SEMI-SMOOTH SECOND-ORDER TYPE METHODS FOR COMPOSITE CONVEX PROGRAMS

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Abstract. The goal of this paper is to study approaches to bridge the gap between first-order and second-order type methods for composite convex programs. Our key observations are: i) Many well-known operator splitting methods, such as forward-backward splitting (FBS) and Douglas-Rachford splitting (DRS), actually define a possibly semi-smooth and monotone fixed-point mapping; ii) The optimal solutions of the composite convex program and the solutions of the system of nonlinear equations derived from the fixed-point mapping are equivalent. Solving the system of nonlinear equations rediscovers a paradigm on developing second-order methods. Although these fixed-point mappings may not be differentiable, they are often semi-smooth and its generalized Jacobian matrix is positive semidefinite due to monotonicity. By combining a regularization approach and a known hyperplane projection technique, we propose an adaptive semi-smooth Newton method and establish its convergence to global optimality. A semi-smooth Levenberg-Marquardt (LM) method in terms of handling the nonlinear least squares formulation is further presented. In practice, the second-order methods can be activated until the first-order type methods reach a good neighborhood of the global optimal solution. Preliminary numerical results on Lasso regression, logistic regression, basis pursuit, linear programming and quadratic programming demonstrate that our second-order type algorithms are able to achieve quadratic or superlinear convergence as long as the fixed-point residual of the initial point is small enough.

Key words. composite convex programs, operator splitting methods, proximal mapping, semi-smoothness, Newton method, Levenberg-Marquardt method

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1. Introduction.

This paper aims to solve a composite convex optimization problem in the form

$$\min_{x \in \mathbb{R}^n} \quad f(x) + h(x), \tag{1.1}$$

where f and h are real-valued convex functions. Problem (1.1) arises from a wide variety of applications, such as signal recovery, image processing, machine learning, data analysis, and etc. For example, it becomes the well-known sparse optimization problem when f or h equals to the ℓ_1 norm, which attracts a significant interest in signal or image processing in recent years. If f is a loss function associated with linear predictors and h is a regularization function, problem (1.1) is often referred as the regularized empirical risk minimization problem extensively studied in machine learning and statistics. When f or h is an indicator function onto a convex set, problem (1.1) represents a general convex constrained optimization problem.

Recently, a series of first-order methods, including the forward-backward splitting (FBS) (also known as proximal gradient) methods, Nesterov's accelerated methods, the alternative direction methods of multipliers (ADMM), the Douglas-Rachford splitting (DRS) and Peaceman-Rachford splitting (PRS) methods, have been extensively studied and widely used for solving a subset of problem (1.1). The readers are referred to, for example, [3, 6] and references therein, for a review on some of these first-order methods. One main feature of these

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methods is that they first exploit the underlying problem structures, then construct subproblems that can be solved relatively efficiently. These algorithms are rather simple yet powerful since they are easy to be implemented in many interested applications and they often converge fast to a solution with moderate accuracy. However, a well-known drawback is that they may suffer from a slow tail convergence and a significantly large number of iterations is needed in order to achieve a high accuracy.

A few Newton-type methods for some special instances of problem (1.1) have been investigated to alleviate the inherent weakness of the first-order type methods. Most existing Newton-type methods for problem (1.1) with a differentiable function f and a simple function h whose proximal mapping can be cheaply evaluated are based on the FBS method to some extent. The proximal Newton method [20, 32] can be interpreted as a generalization of the proximal gradient method. It updates in each iteration by a composition of the proximal mapping with a Newton or quasi-Newton step. The semi-smooth Newton methods proposed in [18, 30, 4] solve the nonsmooth formulation of the optimality conditions corresponding to the FBS method. It should be noted that these Newton-type methods are only designed for the case that at least f or h is smooth. Moreover, semi-smooth Newton-CG methods have been developed for semidefinite programming and matrix spectral norm approximation problems in [45, 5] by applying the second-order type methods to solve a sequence of the dual augmented Lagrangian subproblems.

In this paper, we study a few second-order type methods for problem (1.1) in a general setting even if f is nonsmooth and h is an indicator function. Our key observations are that many first-order methods, such as the FBS and DRS methods, can be written as fixed-point iterations and the optimal solutions of (1.1) are also the solutions of the system of nonlinear equations defined by the fixed-point mapping. Consequently, the concept is to develop second-order type algorithms based on solving the system of nonlinear equations. Although the fixed-point mappings are often nondifferentiable, they are monotone and can be semi-smooth due to the properties of the proximal mappings. We first propose a regularized semi-smooth Newton method to solve the system of nonlinear equations. The regularization term is important since the generalized Jacobian matrix corresponding to monotone equations may only be positive semidefinite. In particular, the regularization parameter is updated by a self-adaptive strategy similar to the trust region algorithms. By combining the semi-smooth Newton step and a hyperplane projection technique, we show that the method converges globally to an optimal solution of problem (1.1). The hyperplane projection step is in fact indispensable for the convergence and it is inspired by the globally convergent iterative methods for solving monotone nonlinear equations [39, 47]. We further propose a semi-smooth LM method to handle an equivalent nonlinear least squares formulation. This version is interesting since it provably converges quadratically to an optimal solution as long as the initial point is suitably chosen.

Our main contribution is setting up a bridge between the well-known first-order and second-order type methods. Our semi-smooth Newton and semi-smooth LM methods are able to solve the general convex composite problem (1.1) as long as a fixed-point mapping is well defined. In particular, our methods are applicable to constrained convex programs, such as linear programming and constrained ℓ_1 minimization problem. In contrast, the Newton-type methods in [20, 32, 18, 30, 4] are designed for unconstrained problems. Unlike the methods in [45, 5] applying the Newton-type method to a sequence of subproblems, our target is a single system of nonlinear equations. Another advantage of our methodology is the generation of a class of hybrid first-second-order methods. At the first stage, a first-order type method is applied to produce a moderately accurate approximate solution. Then a second-order type method is activated to obtain a better solution. These two steps can be repeated until a stopping criterion is satisfied. Our preliminary numerical results on logistic regression, basis pursuit, linear programming and quadratic programing show that our proposed methods are able to reach quadratic or superlinear convergence rates when the first-order type methods can produce a good initial point.

