

Statistical inference and hypotheses testing of risk averse stochastic programs

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Abstract

We study statistical properties of the optimal value and optimal solutions of the Sample Average Approximation of risk averse stochastic problems. Central Limit Theorem type results are derived for the optimal value and optimal solutions when the stochastic program is expressed in terms of a law invariant coherent risk measure. The obtained results are applied to hypotheses testing problems aiming at comparing the optimal values of several risk averse convex stochastic programs on the basis of samples of the underlying random vectors. We also consider non-asymptotic tests based on confidence intervals on the optimal values of the stochastic programs obtained using the Stochastic Mirror Descent algorithm. Numerical simulations show how to use our developments to choose among different distributions and show the superiority of the asymptotic tests on a class of risk averse stochastic programs.

Keywords: Stochastic optimization, statistical inference, hypotheses testing, coherent risk measure, Central Limit Theorem, Sample Average Approximation.

AMS subject classifications: 90C15, 90C90, 90C30.

1 Introduction

Consider the following risk averse stochastic program

$$\min_{x \in \mathcal{X}} \{g(x) := \mathcal{R}(G_x)\}. \quad (1.1)$$

Here \mathcal{X} is a nonempty compact subset of \mathbb{R}^m , G_x is a random variable depending on $x \in \mathcal{X}$ and \mathcal{R} is a risk measure. We assume that G_x is given in the form $G_x(\omega) = G(x, \xi(\omega))$, where $G : \mathcal{X} \times \mathbb{R}^d \rightarrow \mathbb{R}$

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and $\xi : \Omega \rightarrow \mathbb{R}^d$ is a random vector defined on a probability space (Ω, \mathcal{F}, P) whose distribution is supported on set $\Xi \subset \mathbb{R}^d$. We also assume that risk measure \mathcal{R} is *law invariant* (we will give precise definitions in Section 2).

Let $\xi_j = \xi_j(\omega)$, $j = 1, \dots, N$, be an i.i.d sample of the random vector ξ defined on the same probability space. Then the respective sample estimate of $g(x)$, denoted $\hat{g}_N(x)$, is obtained by replacing the “true” distribution of the random vector ξ with its empirical estimate. Consequently the true optimization problem (1.1) is approximated by the problem

$$\text{Min}_{x \in \mathcal{X}} \hat{g}_N(x), \quad (1.2)$$

referred to as the Sample Average Approximation (SAA) problem. Note that $\hat{g}_N(x) = \hat{g}_N(x, \omega)$ is a random function, sometimes we suppress dependence on ω in the notation. In particular if \mathcal{R} is the expectation operator, i.e., $g(x) = \mathbb{E}[G_x]$, then $\hat{g}_N(x) = N^{-1} \sum_{j=1}^N G(x, \xi_j)$.

We denote by ϑ_* and $\hat{\vartheta}_N$ the optimal values of problems (1.1) and (1.2), respectively, and study statistical properties of $\hat{\vartheta}_N$. The random sample can be given by collected data or can be generated by Monte Carlo sampling techniques in the goal of solving the true problem by the SAA method. Although conceptually different, both situations lead to the same statistical inference.

The statistical analysis allows us to address the following question of asymptotic tests of hypotheses. Suppose that we are given $K \geq 2$ optimization problems of the form (1.1) with ξ , G , and \mathcal{X} respectively replaced by ξ^i , G_i , and \mathcal{X}_i for problem $i \in \{1, \dots, K\}$. On the basis of samples ξ_1^i, \dots, ξ_N^i , of size N , of ξ^i , $i = 1, \dots, K$, and denoting by ϑ_*^i the optimal value of problem i , we study the statistical tests:

$$\begin{aligned} (a) \quad & H_0 : \vartheta_*^1 = \vartheta_*^2 = \dots = \vartheta_*^K, \\ (b) \quad & H_0 : \vartheta_*^i \leq \vartheta_*^j \text{ for } 1 \leq j \neq i \leq K, \\ (c) \quad & H_0 : \vartheta_*^1 \leq \vartheta_*^2 \leq \dots \leq \vartheta_*^K, \end{aligned} \quad (1.3)$$

against the respective unrestricted alternatives. As a special case, if the feasibility sets of the K optimizations problems are singletons, say $\{x_*^i\}$ for problem i , the above tests aim at comparing the risks $\mathcal{R}(G_{x_*^1}), \dots, \mathcal{R}(G_{x_*^K})$. These tests are useful when we want to choose among K candidate solutions x_*^1, \dots, x_*^K of problem (1.1) for the one with the smallest risk measure value, using risk measure \mathcal{R} to rank the distributions $G_{x_*^i}$, $i = 1, \dots, K$.

Setting $\theta := (\vartheta_*^1, \dots, \vartheta_*^K)$, we also consider the following extension of tests (1.3)(a),(b),(c):

$$H_0 : \theta \in \Theta_0 \text{ against } H_1 : \theta \in \mathbb{R}^K, \quad (1.4)$$

with $\Theta_0 \subset \mathbb{R}^K$ being a linear space or a convex cone, as well as tests on the optimal value ϑ_* of (1.1) of the form

$$\begin{aligned} (a) \quad & H_0 : \vartheta_* = \rho_0 \text{ against } H_1 : \vartheta_* \neq \rho_0, \\ (b) \quad & H_0 : \vartheta_* \leq \rho_0 \text{ against } H_1 : \vartheta_* > \rho_0. \end{aligned} \quad (1.5)$$

Tests (1.3) and (1.5) will also be studied in a nonasymptotic setting.

Finally, numerical simulations illustrate our results: we show how to use our developments to choose, using tests (1.3), among different distributions. We also use these tests to compare the optimal value of several risk averse stochastic programs. It is shown that the Normal (Gaussian) distribution approximates well the distribution of $\hat{\vartheta}_N$ already for $N = 20$ and problem sizes up to $n = 10000$, and that the asymptotic tests yield much smaller type II errors than the considered nonasymptotic tests for small to moderate sample size (N up to 10^5) and problem size (n up to 500).

We use the following notation throughout the paper. By $F_Z(z) := P(Z \leq z)$ we denote the cumulative distribution function (cdf) of a random variable $Z : \Omega \rightarrow \mathbb{R}$. By $F^{-1}(\alpha) = \inf\{t : F(t) \geq \alpha\}$ we denote the left-side α -quantile of the cdf F . By $\mathfrak{Q}_F(\alpha)$ we denote the interval of α -quantiles of cdf F , i.e.,

$$\mathfrak{Q}_F(\alpha) = [a, b], \text{ where } a := F^{-1}(\alpha), \text{ } b := \sup\{t : F(t) \leq \alpha\}. \quad (1.6)$$

By $\mathbf{1}_A(\cdot)$ we denote the indicator function of set A . We consider space $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, of random variables $Z : \Omega \rightarrow \mathbb{R}$ having finite p -th order moments. The dual of space \mathcal{Z} is the space $\mathcal{Z}^* = L_q(\Omega, \mathcal{F}, P)$, where $q \in (1, \infty]$ is such that $1/p + 1/q = 1$. For $Z \in \mathcal{Z}$ and $\zeta \in \mathcal{Z}^*$ their scalar product is defined as the integral $\langle \zeta, Z \rangle = \int_{\Omega} \zeta(\omega)Z(\omega)dP(\omega)$. The notation $Z \succeq Z'$ means that $Z(\omega) \geq Z'(\omega)$ for a.e. $\omega \in \Omega$. We denote by $C[a, b]$ the space of continuous functions $\psi : [a, b] \rightarrow \mathbb{R}$ equipped with the norm $\|\psi\|_{\infty} := \sup_{t \in [a, b]} |\psi(t)|$. It is said that functions $h_k : \mathbb{R}^n \rightarrow \mathbb{R}$ converge to h uniformly on \mathbb{R}^n if $\sup_{x \in \mathbb{R}^n} |h_k(x) - h(x)| \rightarrow 0$ as $k \rightarrow \infty$.

2 Preliminary discussion

Risk measure $\mathcal{R} : \mathcal{Z} \rightarrow \mathbb{R}$ is a functional assigning to a random variable $Z \in \mathcal{Z}$ real value $\mathcal{R}(Z)$. Note that we consider here real valued risk measures, i.e., we do not allow $\mathcal{R}(Z)$ to have an infinite value. In the influential paper of Artzner et al [2] it was suggested that a “good” risk measure should satisfy the following conditions (axioms).

(i) *Monotonicity*: If $Z, Z' \in \mathcal{Z}$ and $Z \succeq Z'$, then $\mathcal{R}(Z) \geq \mathcal{R}(Z')$.

(ii) *Convexity*:

$$\mathcal{R}(tZ + (1-t)Z') \leq t\mathcal{R}(Z) + (1-t)\mathcal{R}(Z')$$

for all $Z, Z' \in \mathcal{Z}$ and all $t \in [0, 1]$.

(iii) *Translation Equivariance*: If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\mathcal{R}(Z + a) = \mathcal{R}(Z) + a$.

(iv) *Positive Homogeneity*: If $t \geq 0$ and $Z \in \mathcal{Z}$, then $\mathcal{R}(tZ) = t\mathcal{R}(Z)$.

Risk measures \mathcal{R} satisfying the above axioms (i)-(iv) were called *coherent* in [2]. If a risk measure satisfies axioms (i)-(iii), but not necessarily (iv), it is called *convex* (cf., [4]). We assume that \mathcal{R} is *law invariant*. That is, $\mathcal{R}(Z)$ depends only on the distribution of Z , i.e., if $Z, Z' \in \mathcal{Z}$ have the same cumulative distribution function then $\mathcal{R}(Z) = \mathcal{R}(Z')$. We also assume that the probability space (Ω, \mathcal{F}, P) is *nonatomic*.

Since a law invariant risk measure \mathcal{R} can be considered as a function of its cdf $F(\cdot) = F_Z(\cdot)$, we also write $\mathcal{R}(F)$ to denote the corresponding value $\mathcal{R}(Z)$. Let Z_1, \dots, Z_N be an i.i.d sample of Z and $\widehat{F}_N = N^{-1} \sum_{j=1}^N \mathbf{1}_{[Z_j, \infty)}$ be the corresponding empirical estimate of the cdf F . By replacing F with its empirical estimate \widehat{F}_N , we obtain the estimate $\mathcal{R}(\widehat{F}_N)$ to which we refer as the *sample* or *empirical* estimate of $\mathcal{R}(F)$. We assume that for every $x \in \mathcal{X}$, the random variable G_x belongs to the space \mathcal{Z} , and hence $g(x) = \mathcal{R}(G_x)$ is well defined for every $x \in \mathcal{X}$. Let F_x be the cdf of random variable G_x , $x \in \mathcal{X}$, and $\widehat{F}_{x,N}$ be the empirical cdf associated with the sample $G(x, \xi_1), \dots, G(x, \xi_N)$. Then we can write $g(x) = \mathcal{R}(F_x)$ and $\widehat{g}_N(x) = \mathcal{R}(\widehat{F}_{x,N})$.

We have the following result about the convergence of the optimal value and optimal solutions of the SAA problem (1.2) to their counterparts of the “true” problem (1.1) (cf., [22, Theorem 3.3]).

Theorem 2.1 Let $\mathcal{R} : \mathcal{Z} \rightarrow \mathbb{R}$ be a law invariant convex risk measure. Suppose that the set \mathcal{X} is nonempty and compact and the following conditions hold: (i) the function $G_x(\omega)$ is random lower semicontinuous, i.e., the epigraphical multifunction $\omega \mapsto \{(x, t) \in \mathbb{R}^{n+1} : G_x(\omega) \leq t\}$ is closed valued and measurable, (ii) for every $\bar{x} \in \mathbb{R}^n$ there is a neighborhood $\mathcal{V}_{\bar{x}}$ of \bar{x} and a function $h \in \mathcal{Z}$ such that $G_x(\cdot) \geq h(\cdot)$ for all $x \in \mathcal{V}_{\bar{x}}$.

Then the optimal value $\hat{\vartheta}_N$ of problem (1.2) converges w.p.1 to the optimal value ϑ_* of the “true” problem (1.1), and the distance from an optimal solution \hat{x}_N of (1.2) to the set of optimal solutions of (1.1) converges w.p.1 to zero as $N \rightarrow +\infty$.

Remark 1 Recall that it is assumed that the probability space (Ω, \mathcal{F}, P) is nonatomic. Then without loss of generality we can assume that Ω is the interval $[0, 1]$ equipped with its Borel sigma algebra and uniform probability distribution P . We refer to this probability space as the *standard probability space*.

- By \mathcal{L}_p , $p \in [1, \infty)$, we denote the space $L_p(\Omega, \mathcal{F}, P)$ defined on the standard probability space (Ω, \mathcal{F}, P) .

Recall that the dual $\mathcal{L}_p^* = \mathcal{L}_q$. For a cdf F we can view F^{-1} as a measurable function defined on the standard probability space. Then F^{-1} is an element of the space \mathcal{L}_p iff $\int_{-\infty}^{+\infty} |z|^p dF(z) < +\infty$. With some abuse of notation we write that a cdf $F \in \mathcal{L}_p$ if $F^{-1} \in \mathcal{L}_p$. Note also that an element $Z \in \mathcal{L}_p$ is distributionally equivalent to F_Z^{-1} , and $F_Z^{-1} \in \mathcal{L}_p$ iff $Z \in \mathcal{L}_p$.

2.1 Dual representations of law invariant coherent risk measures

Every coherent risk measure $\mathcal{R} : \mathcal{L}_p \rightarrow \mathbb{R}$ has the dual representation

$$\mathcal{R}(Z) = \sup_{\zeta \in \mathfrak{A}} \int_0^1 \zeta(t) Z(t) dt, \quad (2.1)$$

where $\mathfrak{A} \subset \mathcal{L}_p^*$ is a convex weakly* compact set of density functions. Since real valued coherent risk measures are continuous in the norm topology of the Banach space \mathcal{L}_p , this dual representation follows from the Fenchel-Moreau Theorem (cf., [19]).

For *law invariant* coherent risk measures the dual representation (2.1) can be written in the following form

$$\mathcal{R}(F) = \sup_{\sigma \in \Upsilon} \int_0^1 \sigma(\tau) F^{-1}(\tau) d\tau, \quad (2.2)$$

where Υ is a weakly* compact subset of \mathcal{L}_p^* consisting of so-called spectral functions. A function $\sigma : [0, 1) \rightarrow [0, +\infty)$ is called *spectral* if $\sigma(\cdot)$ is right side continuous, monotonically nondecreasing and such that $\int_0^1 \sigma(\tau) d\tau = 1$. The representation (2.2) is obtained from (2.1) by noting that $Z \in \mathcal{L}_p$ is distributionally equivalent to F_Z^{-1} , and applying a measure preserving transformation (cf., [23]). In particular if $\Upsilon = \{\sigma\}$ is a singleton, then

$$\mathcal{R}(F) = \int_0^1 \sigma(\tau) F^{-1}(\tau) d\tau \quad (2.3)$$

is called spectral (or distortion) risk measure. The so-called generating set Υ is not defined uniquely. In a sense minimal generating set is formed by the weak* topological closure of the set of spectral functions which are exposed points of the set \mathfrak{A} (cf., [14], [25, Section 6.3.4]). Consider the set of maximizers in the right-hand side of (2.2),

$$\tilde{\Upsilon}(F) := \arg \max_{\sigma \in \Upsilon} \int_0^1 \sigma(\tau) F^{-1}(\tau) d\tau. \quad (2.4)$$

Since $F^{-1} \in \mathcal{L}_p$ and the set Υ is weakly* compact, it follows that the set $\tilde{\Upsilon}(F)$ is nonempty and weakly* compact.

It is also possible to write representation (2.2) in the following equivalent form

$$\mathcal{R}(F) = \sup_{\psi \in \Psi} \left\{ \int_0^{+\infty} [1 - \psi(F(t))] dt - \int_{-\infty}^0 \psi(F(t)) dt \right\}, \quad (2.5)$$

where $\Psi := \mathfrak{V}(\Upsilon)$ with \mathfrak{V} being a mapping from the set of spectral functions into the space $C[0, 1]$, defined as

$$(\mathfrak{V}\sigma)(\alpha) := \int_0^\alpha \sigma(t) dt, \quad \alpha \in [0, 1]. \quad (2.6)$$

Indeed, note that for any spectral function σ , the corresponding $\psi = \mathfrak{V}\sigma$ is convex, continuous monotonically nondecreasing on the interval $[0, 1]$ function with $\psi(0) = 0$ and $\psi(1) = 1$. By change of variables $\tau = F(t)$ and using integration by parts we can write

$$\int_0^{+\infty} [1 - \psi(F(t))] dt = \int_{F(0)}^1 (1 - \psi(\tau)) dF^{-1}(\tau) = \int_{F(0)}^1 F^{-1}(\tau) \psi'(\tau) d\tau. \quad (2.7)$$

Similarly

$$\int_{-\infty}^0 \psi(F(t)) dt = \int_0^{F(0)} \psi(\tau) dF^{-1}(\tau) = - \int_0^{F(0)} F^{-1}(\tau) \psi'(\tau) d\tau, \quad (2.8)$$

and hence (2.5) follows from (2.2).

The function $\psi = \mathfrak{V}\sigma$ is directionally differentiable. Its directional derivative $\psi'(t, h) = \lim_{\tau \downarrow 0} [\psi(t + \tau h) - \psi(t)]/\tau$ is

$$\psi'(t, h) = \begin{cases} \psi'_+(t)h & \text{if } h \geq 0, \\ \psi'_-(t)h & \text{if } h \leq 0, \end{cases} \quad (2.9)$$

where $\psi'_-(t)$ and $\psi'_+(t)$ are the respective left and right side derivatives

$$\psi'_-(t) = \lim_{\tau \uparrow t} \sigma(\tau) \text{ and } \psi'_+(t) = \lim_{\tau \downarrow t} \sigma(\tau). \quad (2.10)$$

In particular, if $\sigma(\cdot)$ is continuous at t , then $\psi(\cdot)$ is differentiable at t and $\psi'(t) = \sigma(t)$. Note that since a spectral function is monotonically nondecreasing, the set of its discontinuous points is countable.

Lemma 2.1 *The mapping $\mathfrak{V} : \Upsilon \rightarrow C[0, 1]$ is continuous with respect to the weak* topology of \mathcal{L}_p^* and norm topology of $C[0, 1]$.*

Proof. The set Υ is a bounded subset of \mathcal{L}_p^* . Since the space \mathcal{L}_p is separable, it follows that the weak* topology on Υ is metrizable. Therefore it suffices to show that if a sequence $\sigma_n \in \Upsilon$ converges to $\sigma \in \Upsilon$ in weak* topology, then $\psi_n = \mathfrak{V}\sigma_n$ converges to $\psi = \mathfrak{V}\sigma$ in the norm topology of $C[0, 1]$. Note that functions ψ_n and ψ are convex continuous monotonically nondecreasing on the interval $[0, 1]$. Let us also observe that the weak* convergence of σ_n to σ implies pointwise convergence $\psi_n(t) \rightarrow \psi(t)$ for all $t \in [0, 1]$. Indeed, $\psi_n(t) = \langle \sigma_n, \mathbf{1}_{[0, t]} \rangle$ and $\mathbf{1}_{[0, t]}$ belongs to the space \mathcal{L}_p . Since functions ψ_n are convex, it follows from the pointwise convergence that $\sup_{t \in I} |\psi_n(t) - \psi(t)|$ tends to zero for any interval $I \subset (0, 1)$ (e.g., [17, Theorem 10.8]). By monotonicity and continuity of ψ this implies that ψ_n converge to ψ uniformly on $[0, 1]$. This completes the proof. \blacksquare

By the above discussion there is a one-to-one correspondence between representations (2.2) and (2.5) defined by $\Psi = \mathfrak{Y}(\Upsilon)$. Since the generating set Υ is weakly* compact, it follows that the set $\Psi = \mathfrak{Y}(\Upsilon)$ is a compact subset of $C[0, 1]$ (cf., [3]). Consider the set of maximizers in the right-hand side of (2.5),

$$\bar{\Psi}(F) := \arg \max_{\psi \in \Psi} \left\{ \int_0^{+\infty} [1 - \psi(F(t))] dt - \int_{-\infty}^0 \psi(F(t)) dt \right\}. \quad (2.11)$$

It follows that $\bar{\Psi}(F) = \mathfrak{Y}(\bar{\Upsilon}(F))$ is a nonempty and compact subset of $C[0, 1]$.

