Monoidal Cut Strengthening and Generalized Mixed-Integer Rounding for Disjunctions and Complementarity Constraints*

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Abstract

In the early 1980s, Balas and Jeroslow presented monoidal disjunctive cuts exploiting the integrality of variables. This article investigates the relation of monoidal cut strengthening to other classes of cutting planes for general two-term disjunctions. We introduce a generalization of mixed-integer rounding cuts and show equivalence to monoidal disjunctive cuts. Moreover, we demonstrate the effectiveness of these cuts via computational experiments on instances involving complementarity constraints. Finally, we present an adaptation of the mixed-integer rounding approach for mixed-complementarity problems.

1 Introduction

Let n be a positive integer and $V = \{1, \dots, n\}$. This article investigates cutting planes for problems involving a disjunction and mixed-integer conditions of the form

$$\bigvee_{j \in D} \Bigl(\sum_{\nu \in V} d_{j\nu}\, x_\nu \geq d_{j0} \Bigr), \quad x \in \mathbb{Z}^I \times \mathbb{R}^J, \tag{1}$$

where D is a finite set, $V = I \cup J$, and the coefficients $d_{j\nu}$ and right hand sides d_{j0} are assumed to be real valued.

Supposing that $x \geq 0$ and $d_{j0} > 0$ for $j \in D$, one can observe that the disjunctive inequality $\delta^T x \geq 1$ with

$$\delta_\nu := \max \left\{ \frac{d_{j\nu}}{d_{j0}} : j \in D \right\} \quad \forall\, \nu \in V \tag{2}$$

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is valid for (1). This inequality has many applications, e.g., for disjunctions arising from binary variables (see Balas and Perregaard [2]), semi-continuous variables, complementarity constraints, or cardinality constraints (see de Farias et al. [8]).

Balas and Jeroslow [1] used the integrality information of variables to strengthen the coefficients δ_ν, $\nu \in I$ by an algebraic method called monoidal cut strengthening. For disjunctions $x_i \leq \kappa \vee x_i \geq \kappa + 1$, $\kappa \in \mathbb{Z}$, arising from integer variables, it is known that monoidal strengthened disjunctive, Gomory mixed-integer, and mixed-integer rounding cuts are essentially the same, see Cornuéjols and Li [7], Conforti et al. [6], and Balas and Jeroslow [1].

In this article, we discuss the relation of monoidal cut strengthening to a class of generalized mixed-integer rounding cuts for general two-term disjunctions. Our computational results confirm the practical use of monoidal cut strengthening on the example of disjunctions arising from complementarity constraints. This also shows the importance of generalized mixed-integer rounding cuts for these disjunctions. As a further contribution, we present a mixed-complementarity rounding inequality that is similar to the mixed-integer rounding inequality, but exploits complementarity constraints instead of integrality constraints.

Disjunctive cuts are widely used in literature. Balas et al. [4] investigated disjunctive cuts for sets described by a single disjunction and an additional inequality system $Ax \leq b$. These cutting planes are separated by solving a (relatively large) cut generating linear program (CGLP), which lives in a lifted space. However, in practice, explicitly solving CGLP is often too time consuming. To work around with this, Balas and Perregaard [2] investigated a separation procedure for disjunctions modeled as binary variables, which solves CGLP implicitly by executing pivot operations in the simplex tableau of the original LP. Later, Kis [13] generalized this approach to two-term disjunctions and tested it computationally for disjunctions modeled as complementarity constraints. Balas and Perregaard [2] also used monoidal cut strengthening to improve CGLP cuts. Moreover, monoidal cut strengthening is a subject in an article of Balas and Qualizza [3], who investigated an enhanced monoidal cut strengthening procedure. For disjunctions modeled as binary variables, the enhanced procedure sometimes yields cutting planes that are stronger than Gomory mixed-integer cuts. A similar and in the literature often treated as equivalent form of monoidal cut strengthening, called trivial lifting, was studied by Dey and Wolsey [9], who applied it to multiple-term disjunctions arising from lattice free convex sets. To the best of our knowledge, monoidal cut strengthening has not been investigated for complementarity constraints so far.