The rest of this paper is organized as follows. In section 2, we review a few well-known operator splitting methods, derive their equivalent fixed-point iterations and state their convergence properties. Section 3 is devoted to the characterization of the semi-smoothness of proximal mappings from various interesting functions. We propose a semi-smooth Newton method and establish its global convergence in section 4. We apply a semi-smooth LM method and analyze its convergence in section 5. Numerical results on a number of applications are presented in section 6. Finally, we conclude this paper in section 7.

1.1. Notations. Let I be the identity operator or identity matrix of suitable size. Given a function $f : \mathbb{R}^n \to (-\infty, +\infty]$ and a scalar $t > 0$, the *proximal mapping* of f is defined by

$$\mathbf{prox}_{tf}(x) := \arg\min_{u \in \mathbb{R}^n} f(u) + \frac{1}{2t}\|u - x\|_2^2. \tag{1.2}$$

If $f(x) = 1_\Omega(x)$ is the indicator function of a nonempty closed convex set $\Omega \subset \mathbb{R}^n$, then the proximal mapping $\mathbf{prox}_{tf}$ reduces to the *metric projection* which is defined by

$$\mathcal{P}_\Omega(x) := \arg\min_{u \in \Omega} \frac{1}{2}\|u - x\|_2^2. \tag{1.3}$$

The conjugate function f^* of f is

$$f^*(y) := \sup_{x \in \mathbb{R}^n} \{x^T y - f(x)\}. \tag{1.4}$$

A function f is said to be *closed* if its epigraph is closed, or equivalently f is lower semicontinuous. For $k \in \mathbb{N}$, a function with k continuous derivatives is called a $\mathcal{C}^k$ function. For a point $x \in \mathbb{R}^n$ and a set $\Omega \subset \mathbb{R}^n$, the Euclidean distance of x from Ω is defined by

$$\mathrm{dist}(x, \Omega) := \inf_{u \in \Omega} \|u - x\|_2.$$

The graph of a set-valued mapping $\Gamma : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is defined as

$$\mathrm{gph}(\Gamma) := \{(x, y) | y \in \Gamma(x)\}.$$

The closed ball around x with a radius $r > 0$ is denoted by $\mathbb{B}(x, r)$.

2. Operator splitting and fixed-point algorithms. This section reviews some well-known operator splitting algorithms for problem (1.1), including FBS, PRS, DRS, and ADMM. These algorithms are well studied in the literature, see [13, 2, 6, 7] for example. Most of the operator splitting algorithms can also be interpreted as fixed-point algorithms derived from certain optimality conditions.

2.1. FBS. In problem (1.1), let h be a continuously differentiable function whose gradient is Lipschitz continuous with a constant L. The FBS algorithm is the iteration

$$x^{k+1} = \mathbf{prox}_{tf}(x^k - t\nabla h(x^k)), \quad k = 0, 1, \ldots, \tag{2.1}$$

where $t > 0$ is the step size. When $f = 0$, FBS reduces to the *gradient method* for minimizing a function $h(x)$ with a Lipschitz continuous gradient. When $h = 0$, FBS reduces to the *proximal point algorithm* for minimizing a nonsmooth function $f(x)$. When $f(x) = 1_C(x)$, FBS reduces to the *projected gradient method* for solving the constrained program

$$\min_{x\in\mathbb{R}^n} \quad f(x) \quad \text{subject to} \quad x \in C.$$

Define the following operator

$$T_{\text{FBS}} := \mathbf{prox}_{tf} \circ (I - t\nabla h). \tag{2.2}$$

Then FBS can be viewed as a fixed-point iteration

$$x^{k+1} = T_{\text{FBS}}(x^k).$$

For any $t > 0$, it is well known [2] that a fixed-point of T_{FBS} is equivalent to an optimal solution to problem (1.1).

2.2. DRS and PRS. The DRS algorithm solves (1.1) by the following update:

$$x^{k+1} = \mathbf{prox}_{th}(z^k), \tag{2.3}$$

$$y^{k+1} = \mathbf{prox}_{tf}(2x^{k+1} - z^k), \tag{2.4}$$

$$z^{k+1} = z^k + y^{k+1} - x^{k+1}. \tag{2.5}$$

The algorithm is traced back to [10] for solving linear equations and was developed in [23] to deal with monotone inclusion problems. The fixed-point iteration characterization of DRS is in the form of

$$z^{k+1} = T_{\text{DRS}}(z^k),$$

where

$$T_{\text{DRS}} := I + \mathbf{prox}_{tf} \circ (2\mathbf{prox}_{th} - I) - \mathbf{prox}_{th}. \tag{2.6}$$

It has been shown that i) $x^* = \mathbf{prox}_{th}(z^*)$ is a minimizer of (1.1) if z^* is a fixed-point of T_{DRS} and ii) $x^* + u = T_{\text{DRS}}(x^* + u)$ if x^* is a minimizer of (1.1) and $t^{-1}u \in \partial f(x^*) \cap -\partial h(x^*)$. It is necessary to mention that the operator T_{DRS} will be different by switching f and h.

The PRS algorithm, which was originally introduced in [33] for linear equations and was generalized to handle nonlinear equations in [10], can be stated by computing:

$$z^{k+1} = T_{\text{PRS}}(z^k),$$

where

$$T_{\text{PRS}} := (2\mathbf{prox}_{tf} - I) \circ (2\mathbf{prox}_{th} - I). \tag{2.7}$$

The *relaxed* PRS algorithm studied in [7, 8] is described by the fixed-point iteration:

$$z^{k+1} = T_{\lambda_k}(z^k), \quad \text{where } T_{\lambda_k} := (1 - \lambda_k)I + \lambda_k T_{\text{PRS}} \text{ and } \lambda_k \in (0, 1]. \tag{2.8}$$

It is clear that the special cases $\lambda_k \equiv 1/2$ and $\lambda_k \equiv 1$ are DRS and PRS algorithms, respectively.

2.3. Dual operator splitting and ADMM. Consider a linear constrained program:

$$\begin{array}{ll} \min\limits_{x_1\in\mathbb{R}^{n_1},x_2\in\mathbb{R}^{n_2}} & f_1(x_1)+f_2(x_2) \\ \text{subject to} & A_1x_1+A_2x_2=b, \end{array} \tag{2.9}$$

where $A_1 \in \mathbb{R}^{m\times n_1}$ and $A_2 \in \mathbb{R}^{m\times n_2}$. The dual problem of (2.9) is given by

$$\min_{w\in\mathbb{R}^m} \quad d_1(w)+d_2(w), \tag{2.10}$$

where

$$d_1(w) := f_1^*(A_1^T w), \quad d_2(w) := f_2^*(A_2^T w) - b^T w.$$

The dual problem is in the formulation of (1.1). Thus, it is possible to handle the constrained program (2.9) by solving (2.10) with the aforementioned operator splitting algorithms.