An important risk measure is the Average Value-at Risk measure

$$\text{AVaR}_\alpha(F) = \frac{1}{1 - \alpha} \int_\alpha^1 F^{-1}(\tau) d\tau, \quad \alpha \in [0, 1). \quad (2.12)$$

That is, $\text{AVaR}_\alpha(F)$ is a spectral risk measure with the spectral function $\sigma(\cdot) = (1 - \alpha)^{-1} \mathbf{1}_{[\alpha, 1)}(\cdot)$. The corresponding space here is \mathcal{L}_1 , i.e., it is defined for F such that $\int |z| dF(z) < +\infty$. Equivalently $\text{AVaR}_\alpha(Z)$ can be written as

$$\text{AVaR}_\alpha(F) = \inf_{\tau \in \mathbb{R}} \left\{ \tau + (1 - \alpha)^{-1} \int_{-\infty}^{+\infty} [z - \tau]_+ dF(z) \right\}. \quad (2.13)$$

For $\alpha \in (0, 1)$ the minimum in the right-hand side of (2.13) is attained at any point of the interval $\Omega_F(\alpha)$ of α -quantiles of the distribution F , in particular at the left side quantile $\tau = F^{-1}(\alpha)$. For $\alpha = 0$, $\text{AVaR}_0(F) = \mathbb{E}_F[Z]$ although the minimum in (2.13) is not attained if the distribution is unbounded from below. Note that $\text{AVaR}_\alpha(F)$ is monotonically nondecreasing in $\alpha \in [0, 1)$, and tends to $\lim_{t \uparrow 1} F^{-1}(t)$ as $\alpha \rightarrow 1$.

Consider the transformation

$$(\mathbb{T}\mu)(t) := \int_0^t (1 - \alpha)^{-1} d\mu(\alpha), \quad t \in [0, 1), \quad (2.14)$$

from the set of probability distribution functions (measures) $\mu(\cdot)$ on the interval $[0, 1)$ to the set of spectral functions. The inverse of this transformation is (cf., [4, Lemma 4.63], [25, p.307])

$$(\mathbb{T}^{-1}\sigma)(\alpha) = (1 - \alpha)\sigma(\alpha) + (\mathfrak{Y}\sigma)(\alpha), \quad \alpha \in [0, 1), \quad (2.15)$$

where mapping \mathfrak{Y} is defined in (2.6). Then representation (2.2) can be written in the following equivalent form

$$\mathcal{R}(F) = \sup_{\mu \in \mathfrak{M}} \int_0^1 \text{AVaR}_\alpha(F) d\mu(\alpha), \quad (2.16)$$

where $\mathfrak{M} := \mathbb{T}^{-1}(\Upsilon)$. The representation (2.16) is referred to as the Kusuoka representation of \mathcal{R} (cf., [8]).

The mapping \mathbb{T} is one-to-one and continuous¹ (cf., [14, Proposition 3.4]). It follows that the inverse mapping \mathbb{T}^{-1} is also continuous on the set Υ , the set $\mathfrak{M} = \mathbb{T}^{-1}(\Upsilon)$ is compact, and the set

$$\bar{\mathfrak{M}}(F) := \arg \max_{\mu \in \mathfrak{M}} \int_0^1 \text{AVaR}_\alpha(F) d\mu(\alpha) \quad (2.17)$$

is nonempty compact and $\bar{\mathfrak{M}}(F) = \mathbb{T}^{-1}(\bar{\Upsilon}(F))$.

¹We consider here the weak topology of probability measures on the interval $[0, 1]$ and the weak* topology of \mathcal{L}_p^*

Note that since we assume that $F \in \mathcal{L}_p$, $p \in [1, +\infty)$, measures in \mathfrak{M} do not have positive mass at $\alpha = 1$, although they may have positive mass at $\alpha = 0$. It will be convenient to write explicitly measures $\mu \in \mathfrak{M}$ in the form

$$\mu = w\delta(0) + (1-w)\mu', \quad (2.18)$$

where $w \in [0, 1]$ and μ' is the respective probability measure on $[0, 1]$ having zero mass at 0 and 1.

Using variational representation (2.13) of AVaR_α , it is possible to write the Kusuoka representation (2.16) in the following minimax form

$$\mathcal{R}(F) = \sup_{\mu \in \mathfrak{M}} \left\{ w \int_{-\infty}^{+\infty} z dF(z) + (1-w) \int_0^1 \inf_{t \in \mathbb{R}} \left\{ \int_{-\infty}^{+\infty} h_\alpha(z, t) dF(z) \right\} d\mu'(\alpha) \right\}, \quad (2.19)$$

where

$$h_\alpha(z, t) := t + (1-\alpha)^{-1}[z-t]_+, \quad \alpha \in (0, 1). \quad (2.20)$$

By interchanging the integral and minimization operators in the right-hand side of (2.19) (cf., [18, Theorem 14.60]), we can write

$$\mathcal{R}(F) = \sup_{\mu \in \mathfrak{M}} \inf_{\tau \in \mathcal{L}_p} \left\{ w \int_{-\infty}^{+\infty} z dF(z) + (1-w) \int_0^1 \int_{-\infty}^{+\infty} h_\alpha(z, \tau(\alpha)) dF(z) d\mu'(\alpha) \right\} \quad (2.21)$$

$$= \sup_{\mu \in \mathfrak{M}} \inf_{\tau \in \mathcal{L}_p} \left\{ w \int_{-\infty}^{+\infty} z dF(z) + (1-w) \int_0^1 \int_{-\infty}^{+\infty} h_\alpha(z, \tau(\alpha)) d\mu'(\alpha) dF(z) \right\}. \quad (2.22)$$

Note that minimization in (2.21) and (2.22) is performed over functions $\tau \in \mathcal{L}_p$.

By interchanging the ‘sup’ and ‘inf’ operators we can write the dual of problem (2.22):

$$\inf_{\tau \in \mathcal{L}_p} \sup_{\mu \in \mathfrak{M}} \left\{ w \int_{-\infty}^{+\infty} z dF(z) + (1-w) \int_{-\infty}^{+\infty} \int_0^1 h_\alpha(z, \tau(\alpha)) d\mu'(\alpha) dF(z) \right\}. \quad (2.23)$$

The set $\bar{\mathfrak{M}}(F)$, defined in (2.17), is also the set of optimal solutions of the problem (2.22). We denote by $\bar{\mathfrak{T}}(F)$ the set of optimal solutions of the dual problem (2.23). The set $\bar{\mathfrak{T}}(F)$ can be empty. Also the set $\bar{\mathfrak{T}}(F)$ can be unbounded. For example, if $\mathcal{R}(\cdot) := \mathbb{E}[\cdot]$, then $\mathfrak{M} = \{\delta(0)\}$ and the right-hand side of (2.23) does not depend on $\tau \in \mathcal{L}_p$. In that case $\bar{\mathfrak{T}}(F) = \mathcal{L}_p$.

3 Informal analysis

In this section we discuss asymptotics of the empirical estimates $\mathcal{R}(\hat{F}_N)$, in a somewhat informal way, and consider examples. By using (2.22) we can write

$$\mathcal{R}(\hat{F}_N) = \sup_{\mu \in \mathfrak{M}} \inf_{\tau \in \mathcal{L}_p} \left\{ w \int_{-\infty}^{+\infty} z d\hat{F}_N(z) + (1-w) \int_{-\infty}^{+\infty} \int_0^1 h_\alpha(z, \tau(\alpha)) d\mu'(\alpha) d\hat{F}_N(z) \right\} \quad (3.1)$$

$$= \sup_{\mu \in \mathfrak{M}} \inf_{\tau \in \mathcal{L}_p} \left\{ w\bar{Z} + (1-w) \int_0^1 \hat{h}_{\alpha, N}(\tau(\alpha)) d\mu'(\alpha) \right\}, \quad (3.2)$$

where $\bar{Z} := N^{-1} \sum_{j=1}^N Z_j$ and

$$\hat{h}_{\alpha, N}(t) := N^{-1} \sum_{j=1}^N h_\alpha(Z_j, t) = t + (1-\alpha)^{-1} N^{-1} \sum_{j=1}^N [Z_j - t]_+, \quad \alpha \in (0, 1).$$

Suppose that the minimax problem (2.22) has a nonempty set of saddle points, given by $\bar{\mathfrak{M}}(F) \times \bar{\mathfrak{I}}(F)$. Then the minimax representation suggests the following asymptotics

$$\mathcal{R}(\hat{F}_N) = \sup_{\mu \in \bar{\mathfrak{M}}(F)} \inf_{\tau \in \bar{\mathfrak{I}}(F)} \left\{ w\bar{Z} + (1-w) \int_0^1 \hat{h}_{\alpha,N}(\tau(\alpha)) d\mu'(\alpha) \right\} + o_p(N^{-1/2}) \quad (3.3)$$

and

$$N^{1/2}[\mathcal{R}(\hat{F}_N) - \mathcal{R}(F)] \xrightarrow{\mathcal{D}} \sup_{\mu \in \bar{\mathfrak{M}}(F)} \inf_{\tau \in \bar{\mathfrak{I}}(F)} \mathbb{Y}(\mu, \tau), \quad (3.4)$$

where $\mathbb{Y}(\mu, \tau)$, $(\mu, \tau) \in \bar{\mathfrak{M}}(F) \times \bar{\mathfrak{I}}(F)$, is the corresponding Gaussian process (we will discuss this later). In particular, if the set of saddle points is a singleton, $\bar{\mathfrak{M}}(F) = \{\bar{w}\delta(0) + (1-\bar{w})\bar{\mu}'\}$, $\bar{\mathfrak{I}}(F) = \{\bar{\tau}\}$, then $N^{1/2}[\mathcal{R}(\hat{F}_N) - \mathcal{R}(F)]$ converges in distribution to normal $\mathcal{N}(0, \nu^2)$ with variance

$$\nu^2 = \text{Var}_F \left\{ \bar{w}Z + (1-\bar{w}) \int_0^1 (1-\alpha)^{-1} [Z - \bar{\tau}(\alpha)]_+ d\bar{\mu}'(\alpha) \right\}. \quad (3.5)$$

The above derivations are not rigorous. In Theorem 4.1 of the next section we discuss a particular case where formulas (3.3) - (3.5) can be rigorously proved by an application of a finite dimensional minimax asymptotic distribution theorem (cf., [21]). Formula (3.4) suggests that the asymptotic distribution of $\mathcal{R}(\hat{F}_N)$ could be non-normal for two somewhat different reasons. Namely, it could happen that the set $\bar{\mathfrak{M}}(F)$ is not a singleton. Recall that $\bar{\mathfrak{M}}(F) = \mathbb{T}^{-1}(\bar{\Upsilon}(F))$. Therefore $\bar{\mathfrak{M}}(F)$ is a singleton if $\bar{\Upsilon}(F)$ is a singleton, in particular if \mathcal{R} is a spectral risk measure. As it was pointed out, the generating set Υ , and hence the sets \mathfrak{M} and Ψ , are not defined uniquely. Therefore uniqueness of the respective maximizers in (2.4), (2.11) and (2.17) should be verified for the minimal representation. It could also happen that the set $\bar{\mathfrak{I}}(F)$ is not a singleton. Let us discuss some illustrative examples.

Example 1 (AVaR risk measure) Consider $\mathcal{R} := \text{AVaR}_\alpha$, $\alpha \in (0, 1)$. This is a spectral risk measure. Its Kusuoka representation is given by the singleton $\mathfrak{M} = \{\delta(\alpha)\}$, and $\mathfrak{I}(F) = \mathcal{Q}_F(\alpha)$. For this risk measure formula (3.3) becomes

$$\text{AVaR}_\alpha(\hat{F}_N) = \inf_{t \in \mathcal{Q}_F(\alpha)} \left\{ t + \frac{1}{(1-\alpha)N} \sum_{j=1}^N [Z_j - t]_+ \right\} + o_p(N^{-1/2}). \quad (3.6)$$

Suppose that $\mathbb{E}_F[Z^2] < +\infty$. Then by the variational representation (2.13) of AVaR_α , the assertion (3.6) follows from a general result about asymptotics of sample average approximations of stochastic programs (cf., [20, Theorem 3.2], [25, Section 6.6.1]). If, moreover, $\mathcal{Q}_F(\alpha) = \{F^{-1}(\alpha)\}$ is a singleton, then $N^{1/2}[\text{AVaR}_\alpha(\hat{F}_N) - \text{AVaR}_\alpha(F)]$ converges in distribution to normal $\mathcal{N}(0, \nu^2)$ with

$$\nu^2 = (1-\alpha)^2 \text{Var}_F([Z - F^{-1}(\alpha)]_+). \quad (3.7)$$

It follows from (3.6) that uniqueness of the quantile $F^{-1}(\alpha)$ is also a necessary condition for asymptotic normality of $\text{AVaR}_\alpha(\hat{F}_N)$. In Theorem 4.1 (of Section 4) we give a more general result for which AVaR_α risk measure is a particular case.

A different formula for the asymptotic variance of $\text{AVaR}_\alpha(\hat{F}_N)$ was given in [13] and [3]. We are going to show now equivalence of their formula to (3.7). Consider the F -Brownian bridge, denoted \mathbb{B}_F . That is, $\mathbb{B}_F(z)$ is a Gaussian process with mean zero and covariances

$$\mathbb{E}[\mathbb{B}_F(z)\mathbb{B}_F(z')] = F(z \wedge z') - F(z)F(z').$$

Note that for any $\tau \in \mathbb{R}$,

$$\text{Var} \left\{ \int_{\tau}^{+\infty} \mathbb{B}_F(z) dz \right\} = \int_{\tau}^{+\infty} \int_{\tau}^{+\infty} [F(x \wedge y) - F(x)F(y)] dx dy \quad (3.8)$$

$$= \int_{\tau}^{+\infty} \int_{\tau}^{+\infty} [\bar{F}(x \vee y) - \bar{F}(x)\bar{F}(y)] dx dy \quad (3.9)$$

$$= \int_{\alpha}^1 \int_{\alpha}^1 (s \wedge t - st) dF^{-1}(s) dF^{-1}(t), \quad (3.10)$$

where $\bar{F}(\cdot) := 1 - F(\cdot)$.

Lemma 3.1 *Suppose that F has finite second order moment and let $\tau \in \mathbb{R}$. Then*

$$\text{Var}_F([Z - \tau]_+) = \int_{\tau}^{+\infty} \int_{\tau}^{+\infty} [F(x \wedge y) - F(x)F(y)] dx dy. \quad (3.11)$$

Proof. We have

$$\text{Var}_F([Z - \tau]_+) = \int_{\tau}^{+\infty} (z - \tau)^2 dF(z) - \left(\int_{\tau}^{+\infty} (z - \tau) dF(z) \right)^2.$$

Since $\mathbb{E}_F[Z^2] < +\infty$ it follows that $\lim_{z \rightarrow +\infty} z\bar{F}(z) = 0$. Hence using integration by parts we can write

$$\int_{\tau}^{+\infty} (z - \tau) dF(z) = - \int_{\tau}^{+\infty} (z - \tau) d\bar{F}(z) = - \int_{\tau}^{+\infty} \bar{F}(z) dz,$$

and hence

$$\left(\int_{\tau}^{+\infty} (z - \tau) dF(z) \right)^2 = \int_{\tau}^{+\infty} \int_{\tau}^{+\infty} \bar{F}(x)\bar{F}(y) dx dy. \quad (3.12)$$

Now

$$\int_{\tau}^{+\infty} \bar{F}(x \vee y) dx = \int_{\tau}^y \bar{F}(y) dx + \int_y^{+\infty} \bar{F}(x) dx = (y - \tau)\bar{F}(y) + \int_y^{+\infty} \bar{F}(x) dx.$$

Hence

$$\begin{aligned} \int_{\tau}^{+\infty} \int_{\tau}^{+\infty} \bar{F}(x \vee y) dx dy &= \int_{\tau}^{+\infty} (y - \tau)\bar{F}(y) dy + \int_{\tau}^{+\infty} \int_y^{+\infty} \bar{F}(x) dx dy \\ &= \int_{\tau}^{+\infty} (x - \tau)\bar{F}(x) dx + \int_{\tau}^{+\infty} \int_{\tau}^x \bar{F}(x) dy dx \\ &= 2 \int_{\tau}^{+\infty} (x - \tau)\bar{F}(x) dx. \end{aligned}$$

Since $\mathbb{E}_F[Z^2] < +\infty$ it follows that $\lim_{z \rightarrow +\infty} z^2\bar{F}(z) = 0$, and hence using integration by parts we can write

$$\int_{\tau}^{+\infty} (z - \tau)^2 dF(z) = - \int_{\tau}^{+\infty} (z - \tau)^2 d\bar{F}(z) = 2 \int_{\tau}^{+\infty} (z - \tau)\bar{F}(z) dz = \int_{\tau}^{+\infty} \int_{\tau}^{+\infty} \bar{F}(x \vee y) dx dy. \quad (3.13)$$

Noting equivalence of (3.8) and (3.9), we obtain (3.11) by (3.12) and (3.13). \blacksquare

By using (3.11) the right-hand side of (3.7) can be written in terms of the cdf F .

