 by solving*

$$\underset{u}{\text{minimize}}\ \frac{1}{2}\|Bu-b\|^2+\iota_{[0,255]}(u)+\lambda\|\nabla u\|_1, \tag{25}$$

which can be written in the form of (24) with $f=\frac{1}{2}\|B\cdot-b\|^2$, $g=\iota_{[0,255]}$, $A=\nabla$, and $h=\lambda\|\cdot\|_1$.

More examples with the formulation (24) will be given in §4.2. In general, primal-dual methods are capable of solving complicated problems involving constraints and the compositions of proximable and linear maps like $\|\nabla u\|_1$.

In many applications, although h is proximable, $h\circ A$ is generally non-proximable and non-differentiable. To avoid using slow subgradient methods, we can consider the primal-dual splitting approaches to separate h and A so that $\mathbf{prox}_h$ can be applied. We derive that the equivalent form (for convex cases) of (24) is to find x such that

$$0\in(\nabla f+\partial g+A^\top\circ\partial h\circ A)(x). \tag{26}$$

Introducing the dual variable $s\in\mathbb{G}$ and applying the biconjugation prop-

[5]Generalization to the isotropic case is straightforward by grouping variables properly.

erty: $s \in \partial h(Ax) \Leftrightarrow Ax \in \partial h^*(s)$, yields the equivalent condition

$$0 \in \Bigg(\underbrace{\begin{bmatrix} \nabla f & 0 \\ 0 & 0 \end{bmatrix}}_{\text{operator } \mathcal{A}} + \underbrace{\begin{bmatrix} \partial g & 0 \\ 0 & \partial h^* \end{bmatrix} + \begin{bmatrix} 0 & A^\top \\ -A & 0 \end{bmatrix}}_{\text{operator } \mathcal{B}} \Bigg) \underbrace{\begin{bmatrix} x \\ s \end{bmatrix}}_{z}, \tag{27}$$

which we shorten as $0 \in \mathcal{A}z + \mathcal{B}z$, with $z \in \mathbb{H} \times \mathbb{G} =: \mathbb{F}$.

Problem (27) can be solved by the Condat-Vũ algorithm [20, 76]:

$$\begin{cases} s^{k+1} = \mathbf{prox}_{\gamma h^*}(s^k + \gamma A x^k), \\ x^{k+1} = \mathbf{prox}_{\eta g}(x^k - \eta(\nabla f(x^k) + A^\top(2s^{k+1} - s^k))), \end{cases} \tag{28}$$

which explicitly applies A and $A^\top$ and updates s, x in a Gauss-Seidel style [6]. We introduce an operator $\mathcal{T}_{\text{CV}} : \mathbb{F} \to \mathbb{F}$ and write

$$\text{iteration (28)} \quad \Longleftrightarrow \quad z^{k+1} = \mathcal{T}_{\text{CV}}(z^k).$$

Switching the orders of x and s yields the following algorithm:

$$\begin{cases} x^{k+1} = \mathbf{prox}_{\eta g}(x^k - \eta(\nabla f(x^k) + A^\top s^k)), \\ s^{k+1} = \mathbf{prox}_{\gamma h^*}(s^k + \gamma A(2x^{k+1} - x^k)), \end{cases} \quad \text{as } z^{k+1} = \mathcal{T}'_{\text{CV}} z^k. \tag{29}$$

It is known from [18, 23] that both (28) and (29) reduce to iterations of nonexpansive operators (under a special metric), i.e., $\mathcal{T}_{\text{CV}}$ is nonexpansive; see Appendix C for the reasoning.

Remark 1. *Similar primal-dual algorithms can be used to solve other problems such as saddle point problems [40, 46, 16] and variational inequalities [68]. Our coordinate update algorithms below apply to these problems as well.*

4.1. Primal-dual Coordinate Update Algorithms

In this subsection, we make the following assumption.

Assumption 1. *Functions g and h^* in the problem (24) are separable and proximable. Specifically,*

$$g(x) = \sum_{i=1}^{m} g_i(x_i) \quad \text{and} \quad h^*(y) = \sum_{j=1}^{p} h_i^*(y_i).$$

[6] By the Moreau identity: $\mathbf{prox}_{\gamma h^*} = \mathcal{I} - \gamma \mathbf{prox}_{\frac{1}{\gamma} h}(\frac{\cdot}{\gamma})$, one can compute $\mathbf{prox}_{\frac{1}{\gamma} h}$ instead of $\mathbf{prox}_{\gamma h^*}$, which inherits the same separability properties from $\mathbf{prox}_{\frac{1}{\gamma} h}$.

Furthermore, ∇f is CF.

Proposition 1. *Under Assumption 1, the followings hold:*

(a) when $p = O(m)$, the Condat-Vu operator $\mathcal{T}_{\mathrm{CV}}$ in (28) is CF, more specifically,

$$\mathfrak{M}\left[\{z^k, Ax\} \mapsto \{z^+, Ax^+\}\right] = O\left(\frac{1}{m+p}\mathfrak{M}\left[z^k \mapsto \mathcal{T}_{\mathrm{CV}} z^k\right]\right);$$

(b) when $m \ll p$ and $\mathfrak{M}\left[x \mapsto \nabla f(x)\right] = O(m)$, the Condat-Vu operator $\mathcal{T}'_{\mathrm{CV}}$ in (29) is CF, more specifically,

$$\mathfrak{M}\left[\{z^k, A^\top s\} \mapsto \{z^+, A^\top s^+\}\right] = O\left(\frac{1}{m+p}\mathfrak{M}\left[z^k \mapsto \mathcal{T}'_{\mathrm{CV}} z^k\right]\right).$$

Proof. Computing $z^{k+1} = \mathcal{T}_{\mathrm{CV}} z^k$ involves evaluating ∇f, $\mathbf{prox}_g$, and $\mathbf{prox}_{h^*}$, applying A and $A^\top$, and adding vectors. It is easy to see $\mathfrak{M}\left[z^k \mapsto \mathcal{T}_{\mathrm{CV}} z^k\right] = O(mp + m + p) + \mathfrak{M}[x \to \nabla f(x)]$, and $\mathfrak{M}\left[z^k \mapsto \mathcal{T}'_{\mathrm{CV}} z^k\right]$ is the same.
(a) We assume $\nabla f \in \mathcal{F}_1$ for simplicity, and other cases are similar.

1. If $(\mathcal{T}_{\mathrm{CV}} z^k)_j = s_i^{k+1}$, computing it involves: adding s_i^k and $\gamma(Ax^k)_i$, and evaluating $\mathbf{prox}_{\gamma h_i^*}$. In this case $\mathfrak{M}\left[\{z^k, Ax\} \mapsto \{z^+, Ax^+\}\right] = O(1)$.
2. If $(\mathcal{T}_{\mathrm{CV}} z^k)_j = x_i^{k+1}$, computing it involves evaluating: the entire s^{k+1} for $O(p)$ operations, $(A^\top(2s^{k+1} - s^k))_i$ for $O(p)$ operations, $\mathbf{prox}_{\eta g_i}$ for $O(1)$ operations, $\nabla_i f(x^k)$ for $O(\frac{1}{m}\mathfrak{M}\left[x \mapsto \nabla f(x)\right])$ operations, as well as updating Ax^+ for $O(p)$ operations. In this case $\mathfrak{M}\left[\{z^k, Ax\} \mapsto \{z^+, Ax^+\}\right] = O(p + \frac{1}{m}\mathfrak{M}\left[x \mapsto \nabla f(x)\right])$.

Therefore, $\mathfrak{M}\left[\{z^k, Ax\} \mapsto \{z^+, Ax^+\}\right] = O\big(\frac{1}{m+p}\mathfrak{M}\left[z^k \mapsto \mathcal{T}_{\mathrm{CV}} z^k\right]\big)$.
(b) When $m \ll p$ and $\mathfrak{M}\left[x \mapsto \nabla f(x)\right] = O(m)$, following arguments similar to the above, we have
$\mathfrak{M}\left[\{z^k, A^\top s\} \mapsto \{z^+, A^\top s^+\}\right] = O(1) + \mathfrak{M}\left[x \mapsto \nabla_i f(x)\right]$ if $(\mathcal{T}'_{\mathrm{CV}} z^k)_j = x_i^{k+1}$;
and $\mathfrak{M}\left[\{z^k, A^\top s\} \mapsto \{z^+, A^\top s^+\}\right] = O(m) + \mathfrak{M}\left[x \mapsto \nabla f(x)\right]$ if $(\mathcal{T}'_{\mathrm{CV}} z^k)_j = s_i^{k+1}$.
In both cases $\mathfrak{M}\left[\{z^k, A^\top s\} \mapsto \{z^+, A^\top s^+\}\right] = O(\frac{1}{m+p}\mathfrak{M}\left[z^k \mapsto \mathcal{T}'_{\mathrm{CV}} z^k\right])$. □

4.2. Extended Monotropic Programming

We develop a primal-dual coordinate update algorithm for the extended monotropic program:

$$(30)\qquad \begin{array}{ll} \underset{x\in\mathbb{H}}{\text{minimize}} & g_1(x_1)+g_2(x_2)+\cdots+g_m(x_m)+f(x),\\ \text{subject to} & A_1x_1+A_2x_2+\cdots+A_mx_m=b,\end{array}$$

where $x=(x_1,\ldots,x_m)\in\mathbb{H}=\mathbb{H}_1\times\ldots\times\mathbb{H}_m$ with $\mathbb{H}_i$ being Euclidean spaces. It generalizes linear, quadratic, second-order cone, semi-definite programs by allowing extended-valued objective functions g_i and f. It is a special case of (24) by letting $g(x)=\sum_{i=1}^m g_i(x_i)$, $A=[A_1,\cdots,A_m]$ and $h=\iota_{\{b\}}$.

Example 16 (quadratic programming)**.** *Consider the quadratic program*

$$(31)\qquad \underset{x\in\mathbb{R}^m}{\text{minimize}}\ \frac{1}{2}x^\top Ux+c^\top x,\ \text{subject to}\ Ax=b,\ x\in X,$$

where U is a symmetric positive semidefinite matrix and $X=\{x: x_i\geq 0\ \forall i\}$. Then, (31) is a special case of (30) with $g_i(x_i)=\iota_{\geq 0}(x_i)$, $f(x)=\frac{1}{2}x^\top Ux+c^\top x$ and $h=\iota_{\{b\}}$.

Example 17 (Second Order Cone Programming (SOCP))**.** *The SOCP*

$$\begin{array}{ll}\underset{x\in\mathbb{R}^m}{\text{minimize}} & c^\top x,\quad \text{subject to } Ax=b,\\ & x\in X=Q_1\times\cdots\times Q_n,\end{array}$$

(where the number of cones n may not be equal to the number of blocks m,) can be written in the form of (30): $\text{minimize}_{x\in\mathbb{R}^m}\ \iota_X(x)+c^\top x+\iota_{\{b\}}(Ax)$.

Applying iteration (28) to problem (30) and eliminating s^{k+1} from the second row yield the Jacobi-style update (denoted as $\mathcal{T}_{\text{emp}}$):

$$(32)\qquad \left\{\begin{array}{l} s^{k+1}=s^k+\gamma(Ax^k-b),\\ x^{k+1}=\mathbf{prox}_{\eta g}(x^k-\eta(\nabla f(x^k)+A^\top s^k+2\gamma A^\top Ax^k-2\gamma A^\top b)).\end{array}\right.$$

To the best of our knowledge, this update is never found in the literature. Note that x^{k+1} no longer depends on s^{k+1}, making it more convenient to perform coordinate updates.

Remark 2. *In general, when the s update is affine, we can decouple s^{k+1} and x^{k+1} by plugging the s update into the x update. It is the case when h is affine or quadratic in problem (24).*

A sufficient condition for $\mathcal{T}_{\text{emp}}$ to be CF is $\mathbf{prox}_g \in \mathcal{C}_1$ i.e., separable. Indeed, we have $\mathcal{T}_{\text{emp}} = \mathcal{T}_1 \circ \mathcal{T}_2$, where

$$\mathcal{T}_1 = \begin{bmatrix} \mathcal{I} & 0 \\ 0 & \mathbf{prox}_{\eta g} \end{bmatrix}, \mathcal{T}_2 \begin{bmatrix} s \\ x \end{bmatrix} = \begin{bmatrix} s + \gamma(Ax - b) \\ x - \eta(\nabla f(x) + A^\top s + 2\gamma A^\top Ax - 2\gamma A^\top b) \end{bmatrix}.$$

Following Case 5 of Table 2, $\mathcal{T}_{\text{emp}}$ is CF. When $m = \Theta(p)$, the separability condition on $\mathbf{prox}_g$ can be relaxed to $\mathbf{prox}_g \in \mathcal{F}_1$ since in this case $\mathcal{T}_2 \in \mathcal{F}_2$, and we can apply Case 7 of Table 2 (by maintaining $\nabla f(x)$, $A^\top s$, Ax and $A^\top Ax$.)

4.3. Overlapping-Block Coordinate Updates

In the coordinate update scheme based on (28), if we select x_i to update then we must first compute s^{k+1}, because the variables x_i's and s_j's are coupled through the matrix A. However, once x_i^{k+1} is obtained, s^{k+1} is discarded. It is not used to update s or cached for further use. This subsection introduces ways to utilize the otherwise wasted computation.