Assume that f_1 is closed and strongly convex (which implies that ∇d_1 is Lipschitz [36, Proposition 12.60]) and f_2 is convex. The FBS iteration for the dual problem (2.10) can be expressed in terms of the variables in the original problems as

$$\begin{aligned} x_1^{k+1} &= \arg\min_{x_1\in\mathbb{R}^{n_1}}\{f_1(x_1)+\langle w^k, A_1x_1\rangle\} \\ x_2^{k+1} &\in \arg\min_{x_2\in\mathbb{R}^{n_2}}\{f_2(x_2)+\langle w^k, A_2x_2\rangle+\tfrac{t}{2}\|A_1x_1^{k+1}+A_2x_2-b\|_2^2\} \\ w^{k+1} &= w^k - t(b-A_1x_1^{k+1}-A_2x_2^{k+1}). \end{aligned}$$

In fact, the dual FBS is studied in [41] under the name *alternating minimization* (AM) algorithm. Clearly, AM is equivalent to the fixed-point iteration

$$w^{k+1} = T_{\text{AM}}(w^k),$$

where

$$T_{\text{AM}} := \mathbf{prox}_{td_2}(I - t\nabla d_1). \tag{2.11}$$

Assume that f_1 and f_2 are convex. It is well known that the DRS iteration for dual problem (2.10) is the ADMM [16, 15]. It is regarded as a variant of augmented Lagrangian method and has attracted much attention in numerous fields. A recent survey paper [3] describes the applications of ADMM to statistics and machine learning. ADMM is equivalent to the following fixed-point iteration

$$z^{k+1} = T_{\text{DRS}}(z^k),$$

where T_{DRS} is the DRS fixed-point mapping for problem (2.10).

2.4. Error bound condition. Error bound condition is a useful property for establishing the linear convergence of a class of first-order methods including FBS method and ADMM, see [12, 25, 42, 19] and the references therein. However, it is usually non-trivial to verify the error bound condition for a given residual function. In this subsection, our special attention is to study the existing results on error bound condition for residual function corresponding to FBS.

Let X^* be the optimal solution set of problem (1.1) and $F(x) \in \mathbb{R}^n$ be a residual function satisfying $F(x) = 0$ if and only if $x \in X^*$. The definition of error bound condition is given as follows.

DEFINITION 2.1. *The error bound condition holds for some test set T and some residual function $F(x)$ if there exists a constant $\kappa > 0$ such that*

$$\text{dist}(x, X^*) \le \kappa \|F(x)\|_2 \quad \textit{for all } x \in T. \tag{2.12}$$

In particular,

- *(i) it is said that error bound condition with residual-based test set (*EBR*) holds if the test set in (2.12) is selected by $T := \{x \in \mathbb{R}^n | f(x) + h(x) \le v, \|F(x)\|_2 \le \varepsilon\}$ for some constant $\varepsilon \ge 0$ and any $v \ge v^* := \min_x f(x) + h(x)$;*
- *(ii) we say that error bound condition with neighborhood-based test set (*EBN*) holds if the test set in (2.12) is chosen by $T := \{x \in \mathbb{R}^n | \text{dist}(x, X^*) \le \rho\}$ for some constant $\rho \ge 0$.*

The error bound condition (EBR) was originated proposed in [25] and was extensively used to establish the convergence rates of iterative algorithms, see [42] for example. The error bound condition (EBN) is a critical condition to study the fast local convergence of the LM method without nonsingularity assumptions [44, 14].

Error bound condition is closely related to the notion of calmness.

DEFINITION 2.2. *[9, Chapter 3.8] A set-valued mapping $\Theta : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be calm at $\bar{u}$ for $\bar{v}$ if $(\bar{u}, \bar{v}) \in \text{gph}(\Theta)$, and there exists a constants $\kappa \ge 0$ along with neighborhoods U of $\bar{u}$ and V of $\bar{v}$ such that*

$$\Theta(u) \cap V \subseteq \Theta(\bar{u}) + \kappa \|u - \bar{u}\|_2 \mathbb{B}, \quad \textit{for all } u \in U.$$

Let $F_{\text{FBS}}(x) := x - T_{\text{FBS}}(x)$. We make the following assumptions on the structure of problem (1.1).

ASSUMPTION 2.3.

- *(i) The function $f : \mathbb{R}^n \to (-\infty, +\infty]$ is convex, closed and proper.*
- *(ii) The function $h : \mathbb{R}^n \to (-\infty, +\infty]$ takes the form $h(x) = \ell(\mathcal{A}(x)) + c^T x$, where $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator, $c \in \mathbb{R}^n$ and $\ell : \mathbb{R}^m \to (-\infty, +\infty]$ satisfies the properties that ℓ is continuously differentiable, strongly convex and its gradient is Lipschitz continuous.*
- *(iii) The optimal solution set X^* is nonempty and compact. The optimal value $v^* > -\infty$.*

Under Assumption 2.3, it is demonstrated in [48] that there exists a $\bar{y} \in \mathbb{R}^m$ such that

$$\mathcal{A}(x^*) = \bar{y}, \quad \nabla h(x^*) = \bar{g} \quad \text{for all } x^* \in X^*, \tag{2.13}$$

where $\bar{g} = \mathcal{A}^* \nabla \ell(\bar{y}) + c$. Define a set-valued mapping $\Gamma : \mathbb{R}^m \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$\Gamma(y, g) = \{x \in \mathbb{R}^n | \mathcal{A}(x) = y, -g \in \partial f(x)\}.$$

It is easy to show that $\Gamma(\bar{y}, \bar{g}) = X^*$. The following results reveal the equivalence of error bound condition for F_{FBS} with calmness of Γ.

PROPOSITION 2.4. *[48] Suppose that Assumption 2.3 holds. Let the residual function in Definition 2.1 be F_{FBS}, and $(\bar{y}, \bar{g})$ be given in (2.13). Then, the following properties are equivalent:*

- *(a) the error bound condition (EBR) holds;*
- *(b) the error bound condition (EBN) holds;*
- *(c) the set-valued mapping Γ is calm at $(\bar{y}, \bar{g})$ for any $\bar{x} \in X^*$.*

Therefore, the problem of establishing error bound condition for F_{FBS} is reduced to verify the calmness of set-valued mapping Γ. The convenience of this reduction is that the calmness property has been intensively studied in set-valued analysis, see [36, 9] for example.