Example 2 (Absolute semideviation risk measure) Consider risk measure

$$\mathcal{R}_c(F) := \mathbb{E}_F[Z] + c\mathbb{E}_F[Z - \mathbb{E}_F(Z)]_+, \quad c \in (0, 1]. \quad (3.14)$$

We assume that cdf F has finite first order moment, i.e., $\mathcal{R}_c(\cdot)$ is defined on \mathcal{L}_1 . This risk measure has the following representation (cf., [23])

$$\mathcal{R}_c(F) = \sup_{\gamma \in [0,1]} \{(1 - c\gamma)\mathbb{E}_F(Z) + c\gamma \text{AVaR}_{1-\gamma}(F)\} \quad (3.15)$$

$$= \sup_{\gamma \in [0,1]} \inf_{t \in \mathbb{R}} \mathbb{E}_F\{(1 - c\gamma)Z + c\gamma t + c[Z - t]_+\} \quad (3.16)$$

$$= \inf_{t \in \mathbb{R}} \sup_{\gamma \in [0,1]} \mathbb{E}_F\{(1 - c\gamma)Z + c\gamma t + c[Z - t]_+\}. \quad (3.17)$$

Representation (3.15) is the (minimal) Kusuoka representation (2.16) of \mathcal{R}_c with the corresponding set $\mathfrak{M} = \cup_{\gamma \in [0,1]} \{(1 - c\gamma)\delta(0) + c\gamma\delta(1 - \gamma)\}$. Since

$$\sup_{\gamma \in [0,1]} \mathbb{E}_F\{(1 - c\gamma)Z + c\gamma t + c[Z - t]_+\} = \mathbb{E}[Z] + c \max\{\mathbb{E}[Z - t]_+, \mathbb{E}[t - Z]_+\},$$

it follows that problem (3.17) has unique optimal solution $t^* = \mathbf{m}$, where $\mathbf{m} := \mathbb{E}_F[Z]$. Now the set of minimizers of $\gamma t + \mathbb{E}[Z - t]_+$, over $t \in \mathbb{R}$, is defined by the equation $F(t) = 1 - \gamma$. It follows that the set of saddle points of the minimax representation (3.16) is $[\underline{\gamma}, \bar{\gamma}] \times \{\mathbf{m}\}$, where

$$\underline{\gamma} := 1 - \Pr(Z \leq \mathbf{m}), \quad \bar{\gamma} := 1 - \Pr(Z < \mathbf{m})$$

(cf., [25, Section 6.6.2]). In other words here

$$\bar{\mathfrak{M}}(F) = \cup_{\gamma \in [\underline{\gamma}, \bar{\gamma}]} \{(1 - c\gamma)\delta(0) + c\gamma\delta(1 - \gamma)\}$$

and $\bar{\mathfrak{X}}(F) = \{\bar{\tau}(\alpha)\}$ is the singleton with $\bar{\tau}(\cdot) \equiv \mathbb{E}_F[Z]$.

The minimax representation (3.16) leads to the following asymptotics. Suppose that $\mathbb{E}_F[Z^2] < +\infty$. Then by a finite dimensional minimax asymptotics theorem (cf., [21])

$$\mathcal{R}_c(\hat{F}_N) = \sup_{\gamma \in [\underline{\gamma}, \bar{\gamma}]} \left\{ c\gamma \mathbf{m} + (1 - c\gamma)\bar{Z} + cN^{-1} \sum_{j=1}^N [Z_j - \mathbf{m}]_+ \right\} + o_p(N^{-1/2}), \quad (3.18)$$

where $\bar{Z} := N^{-1} \sum_{j=1}^N Z_j$. In particular if the cdf $F(\cdot)$ is continuous at $\mathbf{m} = \mathbb{E}_F[Z]$, then $N^{1/2}[\mathcal{R}_c(\hat{F}_N) - \mathcal{R}_c(F)]$ converges in distribution to normal $\mathcal{N}(0, \nu^2)$ with variance

$$\nu^2 = \text{Var}_F\{(1 - c\gamma^*)Z + c[Z - \mathbf{m}]_+\}, \quad (3.19)$$

where $\gamma^* := 1 - F(\mathbf{m}) = \bar{F}(\mathbf{m})$.

3.1 Von Mises statistical functionals

Let $G \in \mathcal{L}_p$ be an arbitrary cdf and consider convex combination $(1 - t)F + tG = F + tH$, where $H := G - F$. Suppose that the risk measure \mathcal{R} is directionally differentiable at F in direction H , i.e., the following limit exists

$$\mathcal{R}'_F(H) := \lim_{t \downarrow 0} \frac{\mathcal{R}(F + tH) - \mathcal{R}(F)}{t}. \quad (3.20)$$

If moreover the directional derivative $\mathcal{R}'_F(\cdot)$ is linear, then it is said that \mathcal{R} is Gâteaux differentiable at F .

Consider the approximation

$$\mathcal{R}(\widehat{F}_N) - \mathcal{R}(F) \approx \mathcal{R}'_F(\widehat{F}_N - F), \quad (3.21)$$

and hence

$$N^{1/2}[\mathcal{R}(\widehat{F}_N) - \mathcal{R}(F)] \approx \mathcal{R}'_F \left(N^{1/2}(\widehat{F}_N - F) \right). \quad (3.22)$$

Since $N^{1/2}(\widehat{F}_N - F)$ converges in distribution to the F -Brownian bridge \mathbb{B}_F , the approximation (3.22) suggests that $N^{1/2}[\mathcal{R}(\widehat{F}_N) - \mathcal{R}(F)]$ converges in distribution to $\mathcal{R}'_F(\mathbb{B}_F)$.

If moreover \mathcal{R} is Gâteaux differentiable at F , then

$$\mathcal{R}'_F(\widehat{F}_N - F) = \frac{1}{N} \sum_{j=1}^N IF(Z_j), \quad (3.23)$$

where

$$IF(z) := \mathcal{R}'_F(\delta(z) - F) \quad (3.24)$$

is the so-called *influence function*. Consequently $N^{1/2}[\mathcal{R}(\widehat{F}_N) - \mathcal{R}(F)]$ converges in distribution to normal $\mathcal{N}(0, \nu^2)$, with $\nu^2 = \mathbb{E}_F[IF(Z)^2]$. Of course, the above are heuristic arguments which require a rigorous justification.

Now consider representation (2.5) of the risk measure \mathcal{R} . Recall that each function $\psi = \mathfrak{V}\sigma$ is directionally differentiable with directional derivative (2.9). We have that

$$\lim_{t \downarrow 0} \int_{-\infty}^0 \frac{\psi(F(z) + tH(z)) - \psi(F(z))}{t} dz = \int_{-\infty}^0 \psi'(F(z), H(z)) dz,$$

provided the limit and integral operators can be interchanged. Similar arguments can be applied to the first integral term in the right-hand side of (2.5). It follows that if the set $\Psi = \{\psi\}$ is a singleton, i.e., \mathcal{R} is a spectral risk measure, then

$$\mathcal{R}'_F(H) = - \int_{-\infty}^{+\infty} \psi'(F(z), H(z)) dz, \quad (3.25)$$

provided that the limit and integral operators can be interchanged. This suggests the asymptotics

$$N^{1/2} [\mathcal{R}(\widehat{F}_N) - \mathcal{R}(F)] \xrightarrow{\mathcal{D}} - \int_{-\infty}^{+\infty} \psi'(F(z), \mathbb{B}_F(z)) dz. \quad (3.26)$$

Now suppose that every spectral function $\sigma \in \widetilde{\Upsilon}(F)$ is continuous at every point where F^{-1} is discontinuous. Then

$$\mathcal{R}'_F(H) = \sup_{\sigma \in \widetilde{\Upsilon}(F)} \left\{ - \int_{-\infty}^{+\infty} \sigma(F(z)) H(z) dz \right\}. \quad (3.27)$$

In that case the suggested asymptotics are

$$N^{1/2} [\mathcal{R}(\widehat{F}_N) - \mathcal{R}(F)] \xrightarrow{\mathcal{D}} \sup_{\sigma \in \widetilde{\Upsilon}(F)} \left\{ - \int_{-\infty}^{+\infty} \sigma(F(z)) \mathbb{B}_F(z) dz \right\}. \quad (3.28)$$

For example, consider the mean-semideviation risk measure

$$\mathcal{R}_c(F) := \mathbb{E}_F[Z] + c \left(\mathbb{E}_F[Z - \mathbb{E}_F[Z]]_+^2 \right)^{1/2}, \quad c \in (0, 1]. \quad (3.29)$$

If $F(\cdot)$ is continuous at $\mathbf{m} := \mathbb{E}_F[Z]$, then $\mathcal{R}_c(\cdot)$ is Gâteaux differentiable at F and the corresponding influence function is

$$IF(z) = z + c(2\theta)^{-1} ([z - \mathbf{m}]_+^2 - \theta^2 + 2\kappa(1 - F(\mathbf{m}))(z - \mathbf{m})), \quad (3.30)$$

where $\theta := (\mathbb{E}_F[Z - \mathbb{E}_F[Z]]_+^2)^{1/2}$ and $\kappa := \mathbb{E}_F[Z - \mathbf{m}]_+$. This indicates that continuity of $F(\cdot)$ at \mathbf{m} is a necessary condition for $\mathcal{R}_c(\cdot)$ to be Gâteaux differentiable at F , and hence for $\mathcal{R}(\widehat{F}_N)$ to be asymptotically normal.

4 Discrete Kusuoka case

4.1 Asymptotics of risk measures

In this section we discuss asymptotics of empirical estimates $\mathcal{R}(\widehat{F}_N)$, and more generally of the optimal values $\widehat{\vartheta}_N$ of the SAA problem (1.2), for risk measures of the following form. For this class of risk measures some of the required results are readily available.

Consider Kusuoka representation (2.16) and suppose that the set \mathfrak{M} consists of measures supported on finite set $\{\alpha_0, \alpha_1, \dots, \alpha_k\}$, where $0 = \alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_k < 1$. That is

$$\mathcal{R}(Z) = \sup_{w \in \mathfrak{W}} \left\{ w_0 \mathbb{E}[Z] + \sum_{i=1}^k w_i \text{AVaR}_{\alpha_i}(Z) \right\}, \quad (4.1)$$

where \mathfrak{W} is a nonempty subset of $\Delta_{k+1} := \{w \in \mathbb{R}_+^{k+1} : w_0 + \dots + w_k = 1\}$. Recall that $\mathbb{E}[Z] = \text{AVaR}_0[Z]$. Note that $\mathcal{R}(Z)$ is finite valued for every $Z \in \mathcal{L}_1$. Therefore we assume that \mathcal{R} is defined on \mathcal{L}_1 , i.e., $\mathcal{R} : \mathcal{L}_1 \rightarrow \mathbb{R}$. Note that \mathcal{R} is not changed if \mathfrak{W} is replaced by the topological closure of its convex hull. Therefore we assume that \mathfrak{W} is *convex* and *closed*.

By making transformation (2.14) the above risk measure \mathcal{R} can be written in the form (2.2) with the corresponding set of spectral functions

$$\Upsilon = \left\{ \sigma : \sigma = w_0 + \sum_{i=1}^k w_i (1 - \alpha_i)^{-1} \mathbf{1}_{[\alpha_i, 1]}, w \in \mathfrak{W} \right\}. \quad (4.2)$$

Using representation (2.13) we can write $\mathcal{R}(F)$ in the form

$$\mathcal{R}(F) = \sup_{w \in \mathfrak{W}} \inf_{\tau \in \mathbb{R}^k} \mathbb{E}_F[\phi(Z, w, \tau)] = \inf_{\tau \in \mathbb{R}^k} \sup_{w \in \mathfrak{W}} \mathbb{E}_F[\phi(Z, w, \tau)], \quad (4.3)$$

where

$$\phi(z, w, \tau) := w_0 z + \sum_{i=1}^k w_i (\tau_i + (1 - \alpha_i)^{-1} [z - \tau_i]_+). \quad (4.4)$$

This representation is a particular case of the general minimax formula (2.21)-(2.22). The ‘inf’ and ‘sup’ in (4.3) can be interchanged since the objective function is linear in w and convex in τ and the set \mathfrak{W} is compact.

We make the following assumption.

- (A) For every $i \in \{1, \dots, k\}$ there exists $w \in \mathfrak{W}$ such that $w_i \neq 0$.

This is a natural condition. Otherwise there is $i \in \{1, \dots, k\}$ such that $w_i = 0$ for all $w \in \mathfrak{W}$. In that case we can reduce the considered set $\{\alpha_0, \alpha_1, \dots, \alpha_k\}$ by removing the corresponding point α_i . Since the set $\mathfrak{W} \subset \Delta_{k+1}$ is convex, condition (A) means that the relative interior of \mathfrak{W} consists of points with all their coordinates being positive.

Consider the set

$$\overline{\mathfrak{W}} := \arg \max_{w \in \mathfrak{W}} \sum_{i=0}^k w_i \text{AVaR}_{\alpha_i}(Z), \quad (4.5)$$

of maximizers in (4.1). Since the set \mathfrak{W} is nonempty and compact, the set $\overline{\mathfrak{W}}$ is nonempty. This is also the set of maximizers in (4.3). Note also that, under condition (A),

$$\arg \min_{\tau \in \mathbb{R}^k} \sup_{w \in \mathfrak{W}} \mathbb{E}_F[\phi(Z, w, \tau)] = \mathfrak{Q}_F(\alpha_1) \times \dots \times \mathfrak{Q}_F(\alpha_k) =: \overline{\mathfrak{T}}. \quad (4.6)$$

Indeed for every $\bar{w} \in \mathfrak{W}$ such that $\bar{w}_i \neq 0$, $i = 1, \dots, k$, we have that $\arg \min_{\tau \in \mathbb{R}^k} \mathbb{E}_F[\phi(Z, \bar{w}, \tau)] = \overline{\mathfrak{T}}$. The maximum in (4.6) will not be changed if we replace the set \mathfrak{W} by its relative interior. Since the relative interior of the set \mathfrak{W} consists of points with all nonzero coordinates, the equality (4.6) follows. It follows that the set of saddle points of the minimax problem (4.3) is $\overline{\mathfrak{W}} \times \overline{\mathfrak{T}}$.

Theorem 4.1 *Suppose that \mathcal{R} is of the form (4.1), condition (A) holds and $\mathbb{E}_F[Z^2] < +\infty$. Then*

$$\mathcal{R}(\hat{F}_N) = \sup_{w \in \overline{\mathfrak{W}}} \inf_{\tau \in \overline{\mathfrak{T}}} \left\{ w_0 \bar{Z} + \sum_{i=1}^k w_i \left(\tau_i + \frac{1}{N(1-\alpha_i)} \sum_{j=1}^N [Z_j - \tau_i]_+ \right) \right\} + o_p(N^{-1/2}) \quad (4.7)$$

and

$$N^{1/2} [\mathcal{R}(\hat{F}_N) - \mathcal{R}(F)] \xrightarrow{\mathcal{D}} \sup_{w \in \overline{\mathfrak{W}}} \inf_{\tau \in \overline{\mathfrak{T}}} \mathbb{Y}(w, \tau), \quad (4.8)$$

where $\mathbb{Y}(w, \tau)$ is a Gaussian process with mean zero and covariances

$$\mathbb{E}_F[\mathbb{Y}(w, \tau) \mathbb{Y}(w', \tau')] = \text{Cov}_F \left(w_0 Z + \sum_{i=1}^k \frac{w_i}{1-\alpha_i} [Z - \tau_i]_+, w'_0 Z + \sum_{i=1}^k \frac{w'_i}{1-\alpha_i} [Z - \tau'_i]_+ \right). \quad (4.9)$$

Moreover, if the sets $\overline{\mathfrak{W}} = \{\bar{w}\}$ and $\overline{\mathfrak{T}} = \{\bar{\tau}\}$ are singletons, then $N^{1/2} [\mathcal{R}(\hat{F}_N) - \mathcal{R}(F)]$ converges in distribution to normal $\mathcal{N}(0, \nu^2)$ with variance

$$\nu^2 = \text{Var}_F \left\{ \bar{w}_0 Z + \sum_{i=1}^k \frac{\bar{w}_i}{1-\alpha_i} [Z - \bar{\tau}_i]_+ \right\}. \quad (4.10)$$

Proof. Consider function $\phi(Z, w, \tau)$ defined in (4.4). Together with (4.3) we have that

$$\mathcal{R}(\hat{F}_N) = \sup_{w \in \mathfrak{W}} \inf_{\tau \in \mathbb{R}^k} N^{-1} \sum_{j=1}^N \phi(Z_j, w, \tau) = \inf_{\tau \in \mathbb{R}^k} \sup_{w \in \mathfrak{W}} N^{-1} \sum_{j=1}^N \phi(Z_j, w, \tau). \quad (4.11)$$

The set $\overline{\mathfrak{T}}$ is nonempty and compact. We have that the distance from a minimizer $\hat{\tau}_N$ in (4.11) to the set $\overline{\mathfrak{T}}$ tends to zero w.p.1 as $N \rightarrow +\infty$. Therefore as far as the asymptotics is concerned, the minimization in τ in (4.11) can be reduced to a compact set $\mathfrak{S} \subset \mathbb{R}^k$ containing the set $\overline{\mathfrak{T}}$ in its interior. We can view $\hat{\phi}_N(w, \tau) := N^{-1} \sum_{j=1}^N \phi(Z_j, w, \tau)$ as a random element of $C(\mathfrak{W}, \mathfrak{S})$. Note that

$$|\phi(Z, w, \tau) - \phi(Z, w', \tau')| \leq C(Z)(\|w - w'\| + \|\tau - \tau'\|),$$

where $C(\cdot)$ is a piecewise linear function. Hence it follows from the condition $\mathbb{E}[Z^2] < +\infty$ that $\mathbb{E}[C(Z)^2] < +\infty$. Consequently $N^{1/2} \left[\hat{\phi}_N(w, \tau) - \mathbb{E}_F[\phi(w, \tau, Z)] \right]$ converges in distribution (weakly) to a random element of $C(\mathfrak{W}, \mathfrak{S})$ with the respective covariance structure of the Gaussian process $\mathbb{Y}(w, \tau)$ (e.g., [28, Example 19.7, p.271]).

The minimax problem (4.3) is convex in τ and concave (linear) in w , and $\overline{\mathfrak{W}} \times \overline{\mathfrak{T}}$ is its set of saddle points. Now proof can be completed by applying a general result about asymptotics of minimax SAA problems (cf., [21], [25, Section 5.1.4]). \blacksquare

Compared with the corresponding results in [13] and [3], no assumptions about tail behavior of the distribution F and uniqueness of the respective quantiles were made in Theorem 4.1 apart from the assumption of existence of the second order moments. Also note that

$$\text{Cov}_F([Z - t]_+, [Z - s]_+) = \int_t^{+\infty} \int_s^{+\infty} [F(x \wedge y) - F(x)F(y)] dx dy \quad (4.12)$$

(see Lemma 3.1).