We define, for each i, $\mathbb{J}(i) \subset [p]$ as the set of indices j such that $A_{i,j}^\top \neq 0$, and, for each j, $\mathbb{I}(j) \subset [m]$ as the set of indices of i such that $A_{i,j}^\top \neq 0$. We also let $m_j := |\mathbb{I}(j)|$, and assume $m_j \neq 0, \forall j \in [p]$ without loss of generality.

We arrange the coordinates of $z = [x; s]$ into m overlapping blocks. The ith block consists of the coordinate x_i and all s_j's for $j \in \mathbb{J}(i)$. This way, each s_j may appear in more than one block. We propose a block coordinate update scheme based on (28). Because the blocks overlap, each s_j may be updated in multiple blocks, so the s_j update is relaxed with parameters $\rho_{i,j} \geq 0$ (see (33) below) that satisfy $\sum_{i \in \mathbb{I}(j)} \rho_{i,j} = 1, \quad \forall j \in [p]$. The aggregated effect is to update s_j without scaling. (Following the KM iteration [39], we can also assign a relaxation parameter η_k for the x_i update; then, the s_j update should be relaxed with $\rho_{i,j}\eta_k$.)

We propose the following update scheme:

$$(33) \quad \begin{cases} \text{select } i \in [m], \text{ and then compute} \\ \quad \tilde{s}_j^{k+1} = \mathbf{prox}_{\gamma h_j^*}(s_j^k + \gamma(Ax^k)_j), \text{ for all } j \in \mathbb{J}(i), \\ \quad \tilde{x}_i^{k+1} = \mathbf{prox}_{\eta g_i}(x_i^k - \eta(\nabla_i f(x^k) + \sum_{j \in \mathbb{J}(i)} A_{i,j}^\top (2\tilde{s}_j^{k+1} - s_j^k))), \\ \quad \text{update } x_i^{k+1} = x_i^k + (\tilde{x}_i^{k+1} - x_i^k), \\ \quad \text{update } s_j^{k+1} = s_j^k + \rho_{i,j}(\tilde{s}_j^{k+1} - s_j^k), \text{ for all } j \in \mathbb{J}(i). \end{cases}$$

Remark 3. *The use of relaxation parameters $\rho_{i,j}$ makes our scheme different from that in [56].*

Following the assumptions and arguments in §4.1, if we maintain Ax, the cost for each block coordinate update is $O(p) + \mathfrak{M}[x \mapsto \nabla_i f(x)]$, which is $O(\frac{1}{m}\mathfrak{M}[z \mapsto \mathcal{T}_{\mathrm{CV}} z])$. Therefore the coordinate update scheme (33) is computationally worthy.

Typical choices of $\rho_{i,j}$ include: (1) one of the $\rho_{i,j}$'s is 1 for each j, others all equal to 0. This can be viewed as assigning the update of s_j solely to a block containing x_i. (2) $\rho_{i,j} = \frac{1}{m_j}$ for all $i \in \mathbb{I}(j)$. This approach spreads the update of s_j over all the related blocks.

Remark 4. *The recent paper [28] proposes a different primal-dual coordinate update algorithm. The authors produce a new matrix $\bar{A}$ based on A, with only one nonzero entry in each row, i.e. $m_j = 1$ for each j. They also modify h to $\bar{h}$ so that the problem*

$$\underset{x \in \mathbb{H}}{\text{minimize}} \; f(x) + g(x) + \bar{h}(\bar{A}x) \tag{34}$$

has the same solution as (24). Then they solve (34) by the scheme (33). Because they have $m_j = 1$, every dual variable coordinate is only associated with one primal variable coordinate. They create non-overlapping blocks of z by duplicating each dual variable coordinate s_j multiple times. The computation cost for each block coordinate update of their algorithm is the same as (33), but more memory is needed for the duplicated copies of each s_j.

4.4. Async-Parallel Primal-Dual Coordinate Update Algorithms and Their Convergence

In this subsection, we propose two async-parallel primal-dual coordinate update algorithms using the algorithmic framework of [54] and state their convergence results. When there is only one agent, all algorithms proposed in this section reduce to stochastic coordinate update algorithms [19], and their convergence is a direct consequence of Theorem 1. Moreover, our convergence analysis also applies to sync-parallel algorithms.

The two algorithms are based on §4.1 and §4.3, respectively.

Algorithm 1: Async-parallel primal-dual coordinate update algorithm using $\mathcal{T}_{\rm CV}$

Input : $z^0 \in \mathbb{F}$, $K > 0$, a discrete distribution $(q_1, \ldots, q_{m+p})$ with $\sum_{i=1}^{m+p} q_i = 1$ and $q_i > 0, \forall i$,
set global iteration counter $k = 0$;
while $k < K$, *every agent asynchronously and continuously* **do**
  select $i_k \in [m+p]$ with $\text{Prob}(i_k = i) = q_i$;
  perform an update to z_{i_k} according to (35);
  update the global counter $k \leftarrow k + 1$;

Whenever an agent updates a coordinate, the global iteration number k increases by one. The kth update is applied to z_{i_k}, with i_k being independent random variables: $z_i = x_i$ when $i \leq m$ and $z_i = s_{i-m}$ when $i > m$. Each coordinate update has the form:

$$\text{(35)} \qquad \begin{cases} z_{i_k}^{k+1} = z_{i_k}^k - \frac{\eta_k}{(m+p)q_{i_k}} (\hat{z}_{i_k}^k - (\mathcal{T}_{\rm CV}\hat{z}^k)_{i_k}), \\ z_i^{k+1} = z_i^k, \quad \forall i \neq i_k, \end{cases}$$

where η_k is the step size, z^k denotes the state of z in global memory just before the update (35) is applied, and $\hat{z}^k$ is the result that z in global memory is read by an agent to its local cache (see [54, §1.2] for both consistent and inconsistent cases). While $(\hat{z}_{i_k}^k - (\mathcal{T}_{\rm CV}\hat{z}^k)_{i_k})$ is being computed, asynchronous parallel computing allows other agents to make updates to z, introducing so-called asynchronous delays. Therefore, $\hat{z}^k$ can be different from z^k. We refer the reader to [54, §1.2] for more details.

The async-parallel algorithm using the overlapping-block coordinate update (33) is in Algorithm 2 (recall that the overlapping-block coordinate

update is introduced to save computation).

Algorithm 2: Async-parallel primal-dual overlapping-block coordinate update algorithm using $\mathcal{T}_{\rm CV}$

Input : $z^0 \in \mathbb{F}$, $K > 0$, a discrete distribution $(q_1, \ldots, q_m)$ with $\sum_{i=1}^m q_i = 1$ and $q_i > 0, \forall i$,

set global iteration counter $k = 0$;

while $k < K$, *every agent asynchronously and continuously* **do**

- select $i_k \in [m]$ with $\text{Prob}(i_k = i) = q_i$;
- Compute $\tilde{s}_j^{k+1}, \forall j \in \mathbb{J}(i_k)$ and $\tilde{x}_{i_k}^{k+1}$ according to (33);
- update $x_{i_k}^{k+1} = x_{i_k}^k + \frac{\eta_k}{mq_{i_k}}(\tilde{x}_{i_k}^{k+1} - x_{i_k}^k)$;
- let $x_i^{k+1} = x_i^k$ for $i \neq i_k$;
- update $s_j^{k+1} = s_j^k + \frac{\rho_{i,j}\eta_k}{mq_{i_k}}(\tilde{s}_j^{k+1} - s_j^k)$, for all $j \in \mathbb{J}(i_k)$;
- let $s_j^{k+1} = s_j^k$, for all $j \notin \mathbb{J}(i_k)$;
- update the global counter $k \leftarrow k + 1$;

Here we still allow asynchronous delays, so $\tilde{x}_{i_k}$ and $\tilde{s}_j^{k+1}$ are computed using some $\hat{z}^k$.

Remark 5. *If shared memory is used, it is recommended to set all but one $\rho_{i,j}$'s to 0 for each i.*

Theorem 1. *Let Z^* be the set of solutions to problem (24) and $(z^k)_{k\geq 0} \subset \mathbb{F}$ be the sequence generated by Algorithm 1 or Algorithm 2 under the following conditions:*

(i) f, g, h^* are closed proper convex functions, f is differentiable, and ∇f is Lipschitz continuous with constant β;

(ii) the delay for every coordinate is bounded by a positive number τ, i.e. for every $1 \leq i \leq m + p$, $\hat{z}_i^k = z_i^{k-d_i^k}$ for some $0 \leq d_i^k \leq \tau$;

(iii) $\eta_k \in [\eta_{\min}, \eta_{\max}]$ for certain $\eta_{\min}, \eta_{\max} > 0$.

Then $(z^k)_{k\geq 0}$ converges to a Z^*-valued random variable with probability 1.

The formulas for $\eta_{\min}$ and $\eta_{\max}$, as well as the proof of Theorem 1, are given in Appendix D along with additional remarks. The algorithms can be applied to solve problem (24). A variety of examples are provided in §5.1 and §5.2.

5. Applications

In this section, we provide examples to illustrate how to develop coordinate update algorithms based on CF operators. The applications are categorized into five different areas. The first subsection discusses three well-known machine learning problems: empirical risk minimization, Support Vector Machine (SVM), and group Lasso. The second subsection discusses image processing problems including image deblurring, image denoising, and Computed Tomography (CT) image recovery. The remaining subsections provide applications in finance, distributed computing as well as certain stylized optimization models. Several applications are treated with coordinate update algorithms for the first time.

For each problem, we describe the operator $\mathcal{T}$ and how to efficiently calculate $(\mathcal{T}x)_i$. The final algorithm is obtained after plugging the update in a coordinate update framework in §1.1 along with parameter initialization, an index selection rule, as well as some termination criteria.

5.1. Machine Learning

5.1.1. Empirical Risk Minimization (ERM). We consider the following regularized empirical risk minimization problem

$$\underset{x\in\mathbb{R}^m}{\text{minimize}} \ \frac{1}{p}\sum_{j=1}^{p}\phi_j(a_j^\top x) + f(x) + g(x), \tag{36}$$

where a_j's are sample vectors, ϕ_j's are loss functions, and $f+g$ is a regularization function. We assume that f is differentiable and g is proximable. Examples of (36) include linear SVM, regularized logistic regression, ridge regression, and Lasso. Further information on ERM can be found in [34]. The need for coordinate update algorithms arises in many applications of (36) where the number of samples or the dimension of x is large.

We define $A = [a_1^\top; a_2^\top; \ldots; a_p^\top]$ and $h(y) := \frac{1}{p}\sum_{j=1}^{p}\phi_j(y_j)$. Hence, $h(Ax) = \frac{1}{p}\sum_{j=1}^{p}\phi_j(a_j^\top x)$, and problem (36) reduces to form (24). We can apply the primal-dual update scheme to solve this problem, for which we introduce the dual variable $s = (s_1, ..., s_p)^\top$. We use $p+1$ coordinates, where the 0th coordinate is $x \in \mathbb{R}^m$ and the jth coordinate is s_j, $j \in [p]$. The

operator $\mathcal{T}$ is given in (29). At each iteration, a coordinate is updated:

$$
\text{(37)} \qquad \begin{cases} \text{if } x \text{ is chosen (the index 0), then compute} \\ \quad x^{k+1} = \mathbf{prox}_{\eta g}(x^k - \eta(\nabla f(x^k) + A^\top s^k)), \\ \text{if } s_j \text{ is chosen (an index } j \in [p]\text{), then compute} \\ \quad \tilde{x}^{k+1} = \mathbf{prox}_{\eta g}(x^k - \eta(\nabla f(x^k) + A^\top s^k)), \\ \quad \text{and} \\ \quad s_j^{k+1} = \frac{1}{p}\mathbf{prox}_{p\gamma\phi_j^*}(ps_j^k + p\gamma a_j^\top(2\tilde{x}^{k+1} - x^k)). \end{cases}
$$

We maintain $A^\top s \in \mathbb{R}^m$ in the memory. Depending on the structure of ∇f, we can compute it each time or maintain it. When $\mathbf{prox}_g \in \mathcal{F}_1$, we can consider breaking x into coordinates x_i's and also select an index i to update x_i at each time.

5.1.2. Support Vector Machine. Given the training data $\{(a_i, \beta_i)\}_{i=1}^m$ with $\beta_i \in \{+1, -1\}$, $\forall i$, the kernel support vector machine [65] is

$$
\text{(38)} \qquad \begin{array}{ll} \underset{x,\xi,y}{\text{minimize}} & \frac{1}{2}\|x\|_2^2 + C\sum_{i=1}^m \xi_i, \\ \text{subject to} & \beta_i(x^\top\phi(a_i) - y) \geq 1 - \xi_i,\ \xi_i \geq 0,\ \forall i \in [m], \end{array}
$$

where ϕ is a vector-to-vector map, mapping each data a_i to a point in a (possibly) higher-dimensional space. If $\phi(a) = a$, then (38) reduces to the linear support vector machine. The model (38) can be interpreted as finding a hyperplane $\{w : x^\top w - y = 0\}$ to separate two sets of points $\{\phi(a_i) : \beta_i = 1\}$ and $\{\phi(a_i) : \beta_i = -1\}$.