2.5. Convergence of the fixed-point algorithms. We summarize the relation between the aforementioned fixed-points and the optimal solution of problem (1.1), and review the existing convergence results on the fixed-point algorithms.

The following lemma is well known, and its proof is omitted.

LEMMA 2.5. *Let the fixed-point mappings T_{FBS}, T_{DRS} and T_{PRS} be defined in (2.2), (2.6) and (2.7), respectively.*

(i) Suppose that f is closed proper and convex, and h is convex and continuously differentiable. A fixed-point of T_{FBS} is equivalent to an optimal solution to problem (1.1).

(ii) Suppose that f and h are both closed proper and convex. Let z^* be a fixed-point of T_{DRS} or T_{PRS}, then $\mathbf{prox}_{th}(z^*)$ is an optimal solution to problem (1.1).

The next proposition [7, Theorem 3] shows that the fixed-point algorithm corresponding to FBS converges in a sublinear rate.

PROPOSITION 2.6 (Sublinear convergence of FBS). *Assume that f is closed proper and convex, and h is convex and continuously differentiable with Lipschitz continuous gradient (with Lipschitz constant L). Let $x^0 \in \mathrm{dom}(f) \cap \mathrm{dom}(h)$ and let x^* be an optimal solution to problem (1.1). Suppose that $\{x^k\}$ be generated by the fixed-point iteration $x^{k+1} = T_{FBS}(x^k)$ with tuning parameter $t = 1/L$. Then for all $k \geq 1$,*

$$f(x^k) + h(x^k) - f(x^*) - h(x^*) \leq \frac{L\|x^0 - x^*\|_2^2}{2k}.$$

In addition, for all $k \geq 1$, we have $\|T_{FBS}(x^k) - x^k\|_2 = o(1/k)$.

Under the error bound condition, the fixed-point iteration of FBS is proved to converge linearly, see [11, Theorem 3.2] for example.

PROPOSITION 2.7 (Linear convergence of FBS). *Suppose that error bound condition (EBR) holds with parameter κ for residual function F_{FBS}. Let x^* be the limit point of the sequence $\{x^k\}$ generated by the fixed-point iteration $x^{k+1} = T_{FBS}(x^k)$ with $t \leq \beta^{-1}$ for some constant $\beta > 0$. Then there exists an index r such that for all $k \geq 1$,*

$$\|x^{r+k} - x^*\|_2^2 \leq \left(1 - \frac{1}{2\kappa\beta}\right)^k C \cdot (f(x^k) + h(x^k) - f(x^*) - h(x^*)),$$

where $C := \frac{2}{\beta(1-\sqrt{1-(2\beta\gamma)^{-1}})^2}$.

In [21], the authors studied the relationship between error bound condition and Kurdyka-Łojasiewicz (KL) exponent, and showed that FBS converges linearly if $f(x) + h(x)$ is a KL function with an exponent of $1/2$.

Finally, we introduce the sublinear convergence rate of the general fixed-point iteration.

PROPOSITION 2.8. *[7, Theorem 1] Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a non-expansive operator. Let $\{\lambda_k \in (0,1]\}$ be a sequence of positive numbers and $\tau_k := \lambda_k(1-\lambda_k)$. Let $z^0 \in \mathbb{R}^n$ and let z^* be a fixed point of the mapping $T_{\lambda_k} := (1-\lambda_k)I + \lambda_k T$. Suppose that $\{z^k\}$ be generated by the fixed-point iteration $z^{k+1} = T_{\lambda_k}(z^k)$. Then, if $\tau_k > 0$ for all $k \geq 0$,*

$$\|T_{\lambda_k}(z^k) - z^k\|_2^2 \leq \frac{\|z^0 - z^*\|_2^2}{\sum_{i=0}^k \tau_i}.$$

In particular, if $\tau_k \in (\varepsilon, \infty)$ for all $k \geq 0$ and some $\varepsilon > 0$, then $\|T_{\lambda_k}(z^k) - z^k\|_2 = o(1/(k+1))$.

3. Semi-smoothness of proximal mapping. We now discuss the semi-smoothness of proximal mappings. This property often implies that the fixed-point mappings corresponding to operator splitting algorithms are semi-smooth.

Let $\mathcal{O} \subseteq \mathbb{R}^n$ be an open set and $F : \mathcal{O} \to \mathbb{R}^m$ be a locally Lipschitz continuous function. Rademacher's theorem says that F is almost everywhere differentiable. Let D_F be the set of differentiable points of F in $\mathcal{O}$. We now introduce the concepts of generalized differential.

DEFINITION 3.1. *Let $F : \mathcal{O} \to \mathbb{R}^m$ be locally Lipschitz continuous at $x \in \mathcal{O}$. The* B-subdifferential *of F at x is defined by*

$$\partial_B F(x) := \left\{ \lim_{k\to\infty} F'(x^k) | x^k \in D_F, x^k \to x \right\}.$$

The set

$$\partial F(x) = \mathrm{co}(\partial_B F(x))$$

is called Clarke's generalized Jacobian*, where* co *denotes the convex hull.*

The notion of semi-smoothness plays a key role on establishing locally superlinear convergence of the nonsmooth Newton-type method. Semi-smoothness was originally introduced by Mifflin [29] for real-valued functions and extended to vector-valued mappings by Qi and Sun [35].

DEFINITION 3.2. *Let $F : \mathcal{O} \to \mathbb{R}^m$ be a locally Lipschitz continuous function. We say that F is semi-smooth at $x \in \mathcal{O}$ if*

(a) F is directionally differentiable at x; and

(b) for any $d \in \mathcal{O}$ and $J \in \partial F(x + d)$,

$$\|F(x + d) - F(x) - Jd\|_2 = o(\|d\|_2) \quad \text{as } d \to 0.$$

Furthermore, F is said to be strongly semi-smooth at $x \in \mathcal{O}$ if F is semi-smooth and for any $d \in \mathcal{O}$ and $J \in \partial F(x + d)$,

$$\|F(x + d) - F(x) - Jd\|_2 = O(\|d\|_2^2) \quad \text{as } d \to 0.$$

(Strongly) semi-smoothness is closed under scalar multiplication, summation and composition. The examples of semi-smooth functions include the smooth functions, all convex functions (thus norm), and the piecewise differentiable functions. Differentiable functions with Lipschitz gradients are strongly semi-smooth. For every $p \in [1, \infty]$, the norm $\| \cdot \|_p$ is strongly semi-smooth. Piecewise affine functions are strongly semi-smooth, such as $[x]_+ = \max\{0, x\}$. A vector-valued function is (strongly) semi-smooth if and only if each of its component functions is (strongly) semi-smooth. Examples of semi-smooth functions are thoroughly studied in [13, 43].