Corollary 4.1 *Suppose that \mathcal{R} is of the form (4.1), condition (A) holds and $\mathbb{E}_F[Z^2] < +\infty$. Then*

$$N^{1/2} \left[\mathcal{R}(\hat{F}_N) - \mathcal{R}(F) \right] \xrightarrow{\mathcal{D}} \sup_{w \in \overline{\mathfrak{W}}} \inf_{\tau \in \overline{\mathfrak{T}}} \int_{-\infty}^{+\infty} \kappa_{w, \tau}(z) \mathbb{B}_F(z) dz, \quad (4.13)$$

where

$$\kappa_{w, \tau}(z) := w_0 + \sum_{i=1}^k (1 - \alpha_i)^{-1} w_i \mathbf{1}_{[\tau_i, \infty)}(z). \quad (4.14)$$

Proof. By Lemma 3.1 we have that $\int_{-\infty}^{+\infty} \mathbf{1}_{[\tau, \infty)}(z) \mathbb{B}_F(z) dz$ has the same variance as $[Z - \tau]_+$. In a similar way it can be shown that the covariance structure of the process $\int_{-\infty}^{+\infty} \kappa_{w, \tau}(z) \mathbb{B}_F(z) dz$ is the same as the process $\mathbb{Y}(w, \tau)$ in Theorem 4.1. Hence (4.13) follows from (4.8). \blacksquare

Recall that $\mathbb{B}_F(z) = \mathbb{B}(F(z))$, where \mathbb{B} is the standard Brownian bridge corresponding to the uniform distribution on the interval $[0, 1]$. Hence

$$\inf_{\tau_i \in \mathfrak{Q}_F(\alpha_i)} \int_{-\infty}^{+\infty} \mathbf{1}_{[\tau_i, \infty)}(z) \mathbb{B}_F(z) dz = \begin{cases} \int_{b_i}^{\infty} \mathbb{B}(F(z)) dz & \text{if } \mathbb{B}(\alpha_i) > 0, \\ \int_{a_i}^{\infty} \mathbb{B}(F(z)) dz & \text{if } \mathbb{B}(\alpha_i) \leq 0, \end{cases}$$

where $[a_i, b_i] = \mathfrak{Q}_F(\alpha_i)$. Therefore the asymptotics (4.13) can be written as

$$N^{1/2} \left[\mathcal{R}(\hat{F}_N) - \mathcal{R}(F) \right] \xrightarrow{\mathcal{D}} \sup_{w \in \overline{\mathfrak{W}}} \int_0^1 \sigma_w(z) \mathbb{B}(F(z)) dz, \quad (4.15)$$

where

$$\sigma_w(t) := w_0 + \sum_{i=1}^k (1 - \alpha_i)^{-1} w_i \mathbf{1}_{[\tau_i, \infty)}(z) \quad (4.16)$$

with $\tau_i = b_i$ if $\mathbb{B}(\alpha_i) > 0$, and $\tau_i = a_i$ if $\mathbb{B}(\alpha_i) \leq 0$. Note that unless all quantile sets $\mathfrak{Q}_F(\alpha_i)$, $i = 1, \dots, k$, are singletons, σ_w depends on \mathbb{B} .

In particular, if all quantile sets $\mathfrak{Q}_F(\alpha_i)$, $i = 1, \dots, k$, are singletons, then by making change of variables $z = F(t)$ we can write

$$N^{1/2} \left[\mathcal{R}(\hat{F}_N) - \mathcal{R}(F) \right] \xrightarrow{\mathcal{D}} \sup_{\sigma \in \overline{\mathfrak{T}}} \int_0^1 \sigma(t) \mathbb{B}(t) dF^{-1}(t), \quad (4.17)$$

where $\overline{\mathfrak{T}}$ is the set of maximizers in the corresponding representation (2.2).

4.2 Asymptotics of the optimization problem

Consider optimization problem (1.1) and its sample counterpart (1.2). Suppose that \mathcal{R} is of the form (4.1), the set \mathcal{X} is nonempty convex compact, $G(x, \xi)$ is convex in x for all $\xi \in \Xi$, and $\mathbb{E}|G_x| < +\infty$ for all $x \in \mathcal{X}$. It follows that functions $g(x)$ and $\hat{g}_N(x)$ are convex and finite valued, and hence the respective optimization problems (1.1) and (1.2) are convex. Since \mathcal{R} is of the form (4.1), the optimal value ϑ_* of problem (1.1) can be written as

$$\vartheta_* = \inf_{x \in \mathcal{X}} \sup_{w \in \mathfrak{W}} \left\{ w_0 \mathbb{E}[G_x] + \sum_{i=1}^k w_i \text{AVaR}_{\alpha_i}(G_x) \right\} \quad (4.18)$$

$$= \sup_{w \in \mathfrak{W}} \inf_{x \in \mathcal{X}} \left\{ w_0 \mathbb{E}[G_x] + \sum_{i=1}^k w_i \text{AVaR}_{\alpha_i}(G_x) \right\}. \quad (4.19)$$

Let $\overline{\mathcal{X}}$ and $\overline{\mathfrak{W}}$ be the sets of optimal solutions of problems (4.18) and (4.19), respectively.

We also can write

$$\vartheta_* = \inf_{(x, \tau) \in \mathcal{X} \times \mathbb{R}^k} \sup_{w \in \mathfrak{W}} \mathbb{E}[\phi(G_x, w, \tau)] \quad (4.20)$$

$$= \sup_{w \in \mathfrak{W}} \inf_{(x, \tau) \in \mathcal{X} \times \mathbb{R}^k} \mathbb{E}[\phi(G_x, w, \tau)], \quad (4.21)$$

where function $\phi(\cdot, \cdot, \cdot)$ is defined in (4.4). Denote $\mathcal{Y} := \mathcal{X} \times \mathbb{R}^k$ and let $\overline{\mathcal{Y}} \subset \mathcal{Y}$ be the set of optimal solutions of problem (4.20). Assuming that condition (A) holds, the set $\overline{\mathcal{Y}}$ is nonempty and compact. The minimax problem (4.20)–(4.21) is convex in $(x, \tau) \in \mathcal{Y}$ and concave (linear) in $w \in \mathbb{R}^k$. The set of saddle points of this minimax problem is $\overline{\mathfrak{W}} \times \overline{\mathcal{Y}}$. The SAA problem for (4.20) writes

$$\hat{\vartheta}_N = \inf_{(x, \tau) \in \mathcal{X} \times \mathbb{R}^k} \sup_{w \in \mathfrak{W}} \frac{1}{N} \sum_{j=1}^N \phi(G(x, \xi_j), w, \tau). \quad (4.22)$$

The following theorem can be proved in a way similar to the proof of Theorem 4.1.

Theorem 4.2 *Suppose that: (i) \mathcal{R} is of the form (4.1), (ii) the set \mathcal{X} is convex and $G(x, \xi)$ is convex in x , (iii) condition (A) holds, (iv) $\mathbb{E}[G_x^2]$ is finite for some $x \in \mathcal{X}$, (v) there is a measurable function $C(\xi)$ such that $\mathbb{E}[C(\xi)^2]$ is finite and*

$$|G(x, \xi) - G(x', \xi)| \leq C(\xi) \|x - x'\|, \quad \forall x, x' \in \mathcal{X}, \quad \forall \xi \in \Xi. \quad (4.23)$$

Then

$$\hat{\vartheta}_N = \inf_{(x, \tau) \in \overline{\mathcal{Y}}} \sup_{w \in \overline{\mathfrak{W}}} \left\{ \frac{w_0}{N} \sum_{j=1}^N G(x, \xi_j) + \sum_{i=1}^k w_i \left(\tau_i + \frac{1}{N(1 - \alpha_i)} \sum_{j=1}^N [G(x, \xi_j) - \tau_i]_+ \right) \right\} + o_p(N^{-1/2}). \quad (4.24)$$

Moreover, if the sets $\overline{\mathfrak{W}} = \{w_*\}$ and $\overline{\mathcal{Y}} = \{(x_*, \tau_*)\}$ are singletons, then $N^{1/2}(\hat{\vartheta}_N - \vartheta_*)$ converges in distribution to normal $\mathcal{N}(0, \nu_*^2)$ with variance

$$\nu_*^2 := \text{Var}[\phi(G_{x_*}, w_*, \tau_*)] = \text{Var} \left\{ w_{*0} G_{x_*} + \sum_{i=1}^k \frac{w_{*i}}{1 - \alpha_i} [G_{x_*} - \tau_{*i}]_+ \right\}. \quad (4.25)$$

Let us discuss now estimation of the variance ν_*^2 given in (4.25). Let $(\hat{x}_N, \hat{\tau}_N, \hat{w}_N)$ be a saddle point of the SAA problem (4.22). Suppose that the sets $\overline{\mathfrak{W}} = \{w_*\}$ and $\overline{\mathfrak{Y}} = \{(x_*, \tau_*)\}$ are singletons. Since the sets \mathfrak{Y} and \mathfrak{W} are convex and the function $\phi(G(x, \xi), w, \tau)$ is convex in (x, τ) and concave (linear) in w , it follows that $(\hat{x}_N, \hat{\tau}_N)$ converges w.p.1 to (x_*, τ_*) and \hat{w}_N converges w.p.1 to w_* as $N \rightarrow \infty$ (e.g., [25, Theorem 5.4]). It follows that the variance ν_*^2 can be consistently estimated by its sample counterpart, i.e., the estimator

$$\hat{\nu}_N^2 = \frac{1}{N-1} \sum_{j=1}^N \left[\phi(G(\hat{x}_N, \xi_j), \hat{w}_N, \hat{\tau}_N) - \frac{1}{N} \sum_{j=1}^N \phi(G(\hat{x}_N, \xi_j), \hat{w}_N, \hat{\tau}_N) \right]^2 \quad (4.26)$$

converges w.p.1 to ν_*^2 . Then employing Slutsky's theorem we obtain that under the assumptions of Theorem 4.2, it follows that

$$\frac{N^{1/2}(\hat{\nu}_N - \nu_*)}{\hat{\nu}_N} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1). \quad (4.27)$$

5 Hypotheses testing

On the basis of samples $\xi^{N,i} = (\xi_1^i, \dots, \xi_N^i)$ of ξ^i for $i = 1, \dots, K$, we propose nonasymptotic rejection regions for tests (1.3) and (1.5) (in Section 5.1) and asymptotic rejection regions for tests (1.3), (1.4), and (1.5) (in Section 5.2). For the nonasymptotic tests, we show that the probability of type II error can be controlled under some assumptions. We will denote by $0 < \beta < 1$ the maximal type I error.

5.1 Nonasymptotic tests

5.1.1 Risk-neutral case

Let us consider $K \geq 2$ optimization problems of the form (1.1) with $\mathcal{R} := \mathbb{E}$ the expectation. In this situation, several papers have derived nonasymptotic confidence intervals on the optimal value of (1.1): [12] using Talagrand inequality ([26], [27]), [24], [6] using large-deviation type results, and [10], [9], [5] using Robust Stochastic Approximation (RSA) [15], [16], Stochastic Mirror Descent (SMD) [10] and variants of SMD. In all cases, the confidence interval depends on a sample $\xi^N = (\xi_1, \dots, \xi_N)$ of ξ and of parameters. For instance, the confidence interval $[\text{Low}(\Theta_2, \Theta_3, N), \text{Up}(\Theta_1, N)]$ with confidence level $1 - \beta$ from [5] obtained using RSA depends on parameters $\Theta_1 = 2\sqrt{\ln(2/\beta)}$, $\Theta_3 = 2\sqrt{\ln(4/\beta)}$, Θ_2 satisfying $e^{1-\Theta_2^2} + e^{-\Theta_2^2/4} = \frac{\beta}{4}$, and $L, M_1, M_2, D(\mathcal{X})$ with $D(\mathcal{X})$ the maximal Euclidean distance in \mathcal{X} to x_1 (the initial point of the RSA algorithm), L a uniform upper bound on \mathcal{X} on the $\|\cdot\|_2$ -norm of some selection (say, selection $g'(x) \in \partial g(x)$ at x) of subgradients of g , and $M_1, M_2 < +\infty$ such that for all $x \in \mathcal{X}$ it holds

$$\begin{aligned} (a) \quad & \mathbb{E} \left[(G(x, \xi) - g(x))^2 \right] \leq M_1^2, \\ (b) \quad & \mathbb{E} \left[\|G'_x(x, \xi) - \mathbb{E}[G'_x(x, \xi)]\|_2^2 \right] \leq M_2^2, \end{aligned} \quad (5.28)$$

for some selection $G'_x(x, \xi)$ belonging to the subdifferential $\partial_x G(x, \xi)$.

With this notation, on the basis of a sample $\xi^N = (\xi_1, \dots, \xi_N)$ of size N of ξ and of the trajectory x_1, \dots, x_N of the RSA algorithm, setting

$$a(\Theta, N) = \frac{\Theta M_1}{\sqrt{N}} \text{ and } b(\Theta, \mathcal{X}, N) = \frac{K_1(\mathcal{X}) + \Theta(K_2(\mathcal{X}) - M_1)}{\sqrt{N}}, \quad (5.29)$$

where the constants $K_1(\mathcal{X})$ and $K_2(\mathcal{X})$ are given by

$$K_1(\mathcal{X}) = \frac{D(\mathcal{X})(M_2^2 + 2L^2)}{\sqrt{2(M_2^2 + L^2)}} \text{ and } K_2(\mathcal{X}) = \frac{D(\mathcal{X})M_2^2}{\sqrt{2(M_2^2 + L^2)}} + 2D(\mathcal{X})M_2 + M_1,$$

the lower bound $\text{Low}(\Theta_2, \Theta_3, N)$ is

$$\text{Low}(\Theta_2, \Theta_3, N) = \frac{1}{N} \sum_{t=1}^N G(x_t, \xi_t) - b(\Theta_2, \mathcal{X}, N) - a(\Theta_3, N), \quad (5.30)$$

and the upper bound $\text{Up}(\Theta_1, N)$ is

$$\text{Up}(\Theta_1, N) = \frac{1}{N} \sum_{t=1}^N G(x_t, \xi_t) + a(\Theta_1, N). \quad (5.31)$$

More precisely, we have $\mathbb{P}(\vartheta_* < \text{Low}(\Theta_2, \Theta_3, N)) \leq \beta/2$ and $\mathbb{P}(\vartheta_* > \text{Up}(\Theta_1, N)) \leq \beta/2$.

Test (1.3)-(a). Using these bounds Low and Up or one of the aforementioned cited procedures, we can determine for optimization problem $i \in \{1, \dots, K\}$ (stochastic) lower and upper bounds on ϑ_*^i that we will denote by Low_i and Up_i respectively for short, such that $\mathcal{P}(\vartheta_*^i < \text{Low}_i) \leq \beta/2K$ and $\mathcal{P}(\vartheta_*^i > \text{Up}_i) \leq \beta/2K$.

We define for test (1.3)-(a) the rejection region $\mathcal{W}_{(1.3)-(a)}$ to be the set of samples such that the realizations of the confidence intervals $[\text{Low}_i, \text{Up}_i]$, $i = 1, \dots, K$, on the optimal values have no intersection, i.e.,

$$\begin{aligned} \mathcal{W}_{(1.3)-(a)} &= \left\{ (\xi^{N,1}, \dots, \xi^{N,K}) : \bigcap_{i=1}^K [\text{Low}_i, \text{Up}_i] = \emptyset \right\} \\ &= \left\{ (\xi^{N,1}, \dots, \xi^{N,K}) : \max_{i=1, \dots, K} \text{Low}_i > \min_{i=1, \dots, K} \text{Up}_i \right\}. \end{aligned}$$

If H_0 holds, denoting $\vartheta_* = \vartheta_*^1 = \vartheta_*^2 = \dots = \vartheta_*^K$, we have

$$\begin{aligned} \mathbb{P}\left(\max_{i=1, \dots, K} \text{Low}_i > \min_{i=1, \dots, K} \text{Up}_i\right) &= \mathbb{P}\left(\max_{i=1, \dots, K} [\text{Low}_i - \vartheta_*] + \max_{i=1, \dots, K} [\vartheta_* - \text{Up}_i] > 0\right) \\ &\leq \sum_{i=1}^K \left[\mathbb{P}(\text{Low}_i - \vartheta_*^i > 0) + \mathbb{P}(\vartheta_*^i - \text{Up}_i > 0)\right] \leq \beta \end{aligned}$$

and $\mathcal{W}_{(1.3)-(a)}$ is a rejection region for (1.3)-(a) yielding a type I error of at most β . Moreover, as stated in the following lemma, if H_0 does not hold and if two optimal values are sufficiently distant then the probability to accept H_0 will be small:

Lemma 5.1 *Consider test (1.3)-(a) with rejection region $\mathcal{W}_{(1.3)-(a)}$. If for some $i, j \in \{1, \dots, K\}$ with $i \neq j$, we have almost surely $\vartheta_*^i > \vartheta_*^j + \text{Up}_i - \text{Low}_i + \text{Up}_j - \text{Low}_j$ then the probability to accept H_0 is not larger than $\frac{\beta}{K}$.*

Proof. We first check that

$$\left\{ \begin{array}{l} \vartheta_*^i > \vartheta_*^j + \text{Up}_j - \text{Low}_j + \text{Up}_i - \text{Low}_i \quad (a) \\ \text{Low}_j \leq \vartheta_*^j \quad (b) \\ \vartheta_*^i \leq \text{Up}_i \quad (c) \end{array} \right\} \Rightarrow \text{Up}_j < \text{Low}_i. \quad (5.32)$$

Indeed, if (5.32)-(a), (b), and (c) hold then

$$\text{Up}_j = \text{Low}_j + \text{Up}_j - \text{Low}_j \stackrel{(5.32)-(b)}{\leq} \vartheta_*^j + \text{Up}_j - \text{Low}_j \stackrel{(5.32)-(a)}{<} \vartheta_*^i + \text{Low}_i - \text{Up}_i \stackrel{(5.32)-(c)}{\leq} \text{Low}_i.$$

Assume now that $\vartheta_*^i > \vartheta_*^j + \text{Up}_j - \text{Low}_j + \text{Up}_i - \text{Low}_i$. Since $\text{Up}_j < \text{Low}_i$ implies that H_0 is rejected, we get

$$\begin{aligned} \mathbb{P}(\text{reject } H_0) &\geq \mathbb{P}(\text{Up}_j < \text{Low}_i) \stackrel{(5.32)}{\geq} \mathbb{P}(\{\text{Low}_j \leq \vartheta_*^j\} \cap \{\vartheta_*^i \leq \text{Up}_i\}) \\ &\geq \mathbb{P}(\text{Low}_j \leq \vartheta_*^j) + \mathbb{P}(\vartheta_*^i \leq \text{Up}_i) - 1 \geq 1 - \frac{\beta}{K} \end{aligned}$$

which achieves the proof of the lemma. \blacksquare

Test (1.3)-(b). We now consider the test

$$H_0^i : \vartheta_*^i \leq \vartheta_*^j \text{ for } 1 \leq j \neq i \leq K \text{ against unrestricted } H_1^i.$$

Let $[\text{Low}_i, \text{Up}_i]$ be a confidence interval with confidence level at least $1 - \beta/2(K - 1)$ for problem i :

$$\mathbb{P}(\vartheta_*^i < \text{Low}_i) \leq \beta/2(K - 1) \text{ and } \mathbb{P}(\vartheta_*^i > \text{Up}_i) \leq \beta/2(K - 1). \quad (5.33)$$

We define for test (1.3)-(b) the rejection region

$$\mathcal{W}_{(1.3)-(b)} = \{(\xi^{N,1}, \dots, \xi^{N,K}) : \exists 1 \leq j \neq i \leq K \text{ such that } \text{Low}_i > \text{Up}_j\}.$$

If H_0 holds, we have

$$\begin{aligned} \mathbb{P}(\exists 1 \leq j \neq i \leq K : \text{Low}_i > \text{Up}_j) &\leq \sum_{1 \leq j \neq i \leq K} \mathbb{P}(\text{Low}_i > \text{Up}_j) \\ &\leq \sum_{1 \leq j \neq i \leq K} \mathbb{P}(\text{Low}_i - \vartheta_*^i + \vartheta_*^j - \text{Up}_j > 0) \\ &\leq \sum_{1 \leq j \neq i \leq K} \left(\mathbb{P}(\text{Low}_i - \vartheta_*^i > 0) + \mathbb{P}(\vartheta_*^j - \text{Up}_j > 0) \right) \leq \beta \end{aligned}$$

and $\mathcal{W}_{(1.3)-(b)}$ is a rejection region for (1.3)-(b) yielding a type I error of at most β . We also have an analog of Lemma 5.1:

Lemma 5.2 *Consider test (1.3)-(b) with rejection region $\mathcal{W}_{(1.3)-(b)}$. If for some $j \in \{1, \dots, K\}$ with $i \neq j$, we have almost surely $\vartheta_*^i > \vartheta_*^j + \text{Up}_i - \text{Low}_i + \text{Up}_j - \text{Low}_j$ then the probability to accept H_0 is not larger than $\frac{\beta}{K-1}$.*

Proof. The proof is analogue to the proof of Lemma 5.1. \blacksquare

Test (1.3)-(c). Consider test (1.3)-(c):

$$H_0 : \vartheta_*^1 \leq \vartheta_*^2 \leq \dots \leq \vartheta_*^K \text{ against unrestricted } H_1.$$

Let $[\text{Low}_i, \text{Up}_i]$ be a confidence interval on ϑ_*^i satisfying (5.33). We define the rejection region

$$\mathcal{W}_{(1.3)-(c)} = \{(\xi^{N,1}, \dots, \xi^{N,K}) : \exists i \in \{1, \dots, K-1\} \text{ such that } \text{Low}_i > \text{Up}_{i+1}\}.$$

If H_0 holds, we have

$$\begin{aligned} \mathbb{P}\left(\exists i \in \{1, \dots, K-1\} : \text{Low}_i > \text{Up}_{i+1}\right) &\leq \sum_{i=1}^{K-1} \mathbb{P}\left(\text{Low}_i - \vartheta_*^i + \vartheta_*^{i+1} - \text{Up}_{i+1} > 0\right) \\ &\leq \sum_{i=1}^{K-1} \mathbb{P}\left(\text{Low}_i - \vartheta_*^i > 0\right) + \mathbb{P}\left(\vartheta_*^{i+1} - \text{Up}_{i+1} > 0\right) \leq \beta \end{aligned}$$

and $\mathcal{W}_{(1.3)-(c)}$ is a rejection region for (1.3)-(c) yielding a type I error of at most β . As for test (1.3)-(a), we can bound from above the probability of type II error under some assumptions:

Lemma 5.3 *Consider test (1.3)-(c) with rejection region $\mathcal{W}_{(1.3)-(c)}$. If for some $i \in \{1, \dots, K-1\}$ we have almost surely $\vartheta_*^i > \vartheta_*^{i+1} + \text{Up}_i - \text{Low}_i + \text{Up}_{i+1} - \text{Low}_{i+1}$ then the probability to accept H_0 is not larger than $\frac{\beta}{K-1}$.*

Proof. The proof is analogue to the proof of Lemma 5.1. ■

Remark 5.1 *Though Low and Up are stochastic, for bounds (5.30) and (5.31), the difference $\text{Up} - \text{Low} = a(\Theta_1, N) + b(\Theta_2, \mathcal{X}, N) + a(\Theta_3, N)$ is deterministic and inequality $\vartheta_*^i > \vartheta_*^j + \text{Up}_i - \text{Low}_i + \text{Up}_j - \text{Low}_j$ in Lemmas 5.1 and 5.2 is deterministic too. With this choice of lower and upper bounds, inequality $\vartheta_*^i > \vartheta_*^{i+1} + \text{Up}_i - \text{Low}_i + \text{Up}_{i+1} - \text{Low}_{i+1}$ in Lemma 5.3 is also deterministic.*

Tests (1.5). For tests

$$\begin{aligned} H_0 : \vartheta_* &= \rho_0 \text{ against } H_1 : \vartheta_* \neq \rho_0 & (a) \\ H_0 : \vartheta_* &\leq \rho_0 \text{ against } H_1 : \vartheta_* > \rho_0 & (b) \\ H_0 : \vartheta_* &\geq \rho_0 \text{ against } H_1 : \vartheta_* < \rho_0, & (c) \end{aligned} \tag{5.34}$$

we define rejection regions which are respectively of the form

$$\begin{aligned} \mathcal{W}_{(5.34)-(a)} &= \left\{ (\xi_1, \dots, \xi_N) : \left\{ \rho_0 > \text{Up} \right\} \cup \left\{ \rho_0 < \text{Low} \right\} \right\}, \\ \mathcal{W}_{(5.34)-(b)} &= \left\{ (\xi_1, \dots, \xi_N) : \rho_0 < \text{Low} \right\}, \\ \mathcal{W}_{(5.34)-(c)} &= \left\{ (\xi_1, \dots, \xi_N) : \rho_0 > \text{Up} \right\}. \end{aligned}$$

To ensure a type I error of at most β , the confidence interval $[\text{Low}, \text{Up}]$ on ϑ_* satisfies (i) $\mathbb{P}(\vartheta_* > \text{Up}) \leq \beta/2$ and $\mathbb{P}(\vartheta_* < \text{Low}) \leq \beta/2$ for test (5.34)-(a), (ii) $\mathbb{P}(\vartheta_* < \text{Low}) \leq \beta$ for test (5.34)-(b), and (iii) $\mathbb{P}(\vartheta_* > \text{Up}) \leq \beta$ for test (5.34)-(c).

5.1.2 Risk averse case

Consider $K \geq 2$ optimization problems of the form (1.1). For such problems, nonasymptotic confidence intervals $[\text{Low}, \text{Up}]$ on the optimal value ϑ_* were derived in [5] and [9] using RSA and SMD, taking for \mathcal{R} an extended polyhedral risk measure (introduced in [7]) in [5] and $\mathcal{R} = \text{AVaR}_\alpha$ and $G(x, \xi) = \xi^T x$ in [9]. With such confidence intervals at hand, we can use the developments of the previous section for testing hypotheses (1.3) and (1.5). However, the analysis in [5] assumes boundedness of the feasible set of the optimization problem defining the risk measure; an assumption that can be enforced for risk measure \mathcal{R} given by (5.35). We provide in this situation formulas for the constants L, M_1 , and M_2 defined in the previous section, necessary to compute the bounds from [5]. These constants are slightly refined versions of the constants given in Section 4.2 of [9] for the special case $\mathcal{R} = \text{AVaR}_\alpha$ and $G(x, \xi) = \xi^T x$.

We assume here that the set Ξ is compact, $G(\cdot, \cdot)$ is continuous, for every $x \in \mathcal{X}$ the distribution of G_x is continuous, and that the set $\mathfrak{W} = \{w\}$ is a singleton i.e.,

$$\mathcal{R}(Z) = w_0 \mathbb{E}[Z] + \sum_{i=1}^k w_i \text{AVaR}_{\alpha_i}(Z) \quad (5.35)$$

for some $w \in \Delta_{k+1}$. Consequently problem (1.1) can be written as

$$\vartheta_* = \inf_{(x, \tau) \in \mathcal{X} \times \mathbb{R}^k} \left\{ \mathbb{E}[\phi(G_x, \tau)] = \mathbb{E}[H(x, \tau, \xi)] \right\}, \quad (5.36)$$

where $\phi(G_x, \tau)$ is defined in (4.4), with vector w omitted, and

$$H(x, \tau, \xi) := w_0 G(x, \xi) + \sum_{i=1}^k w_i \left(\tau_i + \frac{1}{1 - \alpha_i} [G(x, \xi) - \tau_i]_+ \right).$$

For a given $x \in \mathcal{X}$ the minimum in (5.36) is attained at $\tau_i = F_x^{-1}(\alpha_i)$, $i = 1, \dots, k$, where F_x is the cdf of G_x . Therefore, using the lower and upper bounds from [9] for the quantile of a continuous distribution with finite mean and variance, we can restrict τ to compact set $\mathcal{T} = [\underline{\tau}, \bar{\tau}] \subset \mathbb{R}^k$ where

$$\begin{aligned} \underline{\tau}_i &= \min_{x \in \mathcal{X}} \mathbb{E}[G_x] - \sqrt{\frac{1 - \alpha_i}{\alpha_i}} \sqrt{\max_{x \in \mathcal{X}} \text{Var}(G_x)}, \\ \bar{\tau}_i &= \max_{x \in \mathcal{X}} \mathbb{E}[G_x] + \sqrt{\frac{\alpha_i}{1 - \alpha_i}} \sqrt{\max_{x \in \mathcal{X}} \text{Var}(G_x)}, \end{aligned} \quad (5.37)$$

for $i = 1, \dots, k$. This implies that we can take for $D(\mathcal{X} \times \mathcal{T})$ the quantity $\sqrt{D(\mathcal{X})^2 + \|\bar{\tau} - \underline{\tau}\|_2^2}$.

Computation of M_1 . Setting

$$M_0 := \max_{(x, \xi) \in \mathcal{X} \times \Xi} G(x, \xi) \text{ and } m_0 := \min_{(x, \xi) \in \mathcal{X} \times \Xi} G(x, \xi),$$

we have for $(x, \tau) \in \mathcal{X} \times \mathcal{T}$ that $|G_x - \mathbb{E}[G_x]| \leq M_0 - m_0$ and $|[G_x - \tau_i]_+ - \mathbb{E}[G_x - \tau_i]_+| \leq M_0 - \tau_i$ which implies that almost surely

$$|\phi(G_x, \tau) - \mathbb{E}[\phi(G_x, \tau)]| \leq M_1 := w_0(M_0 - m_0) + \sum_{i=1}^k \frac{w_i}{1 - \alpha_i} (M_0 - \tau_i).$$

Computation of M_2 and L . We have $H'_{x, \tau}(x, \tau, \xi) = [H'_x(x, \tau, \xi); H'_\tau(x, \tau, \xi)]$ with

$$\begin{aligned} H'_x(x, \tau, \xi) &= w_0 G'_x(x, \xi) + \sum_{i=1}^k \frac{w_i}{1 - \alpha_i} G'_x(x, \xi) \mathbf{1}_{G(x, \xi) \geq \tau_i}, \\ H'_\tau(x, \tau, \xi) &= (w_i (1 - \frac{1}{1 - \alpha_i} \mathbf{1}_{G(x, \xi) \geq \tau_i}))_{i=1, \dots, k}. \end{aligned}$$

We assume that for every $x \in \mathcal{X}$, the stochastic subgradients $G'_x(x, \xi)$ are almost surely bounded and we denote by \underline{m} and \overline{M} vectors such that almost surely $\underline{m} \leq G'_x(x, \xi) \leq \overline{M}$. Then for $(x, \tau) \in \mathcal{X} \times \mathcal{T}$, setting

$$a_i = w_0 \overline{M}_i + \sum_{i=1}^k \frac{w_i}{1 - \alpha_i} \max(0, \overline{M}_i) \text{ and } b_i = w_0 \underline{m}_i + \sum_{i=1}^k \frac{w_i}{1 - \alpha_i} \min(0, \underline{m}_i)$$

we have

$$\begin{aligned} \|\mathbb{E}[H'_{x, \tau}(x, \tau, \xi)]\|_2^2 &\leq L^2 := \sum_{i=1}^m \max(a_i^2, b_i^2) + \sum_{i=1}^k w_i^2 \max\left(1, \frac{\alpha_i^2}{(1 - \alpha_i)^2}\right), \\ \mathbb{E}\|H'_{x, \tau}(x, \tau, \xi) - \mathbb{E}[H'_{x, \tau}(x, \tau, \xi)]\|_2^2 &\leq M_2^2 := \sum_{i=1}^m (a_i - b_i)^2 + \sum_{i=1}^k \left(\frac{w_i}{1 - \alpha_i}\right)^2. \end{aligned}$$

In some cases, the above formulas for $\bar{\tau}, \underline{\tau}, L, M_1$, and M_2 can be simplified:

Example 3 Let $k = 1$ in (5.35) and $G(x, \xi) = \xi^T x$ where ξ is a random vector with mean μ and covariance matrix Σ . In this case $\min_{x \in \mathcal{X}} \mathbb{E}[G_x]$ and $\max_{x \in \mathcal{X}} \mathbb{E}[G_x]$ are convex optimization problems with linear objective functions and denoting by U_1 the quantity $\max_{x \in \mathcal{X}} \|x\|_1$ or an upper bound on this quantity, we can replace $\max_{x \in \mathcal{X}} \text{Var}(G_x)$ by $U_1^2 \max_i \Sigma(i, i)$ in the expressions of τ_i and $\bar{\tau}_i$. Computing M_0 and m_0 also amounts to solve convex optimization problems with linear objective. Assume also that almost surely $\|\xi\|_\infty \leq U_2$ for some $0 < U_2 < +\infty$. We have $|G_x - \mathbb{E}[G_x]| \leq 2U_1U_2$ and $|(G_x - \tau)_+ - \mathbb{E}[G_x - \tau]_+| \leq U_1U_2 - \tau$ which shows that we can take $M_1 = 2w_0U_1U_2 + \frac{w_1}{1-\alpha_1}(U_1U_2 - \tau)$. We have $\mathbb{E}[H'_\tau(x, \tau, \xi)] = w_1(1 - \frac{\mathbb{P}(\xi^T x \geq \tau)}{1-\alpha_1})$ so that $|\mathbb{E}[H'_\tau(x, \tau, \xi)]| \leq w_1 \max(1, \frac{\alpha_1}{1-\alpha_1})$ and $\|\mathbb{E}[H'_\tau(x, \tau, \xi)]\|_2^2 \leq n(w_0 + \frac{w_1}{1-\alpha_1})^2 U_2^2$, i.e., we can take $L^2 = w_1^2 \max(1, \frac{\alpha_1^2}{(1-\alpha_1)^2}) + n(w_0 + \frac{w_1}{1-\alpha_1})^2 U_2^2$. Next, for all $\xi_0 \in \Xi$ we have

$$\begin{aligned} |H'_\tau(x, \tau, \xi_0) - \mathbb{E}[H'_\tau(x, \tau, \xi)]| &= \frac{w_1(1 - \mathbb{P}(\xi^T x \geq \tau))}{1-\alpha_1} \text{ if } \xi_0^T x \geq \tau, \\ &= \frac{w_1 \mathbb{P}(\xi^T x \geq \tau)}{1-\alpha_1} \text{ otherwise,} \end{aligned}$$

implying that $|H'_\tau(x, \tau, \xi_0) - \mathbb{E}[H'_\tau(x, \tau, \xi)]| \leq \frac{w_1}{1-\alpha_1}$. Since $\|H'_\tau(x, \tau, \xi_0) - \mathbb{E}[H'_\tau(x, \tau, \xi)]\|_\infty \leq 2(w_0 + \frac{w_1}{1-\alpha_1})U_2$, we can take $M_2^2 = \frac{w_1^2}{(1-\alpha_1)^2} + 4n(w_0 + \frac{w_1}{1-\alpha_1})^2 U_2^2$. In the special case when $\mathcal{X} = \{x_*\}$ is a singleton, denoting $\eta = \xi^T x_*$, we have $\vartheta_* = \mathcal{R}(\eta)$ and the above computations show that we can take

$$L = w_1 \max(1, \frac{\alpha_1}{1-\alpha_1}), M_1 = w_0(b_0 - a_0) + \frac{w_1}{1-\alpha_1}(b_0 - \tau), \text{ and } M_2 = \frac{w_1}{1-\alpha_1}, \quad (5.38)$$

where $\tau = \mathbb{E}[\eta] - \sqrt{\frac{1-\alpha_1}{\alpha_1}} \sqrt{\text{Var}(\eta)}$ with a_0, b_0 satisfying $a_0 \leq \eta \leq b_0$ almost surely.

Finally, note that the nonasymptotic tests of this and the previous section do not require the independence of $\xi^{N,1}, \dots, \xi^{N,K}$ and are valid for any sample size N . However, they use conservative confidence bounds and rejection regions meaning that they can lead to large probabilities of type II errors. The asymptotic tests to be presented in the next section are valid as the sample size tends to infinity but work well in practice for small sample sizes ($N = 20$) for problems of small to moderate size (n up to 500); see the numerical simulations of Section 6.

5.2 Asymptotic tests

Test (1.5). Consider optimization problem (1.1) and the SAA approximation $\hat{\vartheta}_N$ of its optimal value ϑ_* obtained using a sample (ξ_1, \dots, ξ_N) of ξ . Let also $\hat{\nu}_N^2$ be the empirical estimator (4.26) of the variance (4.25). Under the assumptions of Theorem 4.2, we have the asymptotics (4.27). Therefore for N large, we can approximate the distribution of $\frac{N^{1/2}(\hat{\vartheta}_N - \vartheta_*)}{\hat{\nu}_N}$ by the standard normal $\mathcal{N}(0, 1)$.

It follows that for tests (1.5)-(a) and (1.5)-(b), we obtain respectively the asymptotic rejection regions

$$\begin{aligned} \mathcal{W}_{(1.5)-(a)}^{As} &= \left\{ (\xi_1, \dots, \xi_N) : |\hat{\vartheta}_N - \rho_0| > \frac{\hat{\nu}_N}{\sqrt{N}} \Phi^{-1}(1 - \frac{\beta}{2}) \right\} \text{ and} \\ \mathcal{W}_{(1.5)-(b)}^{As} &= \left\{ (\xi_1, \dots, \xi_N) : \hat{\vartheta}_N > \rho_0 + \frac{\hat{\nu}_N}{\sqrt{N}} \Phi^{-1}(1 - \beta) \right\}, \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Tests (1.3) and (1.4). Let us now consider $K > 1$, optimization problems of the form (1.1) with ξ, g , and \mathcal{X} respectively replaced by ξ^i, g_i , and \mathcal{X}_i for problem i . For $i = 1, \dots, K$, let $(\xi_1^i, \dots, \xi_N^i)$ be a sample from the distribution of ξ^i , let ϑ_*^i be the optimal value of problem i and $z_*^i = (x_*^i, \tau_*^i)$ the

optimal solution. Let also $\hat{\vartheta}_N^i$ be the SAA estimator of the optimal value for problem $i = 1, \dots, K$, and $\hat{\nu}_N^i$ be the empirical estimator of the variance $\text{Var}[H_i(z_*^i, \xi_i)]$ based on the sample for problem i . We assume that the samples are i.i.d. and that $\xi^{N,1}, \dots, \xi^{N,K}$ are independent. Under the assumptions of Theorem 4.2 for N large we can approximate the distribution of $\frac{N^{1/2}(\hat{\vartheta}_N^i - \vartheta_*^i)}{\hat{\nu}_N^i}$ by the standard normal $\mathcal{N}(0, 1)$.

Let us first consider the statistical tests (1.3)-(a) and (1.3)-(b) with $K = 2$:

$$\begin{aligned} H_0 : \vartheta_*^1 &= \vartheta_*^2 \text{ against } H_1 : \vartheta_*^1 \neq \vartheta_*^2 \\ H_0 : \vartheta_*^1 &\leq \vartheta_*^2 \text{ against } H_1 : \vartheta_*^1 > \vartheta_*^2. \end{aligned}$$

For N large, we approximate the distribution of $\frac{(\hat{\vartheta}_N^1 - \hat{\vartheta}_N^2) - (\vartheta_*^1 - \vartheta_*^2)}{\sqrt{\frac{(\hat{\nu}_N^1)^2}{N} + \frac{(\hat{\nu}_N^2)^2}{N}}}$ by the standard normal $\mathcal{N}(0, 1)$

and we obtain the rejection regions

$$\begin{cases} \left\{ (\xi_N^1, \xi_N^2) : |\hat{\vartheta}_N^1 - \hat{\vartheta}_N^2| > \sqrt{\frac{(\hat{\nu}_N^1)^2}{N} + \frac{(\hat{\nu}_N^2)^2}{N}} \Phi^{-1}(1 - \frac{\beta}{2}) \right\} & \text{for test (1.3)-(a) with } K = 2, \\ \left\{ (\xi_N^1, \xi_N^2) : \hat{\vartheta}_N^1 > \hat{\vartheta}_N^2 + \sqrt{\frac{(\hat{\nu}_N^1)^2}{N} + \frac{(\hat{\nu}_N^2)^2}{N}} \Phi^{-1}(1 - \beta) \right\} & \text{for test (1.3)-(b) with } K = 2. \end{cases} \quad (5.39)$$

We finally consider test (1.4):

$$H_0 : \theta \in \Theta_0 \text{ against } H_1 : \theta \notin \Theta_0$$

for $\theta = (\vartheta_*^1, \dots, \vartheta_*^K)^T$ with Θ_0 a linear space or a closed convex cone.