The dual problem of (38) is

$$
\text{(39)} \qquad \underset{s}{\text{minimize}}\, \frac{1}{2}s^\top Q s - e^\top s, \text{ subject to } 0 \leq s_i \leq C,\ \forall i,\ \sum_i \beta_i s_i = 0,
$$

where $Q_{ij} = \beta_i\beta_j k(a_i, a_j)$, $k(\cdot,\cdot)$ is a so-called *kernel function*, and $e = (1, ..., 1)^\top$. If $\phi(a) = a$, then $k(a_i, a_j) = a_i^\top a_j$.

Unbiased case. If $y = 0$ is enforced in (38), then the solution hyperplane $\{w : x^\top w = 0\}$ passes through the origin and is called *unbiased*. Consequently, the dual problem (39) will no longer have the linear constraint $\sum_i \beta_i s_i = 0$, leaving it with the coordinate-wise separable box constraints $0 \leq s_i \leq C$. To solve (39), we can apply the FBS operator $\mathcal{T}$ defined by (14).

Let $d(s) := \frac{1}{2}s^\top Qs - e^\top s$, $\mathcal{A} = \mathbf{prox}_{[0,C]}$, and $\mathcal{C} = \nabla d$. The coordinate update based on FBS is

$$s_i^{k+1} = \mathbf{proj}_{[0,C]}(s_i^k - \gamma_i \nabla_i d(s^k)),$$

where we can take $\gamma_i = \frac{1}{Q_{ii}}$.

Biased (general) case. In this case, the mode (38) has $y \in \mathbb{R}$, so the hyperplane $\{w : x^\top w - y = 0\}$ may not pass the origin and is called *biased*. Then, the dual problem (39) retains the linear constraint $\sum_i \beta_i s_i = 0$. In this case, we apply the primal-dual splitting scheme (28) or the three-operator splitting scheme (13).

The coordinate update based on the full primal-dual splitting scheme (28) is:

$$t^{k+1} = t^k + \gamma \sum_{i=1}^m \beta_i s_i^k, \tag{40a}$$

$$s_i^{k+1} = \mathbf{proj}_{[0,C]}\left(s_i^k - \eta\big(\nabla_i d(s^k) + \beta_i(2t^{k+1} - t^k)\big)\right), \tag{40b}$$

where t, s are the primal and dual variables, respectively. Note that we can let $w := \sum_{i=1}^m \beta_i s_i$ and maintain it. With variable w and substituting (40a) into (40b), we can equivalently write (40) into

$$\left\{\begin{array}{l} \text{if } t \text{ is chosen (the index 0), then compute} \\ \quad t^{k+1} = t^k + \gamma w^k, \\ \text{if } s_i \text{ is chosen (an index } i \in [m]\text{), then compute} \\ \quad s_i^{k+1} = \mathbf{proj}_{[0,C]}\left(s_i^k - \eta\big(q_i^\top s^k - 1 + \beta_i(2\gamma w^k + t^k)\big)\right) \\ \quad w^{k+1} = w^k + \beta_i(s_i^{k+1} - s_i^k). \end{array}\right. \tag{41}$$

We can also apply the three-operator splitting (13) as follows. Let $D_1 := [0, C]^m$ and $D_2 := \{s : \sum_{i=1}^m \beta_i s_i = 0\}$. Let $\mathcal{A} = \mathbf{proj}_{D_2}$, $\mathcal{B} = \mathbf{proj}_{D_1}$, and $\mathcal{C}(x) = Qx - e$, The full update corresponding to $\mathcal{T} = (I - \eta_k)\mathcal{I} + \eta_k \mathcal{T}_{3S}$ is

$$s^{k+1} = \mathbf{proj}_{D_2}(u^k), \tag{42a}$$

$$u^{k+1} = u^k + \eta_k \left(\mathbf{proj}_{D_1}\big(2s^{k+1} - u^k - \gamma(Qs^{k+1} - e)\big) - s^{k+1}\right), \tag{42b}$$

where s is just an intermediate variable. Let $\tilde{\beta} := \frac{\beta}{\|\beta\|_2}$ and $w := \tilde{\beta}^\top u$. Then $\mathbf{proj}_{D_2}(u) = (I - \tilde{\beta}\tilde{\beta}^\top)u$. Hence, $s^{k+1} = u^k - w^k\tilde{\beta}$. Plugging it into (42b)

yields the following coordinate update scheme:

$$\begin{cases} \text{if } i \in [m] \text{ is chosen, then compute} \\ \quad s_i^{k+1} = u_i^k - w^k \tilde{\beta}_i, \\ \quad u_i^{k+1} = u_i^k + \eta_k \left(\mathbf{proj}_{[0,C]} \left(2s_i^{k+1} - u_i^k - \gamma \big(q_i^\top u^k - w^k (q_i^\top \tilde{\beta}) - 1 \big) \right) - s_i^{k+1} \right) \\ \quad w^{k+1} = w^k + \tilde{\beta}_i (u_i^{k+1} - u_i^k), \end{cases}$$

where w^k is the maintained variable and s^k is the intermediate variable.

5.1.3. Group Lasso. The group Lasso regression problem [84] is

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \, f(x) + \sum_{i=1}^m \lambda_i \|x_i\|_2, \tag{43}$$

where f is a differentiable convex function, often bearing the form $\frac{1}{2}\|Ax - b\|_2^2$, and $x_i \in \mathbb{R}^{n_i}$ is a subvector of $x \in \mathbb{R}^n$ supported on $\mathbb{I}_i \subset [n]$, and $\cup_i \mathbb{I}_i = [n]$. If $\mathbb{I}_i \cap \mathbb{I}_j = \emptyset$, $\forall i \neq j$, it is called *non-overlapping group Lasso*, and if there are two different groups $\mathbb{I}_i$ and $\mathbb{I}_j$ with a non-empty intersection, it is called *overlapping group Lasso*. The model finds a coefficient vector x that minimizes the fitting (or loss) function $f(x)$ and that is group sparse: all but a few x_i's are zero.

Let U_i be formed by the columns of the identity matrix I corresponding to the indices in $\mathbb{I}_i$, and let $U = [U_1^\top; \ldots; U_m^\top] \in \mathbb{R}^{(\Sigma_i n_i) \times n}$. Then, $x_i = U_i^\top x$. Let $h_i(y_i) = \lambda_i \|y_i\|_2$, $y_i \in \mathbb{R}^{n_i}$ for $i \in [m]$, and $h(y) = \sum_{i=1}^m h_i(y_i)$ for $y = [y_1; \ldots; y_m] \in \mathbb{R}^{\Sigma_i n_i}$. In this way, (43) becomes

$$\underset{x}{\text{minimize}} \, f(x) + h(Ux). \tag{44}$$

Non-overlapping case [84]. In this case, we have $\mathbb{I}_i \cap \mathbb{I}_j = \emptyset$, $\forall i \neq j$, and can apply the FBS scheme (14) to (44). Specifically, let $\mathcal{T}_1 = \partial(h \circ U)$ and $\mathcal{T}_2 = \nabla f$. The FBS full update is

$$x^{k+1} = \mathcal{J}_{\gamma \mathcal{T}_1} \circ (\mathcal{I} - \gamma \mathcal{T}_2)(x^k).$$

The corresponding coordinate update is the following

$$\begin{cases} \text{if } i \in [m] \text{ is chosen, then compute} \\ \quad x_i^{k+1} = \arg\min_{x_i} \frac{1}{2}\|x_i - x_i^k + \gamma_i \nabla_i f(x^k)\|_2^2 + \gamma_i h_i(x_i), \end{cases} \tag{45}$$

where $\nabla_i f(x^k)$ is the partial derivative of f with respect to x_i and the step size can be taken to be $\gamma_i = \frac{1}{\|A_{:,i}\|^2}$. When ∇f is either cheap or easy-to-maintain, the coordinate update in (45) is inexpensive.

Overlapping case [38]. This case allows $\mathbb{I}_i \cap \mathbb{I}_j \neq \emptyset$ for some $i \neq j$, causing the evaluation of $\mathcal{J}_{\gamma\mathcal{T}_1}$ to be generally difficult. However, we can apply the primal-dual update (28) to this problem as

$$s^{k+1} = \mathbf{prox}_{\gamma h^*}(s^k + \gamma U x^k), \tag{46a}$$

$$x^{k+1} = x^k - \eta(\nabla f(x^k) + U^\top(2s^{k+1} - s^k)), \tag{46b}$$

where s is the dual variable. Note that

$$h^*(s) = \begin{cases} 0, & \text{if } \|s_i\|_2 \leq \lambda_i, \forall i, \\ +\infty, & \text{otherwise,} \end{cases}$$

is cheap. Hence, the corresponding coordinate update of (46) is

(47)

$$\begin{cases} \text{if } s_i \text{ is chosen for some } i \in [m], \text{ then compute} \\ \quad s_i^{k+1} = \mathbf{proj}_{B_{\lambda_i}}(s_i^k + \gamma x_i^k) \\ \text{if } x_i \text{ is chosen for some } i \in [m], \text{ then compute} \\ \quad x_i^{k+1} = x_i^k - \eta\left(U_i^T \nabla f(x^k) + U_i^T \sum_{j, U_i^T U_j \neq 0} U_j (2\mathbf{proj}_{B_{\lambda_j}}(s_j^k + \gamma x_j^k) - s_j^k)\right), \end{cases}$$

where B_λ is the Euclidean ball of radius λ. When ∇f is easy-to-maintain, the coordinate update in (47) is inexpensive. To the best of our knowledge, the coordinate update method (47) is new.

5.2. Imaging

5.2.1. DRS for Image Processing in the Primal-dual Form [50]. Many convex image processing problems have the general form

$$\underset{x}{\text{minimize}}\, f(x) + g(Ax),$$

where A is a matrix such as a dictionary, sampling operator, or finite difference operator. We can reduce the problem to the system: $0 \in \mathcal{A}(z) + \mathcal{B}(z)$, where $z = [x; s]$,

$$\mathcal{A}(z) := \begin{bmatrix} \partial f(x) \\ \partial g^*(s) \end{bmatrix}, \quad \text{and} \quad \mathcal{B}(z) := \begin{bmatrix} 0 & A^\top \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix}.$$

(see Appendix C for the reduction.) The work [50] gives their resolvents

$$\mathcal{J}_{\gamma\mathcal{A}} = \begin{bmatrix} \mathbf{prox}_{\gamma f} & 0 \\ 0 & \mathbf{prox}_{\gamma g^*} \end{bmatrix},$$
$$\mathcal{J}_{\gamma\mathcal{B}} = (I + \gamma\mathcal{B})^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I \\ \gamma A \end{bmatrix} (I + \gamma^2 A^\top A)^{-1} \begin{bmatrix} I \\ -\gamma A \end{bmatrix}^\top,$$

where $\mathcal{J}_{\gamma\mathcal{A}}$ is often cheap or separable and we can *explicitly form* $\mathcal{J}_{\gamma\mathcal{B}}$ as a matrix or implement it based on a fast transform. With the defined $\mathcal{J}_{\gamma\mathcal{A}}$ and $\mathcal{J}_{\gamma\mathcal{B}}$, we can apply the RPRS method as $z^{k+1} = \mathcal{T}_{\text{RPRS}} z^k$. The resulting RPRS operator is CF when $\mathcal{J}_{\gamma\mathcal{B}}$ is CF. Hence, we can derive a new RPRS coordinate update algorithm. We leave the derivation to the readers. Derivations of coordinate update algorithms for more specific image processing problems are shown in the following subsections.