The contents of this article are organized as follows: In Section 2, we review the monoidal cut strengthening method of Balas and Jeroslow. Afterwards, in Section 3, we introduce a generalized version of the mixed-integer rounding cut that strengthens inequalities of Nemhauser and Wolsey [15]. Our main result proves that for two-term disjunctions with $|D| = 2$, the elementary closures of monoidal disjunctive cuts and generalized mixed-integer rounding cuts are identical. In Section 5, we demonstrate the effectiveness of monoidal cut strengthening on randomly generated instances involving disjunctions modeled as complementarity constraints. Finally, in Section 4, we present an adaptation of the MIR approach to generate a mixed-complementarity rounding inequality.

2 Monoidal Cut Strengthening

Monoidal cut strengthening is a method that was introduced by Balas and Jeroslow [1] to improve the classical disjunctive cut $\delta^T x \geq 1$ with δ_ν given by (2). The method intends to strengthen the cut coefficients δ_ν belonging to integer variables $\nu \in I$ by finding optimal solutions over a monoid $M := \{m \in \mathbb{Z}^D \ : \ \sum_{j \in D} m_j \geq 0\}$. Although monoidal cut strengthening can be applied to to general disjunctions of the form (1), it is usually investigated in the context of integer variables. In the upcoming part of this article, we discuss them in a more general setting.

We first review monoidal cut strengthening in the following two theorems:

Theorem 2.1 (Balas and Jeroslow [1]). *Let a problem $\mathcal{P}$ involving a disjunction* (1) *be given, where $x \geq 0$ and $d_{j0} > 0$ for $j \in D$. Suppose that for all $j \in D$, a finite upper bound $u_j > 0$ of $d_{j0} - \sum_{\nu \in V} d_{j\nu} x_\nu$ is known. Then the inequality $\overline{\delta}^\top x \geq 1$ with*

$$\overline{\delta}_\nu := \begin{cases} \min\limits_{m \in M} \max \left\{ \frac{d_{j\nu} + u_j m_j}{d_{j0}}, \, j \in D \right\}, & \text{if } \nu \in I, \\ \delta_\nu, & \text{otherwise}, \end{cases} \tag{3}$$

is valid for $\mathcal{P}$.

An optimal solution to (3) can be computed algorithmically with a complexity that is linear in $|D|$, see [1]. Nevertheless, there does not exist a general closed-form expression for the coefficients $\overline{\delta}_\nu$, $\nu \in I$. This makes it difficult to recognize a general correspondence of monoidal strengthened disjunctive cuts to other classes of cutting planes. However, for the special case of two-term disjunctions, i.e., $|D| = 2$, the coefficients $\overline{\delta}_\nu$, $\nu \in I$, are given by the following explicit formula.

Theorem 2.2 (Balas and Jeroslow [1]). *Let $D = \{i, k\}$. Then the coefficients of* (3) *can be determined as*

$$\overline{\delta}_\nu = \min \left\{ \frac{d_{i\nu} - u_i \lfloor m_\nu^* \rfloor}{d_{i0}}, \frac{d_{k\nu} + u_k \lceil m_\nu^* \rceil}{d_{k0}} \right\} \quad \forall \, \nu \in I,$$

where

$$m_\nu^* := \frac{d_{i\nu}\, d_{k0} - d_{k\nu}\, d_{i0}}{u_i\, d_{k0} + u_k\, d_{i0}}.$$

Note that $u_i \lfloor m_\nu^* \rfloor$ and $u_k \lceil m_\nu^* \rceil$ tend to zero if u_k and u_i tend to infinity. Thus, in order to generate strong cutting planes, the bounds u_i and u_k should be small.

For disjunctions $x_i \leq \kappa \vee x_i \geq \kappa{+}1$, $\kappa \in \mathbb{Z}$, arising from integer variables, Balas and Jeroslow [1] as well as Dey and Wolsey [9] demonstrated that Gomory Mixed-Integer (GMI) cuts can be derived as monoidal strengthened disjunctive cuts.

A natural question is whether monoidal cut strengthening can be traced back to known theory in mixed-integer programming for general two-term disjunctions. This is studied in the next section.

3 Relation to Mixed-Integer Rounding

The mixed-integer rounding (MIR) procedure was introduced by Nemhauser and Wolsey [15] in 1988. The idea is to exploit the integrality information of variables via rounding. Later, Wolsey [18] explicitly defined the MIR inequality. This inequality has many applications in mixed-integer programming, e.g., for the derivation of flow cover inequalities, see Marchand and Wolsey [14]. In this section, we introduce a generalized mixed-integer rounding inequality. We will show that for two-term disjunctions with $|D| = 2$, the monoidal disjunctive cuts of Theorem 2.1 are generalized mixed-integer rounding inequalities, and vice versa.