Let Θ_0 be the subspace

$$\Theta_0 = \{\theta \in \mathbb{R}^K : A\theta = 0\} \quad (5.40)$$

where A is a $k_0 \times K$ matrix of full rank k_0 . Note that test (1.3)-(a) can be written under this form with A a $(K - 1) \times K$ matrix of rank $K - 1$. We have for θ the estimator $\hat{\theta}_N = (\hat{\vartheta}_N^1, \dots, \hat{\vartheta}_N^K)^T$.

Fixing N large, since $\xi^{N,1}, \dots, \xi^{N,K}$ are independent, using the fact that $\frac{N^{1/2}(\hat{\vartheta}_N^i - \vartheta_*^i)}{\hat{\nu}_N^i} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$, the distribution of $\hat{\theta}_N$ can be approximated by the Gaussian $\mathcal{N}(\theta, \Sigma)$ distribution with Σ the diagonal matrix $\Sigma = (1/N)\text{diag}(\text{Var}(H_1(z_*^1, \xi_1)), \dots, \text{Var}(H_K(z_*^K, \xi_K)))$. The log-likelihood ratio

statistic for test (1.4) is $\Lambda = \frac{\sup_{\theta \in \Theta_0, \Sigma > 0} \mathcal{L}(\theta, \sigma)}{\sup_{\theta, \Sigma > 0} \mathcal{L}(\theta, \sigma)}$ where $\mathcal{L}(\theta, \Sigma)$ is the likelihood function for a Gaussian

multivariate model. For a sample $(\tilde{\theta}_1, \dots, \tilde{\theta}_M)$ of $\hat{\theta}_N$, introducing the estimators

$$\hat{\theta} = \frac{1}{M} \sum_{i=1}^M \tilde{\theta}_i \text{ and } \hat{\Sigma} = \frac{1}{M-1} \sum_{i=1}^M (\tilde{\theta}_i - \hat{\theta})(\tilde{\theta}_i - \hat{\theta})^T$$

of respectively θ and Σ , we have

$$-2 \ln \Lambda = K \ln \left(1 + \frac{T^2}{M-1} \right) \text{ where } T^2 = K \min_{\theta \in \Theta_0} (\hat{\theta} - \theta)^T \hat{\Sigma}^{-1} (\hat{\theta} - \theta) \quad (5.41)$$

and when Θ_0 is of the form (5.40), under H_0 , we have $T^2 \sim \frac{k_0(M-1)}{M-k_0} F_{k_0, M-k_0}$ where $F_{p,q}$ is the Fisher-Snedecor distribution with parameters p and q . For asymptotic test (1.4) at confidence level

β with Θ_0 given by (5.40), we then reject H_0 if $T^2 \geq \frac{k_0(M-1)}{M-k_0} F_{k_0, M-k_0}^{-1}(1-\beta)$ where $F_{p,q}^{-1}(\beta)$ is the β -quantile of the Fisher-Snedecor distribution with parameters p and q .

Now take for Θ_0 the convex cone $\Theta_0 = \{\theta \in \mathbb{R}^K : A\theta \leq 0\}$ where A is a $k_0 \times K$ matrix of full rank k_0 (tests (1.3)-(b), (c) are special cases) and assume that $M \geq K+1$. Since the corresponding null hypothesis is θ belongs to a one-sided cone, on the basis of the sample $(\hat{\theta}_1, \dots, \hat{\theta}_M)$ of $\hat{\theta}_N$, we can use [11] and we reject H_0 for large values of the statistic

$$\mathcal{U}(\Theta_0) = \|\hat{\theta}\|_S^2 - \|\Pi_S(\hat{\theta}|\Theta_0)\|_S^2 = \|\hat{\theta} - \Pi_S(\hat{\theta}|\Theta_0)\|_S^2$$

where $S = \frac{M-1}{M} \hat{\Sigma}$, $\|x\|_S = \sqrt{x^T S^{-1} x}$, and $\Pi_S(x|A)$ is any point in A minimizing $\|y - x\|_S$ among all $y \in A$. For a type I error of at most $0 < \beta < 1$, knowing that [11]

$$\sup_{\theta \in \Theta_0, \Sigma \succ 0} \mathbb{P}(\mathcal{U}(\Theta_0) \geq u | \theta, \Sigma) \leq \mathbf{Err}(u) := \frac{1}{2} \left[\mathbb{P}(G_{K-1, M-K-1} \geq u) + \mathbb{P}(G_{K, M-K} \geq u) \right], \quad (5.42)$$

where $G_{p,q} = (p/q)F_{p,q}$, we reject H_0 if $\mathcal{U}(\Theta_0) \geq u_\beta$ where u_β satisfies $\beta = \mathbf{Err}(u_\beta)$ with $\mathbf{Err}(\cdot)$ given by (5.42).

6 Numerical experiments

6.1 Comparing the risk of two distributions

We consider test (1.3) with $K = 2$ and \mathcal{X} a singleton. We use the rejection regions given in Section 5.1 (resp. given by (5.39)) in the nonasymptotic (resp. asymptotic) case. In this situation, the test aims at comparing the risk of two distributions. We use the notation $\mathcal{N}(m, \sigma^2; a_0, b_0)$ for the normal distribution with mean m and variance σ^2 conditional on this random variable being in $[a_0, b_0]$ (truncated normal distribution with support $[a_0, b_0]$). More precisely, we compare the risks $\mathcal{R}(\xi_1)$ and $\mathcal{R}(\xi_2)$ of two truncated normal (loss) distributions ξ_1 and ξ_2 with support $[a_0, b_0] = [0, 30]$ in three cases: (I) $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$, $\xi_2 \sim \mathcal{N}(20, 1; 0, 30)$, (II) $\xi_1 \sim \mathcal{N}(5, 1; 0, 30)$, $\xi_2 \sim \mathcal{N}(10, 25; 0, 30)$, and (III) $\xi_1 \sim \mathcal{N}(10, 49; 0, 30)$, $\xi_2 \sim \mathcal{N}(14, 0.25; 0, 30)$. For these three cases, the densities of ξ_1 and ξ_2 are represented in Figure 1 (top left for (I), top right for (II), and bottom for (III)).

We take for \mathcal{R} the risk measure $\mathcal{R}(\xi) = w_0 \mathbb{E}[\xi] + w_1 \text{AVaR}_\alpha(\xi)$ for $0 < \alpha < 1$ where $w_0, w_1 \geq 0$ with $w_0 + w_1 = 1$. We assume that only the support $[a_0, b_0]$ of ξ_1 and ξ_2 and two samples ξ_1^N and ξ_2^N of size N of respectively ξ_1 and ξ_2 are known. Since the distribution of ξ has support $[a_0, b_0]$, we can write

$$\mathcal{R}(\xi) = \min_{\tau \in [a_0, b_0]} w_0 \mathbb{E}[\xi] + w_1 \left(\tau + \frac{1}{1-\alpha} \mathbb{E}[\xi - \tau]_+ \right) \quad (6.43)$$

which is of form (1.1) with a risk-neutral objective function, $G(\tau, \xi) = w_0 \xi + w_1 \tau + \frac{w_1}{1-\alpha} [\xi - \tau]_+$, and \mathcal{X} the compact set $\mathcal{X} = [a_0, b_0] = [0, 30]$. It follows that the RSA algorithm can be used to estimate $\mathcal{R}(\xi_1)$ and $\mathcal{R}(\xi_2)$ and to compute the confidence bounds (5.30) and (5.31) with L, M_1 , and M_2 given by (5.38). In these formulas, we replace $\underline{\tau}$ by its lower bound 0 since we do not assume the mean and standard deviation of ξ_1 and ξ_2 known. We obtain $L = w_1 \max(1, \frac{\alpha}{1-\alpha})$, $M_2 = \frac{w_1}{1-\alpha}$, and $M_1 = 30(w_0 + \frac{w_1}{1-\alpha})$.

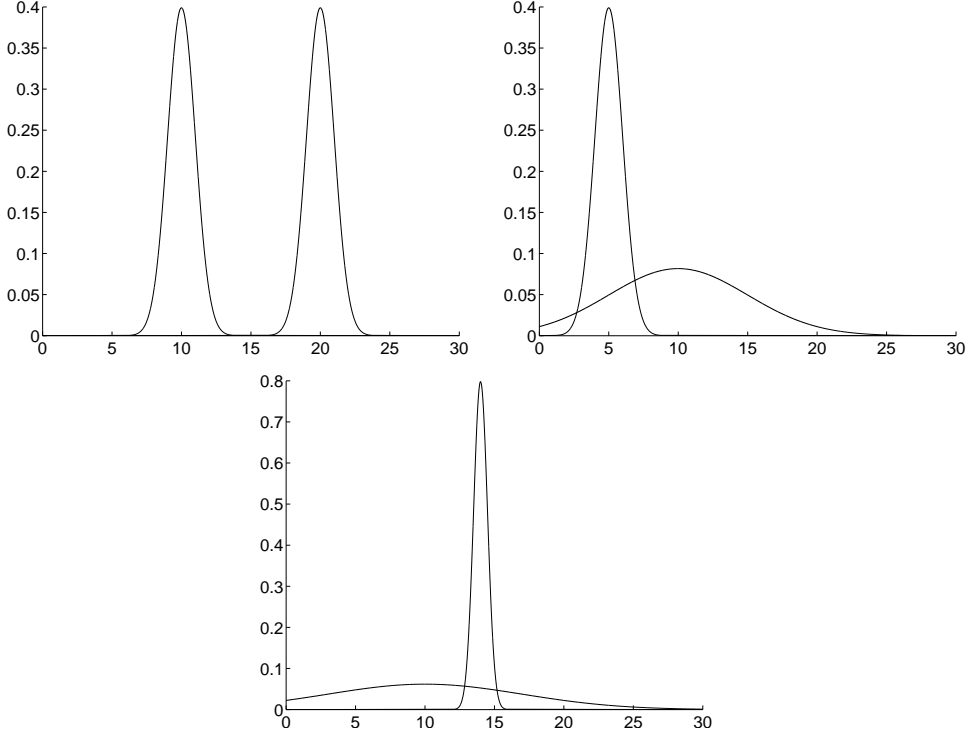


Figure 1: Densities of truncated normal loss distributions ξ_1 and ξ_2 . Top left plot: $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$ and $\xi_2 \sim \mathcal{N}(20, 1; 0, 30)$. Top right plot: $\xi_1 \sim \mathcal{N}(5, 1; 0, 30)$ and $\xi_2 \sim \mathcal{N}(10, 25; 0, 30)$. Bottom plot: $\xi_1 \sim \mathcal{N}(10, 49; 0, 30)$ and $\xi_2 \sim \mathcal{N}(14, 0.25; 0, 30)$.

We first illustrate Theorem 4.1 computing the empirical estimation $\mathcal{R}(\hat{F}_{N,1})$ of $\mathcal{R}(\xi_1)$ on 200 samples of size N of $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$ for $w_0 = 0.1$, $w_1 = 0.9$, and various values of α and of the sample size N . For this experiment, the QQ-plots of the empirical distribution of $\mathcal{R}(\hat{F}_{N,1})$ versus the normal distribution with parameters the empirical mean and standard deviation of this empirical distribution are reported in Figure 2. We see that even for small values of $1 - \alpha$ and N as small as 20, the distribution of $\mathcal{R}(\hat{F}_{N,1})$ is well approximated by a Gaussian distribution: for $N = 20$ the Jarque-Bera test accepts the hypothesis of normality at the significance level 0.05 for $1 - \alpha = 0.01$ and $1 - \alpha = 0.5$.

We fix again the distribution $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$ and approximately compute $\mathcal{R}(\xi_1)$ for various values of (w_0, w_1, α, N) using the RSA and SAA methods on samples ξ_1^N of size N of ξ_1 . For a sample of size N of ξ_1 , let $\hat{\mathcal{R}}_{N,\text{RSA}}(\xi_1)$ and $\hat{\mathcal{R}}_{N,\text{SAA}}(\xi_1) = \mathcal{R}(\hat{F}_{N,1})$ be these estimations using respectively RSA and SAA. For fixed (w_0, w_1, α, N) , we generate 200 samples of size N of ξ_1 and for each sample we compute $\hat{\mathcal{R}}_{N,\text{RSA}}(\xi_1)$ and $\hat{\mathcal{R}}_{N,\text{SAA}}(\xi_1)$ and report in Table 1 the average of these values for $N \in \{20, 50, 100, 10^3, 10^4, 10^5, 10^6\}$. Considering that $\mathcal{R}(\xi_1)$ is the value obtained using SAA for $N = 10^6$, we observe that RSA correctly approximates $\mathcal{R}(\xi_1)$ as N grows and that the estimation of $\mathbb{E}[\hat{\mathcal{R}}_{N,\text{SAA}}(\xi_1)]$ (resp. $\mathbb{E}[\hat{\mathcal{R}}_{N,\text{RSA}}(\xi_1)]$) increases (resp. decreases) with the sample size N , as expected. We also naturally observe that the more weight is given to the AVaR and the smaller $1 - \alpha$ the more difficult it is to estimate the risk measure, i.e., the more distant the expectation of the approximation is to the optimal value and the larger the sample size needs to be to obtain an expected approximation with given accuracy.

We now study for case (I) the test

$$H_0 : \mathcal{R}(\xi_1) = \mathcal{R}(\xi_2) \text{ against } H_1 : \mathcal{R}(\xi_1) \neq \mathcal{R}(\xi_2). \quad (6.44)$$

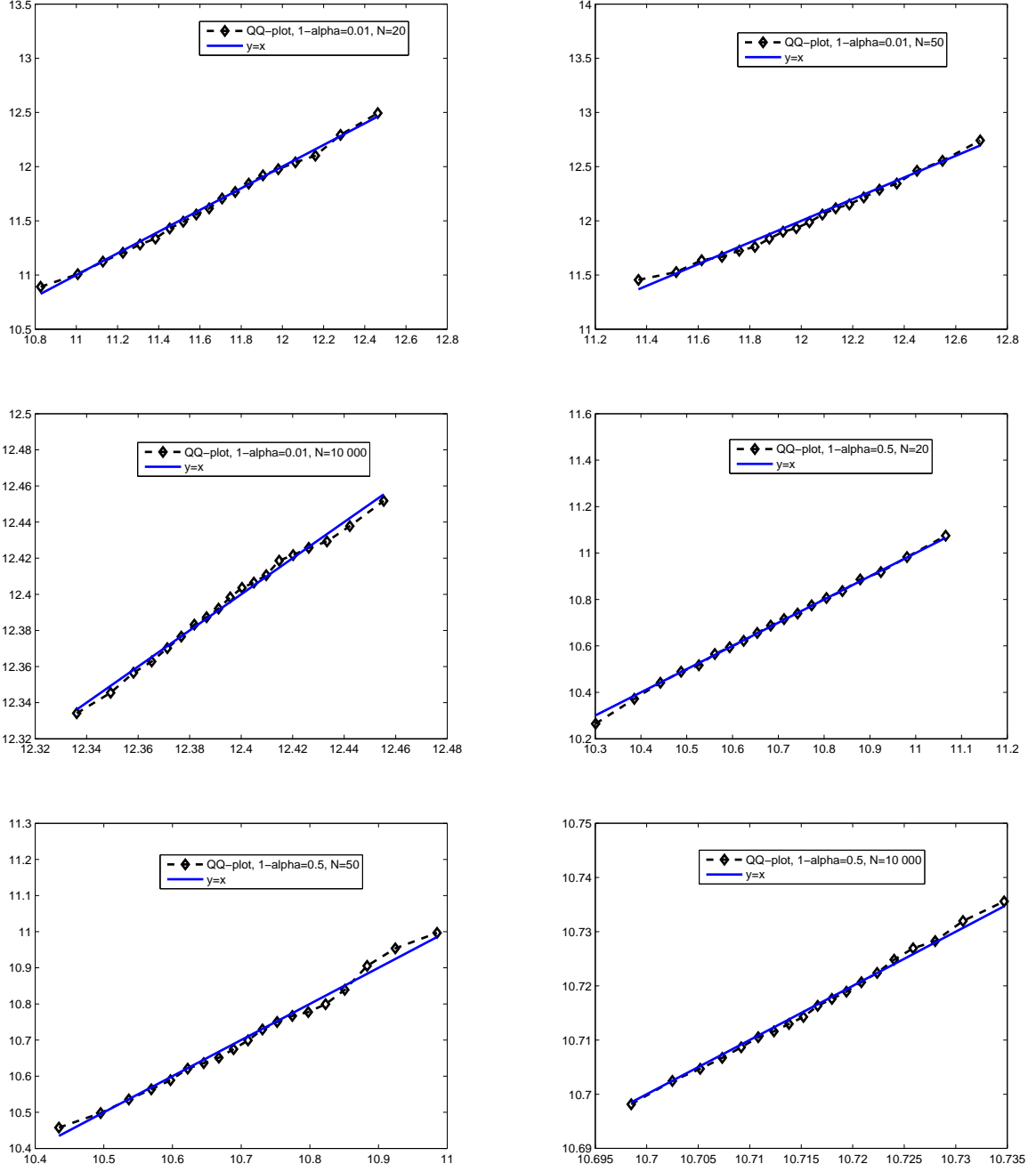


Figure 2: QQ-plots of the empirical distribution of $\mathcal{R}(\hat{F}_{N,1})$ versus the normal distribution with parameters the empirical mean and standard deviation of this empirical distribution for $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$ for $(w_0, w_1) = (0.1, 0.9)$ and various values of (α, N) .

(w_0, w_1)	$1 - \alpha$	Method	Sample size N						
			20	50	10^2	10^3	10^4	10^5	10^6
(0.1, 0.9)	10^{-2}	SAA	11.71	12.00	12.21	12.37	12.40	12.40	12.40
(0.1, 0.9)	10^{-2}	RSA	14.35	14.26	14.16	13.46	12.75	12.51	12.43
(0.1, 0.9)	0.1	SAA	11.51	11.50	11.54	11.58	11.58	11.58	11.58
(0.1, 0.9)	0.1	RSA	20.50	16.78	15.10	12.61	11.90	11.68	11.61
(0.1, 0.9)	0.5	SAA	10.71	10.69	10.72	10.72	10.72	10.72	10.72
(0.1, 0.9)	0.5	RSA	11.42	11.12	11.02	10.81	10.75	10.73	10.72
(0.9, 0.1)	10^{-2}	SAA	10.19	10.23	10.25	10.26	10.27	10.27	10.27
(0.9, 0.1)	10^{-2}	RSA	10.49	10.48	10.47	10.38	10.31	10.28	10.27
(0.9, 0.1)	0.1	SAA	10.17	10.16	10.19	10.18	10.18	10.18	10.18
(0.9, 0.1)	0.1	RSA	10.34	10.28	10.27	10.20	10.18	10.18	10.18
(0.9, 0.1)	0.5	SAA	10.09	10.07	10.08	10.08	10.08	10.08	10.08
(0.9, 0.1)	0.5	RSA	10.17	10.11	10.12	10.09	10.08	10.08	10.08

Table 1: Estimation of the risk measure value $\mathcal{R}(\xi_1)$ for $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$ using SAA and RSA for various values of (w_0, w_1, α) and various sample sizes N .