5.2.2. Total Variation Image Processing. We consider the following Total Variation (TV) image processing model

$$\underset{x}{\text{minimize}} \ \lambda\|x\|_{\text{TV}} + \frac{1}{2}\|A\,x - b\|^2, \tag{48}$$

where $x \in \mathbb{R}^n$ is the vector representation of the unknown image, A is an $m \times n$ matrix describing the transformation from the image to the measurements b. Common A includes sampling matrices in MRI, CT, denoising, deblurring, etc. Let (∇_i^h, ∇_i^v) be the discrete gradient at pixel i and $\nabla x = (\nabla_1^h x, \nabla_1^v x, \ldots, \nabla_n^h x, \nabla_n^v x)^\top$. Then the TV semi-norm $\|\cdot\|_{\text{TV}}$ in the isotropic and anisotropic fashions are, respectively,

$$\|x\|_{\text{TV}} = \sum_i \sqrt{(\nabla_i^h x)^2 + (\nabla_i^v x)^2}, \tag{49a}$$

$$\|x\|_{\text{TV}} = \|\nabla x\|_1 = \sum_i \left(|\nabla_i^h x| + |\nabla_i^v x| \right). \tag{49b}$$

For simplicity, we use the anisotropic TV for analysis and in the numerical experiment in § 6.2. It is slightly more complicated for the isotropic TV. Introducing the following notation

$$B := \begin{pmatrix} \nabla \\ A \end{pmatrix}, \quad h(p,q) := \lambda\|p\|_1 + \frac{1}{2}\|q - b\|^2,$$

we can reformulate (48) as

$$\underset{x}{\text{minimize}} \;\; h(B\,x) = h(\nabla x, A\,x),$$

which reduces to the form of (24) with $f = g = 0$. Based on its definition, the convex conjugate of $h(p, q)$ and its proximal operator are, respectively,

$$h^*(s,t) = \iota_{\|\cdot\|_\infty \le \lambda}(s) + \frac{1}{2}\|t+b\|^2 - \frac{1}{2}\|b\|^2, \tag{50}$$

$$\mathbf{prox}_{\gamma h^*}(s,t) = \mathbf{proj}_{\|\cdot\|_\infty \le \lambda}(s) + \frac{1}{1+\gamma}(t - \gamma b). \tag{51}$$

Let s, t be the dual variables corresponding to ∇x and Ax respectively, then using (51) and applying (29) give the following full update:

$$x^{k+1} = x^k - \eta(\nabla^\top s^k + A^\top t^k), \tag{52a}$$

$$s^{k+1} = \mathbf{proj}_{\|\cdot\|_\infty \le \lambda}\left(s^k + \gamma\nabla(x^k - 2\eta(\nabla^\top s^k + A^\top t^k))\right), \tag{52b}$$

$$t^{k+1} = \frac{1}{1+\gamma}\left(t^k + \gamma A(x^k - 2\eta(\nabla^\top s^k + A^\top t^k)) - \gamma b\right). \tag{52c}$$

To perform the coordinate updates as described in §4, we can maintain $\nabla^\top s^k$ and $A^\top t^k$. Whenever a coordinate of (s, t) is updated, the corresponding $\nabla^\top s^k$ (or $A^\top t^k$) should also be updated. Specifically, we have the following coordinate update algorithm

$$\begin{cases} \text{if } x_i \text{ is chosen for some } i \in [n], \text{ then compute} \\ \quad x_i^{k+1} = x_i^k - \eta(\nabla^\top s^k + A^\top t^k)_i; \\ \text{if } s_i \text{ is chosen for some } i \in [2n], \text{ then compute} \\ \quad s_i^{k+1} = \mathbf{proj}_{\|\cdot\|_\infty \le \lambda}\left(s_i^k + \gamma\nabla_i(x^k - 2\eta(\nabla^\top s^k + A^\top t^k))\right) \\ \quad \text{and update } \nabla^\top s^k \text{ to } \nabla^\top s^{k+1}; \\ \text{if } t_i \text{ is chosen for some } i \in [m], \text{ then compute} \\ \quad t_i^{k+1} = \frac{1}{1+\gamma}\left(t_i^k + \gamma A_{i,:}(x^k - 2\eta(\nabla^\top s^k + A^\top t^k)) - \gamma b_i\right) \\ \quad \text{and update } A^\top t^k \text{ to } A^\top t^{k+1}. \end{cases} \tag{53}$$

5.2.3. 3D Mesh Denoising. Following an example in [60], we consider a 3D mesh described by their nodes $\bar{x}_i = (\bar{x}_i^X, \bar{x}_i^Y, \bar{x}_i^Z), i \in [n]$, and the adjacency matrix $A \in \mathbb{R}^{n\times n}$, where $A_{ij} = 1$ if nodes i and j are adjacent, otherwise $A_{ij} = 0$. We let $\mathcal{V}_i$ be the set of neighbours of node i. Noisy mesh nodes $z_i, i \in [n]$, are observed. We try to recover the original mesh nodes by

solving the following optimization problem [60]:

$$\underset{x}{\text{minimize}} \ \sum_{i=1}^{n} f_i(x_i) + \sum_{i=1}^{n} g_i(x_i) + \sum_{i} \sum_{j \in \mathcal{V}_i} h_{i,j}(x_i - x_j), \tag{54}$$

where f_i's are differentiable data fidelity terms, g_i's are the indicator functions of box constraints, and $\sum_i \sum_{j\in\mathcal{V}_i} h_{i,j}(x_i - x_j)$ is the total variation on the mesh.

We introduce a dual variable s with coordinates $s_{i,j}$, for all ordered pairs of adjacent nodes (i, j), and, based on the overlapping-block coordinate updating scheme (33), perform coordinate update:

$$\begin{cases} \text{select } i \text{ from } [n], \text{ then compute} \\ \quad \tilde{s}_{i,j}^{k+1} = \mathbf{prox}_{\gamma h_{i,j}^*}(s_{i,j}^k + \gamma x_i^k - \gamma x_j^k), \forall j \in \mathcal{V}_i, \\ \quad \tilde{s}_{j,i}^{k+1} = \mathbf{prox}_{\gamma h_{j,i}^*}(s_{j,i}^k + \gamma x_j^k - \gamma x_i^k), \forall j \in \mathcal{V}_i, \\ \quad \text{and update} \\ \quad x_i{}^{k+1} = \mathbf{prox}_{\eta g_i}(x_i^k - \eta(\nabla f_i(x_i^k) + \sum_{j\in\mathcal{V}_i}(2\tilde{s}_{i,j}^{k+1} - 2\tilde{s}_{j,i}^{k+1} - s_{i,j}^k + s_{j,i}^k))), \\ \quad s_{i,j}^{k+1} = s_{i,j}^k + \frac{1}{2}(\tilde{s}_{i,j}^{k+1} - s_{i,j}^k), \forall j \in \mathcal{V}_i, \\ \quad s_{j,i}^{k+1} = s_{j,i}^k + \frac{1}{2}(\tilde{s}_{j,i}^{k+1} - s_{j,i}^k), \forall j \in \mathcal{V}_i. \end{cases}$$

5.3. Finance

5.3.1. Portfolio Optimization. Assume that we have one unit of capital and m assets to invest on. The ith asset has an expected return rate $\xi_i \geq 0$. Our goal is to find a portfolio with the minimal risk such that the expected return is no less than c. This problem can be formulated as

$$\begin{aligned} &\underset{x}{\text{minimize}} \ \frac{1}{2} x^\top Q x, \\ &\text{subject to } x \geq 0, \sum_{i=1}^{m} x_i \leq 1, \ \sum_{i=1}^{m} \xi_i x_i \geq c, \end{aligned}$$

where the objective function is a measure of risk, and the last constraint imposes that the expected return is at least c. Let $a_1 = e/\sqrt{m}$, $b_1 = 1/\sqrt{m}$, $a_2 = \xi/\|\xi\|_2$, and $b_2 = c/\|\xi\|_2$, where $e = (1, \ldots, 1)^\top, \xi = (\xi_1, \ldots, \xi_m)^\top$. The above problem is rewritten as

$$\underset{x}{\text{minimize}} \frac{1}{2} x^\top Q x, \ \text{subject to} \ \ x \geq 0, \ a_1^\top x \leq b_1, \ a_2^\top x \geq b_2. \tag{55}$$

We apply the three-operator splitting scheme (13) to (55). Let $f(x) = \frac{1}{2}x^\top Qx$, $D_1 = \{x : x \geq 0\}$, $D_2 = \{x : a_1^\top x \leq b_1, a_2^\top x \geq b_2\}$, $D_{21} = \{x : a_1^\top x = b_1\}$, and $D_{22} = \{x : a_2^\top x = b_2\}$. Based on (13), the full update is

$$y^{k+1} = \mathbf{proj}_{D_2}(x^k), \tag{56a}$$

$$x^{k+1} = x^k + \eta_k\big(\mathbf{proj}_{D_1}(2y^{k+1} - x^k - \gamma\nabla f(y^{k+1})) - y^{k+1}\big), \tag{56b}$$

where y is an intermediate variable. As the projection to D_1 is simple, we discuss how to evaluate the projection to D_2. Assume that a_1 and a_2 are neither perpendicular nor co-linear, i.e., $a_1^\top a_2 \neq 0$ and $a_1 \neq \lambda a_2$ for any scalar λ. In addition, assume $a_1^\top a_2 > 0$ for simplicity. Let $a_3 = a_2 - \frac{1}{a_1^\top a_2}a_1$, $b_3 = b_2 - \frac{1}{a_1^\top a_2}b_1$, $a_4 = a_1 - \frac{1}{a_1^\top a_2}a_2$, and $b_4 = b_1 - \frac{1}{a_1^\top a_2}b_2$. Then we can partition the whole space into four areas by the four hyperplanes $a_i^\top x = b_i$, $i = 1, \ldots, 4$. Let $P_i = \{x : a_i^\top x \leq b_i, a_{i+1}^\top x \geq b_{i+1}\}$, $i = 1, 2, 3$ and $P_4 = \{x : a_4^\top x \leq b_4, a_1^\top x \geq b_1\}$. Then

$$\mathbf{proj}_{D_2}(x) = \begin{cases} x, & \text{if } x \in P_1, \\ \mathbf{proj}_{D_{22}}(x), & \text{if } x \in P_2, \\ \mathbf{proj}_{D_{21}\cap D_{22}}(x), & \text{if } x \in P_3, \\ \mathbf{proj}_{D_{21}}(x), & \text{if } x \in P_4. \end{cases}$$

Let $w_i = a_i^\top x - b_i, i = 1, 2$, and maintain w_1, w_2. Let $\tilde{a}_2 = \frac{a_2 - a_1(a_1^\top a_2)}{1-(a_1^\top a_2)^2}$, $\tilde{a}_1 = \frac{a_1 - a_2(a_1^\top a_2)}{1-(a_1^\top a_2)^2}$. Then

$$\begin{aligned} \mathbf{proj}_{D_{21}}(x) &= x - w_1 a_1, \\ \mathbf{proj}_{D_{22}}(x) &= x - w_2 a_2, \\ \mathbf{proj}_{D_{21}\cap D_{22}}(x) &= x - w_1\tilde{a}_1 - w_2\tilde{a}_2, \end{aligned}$$

Hence, the coordinate update of (56) is

$$
\begin{aligned}
&\text{(57a)}\\
x^k \in P_1:\ x_i^{k+1} =& (1-\eta_k)x_i^k + \eta_k \max(0, x_i^k - \gamma q_i^\top x^k),\\
x^k \in P_2:\ x_i^{k+1} =& (1-\eta_k)x_i^k + \eta_k w_2^k (a_2)_i + \eta_k \max\left(0,\right.\\
&\text{(57b)} \qquad \left. x_i^k - \gamma q_i^\top x^k - w_2^k(2(a_2)_i - \gamma q_i^\top a_2)\right),\\
x^k \in P_3:\ x_i^{k+1} =& (1-\eta_k)x_i^k + \eta_k \left(w_1^k(\tilde{a}_1)_i + w_2^k(\tilde{a}_2)_i\right) + \eta_k \max\left(0,\right.\\
&\text{(57c)} \qquad \left. x_i^k - \gamma q_i^\top x^k - w_1^k(2(\tilde{a}_1)_i - \gamma q_i^\top \tilde{a}_1) - w_2^k(2(\tilde{a}_2)_i - \gamma q_i^\top \tilde{a}_2)\right),\\
x^k \in P_4:\ x_i^{k+1} =& (1-\eta_k)x_i^k + \eta_k w_1^k (a_1)_i + \eta_k \max\left(0,\right.\\
&\text{(57d)} \qquad \left. x_i^k - \gamma q_i^\top x^k - w_1^k(2(a_1)_i - \gamma q_i^\top a_1)\right),
\end{aligned}
$$

where q_i is the ith column of Q. At each iteration, we select $i \in [m]$, and perform an update to x_i according to (57) based on where x^k is. We then renew $w_j^{k+1} = w_j^k + a_{ij}(x_i^{k+1} - x_i^k), j = 1, 2$. Note that checking x^k in some P_j requires only $O(1)$ operations by using w_1 and w_2, so the coordinate update in (57) is inexpensive.

5.4. Distributed Computing

5.4.1. Network. Consider that m worker agents and one master agent form a star-shaped network, where the master agent at the center connects to each of the worker agents. The $m + 1$ agents collaboratively solve the consensus problem:

$$\underset{x}{\text{minimize}} \sum_{i=1}^{m} f_i(x),$$

where $x \in \mathbb{R}^d$ is the common variable and each proximable function f_i is held privately by agent i. The problem can be reformulated as

$$\text{(58)} \qquad \underset{x_1,\ldots,x_m,y\in\mathbb{R}^d}{\text{minimize}} F(x) := \sum_{i=1}^{m} f_i(x_i), \quad \text{subject to } x_i = y,\ \forall i \in [m],$$

which has the KKT condition

$$\text{(59)} \qquad 0 \in \underbrace{\begin{bmatrix} \partial F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{operator } \mathcal{A}} \begin{bmatrix} x \\ y \\ s \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & I \\ 0 & 0 & -e^\top \\ I & -e & 0 \end{bmatrix}}_{\text{operator } \mathcal{C}} \begin{bmatrix} x \\ y \\ s \end{bmatrix},$$

where s is the dual variable.