Given $\lambda \in \mathbb{R}$ we denote $\lambda^+ := \max\{0, \lambda\}$. The (classical) MIR inequality is given as:

Lemma 3.1 (Wolsey [18])**.** *Consider the mixed-integer set*

$$S = \{x \in \mathbb{Z}_+^I \times \mathbb{R}_+^J : \sum_{\nu \in V} \alpha_\nu\, x_\nu \leq \beta\}.$$

Define $f_0 := \beta - \lfloor \beta \rfloor$ *and* $f_\nu := \alpha_\nu - \lfloor \alpha_\nu \rfloor$ *for* $\nu \in I$*. Then the MIR inequality*

$$\sum_{\nu \in I} \left(\lfloor \alpha_\nu \rfloor + \frac{(f_\nu - f_0)^+}{1 - f_0} \right) x_\nu + \frac{1}{1 - f_0} \sum_{\substack{\nu \in J \\ \alpha_\nu < 0}} \alpha_\nu\, x_\nu \leq \lfloor \beta \rfloor$$

is valid for $\operatorname{conv}(S)$*.*

Marchand and Wolsey [14] showed that the GMI inequality, which is generated from a single simplex tableau row, can be derived from the MIR inequality. The converse of this statement is also correct, see Cornuéjols and Li [7]. In order to show a similar result for cutting planes that are generated from two rows of the simplex tableau, we need to generalize Lemma 3.1 to the mixed-integer set

$$S' = \{x \in \mathbb{Z}_+^I \times \mathbb{R}_+^J : \sum_{\nu \in V} \alpha_\nu^1 \, x_\nu \le \beta^1, \; \sum_{\nu \in V} \alpha_\nu^2 \, x_\nu \le \beta^2\},$$

which is constrained by two inequalities. Define

$$\begin{aligned} \Delta_0 &\coloneqq \beta^2 - \beta^1, & \Delta_\nu &\coloneqq \alpha_\nu^2 - \alpha_\nu^1 & \forall\, \nu \in I, \\ g_0 &\coloneqq \Delta_0 - \lfloor \Delta_0 \rfloor, & g_\nu &\coloneqq \Delta_\nu - \lfloor \Delta_\nu \rfloor & \forall\, \nu \in I. \end{aligned} \tag{4}$$

Nemhauser and Wolsey [15] showed that the inequality

$$\sum_{\nu \in I} \lfloor \Delta_\nu \rfloor \, x_\nu + \frac{1}{1 - g_0} \left(\sum_{\nu \in I} \alpha_\nu^1 \, x_\nu + \sum_{\nu \in J} \min\{\alpha_\nu^1, \alpha_\nu^2\} \, x_\nu - \beta^1 \right) \le \lfloor \Delta_0 \rfloor \tag{5}$$

is valid for $\operatorname{conv}(S')$. One can derive a stronger inequality if $g_\nu > g_0$ for some $\nu \in I$ as follows.

Lemma 3.2. *The generalized mixed-integer rounding (GMIR) inequality*

$$\begin{aligned} \sum_{\nu \in I} \left(\lfloor \Delta_\nu \rfloor + \frac{(g_\nu - g_0)^+}{1 - g_0} \right) x_\nu + \frac{1}{1 - g_0} \Bigg(\sum_{\nu \in I} \alpha_\nu^1 \, x_\nu \\ + \sum_{\nu \in J} \min\{\alpha_\nu^1, \alpha_\nu^2\} \, x_\nu - \beta^1 \Bigg) \le \lfloor \Delta_0 \rfloor \end{aligned} \tag{6}$$

is valid for $\operatorname{conv}(S')$.

Proof. The proof is a modification of a proof from Nemhauser and Wolsey [15] showing the validity of Inequality (5).