We first fix $(w_0, w_1) = (0.1, 0.9)$ and report in Tables 2 and 3 for various values of the pair (α, N) the average nonasymptotic and asymptotic confidence bounds for $\mathcal{R}(\xi_1)$ and $\mathcal{R}(\xi_2)$ when $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$ and $\xi_2 \sim \mathcal{N}(20, 1; 0, 30)$.² We observe that even for small values of the sample size and of the confidence level $1 - \alpha$, the asymptotic confidence interval is of small width and its bounds close to the risk measure value. For RSA, a large sample is needed to obtain a confidence interval of small width, especially when $1 - \alpha$ is small.

For all the remaining tests of this section, we choose $\beta = 0.1$ for the maximal type I error and $1 - \alpha = 0.1$. Since in case (I) we have $\mathcal{R}(\xi_1) \neq \mathcal{R}(\xi_2)$ (see Figure 1), from this experiment we expect to obtain a large probability of type II error using the nonasymptotic tests of Section 5.1 based on the confidence intervals computed using RSA, unless the sample size is very large. More precisely, we compute the probability of type II error for (6.44) considering asymptotic and nonasymptotic rejection regions using various sample sizes $N \in \{20, 50, 100, 1000, 5000, 10000, 20000, 50000, 100000, 130000, 150000\}$, taking $1 - \alpha = 0.1$ and $(w_0, w_1) \in \{(0, 1), (0.1, 0.9), (0.2, 0.8), (0.3, 0.7), (0.4, 0.6), (0.5, 0.5), (0.6, 0.4), (0.7, 0.3), (0.8, 0.2), (0.9, 0.1)\}$. For fixed N , the probability of type II error is estimated using 100 samples of size N of ξ_1 and ξ_2 . Using the asymptotic rejection region, we reject H_0 for all realizations and all parameter combinations, meaning that the probability of type II error is null (since H_1 holds for all parameter combinations). For the nonasymptotic test, the probability of type II errors are reported in Table 4. For sample sizes less than 5000, the probability of type II error is always 1 (the nonasymptotic test always takes the wrong decision) and the larger w_1 the larger the sample size N needs to be to obtain a probability of type II error of zero. In particular, if $w_1 = 1$ (we estimate the AVaR $_\alpha$ of the distribution) as much as 150000 observations are needed to obtain a null probability of type II error. However, if the sample size is sufficiently large, both tests always take the correct decision $\mathcal{R}(\xi_1) \neq \mathcal{R}(\xi_2)$.

Given (possibly small) samples of size N of ξ_1 and ξ_2 , to know which of the two risks $\mathcal{R}(\xi_1)$ and

²The nonasymptotic confidence interval is given by (5.30)-(5.31). Recalling that $\mathcal{R}(\xi)$ is the optimal value of optimization problem (6.43) which is of the form (1.1), we compute for $\mathcal{R}(\xi)$ the asymptotic confidence interval $[\hat{\vartheta}_N - \Phi^{-1}(1 - \beta/2) \frac{\hat{\vartheta}_N}{\sqrt{N}}, \hat{\vartheta}_N + \Phi^{-1}(1 - \beta/2) \frac{\hat{\vartheta}_N}{\sqrt{N}}]$, where $\hat{\vartheta}_N$ is the optimal value of the SAA of (6.43). Note that in this case the optimal value $\hat{\tau}_N$ of the SAA problem is the α -quantile of the distribution of ξ (no optimization step is necessary to solve the SAA problem).

N	Low-As1	Up-As1	Low-RSA1	Up-RSA1	Low-As2	Up-As2	Low-RSA2	Up-RSA2
50	11.20	11.94	-347.30	146.23	21.30	21.97	-335.67	157.97
10^3	11.49	11.67	-68.65	41.71	21.49	21.67	-58.37	51.99
10^4	11.55	11.61	-13.79	21.11	21.55	21.61	-3.71	31.19
10^5	11.57	11.59	3.56	14.59	21.57	21.59	13.58	24.62
1.5×10^5	11.57	11.59	5.03	14.04	21.57	21.59	15.05	24.06

Table 2: Average values of the asymptotic and nonasymptotic confidence bounds for $\mathcal{R}(\xi_1)$ and $\mathcal{R}(\xi_2)$ when $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$ and $\xi_2 \sim \mathcal{N}(20, 1; 0, 30)$, $1 - \alpha = 0.1$. For $\mathcal{R}(\xi_i)$, the average asymptotic confidence interval is [Low-As $_i$, Up-As $_i$] and the average nonasymptotic confidence interval is [Low-RSA $_i$, Up-RSA $_i$].

N	Low-As1	Up-As1	Low-RSA1	Up-RSA1	Low-As2	Up-As2	Low-RSA2	Up-RSA2
50	10.45	10.98	-230.29	40.16	20.47	21.00	-220.25	50.21
10^3	10.65	10.77	-43.18	17.30	20.65	20.77	-33.18	27.29
10^4	10.70	10.74	-6.33	12.80	20.70	20.74	3.68	22.80
10^5	10.71	10.72	5.33	11.38	20.71	20.72	15.33	21.38
1.5×10^5	10.71	10.72	6.32	11.26	20.71	20.72	16.32	21.25

Table 3: Average values of the asymptotic and nonasymptotic confidence bounds for $\mathcal{R}(\xi_1)$ and $\mathcal{R}(\xi_2)$ when $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$ and $\xi_2 \sim \mathcal{N}(20, 1; 0, 30)$, $1 - \alpha = 0.5$. For $\mathcal{R}(\xi_i)$, the average asymptotic confidence interval is [Low-As $_i$, Up-As $_i$] and the average nonasymptotic confidence interval is [Low-RSA $_i$, Up-RSA $_i$].

(w_0, w_1)	Sample size N						
	5 000	10 000	20 000	50 000	100 000	130 000	150 000
(0.0, 1.0)	1	1	1	1	1	1	0
(0.1, 0.9)	1	1	1	1	1	0	0
(0.2, 0.8)	1	1	1	1	0	0	0
(0.3, 0.7)	1	1	1	1	0	0	0
(0.4, 0.6)	1	1	1	1	0	0	0
(0.5, 0.5)	1	1	1	0	0	0	0
(0.6, 0.4)	1	1	1	0	0	0	0
(0.7, 0.3)	1	1	0	0	0	0	0
(0.8, 0.2)	1	0	0	0	0	0	0
(0.9, 0.1)	0	0	0	0	0	0	0

Table 4: Empirical probabilities of type II error for tests (6.44) and (6.45) using a nonasymptotic rejection region when $\xi_1 \sim \mathcal{N}(10, 1; 0, 30)$, $\xi_2 \sim \mathcal{N}(20, 1; 0, 30)$, and $1 - \alpha = 0.1$.

(w_0, w_1)	Sample size N				
	10 000	20 000	50 000	100 000	110 000
(0.0, 1.0)	1	1	1	1	0
(0.1, 0.9)	1	1	1	0	0
(0.2, 0.8)	1	1	1	0	0
(0.3, 0.7)	1	1	1	0	0
(0.4, 0.6)	1	1	1	0	0
(0.5, 0.5)	1	1	1	0	0
(0.6, 0.4)	1	1	0	0	0
(0.7, 0.3)	1	1	0	0	0
(0.8, 0.2)	1	1	0	0	0
(0.9, 0.1)	0.06	0	0	0	0

Table 5: Empirical probabilities of type II error for tests (6.44) and (6.45) using a nonasymptotic rejection region when $\xi_1 \sim \mathcal{N}(5, 1; 0, 30)$, $\xi_2 \sim \mathcal{N}(10, 25; 0, 30)$, and $1 - \alpha = 0.1$.

$\mathcal{R}(\xi_2)$ is the smallest, we now consider the test

$$H_0 : \mathcal{R}(\xi_1) \geq \mathcal{R}(\xi_2) \text{ against } H_1 : \mathcal{R}(\xi_1) < \mathcal{R}(\xi_2). \quad (6.45)$$

Computing $\mathcal{R}(\xi_1)$ and $\mathcal{R}(\xi_2)$ with a very large sample (of size 10^6) of ξ_1 and ξ_2 either with SAA or RSA or looking at Figure 1, we know that $\mathcal{R}(\xi_1) < \mathcal{R}(\xi_2)$. We again analyze the probability of type II error using the asymptotic and nonasymptotic rejection regions when the decision is taken on the basis of a much smaller sample. For the nonasymptotic test, the empirical probabilities of type II error for various sample sizes (estimated, for fixed N , using 100 samples of size N of ξ_1 and ξ_2) are exactly those obtained for test (6.44) and are given in Table 4. The asymptotic test again always takes the correct decision $\mathcal{R}(\xi_1) < \mathcal{R}(\xi_2)$ while a large sample size is needed to always take the correct decision using the nonasymptotic test, as large as 150 000 for $w_1 = 1$.

We now consider tests (6.44) and (6.45) for case (II). In this case, there is a larger overlap between the distributions of ξ_1 and ξ_2 . However, from Figure 1 and computing $\mathcal{R}(\xi_1)$ and $\mathcal{R}(\xi_2)$ with a very large sample (say of size 10^6) of ξ_1 and ξ_2 either using SAA or RSA, we check that we have again $\mathcal{R}(\xi_2) > \mathcal{R}(\xi_1)$ for all values of (w_0, w_1) . The empirical probabilities of type II error are null for the asymptotic test for all sample sizes N tested while for the nonasymptotic test, the probabilities of type II error are given in Table 5 for both tests (6.44) and (6.45). As a result, here again, the asymptotic test always takes the correct decision $\mathcal{R}(\xi_1) < \mathcal{R}(\xi_2)$ while a large sample size is needed to always take the correct decision using the nonasymptotic test (as large as 110 000 for $w_1 = 1$). For sample sizes less than 10 000, the empirical probability of type II error with the nonasymptotic test is 1. We see that for fixed (w_0, w_1) , in most cases, we need a larger sample size than in case (I) to have a null probability of type II error, due the overlap of the two distributions.

We finally consider case (III) where the choice between ξ_1 and ξ_2 is more delicate and depends on the pair (w_0, w_1) . In this case, we have (see Figure 1) $\mathbb{E}[\xi_2] > \mathbb{E}[\xi_1]$ and $\text{AVaR}_\alpha(\xi_2) < \text{AVaR}_\alpha(\xi_1)$ for $1 - \alpha = 0.1$. It follows that for pairs (w_0, w_1) summing to one, when

$$0 \leq w_0 < w_{\text{crit}} = \frac{\text{AVaR}_\alpha(\xi_1) - \text{AVaR}_\alpha(\xi_2)}{\mathbb{E}[\xi_2] - \mathbb{E}[\xi_1] + \text{AVaR}_\alpha(\xi_1) - \text{AVaR}_\alpha(\xi_2)}$$

then $\mathcal{R}(\xi_2) < \mathcal{R}(\xi_1)$ and for $w_0 > w_{\text{crit}}$ then $\mathcal{R}(\xi_2) > \mathcal{R}(\xi_1)$. The empirical estimation of w_{crit} (estimated using a sample of size 10^6) is 0.71. For w_0 close to w_{crit} , $\mathcal{R}(\xi_1)$ and $\mathcal{R}(\xi_2)$

(w_0, w_1)	Sample size N							
	20	50	100	200	500	1000	2000	5000
(0.0, 1.0)	0.13	0.01	0	0	0	0	0	0
(0.1, 0.9)	0.24	0	0	0	0	0	0	0
(0.2, 0.8)	0.32	0.03	0	0	0	0	0	0
(0.3, 0.7)	0.50	0.07	0	0	0	0	0	0
(0.4, 0.6)	0.61	0.11	0	0	0	0	0	0
(0.5, 0.5)	0.71	0.46	0.11	0.01	0	0	0	0
(0.6, 0.4)	0.86	0.69	0.50	0.28	0.01	0	0	0
(0.7, 0.3)	0.83	0.85	0.90	0.91	0.87	0.89	0.69	0.53
(0.8, 0.2)	0.71	0.71	0.65	0.29	0.07	0	0	0
(0.9, 0.1)	0.57	0.34	0.09	0	0	0	0	0

Table 6: Empirical probabilities of type II error for test (6.44) using an asymptotic rejection region when $\xi_1 \sim \mathcal{N}(10, 49; 0, 30)$, $\xi_2 \sim \mathcal{N}(14, 0.25; 0, 30)$, and $1 - \alpha = 0.1$.

(w_0, w_1)	Sample size N					
	100 000	300 000	500 000	700 000	10^6	5×10^6
(0.0, 1.0)	1	0.83	0	0	0	0
(0.1, 0.9)	1	1	0	0	0	0
(0.2, 0.8)	1	1	0	0	0	0
(0.3, 0.7)	1	1	0	0	0	0
(0.4, 0.6)	1	1	1	0	0	0
(0.5, 0.5)	1	1	1	1	0	0
(0.6, 0.4)	1	1	1	1	1	0
(0.7, 0.3)	1	1	1	1	1	1
(0.8, 0.2)	1	1	1	1	1	0
(0.9, 0.1)	0	0	0	0	0	0

Table 7: Empirical probabilities of type II error for test (6.44) using a nonasymptotic rejection region when $\xi_1 \sim \mathcal{N}(10, 49; 0, 30)$, $\xi_2 \sim \mathcal{N}(14, 0.25; 0, 30)$, and $1 - \alpha = 0.1$.

are close and the probability of type II error for test (6.44) can be large even for the asymptotic test if the sample size is not sufficiently large. More precisely, for the asymptotic test, when $(w_0, w_1) = (0.7, 0.3)$, the empirical probabilities of type II error are given in Table 6 for $N \in \{20, 50, 100, 200, 500, 1000, 2000, 5000\}$, and are 0.28, 0.11, 0.01, and 0 for respectively $N = 10000, 20000, 40000$, and 45000 . For the remaining values of w_0 the empirical probabilities of type II error are given in Table 6 for the asymptotic test. For the nonasymptotic test, the empirical probabilities of type II error for test (6.44) are given in Table 7. It is seen that much larger sample sizes are needed in this case to obtain a small probability of type II error. However, for the sample size $N = 5 \times 10^6$, the nonasymptotic test still always takes the wrong decision for the difficult case $w_0 = 0.7$.

For $w_0 < w_{\text{crit}}$ with $w_0 \in \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7\}$, we are interested in the probability of type II error of the test

$$H_0 : \mathcal{R}(\xi_2) \geq \mathcal{R}(\xi_1) \text{ against } H_1 : \mathcal{R}(\xi_2) < \mathcal{R}(\xi_1) \quad (6.46)$$

since H_1 holds in this case. Using the asymptotic rejection region, except for the difficult case

$w_0 = 0.7$ where the probability of type II error is still positive for $N = 30\,000$, the empirical probability of type II error is null for small to moderate (at most 1 000) sample sizes; see Table 8. Using the nonasymptotic rejection region, much larger sample sizes are necessary to obtain a small probability of type II error, see Table 9.

(w_0, w_1)	Sample size N							
	20	100	200	1 000	5 000	10 000	30 000	50 000
(0.0, 1.0)	0.11	0	0	0	0	0	0	0
(0.1, 0.9)	0.26	0	0	0	0	0	0	0
(0.2, 0.8)	0.28	0	0	0	0	0	0	0
(0.3, 0.7)	0.35	0	0	0	0	0	0	0
(0.4, 0.6)	0.51	0	0	0	0	0	0	0
(0.5, 0.5)	0.66	0.2	0.01	0	0	0	0	0
(0.6, 0.4)	0.83	0.53	0.22	0	0	0	0	0
(0.7, 0.3)	0.87	0.88	0.90	0.81	0.61	0.39	0.05	0

Table 8: Empirical probabilities of type II error for test (6.46) using an asymptotic rejection region when $\xi_1 \sim \mathcal{N}(10, 49; 0, 30)$, $\xi_2 \sim \mathcal{N}(14, 0.25; 0, 30)$, and $1 - \alpha = 0.1$.

(w_0, w_1)	Sample size N						
	300 000	400 000	500 000	700 000	900 000	2×10^6	5×10^6
(0.0, 1.0)	0	0	0	0	0	0	0
(0.1, 0.9)	0.85	0	0	0	0	0	0
(0.2, 0.8)	1	0	0	0	0	0	0
(0.3, 0.7)	1	1	0	0	0	0	0
(0.4, 0.6)	1	1	1	0	0	0	0
(0.5, 0.5)	1	1	1	1	0	0	0
(0.6, 0.4)	1	1	1	1	1	0.75	0
(0.7, 0.3)	1	1	1	1	1	1	1

Table 9: Empirical probabilities of type II error for test (6.46) using a nonasymptotic rejection region when $\xi_1 \sim \mathcal{N}(10, 49; 0, 30)$, $\xi_2 \sim \mathcal{N}(14, 0.25; 0, 30)$, and $1 - \alpha = 0.1$.

For $w_0 > w_{\text{crit}}$ with $w_0 \in \{0.8, 0.9\}$, we are interested in the probability of type II error of test (6.45) since H_1 holds in this case. The probability of type II error for this test using the nonasymptotic rejection region is 1 (resp. 0) for $(N, w_0, w_1) = (10^6, 0.8, 0.2)$ (resp. $(N, w_0, w_1) = (10^6, 0.9, 0.1)$), and null for $(N, w_0, w_1) = (5 \times 10^6, 0.8, 0.2)$, $(5 \times 10^6, 0.9, 0.1)$, meaning that we always take the correct decision $\mathcal{R}(\xi_1) < \mathcal{R}(\xi_2)$ for $N = 5 \times 10^6$ and $(w_0, w_1) = (0.8, 0.2), (0.9, 0.1)$. Using the asymptotic rejection region, the probabilities of type II errors are null already for $N = 1\,000$. For $N = 100$, we get probabilities of type II error of 0.09 and 0.42 for respectively $(w_0, w_1) = (0.8, 0.2)$ and $(w_0, w_1) = (0.9, 0.1)$.