Applying the FBFS scheme (18) to (59) yields the following full update:

$$x_i^{k+1} = \mathbf{prox}_{\gamma f_i}(x_i^k - \gamma s_i^k) + \gamma^2 x_i^k - \gamma^2 y^k - 2\gamma s_i^k, \tag{60a}$$

$$y^{k+1} = (1 + m\gamma^2)y^k + 3\gamma \sum_{j=1}^m s_j^k - \gamma^2 \sum_{j=1}^m x_j^k, \tag{60b}$$

$$s_i^{k+1} = s_i^k - 2\gamma x_i^k - \gamma \mathbf{prox}_{\gamma f_i}(x_i^k - \gamma s_i^k) + 3\gamma y^k + \gamma^2 \sum_{j=1}^m s_j^k, \tag{60c}$$

where (60a) and (60c) are applied to all $i \in [m]$. Hence, for each i, we group x_i and s_i together and assign them on agent i. We let the master agent maintain $\sum_j s_j$ and $\sum_j x_j$. Therefore, in the FBFS coordinate update, updating any (x_i, s_i) needs only y and $\sum_j s_j$ from the master agent, and updating y is done on the master agent. In synchronous parallel setting, at each iteration, each worker agent i computes s_i^{k+1}, x_i^{k+1}, then the master agent collects the updates from all of the worker agents and then updates y and $\sum_j s_j$. The above update can be relaxed to be asynchronous. In this case, the master and worker agents work concurrently, the master agent updates y and $\sum_j s_j$ as soon as it receives the updated s_i and x_i from any of the worker agents. It also periodically broadcasts y back to the worker agents.

5.5. Dimension Reduction

5.5.1. Nonnegative Matrix Factorization. Nonnegative Matrix Factorization (NMF) is an important dimension reduction method for nonnegative data. It was proposed by Paatero and his coworkers in [51]. Given a nonnegative matrix $A \in \mathbb{R}_+^{p\times n}$, NMF aims at finding two nonnegative matrices $W \in \mathbb{R}_+^{p\times r}$ and $H \in \mathbb{R}_+^{n\times r}$ such that $WH^\top \approx A$, where r is user-specified depending on the applications, and usually $r \ll \min(p, n)$. A widely used model is

$$\begin{aligned} &\underset{W,H}{\text{minimize}}\, F(W, H) := \frac{1}{2}\|WH^\top - A\|_F^2, \\ &\text{subject to } W \in \mathbb{R}_+^{p\times r},\ H \in \mathbb{R}_+^{n\times r}. \end{aligned} \tag{61}$$

Applying the projected gradient method (21) to (61), we have

$$W^{k+1} = \max\left(0, W^k - \eta_k \nabla_W F(W^k, H^k)\right), \tag{62a}$$

$$H^{k+1} = \max\left(0, H^k - \eta_k \nabla_H F(W^k, H^k)\right). \tag{62b}$$

In general, we do not know the Lipschitz constant of ∇F, so we have to choose η_k by line search such that the Armijo condition is satisfied.

Partitioning the variables into $2r$ block coordinates: $(w_1, \ldots, w_r, h_1, \ldots, h_r)$ where w_i and h_i are the ith columns of W and H, respectively, we can apply the coordinate update based on the projected-gradient method:

$$\left\{\begin{array}{l} \text{if } w_{i_k} \text{ is chosen for some } i_k \in [r], \text{ then compute} \\ \quad w_{i_k}^{k+1} = \max\left(0, w_{i_k}^k - \eta_k \nabla_{w_{i_k}} F(W^k, H^k)\right); \\ \text{if } h_{i_k - r} \text{ is chosen for some } i_k \in \{r+1, ..., 2r\}, \text{ then compute} \\ \quad h_{i_k - r}^{k+1} = \max\left(0, h_{i_k - r}^k - \eta_k \nabla_{h_{i_k - r}} F(W^k, H^k)\right). \end{array}\right. \tag{63}$$

It is easy to see that $\nabla_{w_i} F(W^k, H^k)$ and $\nabla_{h_i} F(W^k, H^k)$ are both Lipschitz continuous with constants $\|h_i^k\|_2^2$ and $\|w_i^k\|_2^2$ respectively. Hence, we can set

$$\eta_k = \left\{\begin{array}{ll} \frac{1}{\|h_{i_k}^k\|_2^2}, & \text{if } 1 \le i_k \le r, \\ \frac{1}{\|w_{i_k - r}^k\|_2^2}, & \text{if } r+1 \le i_k \le 2r. \end{array}\right.$$

However, it is possible to have $w_i^k = 0$ or $h_i^k = 0$ for some i and k, and thus the setting in the above formula may have trouble of being divided by zero. To overcome this problem, one can first modify the problem (61) by restricting W to have unit-norm columns and then apply the coordinate update method in (63). Note that the modification does not change the optimal value since $WH^\top = (WD)(HD^{-1})^\top$ for any $r \times r$ invertible diagonal matrix D. We refer the readers to [82] for more details.

Note that $\nabla_W F(W, H) = (WH^\top - A)H, \nabla_H F(W, H) = (WH^\top - A)^\top W$ and $\nabla_{w_i} F(W, H) = (WH^\top - A)h_i, \nabla_{h_i} F(W, H) = (WH^\top - A)^\top w_i, \forall i$. Therefore, the coordinate updates given in (63) are computationally worthy (by maintaining the residual $W^k(H^k)^\top - A$).

5.6. Stylized Optimization

5.6.1. Second-Order Cone Programming (SOCP). SOCP extends LP by incorporating second-order cones. A second-order cone in $\mathbb{R}^n$ is

$$Q = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \|(x_2, \ldots, x_n)\|_2 \le x_1\right\}.$$

Given a point $v \in \mathbb{R}^n$, let $\rho_1^v := \|(v_2, \dots, v_n)\|_2$ and $\rho_2^v := \frac{1}{2}(v_1 + \rho_1^v)$. Then, the projection of v to Q returns 0 if $v_1 < -\rho_1^v$, returns v if $v_1 \geq \rho_1^v$, and returns $(\rho_2^v, \frac{\rho_2^v}{\rho_1^v} \cdot (v_2, \dots, v_n))$ otherwise. Therefore, if we define the scalar couple:

$$(\xi_1^v, \xi_2^v) = \begin{cases} (0,0), & v_1 < -\rho_1^v, \\ (1,1), & v_1 \geq \rho_1^v, \\ \left(\rho_2^v, \frac{\rho_2^v}{\rho_1^v}\right), & \text{otherwise}, \end{cases}$$

then we have $u = \mathbf{proj}_Q(v) = \left(\xi_1^v v_1,\ \xi_2^v \cdot (v_2, \dots, v_n)\right)$. Based on this, we have

Proposition 2. *1. Let $v \in \mathbb{R}^n$ and $v^+ := v + \nu e_i$ for any $\nu \in \mathbb{R}$. Then, given $\rho_1^v, \rho_2^v, \xi_1^v, \xi_2^v$ defined above, it takes $O(1)$ operations to obtain $\rho_1^{v^+}, \rho_2^{v^+}, \xi_1^{v^+}, \xi_2^{v^+}$.*

2. Let $v \in \mathbb{R}^n$ and $A = [a_1\ A_2] \in \mathbb{R}^{m \times n}$, where $a_1 \in \mathbb{R}^m, A_2 \in \mathbb{R}^{m \times (n-1)}$. Given $\rho_1^v, \rho_2^v, \xi_1^v, \xi_2^v$, we have

$$A(2 \cdot \mathbf{proj}_Q(v) - v) = ((2\xi_1^v - 1)v_1) \cdot a_1 + (2\xi_2^v - 1) \cdot A_2(v_2, \dots, v_n)^\top.$$

By the proposition, if $\mathcal{T}_1$ is an affine operator, then in the composition $\mathcal{T}_1 \circ \mathbf{proj}_Q$, the computation of $\mathbf{proj}_Q$ is cheap as long as we maintain $\rho_1^v, \rho_2^v, \xi_1^v, \xi_2^v$.

Given $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$, the standard form of SOCP is

$$\text{(64a)} \qquad \underset{x}{\text{minimize}}\ c^\top x, \quad \text{subject to } Ax = b,$$

$$\text{(64b)} \qquad x \in X = Q_1 \times \cdots \times Q_{\bar{n}},$$

where each Q_i is a second-order cone, and $\bar{n} \neq n$ in general. The problem (64) is equivalent to

$$\underset{x}{\text{minimize}}\ \left(c^\top x + \iota_{A\cdot=b}(x)\right) + \iota_X(x),$$

to which we can apply the DRS iteration $z^{k+1} = \mathcal{T}_{\text{DRS}}(z^k)$ (see (16)), in which $\mathcal{J}_{\gamma\mathcal{A}} = \mathbf{proj}_X$ and $\mathcal{T}_{\gamma\mathcal{B}}$ is a linear operator given by

$$\mathcal{J}_{\gamma\mathcal{B}}(x) = \arg\min_y\ c^\top y + \frac{1}{2\gamma}\|y - x\|^2 \quad \text{subject to } Ay = b.$$

Assume that the matrix A has full row-rank (otherwise, $Ax = b$ has either redundant rows or no solution). Then, in (16), we have $\mathcal{R}_{\gamma\mathcal{B}}(x) = Bx + d$, where $B := I - 2A^\top(AA^\top)^{-1}A$ and $d := 2A^\top(AA^\top)^{-1}(b + \gamma Ac) - 2\gamma c$.

It is easy to apply coordinate updates to $z^{k+1} = \mathcal{T}_{\rm DRS}(z^k)$ following Proposition 2. Specifically, by maintaining the scalars $\rho_1^v, \rho_2^v, \xi_1^v, \xi_2^v$ for each $v = x_i \in Q_i$ during coordinate updates, the computation of the projection can be completely avoided. We pre-compute $(AA^\top)^{-1}$ and cache the matrix B and vector d. Then, $\mathcal{T}_{\rm DRS}$ is CF, and we have the following coordinate update method

$$\left\{\begin{array}{l} \text{select } i \in [\bar{n}], \text{ then compute} \\ \quad y_i^{k+1} = B_i x^k + d_i \\ \quad x_i^{k+1} = \mathbf{proj}_{Q_i}(y_i^{k+1}) + \frac{1}{2}(x_i^k - y_i^{k+1}), \end{array}\right. \tag{65}$$

where $B_i \in \mathbb{R}^{n_i \times n}$ is the ith row block submatrix of B, and y_i^{k+1} is the intermediate variable.

It is trivial to extend this method for SOCPs with a quadratic objective:

$$\underset{x}{\text{minimize}}\ c^\top x + \frac{1}{2} x^\top C x, \quad \text{subject to } Ax = b,\ x \in X = Q_1 \times \cdots \times Q_{\bar{n}},$$

because $\mathcal{J}_2$ is still linear. Clearly, this method applies to linear programs as they are special SOCPs.

Note that many LPs and SOCPs have sparse matrices A, which deserve further investigation. In particular, we may prefer not to form $(AA^\top)^{-1}$ and use the results in §4.2 instead.

6. Numerical Experiments

We illustrate the behavior of coordinate update algorithms for solving portfolio optimization, image processing, and sparse logistic regression problems. Our primary goal is to show the efficiency of coordinate update algorithms compared to the corresponding full update algorithms. We will also illustrate that asynchronous parallel coordinate update algorithms are more scalable than their synchronous parallel counterparts.

Our first two experiments run on Mac OSX 10.9 with 2.4 GHz Intel Core i5 and 8 Gigabytes of RAM. The experiments were coded in Matlab. The sparse logistic regression experiment runs on 1 to 16 threads on a machine with two 2.5 Ghz 10-core Intel Xeon E5-2670v2 (20 cores in total) and 64 Gigabytes of RAM. The experiment was coded in C++ with OpenMP enabled. We use the Eigen library[7] for sparse matrix operations.

[7] http://eigen.tuxfamily.org

6.1. Portfolio Optimization

In this subsection, we compare the performance of the 3S splitting scheme (56) with the corresponding coordinate update algorithm (57) for solving the portfolio optimization problem (55). In this problem, our goal is to distribute our investment resources to all the assets so that the investment risk is minimized and the expected return is greater than c. This test uses two datasets, which are summarized in Table 3. The NASDAQ dataset is collected through Yahoo! Finance. We collected one year (from 10/31/2014 to 10/31/2015) of historical closing prices for 2730 stocks.