Since $x \ge 0$, it follows from $\sum_{\nu \in V} \alpha_\nu^1 \, x_\nu \le \beta^1$ and $\sum_{\nu \in V} \alpha_\nu^2 \, x_\nu \le \beta^2$ that

$$\sum_{\nu \in I} \alpha_\nu^k \, x_\nu + \sum_{\nu \in J} \min\{\alpha_\nu^1, \alpha_\nu^2\} \, x_\nu \le \beta^k,$$

for $k \in \{1, 2\}$, is valid. The inequality for $k = 2$ can be written as

$$\sum_{\nu \in I} (\alpha_\nu^2 - \alpha_\nu^1) \, x_\nu - s \le \beta^2 - \beta^1, \tag{7}$$

where

$$s \coloneqq - \sum_{\nu \in I} \alpha_\nu^1 \, x_\nu - \sum_{\nu \in J} \min\{\alpha_\nu^1, \alpha_\nu^2\} \, x_\nu + \beta^1.$$

From the inequality for $k = 1$, we deduce that $s \ge 0$. Therefore, Inequality (7) is in a form suitable for application of Lemma 3.1 and we obtain the validity of (6). $\square$

From the proof of Lemma 3.2, it follows that every GMIR inequality is an MIR inequality. The converse is also correct, since the GMIR inequality reduces to the MIR inequality if $\alpha^1 = 0$ and $\beta^1 = 0$.

The GMIR inequality allows for a new perspective on monoidal strengthened disjunctive cuts:

Theorem 3.3. *Let $D = \{i, k\}$. Then the cut derived from monoidal cut strengthening (see Theorem 2.1) is a GMIR inequality.*

Proof. We consider the two-term disjunction

$$\sum_{\nu \in V} \bar{d}_{i\nu}\, x_\nu \geq 1 \quad \vee \quad \sum_{\nu \in V} \bar{d}_{k\nu}\, x_\nu \geq 1, \tag{8}$$

where $x \geq 0$. Here, we assume w.l.o.g. that the right hand sides are scaled to be one. At the end of the proof, the cut of Theorem 2.1 can be recovered by replacing $\bar{d}_{i\nu}$ and $\bar{d}_{k\nu}$ with $d_{i\nu}/d_{i0}$ and $d_{k\nu}/d_{k0}$, respectively (note that $d_{i0}, d_{k0} > 0$ by assumption).

Suppose that $u_i, u_k > 0$ are valid upper bounds of $1 - \sum_{\nu \in V} \bar{d}_{i\nu}$ and $1 - \sum_{\nu \in V} \bar{d}_{k\nu}$, respectively. We introduce an auxiliary binary variable $z \in \{0, 1\}$ to transform (8) into

$$1 - \sum_{\nu \in V} \bar{d}_{i\nu}\, x_\nu \leq u_i\,(1 - z), \quad 1 - \sum_{\nu \in V} \bar{d}_{k\nu}\, x_\nu \leq u_k\, z.$$

Multiplying both inequalities with $\gamma := \frac{1}{u_i + u_k} > 0$ yields

$$\begin{aligned} \gamma\, u_i\, z - \sum_{\nu \in V} \gamma\, \bar{d}_{i\nu}\, x_\nu &\leq \gamma\,(u_i - 1), \\ -\gamma\, u_k\, z - \sum_{\nu \in V} \gamma\, \bar{d}_{k\nu}\, x_\nu &\leq -\gamma. \end{aligned}$$

We will apply Lemma 3.2 with

$$\begin{aligned} &\beta^1 = \gamma\,(u_i - 1), \qquad &&\alpha^1_{\nu^*} = \gamma\, u_i, \qquad &&\alpha^1_\nu = -\gamma\, \bar{d}_{i\nu} \;\; \forall \nu \in V, \\ &\beta^2 = -\gamma, &&\alpha^2_{\nu^*} = -\gamma\, u_k, &&\alpha^2_\nu = -\gamma\, \bar{d}_{k\nu} \;\; \forall \nu \in V, \end{aligned}$$

where ν^* is the additional index belonging to z. We use that $0 < \gamma\, u_i \leq 1$ to obtain the following values defined in (4)

$$\begin{aligned} \Delta_0 &= -\gamma - \gamma\,(u_i - 1) = -\gamma\, u_i, \\ g_0 &= -\gamma\, u_i - \lfloor -\gamma\, u_i \rfloor = -\gamma\, u_i + 1 \geq 0, \\ \Delta_{\nu^*} &= -\gamma\, u_k - \gamma\, u_i = -1, \quad g_{\nu^*} = 0, \\ \Delta_\nu &= -\gamma\, \bar{d}_{k\nu} + \gamma\, \bar{d}_{i\nu} = m^*_\nu, \quad g_\nu = m^*_\nu - \lfloor m^*_\nu \rfloor, \quad \forall\, \nu \in I, \end{aligned}$$

where $m_\nu^* := \frac{\bar d_{i\nu} - \bar d_{k\nu}}{u_i + u_k}$ is defined as in Theorem 2.2. Taking into account that $g_{\nu^*} - g_0 \le 0$, we derive from Lemma 3.2 the inequality