The basic properties of proximal mapping is well documented in textbooks such as [36, 2]. The proximal mapping $\mathbf{prox}_f$, corresponding to a proper, closed and convex function $f : \mathbb{R}^n \to \mathbb{R}$, is single-valued, maximal monotone and global Lipschitz continuous with constant one (non-expansive). Moreover, the proximal mappings of many interesting functions are (strongly) semi-smooth. It is worth mentioning that the semi-smoothness of proximal mapping does not hold in general [38]. The following lemma is useful when the proximal mapping of a function is complicate but the proximal mapping of its conjugate is easy.

LEMMA 3.3 (Moreau's decomposition). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper, closed and convex function. Then, for any $t > 0$ and $x \in \mathbb{R}^n$,*

$$x = \mathbf{prox}_{tf}(x) + t\mathbf{prox}_{f^*/t}(x/t).$$

We next review some existing results on the semi-smoothness of proximal mappings of various interesting functions.

EXAMPLE 3.4 (Sparsity inducing norms). *The proximal mapping of ℓ_1 norm $\|x\|_1$, which is the well-known soft-thresholding operator, is component-wise separable and piecewise affine. Hence, the operator $\mathbf{prox}_{\|\cdot\|_1}$ is strongly semi-smooth. According to the Moreau's decomposition, the proximal mapping of ℓ_∞ norm (the conjugate of ℓ_1 norm) is also strongly semi-smooth.*

EXAMPLE 3.5 (Piecewise $\mathcal{C}^k$ functions). *A function $f : \mathcal{O} \to \mathbb{R}^m$ defined on the open set $\mathcal{O} \subseteq \mathbb{R}^n$ is called piecewise $\mathcal{C}^k$ function, $k \in [1, \infty]$, if f is continuous and if at every point $\bar{x} \in \mathcal{O}$ there exist a neighborhood $V \subset \mathcal{O}$ and a finite collection of $\mathcal{C}^k$ functions $f_i : V \to \mathbb{R}^m, i = 1, \ldots, N$, such that*

$$f(x) \in \{f_1(x), \ldots, f_N(x)\} \quad \textit{for all } x \in V.$$

For a comprehensive study on piecewise $\mathcal{C}^k$ functions, the readers are referred to [37].

From [43, Proposition 2.26], if f is a piecewise $\mathcal{C}^1$ (piecewise smooth) function, then f is semi-smooth; if f is a piecewise $\mathcal{C}^2$ function, then f is strongly semi-smooth. As described in [32, Section 5], in many applications the proximal mappings are piecewise $\mathcal{C}^1$ and thus semi-smooth.

EXAMPLE 3.6 (Metric projections). *Metric projection, which is the proximal mapping of an indicator function, plays an important role in the analysis of constrained programs. The projection over a polyhedral set is piecewise linear [36, Example 12.31] and hence strongly semi-smooth. The projections over symmetric cones are proved to be strongly semi-smooth in [40]. Consequently, the strongly semi-smoothness of projections over SDP cone and SOC cone is derived.*

EXAMPLE 3.7 (Piecewise linear-quadratic functions). *Piecewise linear-quadratic functions [36, Definition 10.20] are an important class of functions in convex analysis, see [24]. It is shown in [36, Proposition 12.30] that the proximal mapping of piecewise linear-quadratic function is piecewise linear and hence strongly semi-smooth.*

EXAMPLE 3.8 (Proper, closed and convex functions). *For a proper, closed and convex function $f : \mathbb{R}^n \to (-\infty, +\infty]$, it was proved in [27] that the proximal mapping $\mathbf{prox}_f(x)$ is (strongly) semi-smooth if the projection over the epigraph of f is (strongly) semi-smooth. Therefore, it allow us to study the semi-smoothness of proximal mapping by analyzing the fine properties of the metric projection over closed convex sets.*

EXAMPLE 3.9 (Lagrange dual functions). *Consider the following convex program*

$$\min_{x \in \mathbb{R}^n} \quad f_0(x) \quad \textit{subject to} \quad Ax = b, x \in C, \tag{3.1}$$

where $A \in \mathbb{R}^{m\times n}$ with rank$(A) = m$, $C := \{x \in \mathbb{R}^n | f_i(x) \leq 0, i = 1, \ldots, l\}$ and $f_i, i = 0, 1, \ldots, l$, are convex and smooth. Define the dual function

$$d(w) = -\inf_{x \in C}\{f_0(x) + w^T(Ax - b)\}.$$

The authors in [28] showed that the proximal mapping $\mathbf{prox}_d(w)$ is piecewise smooth and therefore semi-smooth, if the functions $f_i, i = 0, 1 \ldots, l$ either are all affine or all possess positive definite Hessian matrices. This result can be useful to deal with the fixed-point iteration of dual operator splitting algorithms.

4. Semi-smooth Newton method for nonlinear monotone equations. The purpose of this section is to design a Newton-type method for solving the system of nonlinear equations

$$F(z) = 0, \tag{4.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly semi-smooth and monotone. In particular, we are interested in $F(z) = z - T(z)$, where $T(z)$ is a fixed-point mapping corresponding to certain first-order type algorithms. Let Z^* be the solution set of system (4.1). Throughout this section, we assume that Z^* is nonempty.

The system of monotone equations has various applications, see [31] for example. Inspired by a pioneer work [39], a class of iterative methods for solving nonlinear (smooth) monotone equations were proposed in recent years [47, 22, 1]. In [39], the authors proposed a globally convergent Newton method by exploiting the structure of monotonicity, whose primary advantage is that the whole sequence of the distances from the iterates to the solution set is decreasing. Zhou and Toh [46] extended the method to solve monotone equations without nonsingularity assumption.