6.2 Tests on the optimal value of two risk averse stochastic programs

We illustrate the results of Sections 4 and 5 on the risk averse problem

$$\begin{cases} \min w_0 \mathbb{E}[\sum_{i=1}^n \xi_i x_i] + w_1 \left(x_0 + \mathbb{E} \left[\frac{1}{1-\alpha} [\sum_{i=1}^n \xi_i x_i - x_0]_+ \right] \right) + \lambda_0 \| [x_0; x_1; \dots; x_n] \|_2^2 + c_0 \\ -1 \leq x_0 \leq 1, \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, \dots, n, \end{cases} \quad (6.47)$$

Instance	$(w_0, w_1, 1 - \alpha, \lambda_0)$	c_0	n	$(\mathbb{P}(\xi_i = 1))_i$
\mathcal{I}_1	(0.9, 0.1, 0.1, 2)	0	100	Ψ_1
\mathcal{I}_2	(0.9, 0.1, 0.1, 2)	0	100	$0.8\Psi_1$
\mathcal{I}_3	(0.9, 0.1, 0.1, 2)	-3	100	$0.8\Psi_1$
\mathcal{I}_4	(0.9, 0.1, 0.1, 2)	0	500	Ψ_2
\mathcal{I}_5	(0.9, 0.1, 0.1, 2)	0	500	$0.8\Psi_2$
\mathcal{I}_6	(0.9, 0.1, 0.1, 2)	-3	500	$0.8\Psi_2$

Table 10: Definition of instances $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5$, and \mathcal{I}_6 of problem (6.47) (Ψ_1 and Ψ_2 are vectors with entries drawn independently and randomly over $[0, 1]$).

where ξ is a random vector with i.i.d. Bernoulli entries: $\mathbb{P}(\xi_i = 1) = \Psi_i$, $\mathbb{P}(\xi_i = -1) = 1 - \Psi_i$, with Ψ_i randomly drawn over $[0, 1]$.³ This problem amounts to minimizing a linear combination of the expectation and the AVaR $_\alpha$ of $\sum_{i=1}^n \xi_i x_i$ plus a penalty obtained taking $\lambda_0 > 0$. Therefore, it has a unique optimal solution. SAA formulation of this problem as well as the quadratic problems of each iteration of RSA were solved numerically using Mosek Optimization Toolbox [1]. We will again use the rejection regions given in Section 5.1 (resp. given by (5.39)) in the nonasymptotic (resp. asymptotic) case.

To illustrate Theorem 4.2, for several instances of this problem, we report in Figures 3 and 4 the QQ-plots of the empirical distribution of the SAA optimal value for problem (6.47) versus the normal distribution with parameters the empirical mean and standard deviation of this empirical distribution for various sample sizes N . We observe again that this distribution is well approximated by a Gaussian distribution even when the sample size is small ($N = 20$): for all problem sizes ($n = 100$, $n = 500$, $n = 10^3$, and $n = 10^4$) and the smallest sample size tested ($N = 20$), the Jarque-Bera test accepts the null hypothesis (the data comes from a normal distribution with unknown mean and variance) at the 5% significance level.

We now define in Table 10 six instances $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5$, and \mathcal{I}_6 of problem (6.47).

We first compare the estimation of the optimal value of \mathcal{I}_2 using RSA and SAA. For the RSA algorithm, we take $\|\cdot\| = \|\cdot\|_2 = \|\cdot\|_*$ and (see [5]) $L = \sqrt{\left(\frac{w_1\alpha}{1-\alpha}\right)^2 + n(w_0 + \frac{w_1}{1-\alpha})^2 + 2\lambda_0}$, $M_1 = 2(w_0 + \frac{w_1}{1-\alpha})$, $M_2 = \sqrt{\left(\frac{w_1}{1-\alpha}\right)^2 + 4n\left(w_0 + \frac{w_1}{1-\alpha}\right)^2}$. The average approximate optimal value of instance \mathcal{I}_2 (averaging taking 100 samples of ξ^N) using RSA and SAA is given in Table 11 for various sample sizes N . These values increase (resp. decrease) with the sample size for SAA (resp. RSA). With SAA, the optimal value is already well approximated with small sample sizes while large samples are needed to obtain a good approximation with RSA. We also report in Table 12 the average values of the asymptotic and nonasymptotic confidence bounds (computed using 100 samples of ξ^N) on the optimal values of instances \mathcal{I}_1 and \mathcal{I}_2 and various sample sizes.⁴ Knowing that the optimal values of \mathcal{I}_1 and \mathcal{I}_2 , estimated using SAA with a sample of size 10^6 , are respectively $\vartheta_1 = -0.6515$ and $\vartheta_2 = -0.6791$, we observe that the asymptotic confidence interval is in mean much closer to the optimal value and of small width while large samples are needed to obtain a nonasymptotic confidence interval of small width. However, the confidence bounds on the optimal

³Of course c_0 can be ignored to solve the problem. However, it will be used to define several instances and test the equality about their optimal values.

⁴The nonasymptotic confidence interval is $[\text{Low}(\Theta_2, \Theta_3, N), \text{Up}(\Theta_1, N)]$ with $\text{Low}(\Theta_2, \Theta_3, N), \text{Up}(\Theta_1, N)$ given by (5.30), (5.31) and $\Theta_1 = 2\sqrt{\ln(2/\beta)}$, $\Theta_3 = 2\sqrt{\ln(4/\beta)}$, and Θ_2 satisfying $e^{1-\Theta_2^2} + e^{-\Theta_2^2/4} = \frac{\beta}{4}$. The asymptotic confidence interval for (6.47) is $[\hat{\vartheta}_N - \Phi^{-1}(1 - \beta/2)\frac{\hat{\vartheta}_N}{\sqrt{N}}, \hat{\vartheta}_N + \Phi^{-1}(1 - \beta/2)\frac{\hat{\vartheta}_N}{\sqrt{N}}]$.

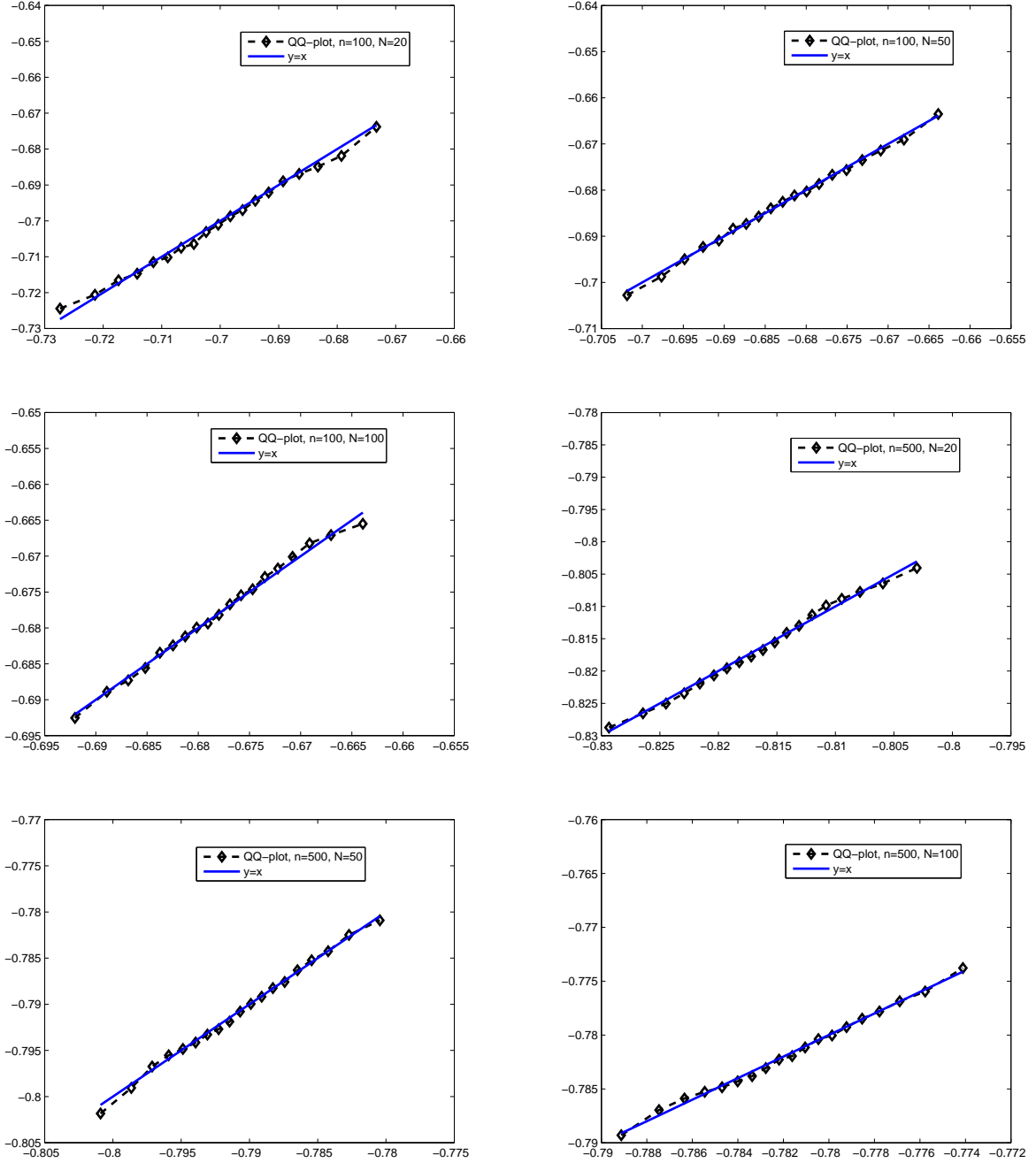


Figure 3: QQ-plots of the empirical distribution of the SAA optimal value for problem (6.47) versus the normal distribution with parameters the empirical mean and standard deviation of this empirical distribution for instances with $w_0 = 0.9, w_1 = 0.1, 1 - \alpha = 0.1, \lambda_0 = 2$, and various sample and problem sizes.

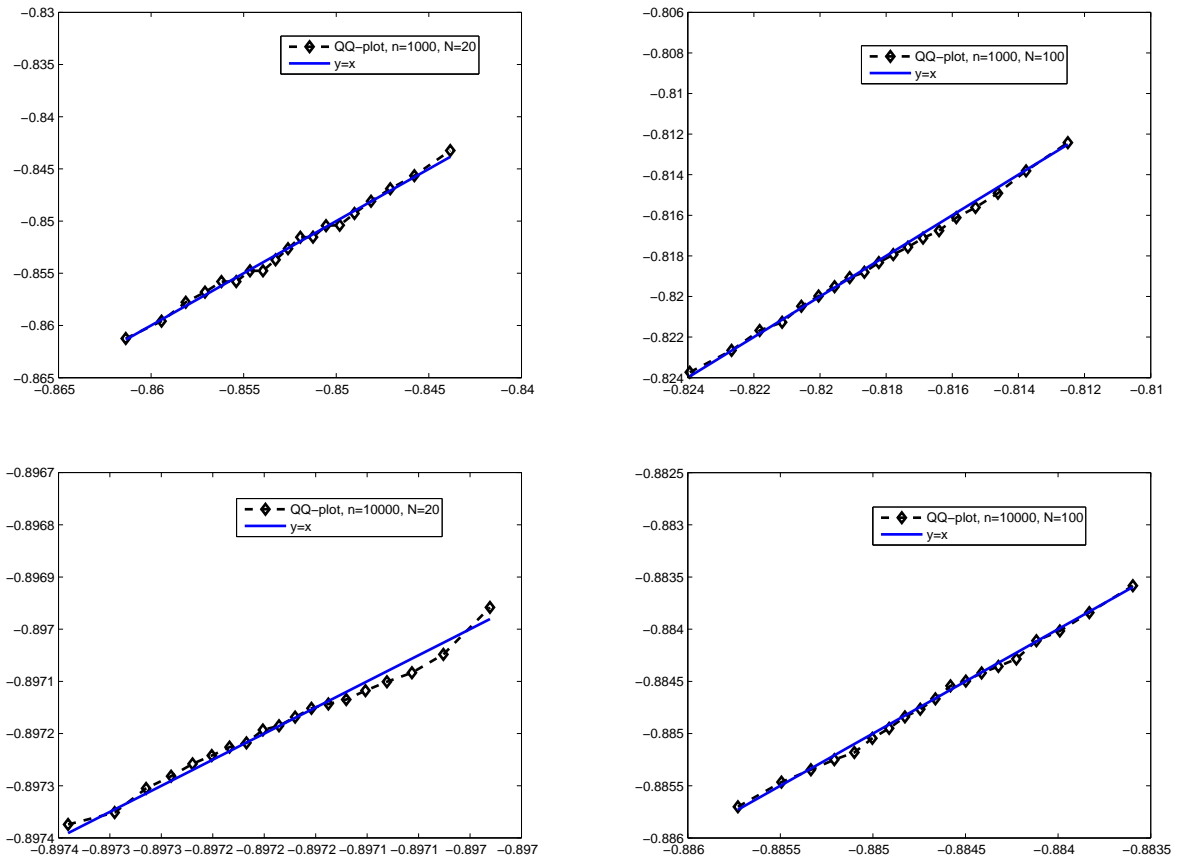


Figure 4: QQ-plots of the empirical distribution of the SAA optimal value for problem (6.47) versus the normal distribution with parameters the empirical mean and standard deviation of this empirical distribution for instances with $w_0 = 0.9, w_1 = 0.1, 1 - \alpha = 0.1, \lambda_0 = 2$, and various sample and problem sizes.

Method	$N = 20$	$N = 50$	$N = 10^2$	$N = 10^3$	$N = 10^4$	$N = 10^5$
SAA	-0.7205	-0.6965	-0.6883	-0.6799	-0.6791	-0.6791
RSA	-0.4615	-0.5274	-0.5646	-0.6389	-0.6654	-0.6738

Table 11: Average approximate optimal value of instance \mathcal{I}_2 (computed using 100 samples of ξ^N) using SAA and RSA for various sample sizes N .

N	Low-As1	Up-As1	Low-RSA1	Up-RSA1	Low-As2	Up-As2	Low-RSA2	Up-RSA2
20	-0.7207	-0.6666	-95.7926	2.5227	-0.7443	-0.6967	-95.8354	2.4799
50	-0.6888	-0.6475	-60.8057	1.3743	-0.7148	-0.6781	-60.8472	1.3329
10^2	-0.6752	-0.6444	-43.1779	0.7900	-0.7019	-0.6746	-43.2171	0.7508
10^3	-0.6573	-0.6474	-14.0952	-0.1913	-0.6843	-0.6755	-14.1269	-0.2230
10^4	-0.6532	-0.6501	-4.9019	-0.5051	-0.6805	-0.6777	-4.9307	-0.5339
10^5	-0.6520	-0.6510	-1.9947	-0.6043	-0.6796	-0.6787	-2.0226	-0.6322

Table 12: Average values of the asymptotic and nonasymptotic confidence bounds (computed using 100 samples of ξ^N) for instances \mathcal{I}_1 and \mathcal{I}_2 and various sample sizes. For instance \mathcal{I}_i , the average asymptotic confidence interval is [Low-As i , Up-As i] and the average nonasymptotic confidence interval is [Low-RSA i , Up-RSA i].

value obtained using RSA are almost independent on the problem size and as for the one dimensional problem of the previous section the sample size $N = 10^5$ provides confidence intervals of small width and allows us to have small probabilities of type I and type II errors for nonasymptotic tests on the optimal value of two instances of (6.47) if their optimal values are sufficiently distant (see Lemmas 5.1, 5.2, and 5.3). To check that and the superiority of the asymptotic tests for problems of moderate sizes ($n = 100$ and $n = 500$), we compare the empirical probabilities of type II error of several tests of form (1.3) with $K = 2$ for which H_1 holds and where ϑ_i is the optimal value of instance \mathcal{I}_i .

More precisely, the empirical probabilities of type II error of asymptotic and nonasymptotic tests of form

$$H_0 : \vartheta_i = \vartheta_j \text{ against } H_1 : \vartheta_i \neq \vartheta_j, \quad (6.48)$$

are reported in Table 13 (for all these tests, we check that H_1 holds computing ϑ_i solving the SAA problem of instance \mathcal{I}_i with a sample of ξ of size 10^6 : $\vartheta_1 = -0.6515$, $\vartheta_2 = -0.6791$, $\vartheta_3 = -3.6791$, $\vartheta_4 = -0.7725$, $\vartheta_5 = -0.7868$, and $\vartheta_6 = -3.7868$).

Though it was observed in [5], [6] that for sample sizes that are not much larger than the problem size the coverage probability of the asymptotic confidence interval is much lower than the coverage probability of the nonasymptotic confidence interval and than the target coverage probability, the asymptotic confidence bounds are much closer to each other and much closer to the optimal value than the nonasymptotic confidence bounds. This explains why the probability of type II error of the asymptotic test is much less than the probability of type II error of the nonasymptotic test, even for small sample sizes and a smaller sample is needed to always take the correct decision H_1 with the asymptotic test, i.e., to obtain a null probability of type II error. Of course, in both cases, for fixed N , the empirical probability of type II error depends on the distance between ϑ_i and ϑ_j .

Similar conclusions can be drawn from Table 14 which reports the empirical probability of type II error for various tests of form

$$H_0 : \vartheta_i \leq \vartheta_j \text{ against } H_1 : \vartheta_j < \vartheta_i. \quad (6.49)$$

			Sample size N					
H_0	H_1	Test type	20	50	10^2	10^3	10^4	10^5
$\vartheta_1 = \vartheta_2$	$\vartheta_1 \neq \vartheta_2$	Asymptotic	0.72	0.45	0.29	0	0	0
$\vartheta_1 = \vartheta_2$	$\vartheta_1 \neq \vartheta_2$	Nonasymptotic	1	1	1	1	1	1
$\vartheta_1 = \vartheta_3$	$\vartheta_1 \neq \vartheta_3$	Asymptotic	0	0	0	0	0	0
$\vartheta_1 = \vartheta_3$	$\vartheta_1 \neq \vartheta_3$	Nonasymptotic	1	1	1	1	1	0
$\vartheta_4 = \vartheta_5$	$\vartheta_4 \neq \vartheta_5$	Asymptotic	0.33	0.36	0.21	0	0	0
$\vartheta_4 = \vartheta_5$	$\vartheta_4 \neq \vartheta_5$	Nonasymptotic	1	1	1	1	1	1
$\vartheta_4 = \vartheta_6$	$\vartheta_4 \neq \vartheta_6$	Asymptotic	0	0	0	0	0	0
$\vartheta_4 = \vartheta_6$	$\vartheta_4 \neq \vartheta_6$	Nonasymptotic	1	1	1	1	1	0

Table 13: Empirical probabilities of type II error for tests of form (6.48).

			Sample size N					
H_0	H_1	Test type	20	50	10^2	10^3	10^4	10^5
$\vartheta_1 \leq \vartheta_2$	$\vartheta_1 > \vartheta_2$	Asymptotic	0.54	0.38	0.16	0	0	0
$\vartheta_1 \leq \vartheta_2$	$\vartheta_1 > \vartheta_2$	Nonasymptotic	1	1	1	1	1	1
$\vartheta_1 \leq \vartheta_3$	$\vartheta_1 > \vartheta_3$	Asymptotic	0	0	0	0	0	0
$\vartheta_1 \leq \vartheta_3$	$\vartheta_1 > \vartheta_3$	Nonasymptotic	1	1	1	1	1	0
$\vartheta_4 \leq \vartheta_5$	$\vartheta_4 > \vartheta_5$	Asymptotic	0.29	0.26	0.15	0	0	0
$\vartheta_4 \leq \vartheta_5$	$\vartheta_4 > \vartheta_5$	Nonasymptotic	1	1	1	1	1	1
$\vartheta_4 \leq \vartheta_6$	$\vartheta_4 > \vartheta_6$	Asymptotic	0	0	0	0	0	0
$\vartheta_4 \leq \vartheta_6$	$\vartheta_4 > \vartheta_6$	Nonasymptotic	1	1	1	1	1	0

Table 14: Empirical probabilities of type II error for tests of form (6.49).

In particular, from these results, we see that we always take the correct decision H_1 with the asymptotic test for sample sizes above $N = 100$.

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