$$\begin{aligned} &-z + \sum_{\nu\in I}\left(\lfloor m_\nu^*\rfloor + \frac{(g_\nu - g_0)^+}{1-g_0}\right)x_\nu + \frac{1}{1-g_0}\Bigg(\gamma\, u_i\, z - \sum_{\nu\in I}\gamma\,\bar d_{i\nu}\,x_\nu \\ &\quad - \sum_{\nu\in J}\max\{\gamma\,\bar d_{i\nu}, \gamma\,\bar d_{k\nu}\}\,x_\nu - \gamma\,(u_i - 1)\Bigg) \le \lfloor \Delta_0 \rfloor. \end{aligned}$$

Since $\lfloor \Delta_0\rfloor = -1$ and $1 - g_0 = \gamma\, u_i$, this inequality simplifies to

$$\begin{aligned} &-z + \sum_{\nu\in I}\left(\lfloor m_\nu^*\rfloor + \frac{(g_\nu - g_0)^+}{\gamma\, u_i}\right)x_\nu + z - \sum_{\nu\in I}\frac{\bar d_{i\nu}}{u_i}\,x_\nu \\ &\quad - \sum_{\nu\in J}\max\left\{\frac{\bar d_{i\nu}}{u_i}, \frac{\bar d_{k\nu}}{u_i}\right\}x_\nu - 1 + \frac{1}{u_i} \le -1. \end{aligned}$$

Multiplication with $-u_i < 0$ and further simplification yields

$$\begin{aligned} &-\sum_{\nu\in I}\left(u_i\,\lfloor m_\nu^*\rfloor + \frac{(g_\nu - g_0)^+}{\gamma}\right)x_\nu + \sum_{\nu\in I}\bar d_{i\nu}\,x_\nu \\ &\quad + \sum_{\nu\in J}\max\left\{\bar d_{i\nu}, \bar d_{k\nu}\right\}x_\nu \ge 1. \end{aligned} \tag{9}$$

Recall that $m_\nu^* := \frac{\bar d_{i\nu} - \bar d_{k\nu}}{u_i + u_k}$ and $\gamma := \frac{1}{u_i + u_k}$. We consider the general case that $g_\nu - g_0 \in \mathbb{R}$ and use that $\gamma\, u_i - 1 = -\gamma\, u_k$ to obtain

$$\begin{aligned} &u_i\,\lfloor m_\nu^*\rfloor + \frac{1}{\gamma}\,(g_\nu - g_0) \\ &= u_i\,\lfloor m_\nu^*\rfloor + (u_i + u_k)\,(m_\nu^* - \lfloor m_\nu^*\rfloor + \gamma\, u_i - 1) \\ &= -\,u_k\,\lfloor m_\nu^*\rfloor + (u_i + u_k)\,(m_\nu^* - \gamma\, u_k) \\ &= -\,u_k\,\lfloor m_\nu^*\rfloor + \bar d_{i\nu} - \bar d_{k\nu} - u_k \\ &= -\,u_k\,(1 + \lfloor m_\nu^*\rfloor) + \bar d_{i\nu} - \bar d_{k\nu}. \end{aligned} \tag{10}$$

Considering the case that $g_\nu = m_\nu^* - \lfloor m_\nu^*\rfloor > g_0$, it follows that $m_\nu^* \notin \mathbb{Z}$ because $g_0 \ge 0$. Then $1 + \lfloor m_\nu^*\rfloor = \lceil m_\nu^*\rceil$ and with the help of (10), Inequality (9) can be reformulated as

$$\begin{aligned} &\sum_{\substack{\nu\in I\\ g_\nu \le g_0}}(\bar d_{i\nu} - u_i\,\lfloor m_\nu^*\rfloor)\,x_\nu + \sum_{\substack{\nu\in I\\ g_\nu > g_0}}(\bar d_{k\nu} + u_k\,\lceil m_\nu^*\rceil)\,x_\nu \\ &\quad + \sum_{\nu\in J}\max\left\{\bar d_{i\nu}, \bar d_{k\nu}\right\}x_\nu \ge 1. \end{aligned}$$