The main concept in [39] is introduced as follows. For an iterate z^k, let d^k be a step such that

$$\left\langle F(u^k), -d^k \right\rangle > 0,$$

where $u^k = z^k + d^k$ is an intermediate iterate. By monotonicity of F, for any $z^* \in Z^*$ one has

$$\left\langle F(u^k), z^* - u^k \right\rangle \leq 0.$$

Therefore, the hyperplane

$$H_k := \{z \in \mathbb{R}^n | \left\langle F(u^k), z - u^k \right\rangle = 0\}$$

strictly separates z^k from the solution set Z^*. By noting this fact, it was advised in [39] that the next iterate is set by

$$z^{k+1} = z^k - \frac{\left\langle F(u^k), z^k - u^k \right\rangle}{\|F(u^k)\|_2^2} F(u^k).$$

It is easy to show that the point z^{k+1} is the projection of z^k onto the hyperplane H_k. The hyperplane projection step is critical to construct a globally convergent method for solving nonlinear monotone equations. By applying the same technique, we develop a globally convergent method for solving semi-smooth monotone equations (4.1).

It is known that each element of the B-subdifferential of a monotone and semi-smooth mapping is positively semidefinite. Hence, for an iterate z^k, by choosing an element $J_k \in \partial_B F(z^k)$, it is natural to apply a regularized Newton's method which computes

$$(J_k + \mu_k I)d = -F^k, \tag{4.2}$$

where $F^k = F(z^k)$, $\mu_k = \lambda_k \|F^k\|_2$ and $\lambda_k > 0$ is a regularization parameter. The regularization term $\mu_k I$ is chosen such that $J_k + \mu_k I$ is invertible. From the computational view, it is practical to solve the linear system (4.2) inexactly. Define

$$r^k := (J_k + \mu_k I)d^k + F^k. \tag{4.3}$$

At each iteration, we seek a step d^k by solving (4.2) approximately such that

$$\|r^k\|_2 \le \tau \min\{1, \lambda_k \|F^k\|_2 \|d^k\|_2\}, \tag{4.4}$$

where $0 < \tau < 1$ is some positive constant. Then a trial point is obtained as

$$u^k = z^k + d^k.$$

Define a ratio

$$\rho_k = \frac{-\langle F(u^k), d^k \rangle}{\|d^k\|_2^2}. \tag{4.5}$$

Select some parameters $0 < \eta_1 \le \eta_2 < 1$ and $1 < \gamma_1 \le \gamma_2$. If $\rho_k \ge \eta_1$, then the iteration is said to be *successful* and we set

$$z^{k+1} = z^k - \frac{\langle F(u^k), z^k - u^k \rangle}{\|F(u^k)\|_2^2} F(u^k). \tag{4.6}$$

Otherwise, the iteration is *unsuccessful* and we set $z^{k+1} = z^k$, that is

$$z^{k+1} = \begin{cases} z^k - \frac{\langle F(u^k), z^k - u^k \rangle}{\|F(u^k)\|_2^2} F(u^k), & \text{if } \rho_k \ge \eta_1, \\ z^k, & \text{otherwise.} \end{cases} \tag{4.7}$$

Then the regularization parameter λ_k is updated as

$$\lambda_{k+1} \in \begin{cases} (\underline{\lambda}, \lambda_k), & \text{if } \rho_k \ge \eta_2, \\ [\lambda_k, \gamma_1 \lambda_k], & \text{if } \eta_1 \le \rho_k < \eta_2, \\ (\gamma_1 \lambda_k, \gamma_2 \lambda_k], & \text{otherwise,} \end{cases} \tag{4.8}$$

where $\underline{\lambda} > 0$ is a small positive constant. These parameters determine how aggressively the regularization parameter is decreased when an iteration is successful or it is increased when an iteration is unsuccessful. The complete approach to solve (4.1) is summarized in Algorithm 1.

Algorithm 1: *Semi-smooth Newton method for nonlinear monotone equations*

0. Give $0 < \tau < 1$, $0 < \eta_1 \le \eta_2 < 1$ and $1 < \gamma_1 \le \gamma_2$. Choose z^0 and some $\varepsilon > 0$. Set $k = 0$.
1. If $d^k = 0$, stop.
2. Select $J_k \in \partial_B F(x^k)$. Solve linear system (4.2) approximately such that d^k satisfies (4.4).
3. Compute $u^k = z^k + d^k$ and calculate the ratio ρ_k as in (4.5).
4. Update z^{k+1} and λ_{k+1} according to (4.7) and (4.8), respectively.
5. Set $k = k + 1$, go to step 1.

4.1. Global convergence. It is clear that a solution is obtained if Algorithm 1 terminates in finitely many iterations. Therefore, we assume that Algorithm 1 always generates an infinite sequence $\{z^k\}$ and $d^k \ne 0$ for any $k \ge 0$. The following assumption is used in this subsection.

ASSUMPTION 4.1. *Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly semi-smooth and monotone. Suppose that there exists a constant $c_1 > 0$ such that $\|J_k\| \le c_1$ for any $k \ge 0$ and any $J_k \in \partial_B F(z^k)$.*

The following lemma demonstrates that the distance from z^k to Z^* decreases in a successful iteration. The proof follows directly from [39, Lemma 2.1], and it is omitted.

LEMMA 4.2. *For any $z^* \in Z^*$ and any successful iteration, indexed by say k, we have that*

$$\|z^{k+1} - z^*\|_2^2 \le \|z^k - z^*\|_2^2 - \|z^{k+1} - z^k\|_2^2. \tag{4.9}$$

Recall that F is strongly semi-smooth. Then for a point $z \in \mathbb{R}^n$ there exists $c_2 > 0$ (dependent on z) such that for any $d \in \mathbb{R}^n$ and any $J \in \partial_B F(z + d)$,

$$\|F(z + d) - F(z) - Jd\|_2 \le c_2\|d\|_2^2, \quad \text{as } \|d\|_2 \to 0. \tag{4.10}$$

In the sequel, we denote the index set of all successful iterations in Algorithm 1 by

$$\mathcal{S} := \{k \ge 0 : \rho_k \ge \eta_1\}.$$

We next show that if there are only finitely many successful iterations, the later iterates are optimal solutions.