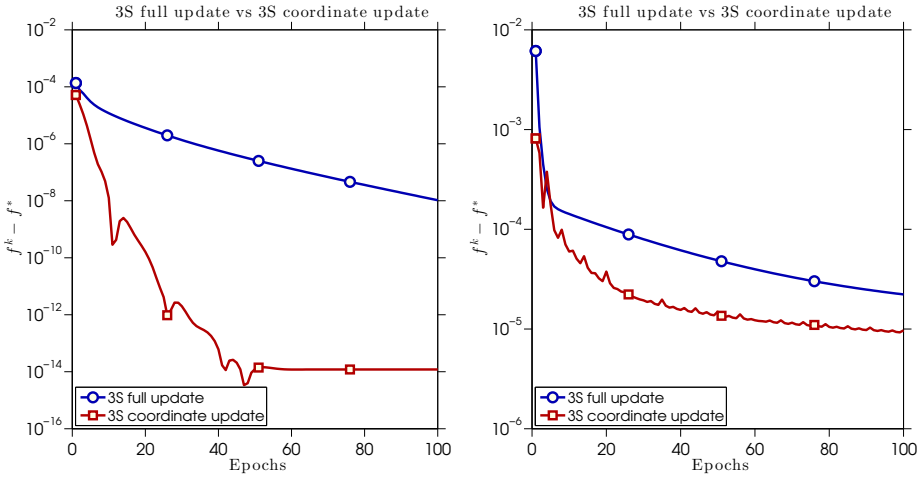
	Synthetic data	NASDAQ data
Number of assets (N)	1000	2730
Expected return rate	0.02	0.02
Asset return rate	`3 * rand(N, 1) - 1`	mean of 30 days return rate
Risk	covariance matrix + $0.01 \cdot I$	positive definite matrix

Table 3: Two datasets for portfolio optimization

In our numerical experiments, for comparison purposes, we first obtain a high accurate solution by solving (55) with an interior point solver. For both full update and coordinate update, η_k is set to 0.8. However, we use different γ. For 3S full update, we used the step size parameter $\gamma_1 = \frac{2}{\|Q\|_2}$, and for 3S coordinate update, $\gamma_2 = \frac{2}{\max\{Q_{11},\ldots,Q_{NN}\}}$. In general, coordinate update can benefit from more relaxed parameters. The results are reported in Figure 3. We can observe that the coordinate update method converges much faster than the 3S method for the synthetic data. This is due to the fact that γ_2 is much larger than γ_1. However, for the NASDAQ dataset, $\gamma_1 \approx \gamma_2$, so 3S coordinate update is only moderately faster than 3S full update.

6.2. Computed Tomography Image Reconstruction

We compare the performance of algorithm (52) and its corresponding coordinate version on Computed Tomography (CT) image reconstruction. We generate a thorax phantom of size 284×284 to simulate spectral CT measurements. We then apply the Siddon's algorithm [66] to form the sinogram data. There are 90 parallel beam projections and, for each projection, there are 362 measurements. Then the sinogram data is corrupted with Gaussian noise. We formulate the image reconstruction problem in the form of (48). The primal-dual full update corresponds to (52). For coordinate update, the



(a) Synthesis dataset (b) NASDAQ dataset

Figure 3: Compare the convergence of 3S full update with 3S coordinate update algorithms.

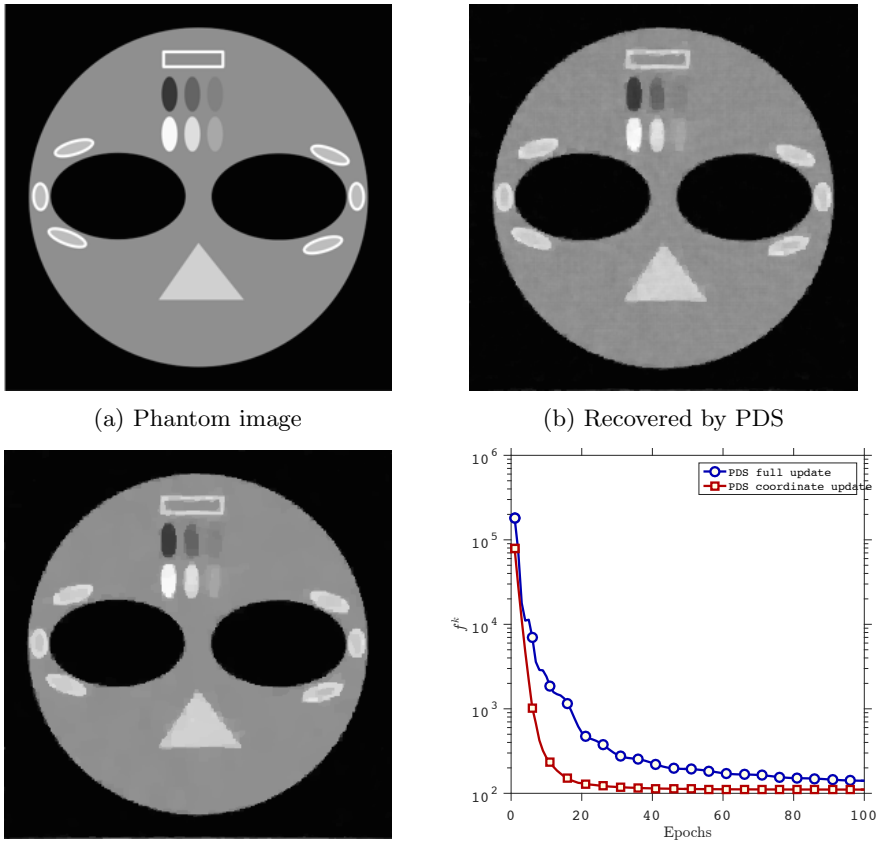
block size for x is set to 284, which corresponds to a column of the image. The dual variables s, t are also partitioned into 284 blocks accordingly. A block of x and the corresponding blocks of s and t are bundled together as a single block. In each iteration, a bundled block is randomly chosen and updated. The reconstruction results are shown in Figure 4. After 100 epochs, the image recovered by the coordinate version is better than that by (52). As shown in Figure 4d, the coordinate version converges faster than (52).

6.3. ℓ_1 Regularized Logistic Regression

In this subsection, we compare the performance of sync-parallel coordinate update and async-parallel coordinate update for solving the sparse logistic regression problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \lambda\|x\|_1 + \frac{1}{N}\sum_{j=1}^{N}\log\left(1+\exp(-b_j\cdot a_j^\top x)\right), \tag{66}$$

where $\{(a_j, b_j)\}_{j=1}^N$ is the set of sample-label pairs with $b_j \in \{1, -1\}$, $\lambda = 0.0001$, and n and N represent the numbers of features and samples, respec-



(a) Phantom image (b) Recovered by PDS

(c) Recovered by PDS coord (d) Objective function value

Figure 4: CT image reconstruction.

tively. This test uses the datasets[8]: real-sim and news20, which are summarized in Table 4.

We let each coordinate hold roughly 50 features. Since the total number of features is not divisible by 50, some coordinates have 51 features. We let each thread draw a coordinate uniformly at random at each iteration. We stop all the tests after 10 epochs since they have nearly identical progress per epoch. The step size is set to $\eta_k = 0.9$, $\forall k$. Let $A = [a_1, \ldots, a_N]^\top$ and $b = [b_1, ..., b_N]^\top$. In global memory, we store A, b and x. We also store the product Ax in global memory so that the forward step can be efficiently computed. Whenever a coordinate of x gets updated, Ax is immediately

[8] http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/

Name	# samples	# features
real-sim	72, 309	20, 958
news20	19,996	1,355,191

Table 4: Two datasets for sparse logistic regression.

updated at a low cost. Note that if Ax is *not* stored in global memory, every coordinate update will have to compute Ax from scratch, which involves the entire x and will be very expensive.

Table 5 gives the running times of the sync-parallel and async-parallel implementations on the two datasets. We can observe that async-parallel achieves almost-linear speedup, but sync-parallel scales very poorly as we explain below.

In the sync-parallel implementation, all the running threads have to wait for the last thread to finish an iteration, and therefore if a thread has a large load, it slows down the iteration. Although every thread is (randomly) assigned to roughly the same number of features (either 50 or 51 components of x) at each iteration, their a_i's have very different numbers of nonzeros, and the thread with the largest number of nonzeros is the slowest. (Sparse matrix computation is used for both datasets, which are very large.) As more threads are used, despite that they altogether do more work at each iteration, the per-iteration time may increase as the slowest thread tends to be slower. On the other hand, async-parallel coordinate update does not suffer from the load imbalance. Its performance grows nearly linear with the number of threads.

Finally, we have observed that the progress toward solving (66) is mainly a function of the number of epochs and does not change appreciably when the number of threads increases or between sync-parallel and async-parallel. Therefore, we always stop at 10 epochs.

7. Conclusions

We have presented a coordinate update method for fixed-point iterations, which updates one coordinate (or a few variables) at every iteration and can be applied to solve linear systems, optimization problems, saddle point problems, variational inequalities, and so on. We proposed a new concept called CF operator. When an operator is CF, its coordinate update is computationally worthy and often preferable over the full update method, in particular in a parallel computing setting. We gave examples of CF operators and also discussed how the properties can be preserved by composing two or more such operators such as in operator splitting and primal-dual splitting schemes. In

# threads	real-sim				news20			
	time (s)		speedup		time (s)		speedup	
	async	sync	async	sync	async	sync	async	sync
1	81.6	82.1	1.0	1.0	591.1	591.3	1.0	1.0
2	45.9	80.6	1.8	1.0	304.2	590.1	1.9	1.0
4	21.6	63.0	3.8	1.3	150.4	557.0	3.9	1.1
8	16.1	61.4	5.1	1.3	78.3	525.1	7.5	1.1
16	7.1	46.4	11.5	1.8	41.6	493.2	14.2	1.2

Table 5: Running times of async-parallel and sync-parallel FBS implementations for ℓ_1 regularized logistic regression on two datasets. Sync-parallel has very poor speedup due to the large distribution of coordinate sparsity and thus the large load imbalance across threads.

addition, we have developed CF algorithms for problems arising in several different areas including machine learning, imaging, finance, and distributed computing. Numerical experiments on portfolio optimization, CT imaging, and logistic regression have been provided to demonstrate the superiority of CF methods over their counterparts that update all coordinates at every iteration.

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Appendix A. Some Key Concepts of Operators

In this section, we go over a few key concepts in monotone operator theory and operator splitting theory.

Definition 8 (monotone operator)**.** *A* set-valued *operator* $\mathcal{T} : \mathbb{H} \rightrightarrows \mathbb{H}$ *is* monotone *if* $\langle x - y, u - v\rangle \geq 0,\ \forall x, y \in \mathbb{H},\ u \in \mathcal{T}x,\ v \in \mathcal{T}y$*. Furthermore,* $\mathcal{T}$ *is* maximally monotone *if its graph* $\mathrm{Grph}(T) = \{(x, u) \in \mathbb{H} \times \mathbb{H} : u \in \mathcal{T}x\}$ *is not strictly contained in the graph of any other monotone operator.*

Example 18. *An important maximally monotone operator is the subdifferential* ∂f *of a closed proper convex function* f*.*

Definition 9 (nonexpansive operator)**.** *An operator* $\mathcal{T} : \mathbb{H} \to \mathbb{H}$ *is* nonexpansive *if* $\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|,\ \forall x, y \in \mathbb{H}$*. We say* $\mathcal{T}$ *is averaged, or* α*-averaged, if there is one nonexpansive operator* $\mathcal{R}$ *such that* $\mathcal{T} = (1 - \alpha)\mathcal{I} + \alpha\mathcal{R}$ *for some* $0 < \alpha < 1$*. A* $\frac{1}{2}$*-averaged operator* $\mathcal{T}$ *is also called* firmly-nonexpansive*.*

By definition, a nonexpansive operator is single-valued. Let $\mathcal{T}$ be averaged. If $\mathcal{T}$ has a fixed point, the iteration (2) converges to a fixed point; otherwise, the iteration diverges unboundedly. Now let $\mathcal{T}$ be nonexpansive. The damped update of $\mathcal{T}$: $x^{k+1} = x^k - \eta(x^k - \mathcal{T}x^k)$, is equivalent to applying the averaged operator $(1-\eta)\mathcal{I} + \eta\mathcal{T}$.

Example 19. *A common firmly-nonexpansive operator is the resolvent of a maximally monotone map $\mathcal{T}$, written as*

$$\mathcal{J}_{\mathcal{A}} := (\mathcal{I} + \mathcal{A})^{-1}. \tag{67}$$

Given $x \in \mathbb{H}$, $\mathcal{J}_{\mathcal{A}}(x) = \{y : x \in y + \mathcal{A}y\}$. (By monotonicity of $\mathcal{A}$, $\mathcal{J}_{\mathcal{A}}$ is a singleton, and by maximality of $\mathcal{A}$, $\mathcal{J}_{\mathcal{A}}(x)$ is well defined for all $x \in \mathbb{H}$.) A reflective resolvent is

$$\mathcal{R}_{\mathcal{A}} := 2\mathcal{J}_{\mathcal{A}} - \mathcal{I}. \tag{68}$$

Definition 10 (proximal map)**.** *The proximal map for a function f is a special resolvent defined as:*

$$\mathbf{prox}_{\gamma f}(y) = \arg\min_x \big\{ f(x) + \frac{1}{2\gamma}\|x - y\|^2 \big\}, \tag{69}$$

where $\gamma > 0$. The first-order variational condition of the minimization yields $\mathbf{prox}_{\gamma f}(y) = (\mathcal{I} + \gamma\partial f)^{-1}$; hence, $\mathbf{prox}_{\gamma f}$ is firmly-nonexpansive. When $x \in \mathbb{R}^m$ and $\mathbf{prox}_{\gamma f}$ can be computed in $O(m)$ or $O(m \log m)$ operations, we call f proximable*.*

Examples of proximable functions include $\ell_1, \ell_2, \ell_\infty$-norms, several matrix norms, the owl-norm [22], (piece-wise) linear functions, certain quadratic functions, and many more.