To show the assertion, it remains to prove that we have

$$\min\left\{\bar d_{i\nu} - u_i\,\lfloor m_\nu^*\rfloor, \bar d_{k\nu} + u_k\,\lceil m_\nu^*\rceil\right\} = \begin{cases} \bar d_{i\nu} - u_i\,\lfloor m_\nu^*\rfloor, & \text{if } g_\nu \le g_0, \\ \bar d_{k\nu} + u_k\,\lceil m_\nu^*\rceil, & \text{otherwise,} \end{cases}$$

for $\nu \in I$, see Theorems 2.1 and 2.2. If $m_\nu \notin \mathbb{Z}$, then this results from the equality

$$\bar{d}_{i\nu} - u_i \lfloor m_\nu^* \rfloor - \frac{g_\nu - g_0}{\gamma} = \bar{d}_{k\nu} + u_k \lceil m_\nu^* \rceil$$

that we derive from (10) using that $1 + \lfloor m_\nu^* \rfloor = \lceil m_\nu^* \rceil$. Conversely, if $m_\nu \in \mathbb{Z}$, then $g_\nu = 0$, and we obtain that

$$\frac{g_\nu - g_0}{\gamma} = \frac{\gamma\, u_i - 1}{\gamma} = \frac{-\gamma\, u_k}{\gamma} = -u_k,$$

and since $\lfloor m_\nu^* \rfloor = \lceil m_\nu^* \rceil$, we derive from (10) that

$$\bar{d}_{i\nu} - u_i \lfloor m_\nu^* \rfloor = \bar{d}_{k\nu} + u_k \lceil m_\nu^* \rceil.$$

This completes the proof. □

As noted by one of the referees, an alternative proof of Theorem 3.3 using split cuts is possible. Moreover, Theorem 3.3 provides a new proof for the validity of the monoidal disjunctive cuts from Theorem 2.1 for the special case of two-term disjunctions.

The converse of Theorem 3.3 is also correct:

Corollary 3.4. *Every GMIR inequality is a monoidal disjunctive inequality.*

Proof. Every GMIR cut is an MIR cut (proof of Lemma 3.2), every MIR cut is a GMI cut (Cornuéjols and Li [7]), and every GMI cut is a monoidal disjunctive cut (Balas and Jeroslow [1]). □

It remains an open question whether the results of this section can be generalized to arbitrary disjunctions including the case $|D| > 2$.

4 Computational Experiments

In this section, we report on computational experience with our implementation of monoidal cut strengthening. One example where disjunctions naturally occur are complementarity constraints $x_i \cdot x_j = 0$, which can be modeled as $x_i \leq 0 \vee x_j \leq 0$ if $x_i, x_j \geq 0$. For the considered instances, the computational results confirm that GMIR cuts are for complementarity constraints just as important as MIR cuts for mixed-integer problems. Our implementation uses the branch-and-cut solver presented in [10] with SCIP 3.2 [16] as framework and CPLEX 12.6.0 as LP-solver. All experiments were run with a time limit of two hours on an Intel core i7-5820K CPU processor with 3.3 GHz and 15 MB cache.

4.1 Instances

The instances arise from an application in telecommunications engineering: Consider a multi-hop wireless network (see, e.g., Shi et al. [17]) represented by a digraph $D = (N, L)$. Here, N denotes a set of base stations that are connected via a set of wireless links L. The goal is to find a maximum data flow from a source to a destination subject to link capacity, flow conservation, and interference constraints.

The model of Gupta and Kumar [12] makes use of the same channel that is split into subchannels for all data transmissions; e.g., by frequencies $t \in T$. The interference constraints state that a base station cannot use the same subchannel for more than one transmission and more than one reception. This can be modeled with the help of complementarity constraints.

Given $u \in N$ we denote by $\delta^+(u)$ and $\delta^-(u)$ the successors and predecessors of u in D, respectively. Furthermore, we denote by E the set of interfering links and by $x^t_\ell \in \mathbb{Z}$ the variable for the integer flow on link $\ell \in L$ at subchannel $t \in T$. Mathematically, the problem can be formulated as follows:

$$\begin{aligned}
\max_{r \in \mathbb{Z}} \quad & r \\
\text{s.t.} \quad & \sum_{t \in T} \Big(\sum_{\ell \in \delta^+(u)} x^t_\ell - \sum_{\ell \in \delta^-(u)} x^t_\ell \Big) = \beta_u && \forall\, u \in N, \\
& 0 \leq x^t_\ell \leq C_\ell,\ x^t_\ell \in \mathbb{Z} && \forall\, \ell \in L,\ t \in T, \\
& x^t_\ell \cdot x^t_{\bar{\ell}} = 0 && \forall\, \{\ell, \bar{\ell}\} \in E,\ t \in T.
\end{aligned}$$

Here, β_u is r if u is the source, $-r$ if u is the destination, and 0 otherwise. The capacities $C_\ell \in \mathbb{Z}_+$, $\ell \in L$, depend on the distance between two nodes and the maximum transmission power, see the formula in Shi et al. [17].