LEMMA 4.3. *Suppose that Assumption 4.1 holds and the successful index set $\mathcal{S}$ is finite. Then $z^k = z^*$ for all sufficiently large k and $F(z^*) = 0$.*

Proof. Denote the index of the last successful iteration by k_0. The construction of the algorithm implies that $z^{k_0+i} = z^{k_0+1} := z^*$, for all $i \ge 1$ and additionally $\lambda_k \to \infty$. Suppose that $a := \|F(z^*)\|_2 > 0$. For all $k > k_0$, it follows from (4.3) that

$$d^k = (J_k + \lambda_k\|F^k\|_2 I)^{-1}(r^k - F^k),$$

which, together with $\lambda_k \to \infty$, $\|r^k\|_2 \le \tau$ and the fact that J_k is positive semidefinite, imply that $d^k \to 0$, and hence $u^k \to z^*$.

We now show that when λ_k is large enough, the ratio ρ_k is not smaller than η_2. For this purpose, we consider an iteration with index $k > k_0$ sufficiently large such that $\|d^k\|_2 \le 1$ and

$$\lambda_k \ge \frac{\eta_2 + c_1 + c_2}{a - \tau a}.$$

Then, it yields that

$$\begin{aligned} -\left\langle F(z^k), d^k\right\rangle &= \left\langle (J_k + \lambda_k\|F^k\|_2 I)d^k), d^k\right\rangle - \left\langle r^k, d^k\right\rangle \\ &\ge \lambda_k\|F^k\|_2\|d^k\|_2^2 - \tau\lambda_k\|F^k\|_2\|d^k\|_2^2 \\ &\ge (\eta_2 + c_1 + c_2)\|d^k\|_2^2. \end{aligned} \tag{4.11}$$

Further, for any $J_{u^k} \in \partial_B F(u^k)$ we obtain

$$\begin{aligned} &-\left\langle F(u^k), d^k\right\rangle \\ &= -\left\langle F(z^k), d^k\right\rangle - \left\langle J_{u^k} d^k, d^k\right\rangle + \left\langle -F(u^k) + F(z^k) + J_{u^k} d^k, d^k\right\rangle \\ &\ge -\left\langle F(z^k), d^k\right\rangle - c_1\|d^k\|_2^2 - \|F(z^* + d^k) - F(z^*) - J_{u^k} d^k\|_2 \\ &\ge (\eta_2 + c_1 + c_2)\|d^k\|_2^2 - c_1\|d^k\|_2^2 - c_2\|d^k\|_2^2 \\ &= \eta_2\|d^k\|_2^2, \end{aligned}$$

where the first inequality is from Assumption 4.1 and $\|d^k\|_2 \leq 1$, and the second inequality comes from (4.11) and (4.10). Hence, we have $\rho_k \geq \eta_2$, which generates a successful iteration and yields a contradiction. This completes the proof. □

We are now ready to prove the main global convergence result. In specific, we show that either in the above case $z^k \in Z^*$ for some finite k, or we obtain an infinite sequence $\{z^k\}$ converging to some solution.

THEOREM 4.4. *Let Assumption 4.1 hold. Then $\{z^k\}$ converges to some point $\bar{z}$ such that $F(\bar{z}) = 0$.*

Proof. If the successful index set $\mathcal{S}$ is finite, the result is directly from Lemma 4.3. In the sequel, we suppose that $\mathcal{S}$ is infinite. Let z^* be any point in solution set Z^*. By Lemma 4.2, for any $k \in \mathcal{S}$, it yields that

$$\|z^{k+1} - z^*\|_2^2 \leq \|z^k - z^*\|_2^2 - \|z^{k+1} - z^k\|_2^2. \tag{4.12}$$

Therefore, the sequence $\{\|z^k - z^*\|_2\}$ is non-increasing and convergent, the sequence $\{z^k\}$ is bounded, and

$$\lim_{k\to\infty} \|z^{k+1} - z^k\|_2 = 0. \tag{4.13}$$

By (4.3) and (4.4), it follows that

$$\|F^k\|_2 \geq \|(J_k + \lambda_k\|F^k\|_2 I)d^k\|_2 - \|r^k\|_2 \geq (1-\tau)\lambda_k\|F^k\|_2\|d^k\|_2,$$

which implies that $\|d^k\|_2 \leq 1/[(1-\tau)\underline{\lambda}]$. This inequality shows that $\{d^k\}$ is bounded, and $\{u^k\}$ is also bounded. By using the continuity of F, there exists a constant $c_3 > 0$ such that

$$\|F(u^k)\|_2^{-1} \geq c_3, \quad \text{for any } \ k \geq 0.$$

Using (4.7), for any $k \in \mathcal{S}$, we obtain that

$$\|z^{k+1} - z^k\|_2 = \frac{-\langle F(u^k), d^k\rangle}{\|F(u^k)\|_2} \geq c_3\rho_k\|d^k\|_2^2,$$

which, together with (4.13), imply that

$$\lim_{k\to\infty, k\in\mathcal{S}} \rho_k\|d^k\|_2^2 = 0. \tag{4.14}$$

Now we consider two possible cases:

$$\liminf_{k\to\infty} \|F^k\|_2 = 0 \quad \text{and} \quad \liminf_{k\to\infty} \|F^k\|_2 = c_4 > 0.$$

In the first case, the continuity of F and the boundedness of $\{z^k\}$ imply that the sequence $\{z^k\}$ has some accumulation point $\hat{z}$ such that $F(\hat{z}) = 0$. Since z^* is an arbitrary point in Z^*, we can choose $z^* = \hat{z}$ in (4.12). Then $\{z^k\}$ converges to $\hat{z}$.

In the second case, by using the continuity of F and the boundedness of $\{z^k\}$ again, there exist constants $c_5 > c_6 > 0$ such that

$$c_6 \leq \|F^k\|_2 \leq c_5, \quad \text{for all } k \geq 0.$$

If λ_k is large enough such that $\|d^k\|_2 \leq 1$ and

$$\lambda_k \geq \frac{\eta_2 + c_1 + c_2}{(1-\tau)c_6},$$

then by a similar proof as in Lemma 4.3 we have that $\rho_k \geq \eta_2$ and consequently $\lambda_{k+1} < \lambda_k$. Hence, it turns out that $\{\lambda_k\}$ is bounded from above, by say $\bar{\lambda} > 0$. Using (4.3), (4.4), Assumption 4.1 and the upper bound of $\{\lambda_k\}$, we have

$$\|F^k\|_2 \leq \|(J_k + \lambda_k \|F^k\|_2 I)d^k\|_2 + \|r^k\|_2 \leq (c_1 + (1+\tau)c_5\bar{\lambda})\|d^k\|_2.$$

Hence, it follows that

$$\liminf_{k\to\infty} \|d^k\|_2 > 0.$$

Then, by (4.14), it must hold that

$$\lim_{k\to\infty, k\in\mathcal{S}} \rho_k = 0,$$

which yields a contradiction to the definition of $\mathcal{S}$. Hence the second case is not possible. The proof is completed. □

Finally, we remark that, if the Jacobian matrix is nonsingular, the superlinear local convergence rate of Algorithm 1 can be derived by following the same lines as in [39, Theorem 2.2]. Moreover, if the hyperplane projection step is removed in a neighborhood of the solution set, it is easy to show that the algorithm locally converges with quadratic rates under certain conditions.