Example 20. *A special proximal map is the projection map. Let X be a nonempty closed convex set, and ι_S be its indicator function. Minimizing $\iota_S(x)$ enforces $x \in S$, so $\mathbf{prox}_{\gamma\iota_S}$ reduces to the projection map $\mathbf{proj}_S$ for any $\gamma > 0$. Therefore, $\mathbf{proj}_S$ is also firmly nonexpansive.*

Definition 11 (β-cocoercive operator)**.** *An operator $\mathcal{T} : \mathbb{H} \to \mathbb{H}$ is* β-cocoercive *if $\langle x - y, \mathcal{T}x - \mathcal{T}y\rangle \geq \beta\|\mathcal{T}x - \mathcal{T}y\|^2$, $\forall x, y \in \mathbb{H}$.*

Example 21. *A special example of cocoercive operator is the gradient of a smooth function. Let f be a differentiable function. Then ∇f is β-Lipschitz continuous* if and only if *∇f is $\frac{1}{\beta}$-cocoercive [5, Corollary 18.16].*

Appendix B. Derivation of ADMM from the DRS Update

We derive the ADMM update in (23) from the DRS update

$$s^k = \mathcal{J}_{\eta\mathcal{B}}(t^k), \tag{70a}$$

$$t^{k+1} = \left(\frac{1}{2}(2\mathcal{J}_{\eta\mathcal{A}} - \mathcal{I}) \circ (2\mathcal{J}_{\eta\mathcal{B}} - \mathcal{I}) + \frac{1}{2}\mathcal{I}\right)(t^k), \tag{70b}$$

where $\mathcal{A} = -\partial f^*(-\cdot)$ and $\mathcal{B} = \partial g^*$.

Note (70a) is equivalent to $t^k \in s^k + \eta\partial g^*(s^k)$, i.e., there is a $y^k \in \partial g^*(s^k)$ such that $t^k = s^k + \eta y^k$, so

$$t^k - \eta y^k = s^k \in \partial g(y^k). \tag{71}$$

In addition, (70b) can be written as

$$\begin{aligned} t^{k+1} &= \mathcal{J}_{\eta\mathcal{A}}(2s^k - t^k) + t^k - s^k \\ &= s^k + (\mathcal{J}_{\eta\mathcal{A}} - \mathcal{I})(2s^k - t^k) \\ &= s^k + (\mathcal{I} - (\mathcal{I} + \eta\partial f^*)^{-1})(t^k - 2s^k) \\ &= s^k + \eta(\eta\mathcal{I} + \partial f)^{-1}(t^k - 2s^k) \\ &= s^k + \eta(\eta\mathcal{I} + \partial f)^{-1}(\eta y^k - s^k), \end{aligned} \tag{72}$$

where in the fourth equality, we have used the Moreau's Identity [63]: $(\mathcal{I} + \partial h)^{-1} + (\mathcal{I} + \partial h^*)^{-1} = \mathcal{I}$ for any closed convex function h. Let

$$x^{k+1} = (\eta\mathcal{I} + \partial f)^{-1}(\eta y^k - s^k) = (\mathcal{I} + \frac{1}{\eta}\partial f)^{-1}(y^k - \frac{1}{\eta}s^k). \tag{73}$$

Then (72) becomes

$$t^{k+1} = s^k + \eta x^{k+1},$$

and

$$s^{k+1} \overset{(71)}{=} t^{k+1} - \eta y^{k+1} = s^k + \eta x^{k+1} - \eta y^{k+1}, \tag{74}$$

which together with $s^{k+1} \in \partial g(y^{k+1})$ gives

$$y^{k+1} = (\eta\mathcal{I} + \partial g)^{-1}(s^k + \eta x^{k+1}) = (\mathcal{I} + \frac{1}{\eta}\partial g)^{-1}(x^{k+1} + \frac{1}{\eta}s^k). \tag{75}$$

Hence, from (73), (74), and (75), the ADMM update in (23) is equivalent to the DRS update in (70) with $\eta = \frac{1}{\gamma}$.

Appendix C. Representing the Condat-Vũ Algorithm as a Nonexpansive Operator

We show how to derive the Condat-Vũ algorithm (28) by applying a forward-backward operator to the optimality condition (27):

$$0 \in \Bigg(\underbrace{\begin{bmatrix} \nabla f & 0 \\ 0 & 0 \end{bmatrix}}_{\text{operator } \mathcal{A}} + \underbrace{\begin{bmatrix} \partial g & 0 \\ 0 & \partial h^* \end{bmatrix} + \begin{bmatrix} 0 & A^\top \\ -A & 0 \end{bmatrix}}_{\text{operator } \mathcal{B}} \Bigg) \underbrace{\begin{bmatrix} x \\ s \end{bmatrix}}_{z}, \tag{76}$$

It can be written as $0 \in \mathcal{A}z + \mathcal{B}z$ after we define $z = \begin{bmatrix} x \\ s \end{bmatrix}$. Let M be a symmetric positive definite matrix, we have

$$\begin{aligned} & 0 \in \mathcal{A}z + \mathcal{B}z \\ \Leftrightarrow & Mz - \mathcal{A}z \in Mz + \mathcal{B}z \\ \Leftrightarrow & z - M^{-1}\mathcal{A}z \in z + M^{-1}\mathcal{B}z \\ \Leftrightarrow & z = (\mathcal{I} + M^{-1}\mathcal{B})^{-1} \circ (\mathcal{I} - M^{-1}\mathcal{A})z. \end{aligned}$$

Convergence and other results can be found in [23]. The last equivalent relation is due to $M^{-1}\mathcal{B}$ being a maximally monotone operator under the norm induced by M. We let

$$M = \begin{bmatrix} \frac{1}{\eta} I & A^\top \\ A & \frac{1}{\gamma} I \end{bmatrix} \succ 0$$

and iterate

$$z^{k+1} = \mathcal{T} z^k = (\mathcal{I} + M^{-1}\mathcal{B})^{-1} \circ (\mathcal{I} - M^{-1}\mathcal{A}) z^k.$$

We have $Mz^{k+1} + \mathcal{B}z^{k+1} = Mz^k - \mathcal{A}z^k$:

$$\begin{cases} \frac{1}{\eta} x^k + A^\top s^k - \nabla f(x^k) & \in \frac{1}{\eta} x^{k+1} + A^\top s^{k+1} + A^\top s^{k+1} + \partial g(x^{k+1}), \\ \frac{1}{\gamma} s^k + A\, x^k & \in \frac{1}{\gamma} s^{k+1} + A\, x^{k+1} \; - A\, x^{k+1} \; + \partial h^*(s^{k+1}), \end{cases}$$

which is equivalent to

$$\begin{cases} s^{k+1} = \mathbf{prox}_{\gamma h^*}(s^k + \gamma A x^k), \\ x^{k+1} = \mathbf{prox}_{\eta g}(x^k - \eta(\nabla f(x^k) + A^\top(2s^{k+1} - s^k))). \end{cases}$$

Now we derived the Condat-Vũ algorithm. With proper choices of η and γ, the forward-backward operator $\mathcal{T} = (\mathcal{I} + M^{-1}\mathcal{B})^{-1} \circ (\mathcal{I} - M^{-1}\mathcal{A})$ can be shown to be α-averaged if we use the inner product $\langle z_1, z_2\rangle_M = z_1^\top M z_2$ and norm $\|z\|_M = \sqrt{z^\top M z}$ on the space of $z = \begin{bmatrix} x \\ s \end{bmatrix}$. More details can be found in [23].

If we change the matrix M to $\begin{bmatrix} \frac{1}{\eta} I & -A^\top \\ -A & \frac{1}{\gamma} I \end{bmatrix}$, the other algorithm (29) can be derived similarly.

Appendix D. Proof of Convergence for Async-parallel Primal-dual Coordinate Update Algorithms

Algorithms 1 and 2 differ from that in [54] in the following aspects:

1. the operator $\mathcal{T}_{\mathrm{CV}}$ is nonexpansive under a norm induced by a symmetric positive definite matrix M (see Appendix C), instead of the standard Euclidean norm;
2. the coordinate updates are no longer orthogonal to each other under the norm induced by M;
3. the block coordinates may overlap each other.

Because of these differences, we make two major modifications to the proof in [54, Section 3]: (i) adjusting parameters in [54, Lemma 2] and modify its proof to accommodate for the new norm; (2) modify the inner product and induced norm used in [54, Theorem 2] and adjust the constants in [54, Theorems 2 and 3].

We assume the same inconsistent case as in [54], i.e., the relationship between $\hat{z}^k$ and z^k is

$$\hat{z}^k = z^k + \sum_{d \in J(k)} (z^d - z^{d+1}), \tag{77}$$

where $J(k) \subseteq \{k-1, ..., k-\tau\}$ and τ is the maximum number of other updates to z during the computation of the update. Let $\mathcal{S} = \mathcal{I} - \mathcal{T}_{\mathrm{CV}}$. Then the coordinate update can be rewritten as $z^{k+1} = z^k - \frac{\eta_k}{(m+p)q_{i_k}} \mathcal{S}_{i_k} \hat{z}^k$, where $\mathcal{S}_i \hat{z}^k = (\hat{z}_1^k, \ldots, \hat{z}_{i-1}^k, (\mathcal{S}\hat{z}^k)_i, \hat{z}_{i+1}^k, \ldots, \hat{z}_{m+p}^k)$ for Algorithm 1. For Algorithm 2, the update is

$$z^{k+1} = z^k - \frac{\eta_k}{m q_{i_k}} \mathcal{S}_{i_k} \hat{z}^k, \tag{78}$$

where

$$\mathcal{S}_i\hat{z}^k = \begin{bmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & & & & \\ & & & \mathcal{I}_{\mathbb{H}_i} & & & & & \\ & & & & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & & \\ & & & & & & & \rho_{i,1}\mathcal{I}_{\mathbb{G}_1} & \\ & & & & & & & & \ddots \\ & & & & & & & & & \rho_{i,p}\mathcal{I}_{\mathbb{G}_p} \end{bmatrix} \mathcal{S}\hat{z}^k.$$

Let $\lambda_{\max}$ and $\lambda_{\min}$ be the maximal and minimal eigenvalues of the matrix M, respectively, and $\kappa = \frac{\lambda_{\max}}{\lambda_{\min}}$ be the condition number. Then we have the following lemma.

Lemma 1. *For both Algorithms 1 and 2,*

$$\sum_i \mathcal{S}_i\hat{z}^k = \mathcal{S}\hat{z}^k, \tag{79}$$

$$\sum_i \|\mathcal{S}_i\hat{z}^k\|_M^2 \le \kappa\|\mathcal{S}\hat{z}^k\|_M^2, \tag{80}$$

where i runs from 1 to $m+p$ for Algorithm 1 and 1 to m for Algorithm 2.

Proof. The first part comes immediately from the definition of $\mathcal{S}$ for both algorithms. For the second part, we have

$$\sum_i \|\mathcal{S}_i\hat{z}^k\|_M^2 \le \sum_i \lambda_{\max}\|\mathcal{S}_i\hat{z}^k\|^2 = \lambda_{\max}\|\mathcal{S}\hat{z}^k\|^2 \le \frac{\lambda_{\max}}{\lambda_{\min}}\|\mathcal{S}\hat{z}^k\|_M^2, \tag{81}$$

for Algorithm 1. For Algorithm 2, the equality is replaced by "$\le$". □

At last we define

$$\bar{z}^{k+1} := z^k - \eta_k\mathcal{S}\hat{z}^k, \tag{82}$$

$q_{\min} = \min_i q_i > 0$, and $|J(k)|$ be the number of elements in $J(k)$. It is shown in [23] that with proper choices of η and γ, $\mathcal{T}_{\mathrm{CV}}$ is nonexpansive under the norm induced by M. Then Lemma 2 shows that $\mathcal{S}$ is 1/2-cocoercive under the same norm.

Lemma 2. *An operator $\mathcal{T}:\mathbb{F}\to\mathbb{F}$ is nonexpansive under the induced norm by M if and only if $\mathcal{S}=\mathcal{I}-\mathcal{T}$ is $1/2$-cocoercive under the same norm, i.e.,*

$$\langle z-\tilde{z},\mathcal{S}z-\mathcal{S}\tilde{z}\rangle_M \geq \frac{1}{2}\|\mathcal{S}z-\mathcal{S}\tilde{z}\|_M^2,\quad \forall\ z,\tilde{z}\in\mathbb{F}. \tag{83}$$

The proof is the same as that of [5, Proposition 4.33].

We state the complete theorem for Algorithm 2. The theorem for Algorithm 1 is similar (we need to change m to $m+p$ when necessary).

Theorem 2. *Let Z^* be the set of optimal solutions of (24) and $(z^k)_{k\geq 0}\subset\mathbb{F}$ be the sequence generated by Algorithm 2 (with proper choices of η and γ such that $\mathcal{T}_{\rm CV}$ is nonexpansive under the norm induced by M), under the following conditions:*

(i) f,g,h^* are closed proper convex functions. In addition, f is differentiable and ∇f is Lipschitz continuous with β;

(ii) $\eta_k\in[\eta_{\min},\eta_{\max}]$ for certain $0<\eta_{\max}<\frac{mq_{\min}}{2\tau\sqrt{\kappa q_{\min}}+\kappa}$ and any $0<\eta_{\min}\leq\eta_{\max}$.

Then $(z^k)_{k\geq 0}$ converges to a Z^*-valued random variable with probability 1.