We generated 30 instances that were produced analogously to [10]. All instances consider $|N| = 80$ nodes and $|T| = 8$ frequencies. The nodes were located randomly on a 1000×1000 grid.

4.2 Experimental Results

We made tests with three different settings: "disj-off" for turning the separation of disjunctive cuts off, "disj-uns" for using unstrengthened disjunctive cuts (2), and "disj-str" for using strengthened disjunctive cuts (3). Disjunctive cuts were generated directly from the simplex tableau for disjunctions $x_i \leq 0 \ \vee\ x_j \leq 0$ arising from complementarity constraints. If disjunctive cuts were turned on, we only separated them in the root node of the branch-and-bound tree. For all three settings, we additionally used bound inequalities that are described in [10]. We separated them with node-depth frequency 10. All other cutting plane separators of SCIP were turned off.

Table 1: Computational experiments on all instances.

settings	solved	cuts	nodes	time
disj-off	21	0.0	273617.0	1912.7
disj-uns	23	67.0	189894.0	1446.3
disj-str	24	65.4	149482.8	1118.6

In order to eliminate the influence of heuristics on the performance, we initialized the algorithm with precomputed optimal solutions. The remaining settings are set to their default in SCIP.

Table 1 shows the aggregated results of all 30 individual instances. Column "solved" lists the number of instances that could be solved within the time limit of two hours, column "cuts" the number of applied disjunctive cuts in arithmetic mean, and columns "nodes" and "time" the number of branching nodes and the CPU time after the solving process terminated both in shifted geometric mean. The shifted geometric mean of values $t_1, \ldots, t_n$ with shift value δ is defined as $\left(\Pi_{i=1}^{n}(t_i + \delta)\right)^{1/n} - \delta$. We use a shift value of 100 for the nodes and 10 for the CPU time.

The computational results clearly demonstrate that "disj-uns" outperforms "disj-off" and that "disj-str" outperforms "disj-uns" for the considered instances. Moreover, in direct comparison of "disj-off" and "disj-uns", a Wilcoxon signed rank test, see [5], confirmed a statistically significant reduction in the time with a p-value of less than 0.05. If we compare "disj-off" and "disj-str", the reduction is even more significant with $p < 0.005$.

5 Mixed-Complementarity Rounding

In the last two sections, we have seen that the MIR approach is useful to generate cuts from a single complementarity constraint. The *mixed-complementarity rounding (MCR)* inequality that we present in this section uses the MIR approach to exploit specific structures in a set of complementarity constraints. The structures we investigate here are SOS1 constraints of the form $\|x_K\|_0 \leq 1$, where $\|x_K\|_0 := |\{i \in K \ : \ x_i \neq 0\}|$, which represent cliques $K \subseteq V$ with $x_i \cdot x_j = 0$ for all $i, j \in K$, $i \neq j$. Consider the problem

$$\begin{aligned} &\sum_{\nu \in V} \alpha_\nu \, x_\nu \leq \beta, \\ &\|x_K\|_0 \leq 1, \\ &0 \leq x_\nu \leq u_\nu \qquad \forall \, \nu \in K, \\ &x \in \mathbb{R}_+^V, \end{aligned} \tag{11}$$

where $\emptyset \neq K \subseteq V$, $\alpha \in \mathbb{R}^V$, $u \in \mathbb{R}_{>0}^K$, $\beta \in \mathbb{R}$.

Using auxiliary variables $y \in \{0,1\}^K$, (11) can be transformed into mixed-integer form:

$$\begin{aligned} &\sum_{\nu \in V} \alpha_\nu \, x_\nu \le \beta, \\ &\sum_{\nu \in K} y_\nu \le 1, \\ &0 \le x_\nu \le u_\nu \, y_\nu \qquad\qquad \forall\, \nu \in K, \\ &x \in \mathbb{R}^V_+,\; y \in \{0,1\}^K. \end{aligned} \tag{12}$$

A valid inequality of (12) can be projected to the space of the x-variables to obtain a valid inequality of (11). This can be done using the following formula:

Lemma 5.1 ([11])**.** *Let* $\sum_{\nu \in V} a_\nu \, x_\nu + \sum_{\nu \in K} b_\nu \, y_\nu \le \beta$ *be valid for* (12)*. Then the inequality*

$$\sum_{\nu \in K} \left(a_\nu + \frac{b_\nu - h}{u_\nu} \right) x_\nu + \sum_{\nu \in V \setminus K} a_\nu \, x_\nu \le \beta - h$$

is valid for (11)*, where* $h \coloneqq \min\{\min\{0, b_\nu\} \;:\; \nu \in K\}$*.*

With the help of this lemma, the following inequality can be derived as an MIR inequality.