The proof directly follows [54, Section 3]. Here we only present the key modifications. Interested readers are referred to [54] for the complete procedure.

The next lemma shows that the conditional expectation of the distance between z^{k+1} and any $z^*\in\mathbf{Fix}\mathcal{T}_{\rm CV}=Z^*$ for given $\mathcal{Z}^k=\{z^0,z^1,\cdots,z^k\}$ has an upper bound that depends on $\mathcal{Z}^k$ and z^* only.

Lemma 3. *Let $(z^k)_{k\geq 0}$ be the sequence generated by Algorithm 2. Then for any $z^*\in\mathbf{Fix}\mathcal{T}_{\rm CV}$, we have*

$$\begin{aligned}\mathbb{E}\big(\|z^{k+1}-z^*\|_M^2\,\big|\,\mathcal{Z}^k\big)\leq&\|z^k-z^*\|_M^2+\frac{\sigma}{m}\sum_{d\in J(k)}\|z^d-z^{d+1}\|_M^2\\&+\frac{1}{m}\left(\frac{|J(k)|}{\sigma}+\frac{\kappa}{mq_{\min}}-\frac{1}{\eta_k}\right)\|z^k-\bar{z}^{k+1}\|_M^2\end{aligned} \tag{84}$$

where $\mathbb{E}(\cdot\,|\,\mathcal{Z}^k)$ denotes conditional expectation on $\mathcal{Z}^k$ and $\sigma>0$ (to be optimized later).

Proof. We have

$$
\begin{aligned}
&\mathbb{E}\left(\|z^{k+1}-z^*\|_M^2 \,|\, \mathcal{Z}^k\right)\\
\overset{(78)}{=}&\mathbb{E}\left(\|z^k-\tfrac{\eta_k}{mq_{i_k}}\mathcal{S}_{i_k}\hat{z}^k-z^*\|_M^2 \,|\, \mathcal{Z}^k\right)\\
=&\|z^k-z^*\|_M^2+\mathbb{E}\left(\tfrac{2\eta_k}{mq_{i_k}}\left\langle\mathcal{S}_{i_k}\hat{z}^k,z^*-z^k\right\rangle_M+\tfrac{\eta_k^2}{m^2q_{i_k}^2}\|\mathcal{S}_{i_k}\hat{z}^k\|_M^2 \,\Big|\, \mathcal{Z}^k\right)\\
=&\|z^k-z^*\|_M^2+\tfrac{2\eta_k}{m}\textstyle\sum_{i=1}^m\left\langle\mathcal{S}_i\hat{z}^k,z^*-z^k\right\rangle_M+\tfrac{\eta_k^2}{m^2}\sum_{i=1}^m\tfrac{1}{q_i}\|\mathcal{S}_i\hat{z}^k\|_M^2\\
=&\|z^k-z^*\|_M^2+\tfrac{2\eta_k}{m}\left\langle\mathcal{S}\hat{z}^k,z^*-z^k\right\rangle_M+\tfrac{\eta_k^2}{m^2}\textstyle\sum_{i=1}^m\tfrac{1}{q_i}\|\mathcal{S}_i\hat{z}^k\|_M^2,
\end{aligned}
\tag{85}
$$

where the third equality holds because the probability of choosing i is q_i.

Note that

$$
\begin{aligned}
\textstyle\sum_{i=1}^m\frac{1}{q_i}\|\mathcal{S}_i\hat{z}^k\|_M^2\le\frac{1}{q_{\min}}\sum_{i=1}^m\|\mathcal{S}_i\hat{z}^k\|_M^2&\overset{(80)}{\le}\frac{\kappa}{q_{\min}}\sum_{i=1}^m\|\mathcal{S}\hat{z}^k\|_M^2\\
&\overset{(82)}{=}\frac{\kappa}{\eta_k^2q_{\min}}\|z^k-\bar{z}^{k+1}\|_M^2,
\end{aligned}
\tag{86}
$$

and

$$
\begin{aligned}
&\langle\mathcal{S}\hat{z}^k,z^*-z^k\rangle_M\\
=&\langle\mathcal{S}\hat{z}^k,z^*-\hat{z}^k+\textstyle\sum_{d\in J(k)}(z^d-z^{d+1})\rangle_M\\
\overset{(82)}{=}&\langle\mathcal{S}\hat{z}^k,z^*-\hat{z}^k\rangle_M+\tfrac{1}{\eta_k}\textstyle\sum_{d\in J(k)}\langle z^k-\bar{z}^{k+1},z^d-z^{d+1}\rangle_M\\
\le&\langle\mathcal{S}\hat{z}^k-\mathcal{S}z^*,z^*-\hat{z}^k\rangle_M+\tfrac{1}{2\eta_k}\textstyle\sum_{d\in J(k)}\left(\frac{1}{\sigma}\|z^k-\bar{z}^{k+1}\|_M^2+\sigma\|z^d-z^{d+1}\|_M^2\right)\\
\overset{(83)}{\le}&-\tfrac{1}{2}\|\mathcal{S}\hat{z}^k\|_M^2+\tfrac{1}{2\eta_k}\textstyle\sum_{d\in J(k)}(\frac{1}{\sigma}\|z^k-\bar{z}^{k+1}\|_M^2+\sigma\|z^d-z^{d+1}\|_M^2)\\
\overset{(82)}{=}&-\tfrac{1}{2\eta_k^2}\|z^k-\bar{z}^{k+1}\|_M^2+\tfrac{|J(k)|}{2\sigma\eta_k}\|z^k-\bar{z}^{k+1}\|_M^2+\tfrac{\sigma}{2\eta_k}\textstyle\sum_{d\in J(k)}\|z^d-z^{d+1}\|_M^2,
\end{aligned}
\tag{87}
$$

where the first inequality follows from the Young's inequality. Plugging (86) and (87) into (85) gives the desired result. □

Let $\mathbb{F}^{\tau+1}=\prod_{i=0}^{\tau}\mathbb{F}$ be a product space and $\langle\cdot\,|\,\cdot\rangle$ be the induced inner product:

$$
\langle(z^0,\ldots,z^\tau)\,|\,(\tilde{z}^0,\ldots,\tilde{z}^\tau)\rangle=\sum_{i=0}^{\tau}\langle z^i,\tilde{z}^i\rangle_M,\quad\forall(z^0,\ldots,z^\tau),(\tilde{z}^0,\ldots,\tilde{z}^\tau)\in\mathbb{F}^{\tau+1}.
$$

Define a $(\tau+1)\times(\tau+1)$ matrix U' by

$$U' := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \sqrt{\frac{q_{\min}}{\kappa}} \begin{bmatrix} \tau & -\tau & & & & \\ -\tau & 2\tau-1 & 1-\tau & & & \\ & 1-\tau & 2\tau-3 & 2-\tau & & \\ & & \ddots & \ddots & \ddots & \\ & & & -2 & 3 & -1 \\ & & & & -1 & 1 \end{bmatrix},$$

and let $U = U' \otimes \mathcal{I}_{\mathbb{F}}$. Here $\otimes$ represents the Kronecker product. For a given $(y^0, \cdots, y^\tau) \in \mathbb{F}^{\tau+1}$, $(z^0, \cdots, z^\tau) = U(y^0, \cdots, y^\tau)$ is given by:

$$\begin{aligned} &z^0 = y^0 + \tau\sqrt{\tfrac{q_{\min}}{\kappa}}(y^0 - y^1), \\ &z^i = \sqrt{\tfrac{q_{\min}}{\kappa}}((i-\tau-1)y^{i-1} + (2\tau-2i+1)y^i + (i-\tau)y^{i+1}), \text{ if } 1 \le i \le \tau-1, \\ &z^\tau = \sqrt{\tfrac{q_{\min}}{\kappa}}(y^\tau - y^{\tau-1}). \end{aligned}$$

Then U is a self-adjoint and positive definite linear operator since U' is symmetric and positive definite, and we define $\langle \cdot \,|\, \cdot \rangle_U = \langle \cdot \,|\, U \cdot \rangle$ as the U-weighted inner product and $\|\cdot\|_U$ the induced norm.

Let

$$\mathbf{z}^k = (z^k, z^{k-1}, \ldots, z^{k-\tau}) \in \mathbb{F}^{\tau+1},\ k \ge 0,\ \mathbf{z}^* = (z^*, \ldots, z^*) \in \mathbf{Z}^* \subseteq \mathbb{F}^{\tau+1},$$

where $z^k = z^0$ for $k < 0$. With

(88)
$$\xi_k(\mathbf{z}^*) := \|\mathbf{z}^k - \mathbf{z}^*\|_U^2 = \|z^k - z^*\|_M^2 + \sqrt{\tfrac{q_{\min}}{\kappa}} \textstyle\sum_{i=k-\tau}^{k-1} (i-(k-\tau)+1)\|z^i - z^{i+1}\|_M^2,$$

we have the following fundamental inequality:

Theorem 3 (fundamental inequality)**.** *Let $(z^k)_{k\ge0}$ be the sequence generated by Algorithm 2. Then for any $\mathbf{z}^* \in \mathbf{Z}^*$, it holds that*

$$\mathbb{E}\left(\xi_{k+1}(\mathbf{z}^*) \,\middle|\, \mathcal{Z}^k\right) \le \xi_k(\mathbf{z}^*) + \frac{1}{m}\left(\frac{2\tau\sqrt{\kappa}}{m\sqrt{q_{\min}}} + \frac{\kappa}{m q_{\min}} - \frac{1}{\eta_k}\right)\|\bar{z}^{k+1} - z^k\|_M^2.$$

Proof. Let $\sigma = m\sqrt{\frac{q_{\min}}{\kappa}}$. We have

$$
\begin{aligned}
&\mathbb{E}(\xi_{k+1}(\mathbf{z}^*)|\mathcal{Z}^k)\\
\overset{(88)}{=}&\mathbb{E}(\|z^{k+1}-z^*\|_M^2|\mathcal{Z}^k)+\sigma\textstyle\sum_{i=k+1-\tau}^{k}\frac{i-(k-\tau)}{m}\mathbb{E}(\|z^i-z^{i+1}\|_M^2|\mathcal{Z}^k)\\
\overset{(78)}{=}&\mathbb{E}(\|z^{k+1}-z^*\|_M^2|\mathcal{Z}^k)+\tfrac{\sigma\tau}{m}\mathbb{E}(\tfrac{\eta_k^2}{m^2q_{i_k}^2}\|S_{i_k}\hat{z}^k\|_M^2|\mathcal{Z}^k)+\sigma\textstyle\sum_{i=k+1-\tau}^{k-1}\frac{i-(k-\tau)}{m}\|z^i-z^{i+1}\|_M^2\\
\leq&\mathbb{E}(\|z^{k+1}-z^*\|_M^2|\mathcal{Z}^k)+\tfrac{\sigma\tau\kappa}{m^3q_{\min}}\|z^k-\bar{z}^{k+1}\|_M^2+\sigma\textstyle\sum_{i=k+1-\tau}^{k-1}\frac{i-(k-\tau)}{m}\|z^i-z^{i+1}\|_M^2\\
\overset{(84)}{\leq}&\|z^k-z^*\|_M^2+\tfrac{1}{m}\left(\tfrac{|J(k)|}{\sigma}+\tfrac{\sigma\tau\kappa}{m^2q_{\min}}+\tfrac{\kappa}{mq_{\min}}-\tfrac{1}{\eta_k}\right)\|z^k-\bar{z}^{k+1}\|_M^2\\
&+\tfrac{\sigma}{m}\textstyle\sum_{d\in J(k)}\|z^d-z^{d+1}\|_M^2+\sigma\sum_{i=k+1-\tau}^{k-1}\frac{i-(k-\tau)}{m}\|z^i-z^{i+1}\|_M^2\\
\leq&\|z^k-z^*\|_M^2+\tfrac{1}{m}\left(\tfrac{\tau}{\sigma}+\tfrac{\sigma\tau\kappa}{m^2q_{\min}}+\tfrac{\kappa}{mq_{\min}}-\tfrac{1}{\eta_k}\right)\|z^k-\bar{z}^{k+1}\|_M^2\\
&+\tfrac{\sigma}{m}\textstyle\sum_{i=k-\tau}^{k-1}\|z^i-z^{i+1}\|_M^2+\sigma\sum_{i=k+1-\tau}^{k-1}\frac{i-(k-\tau)}{m}\|z^i-z^{i+1}\|_M^2\\
\overset{(88)}{=}&\xi_k(\mathbf{x}^*)+\tfrac{1}{m}\left(\tfrac{2\tau\sqrt{\kappa}}{m\sqrt{q_{\min}}}+\tfrac{\kappa}{mq_{\min}}-\tfrac{1}{\eta_k}\right)\|z^k-\bar{z}^{k+1}\|_M^2.
\end{aligned}
$$

The first inequality follows from the computation of the conditional expectation on $\mathcal{Z}^k$ and (86), the third inequality holds because $J(k)\subset\{k-1,k-2,\cdots,k-\tau\}$, and the last equality uses $\sigma = m\sqrt{\frac{q_{\min}}{\kappa}}$, which minimizes $\frac{\tau}{\sigma}+\frac{\sigma\tau\kappa}{m^2q_{\min}}$ over $\sigma>0$. Hence, the desired inequality holds. □

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