Theorem 5.2. *Let* $\bar{f}_0 \coloneqq \beta - \lfloor \beta \rfloor$ *and* $\bar{f}_\nu \coloneqq \alpha_\nu \, u_\nu - \lfloor \alpha_\nu \, u_\nu \rfloor$ *for* $\nu \in K$*. Furthermore, let*

$$b_\nu \coloneqq \lfloor \alpha_\nu \, u_\nu \rfloor + \frac{(\bar{f}_\nu - \bar{f}_0)^+}{1 - \bar{f}_0} - \frac{(\alpha_\nu \, u_\nu)^+}{1 - \bar{f}_0}$$

and $h \coloneqq \min\{\min\{0, b_\nu\} \;:\; \nu \in K\}$*. Then the mixed-complementarity rounding (MCR) inequality*

$$\sum_{\nu \in K} \left(\frac{(\alpha_\nu)^+}{1 - \bar{f}_0} + \frac{b_\nu - h}{u_\nu} \right) x_\nu + \frac{1}{1 - \bar{f}_0} \sum_{\substack{\nu \in V \setminus K \\ \alpha_\nu < 0}} \alpha_\nu \, x_\nu \le \lfloor \beta \rfloor - h$$

is valid for (11)*.*

Proof. With the help of a slack variable $s_\nu \ge 0$, we write $x_\nu \le u_\nu \, y_\nu$ as $x_\nu + s_\nu = u_\nu \, y_\nu$ for all $\nu \in K$. Then $\sum_{\nu \in V} \alpha_\nu \, x_\nu \le \beta$ is equivalent to

$$\sum_{\nu \in K} \alpha_\nu \, u_\nu \, y_\nu - \sum_{\nu \in K} \alpha_\nu \, s_\nu + \sum_{\nu \in V \setminus K} \alpha_\nu \, x_\nu \le \beta.$$

We apply the MIR inequality of Lemma 3.1 to obtain

$$\sum_{\nu \in K} \left(\lfloor \alpha_\nu \, u_\nu \rfloor + \frac{(\bar{f}_\nu - \bar{f}_0)^+}{1 - \bar{f}_0} \right) y_\nu \\ + \frac{1}{1 - \bar{f}_0} \Bigg(\sum_{\substack{\nu \in K \\ \alpha_\nu > 0}} (-\alpha_\nu) \, s_\nu + \sum_{\substack{\nu \in V \setminus K \\ \alpha_\nu < 0}} \alpha_\nu \, x_\nu \Bigg) \leq \lfloor \beta \rfloor.$$

Resubstitution of $s_\nu = u_\nu \, y_\nu - x_\nu$ yields

$$\sum_{\nu \in K} \left(\lfloor \alpha_\nu \, u_\nu \rfloor + \frac{(\bar{f}_\nu - \bar{f}_0)^+}{1 - \bar{f}_0} - \frac{(\alpha_\nu \, u_\nu)^+}{1 - \bar{f}_0} \right) y_\nu \\ + \frac{1}{1 - \bar{f}_0} \Bigg(\sum_{\substack{\nu \in K \\ \alpha_\nu > 0}} \alpha_\nu \, x_\nu + \sum_{\substack{\nu \in V \setminus K \\ \alpha_\nu < 0}} \alpha_\nu \, x_\nu \Bigg) \leq \lfloor \beta \rfloor.$$

Using Lemma 5.1, this inequality can be projected to the x-variables to derive the MCR inequality. □

A generalized MCR inequality that exploits two instead of one inequality can be derived analogously to the proof of Lemma 3.2.

Further research is necessary to investigate practical applications of the generalized MCR inequality: Since disjunctive cuts can be strengthened by exploiting the integrality of variables, a natural question is whether disjunctive cuts can also be strengthened by exploiting complementarity constraints.

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