5. Semi-smooth LM methods.

5.1. Basic Semi-smooth LM method. Assume that $F(z)$ in (4.1) is (strongly) semi-smooth. The system of nonlinear equations (4.1) is equivalent to the nonlinear least squares problem

$$\min_z \phi(z) := \frac{1}{2}\|F(z)\|_2^2. \tag{5.1}$$

For an iterate point z^k, we choose an element $J_k \in \partial_B F(z^k)$ and denote

$$F^k = F(z^k), \quad \phi_k = \phi(z^k), \quad g^k = J_k^T F^k.$$

Consider a quadratic model as follows

$$m_k(s) := \frac{1}{2}\|F^k + J_k s\|_2^2.$$

The step s^k is obtained by solving the minimization problem

$$\min_s \; m_k(s) + \frac{1}{2}\lambda_k \|F^k\|_2 \|s\|_2^2,$$

which is equivalent to solve a linear system

$$[J_k^T J_k + \lambda_k \|F^k\|_2 I]s = -J_k^T F^k, \tag{5.2}$$

where $\lambda_k > 0$ is the regularization parameter. In practice, an inexact step s^k can be obtained by solving (5.2) approximately such that

$$\|r^k\|_2 \leq \tau \min\{\bar{\eta}, \|F^k\|_2^2\}, \tag{5.3}$$

where τ and $\bar{\eta}$ are some positive constants, and r^k is defined by

$$r^k := (J_k^T J_k + \lambda_k \|F^k\|_2 I)s^k + g^k.$$

Define the following ratio between actual reduction and predicted reduction

$$\rho_k = \frac{\phi(z^k) - \phi(z^k + s^k)}{m_k(0) - m_k(s^k)}. \tag{5.4}$$

Then for some constants $0 < \eta_1 \leq \eta_2 < 1$ and $1 < \gamma_1 \leq \gamma_2$, the iterate point is updated as

$$z^{k+1} = \begin{cases} z^k + s^k, & \text{if } \rho_k \geq \eta_1, \\ z^k, & \text{otherwise} \end{cases} \tag{5.5}$$

and the regularization parameter λ_k is updated as

$$\lambda_{k+1} \in \begin{cases} (0, \lambda_k), & \text{if } \rho_k > \eta_2, \\ [\lambda_k, \gamma_1 \lambda_k], & \text{if } \eta_1 \leq \rho_k \leq \eta_2, \\ (\gamma_1 \lambda_k, \gamma_2 \lambda_k], & \text{otherwise.} \end{cases} \tag{5.6}$$

A description of the LM method is summarized in Algorithm 2.

Algorithm 2: *Truncated Semi-smooth LM method*

0. Give $\tau > 0$, $\bar{\eta} \in (0, 1)$, $0 < \eta_1 \leq \eta_2 < 1$ and $1 < \gamma_1 \leq \gamma_2$. Choose $z^0 \in \mathbb{R}^n$ and some $\varepsilon > 0$. Set $k = 0$.
1. If $\|g^k\|_2 \leq \varepsilon$, stop.
2. Select $J_k \in \partial_B F(z^k)$. Solve the linear system (5.2) by the CG method to obtain an approximate solution s^k satisfying (5.3).
3. Calculate the ratio ρ_k as in (5.4).
4. Update z^{k+1} and λ_{k+1} according to (5.5) and (5.6), respectively.
5. Set $k = k + 1$, go to step 1.

The global convergence and local convergence rate of Algorithm 2 follow directly from [34], and the proofs are omitted. We make the following assumption.

ASSUMPTION 5.1. *Suppose that the level set $L(z^0) := \{z \in \mathbb{R}^n : \phi(z) \leq \phi(z^0)\}$ is bounded and $F(z)$ is strongly semi-smooth over $L(z^0)$.*

The following theorem shows that Algorithm 2 is globally convergent.

THEOREM 5.2. *Suppose that Assumption 5.1 holds. Then the sequence $\{z^k\}$ generated by Algorithm 2 satisfies*

$$\lim_{k \to \infty} \|g^k\|_2 = 0.$$

We say that F is *BD-regular* at $z \in \mathbb{R}^n$, if every element in $\partial_B F(z)$ is nonsingular. This property, which is viewed as a generalized nonsingularity condition, is important to establish the local convergence rate of the semi-smooth Newton-type method.

THEOREM 5.3. *Suppose that Assumption 5.1 holds and F is BD-regular at a solution z^* to $F(z) = 0$. Assume that the sequence $\{z^k\}$ lies in a sufficient small neighborhood of z^*, then the sequence $\{z^k\}$ converges to z^* quadratically, i.e.,*

$$\|z^{k+1} - z^*\|_2 = O(\|z^k - z^*\|_2^2).$$

5.2. Globalization of the LM method. Note that generally there are more than one local solution to (5.1) and $\|g\|_2 = \|J^T F\|_2 = 0$ does not imply $F = 0$. It is observed that in some situations the semi-smooth LM algorithm converges to a local but not global optimal solution from an arbitrary initial point. Since our goal is to find a global solution, it is necessary to develop certain globalization technique to escape the local solutions. For example, the first-order methods can help in a two-stage framework. At the first stage, a first-order type method is applied to produce a moderately accurate approximate solution. Then a second-order type method is activated to obtain a better solution. These two steps can be repeated until a stopping criterion is satisfied. An outline of this method is presented in Algorithm 3.

Algorithm 3: *Globalized Semi-smooth LM method*

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0. Give $\tau > 0$, $\mu \in (0,1)$, $\bar{\eta} \in (0,1)$, $0 < \eta_1 \le \eta_2 < 1$ and $1 < \gamma_1 \le \gamma_2$. Choose z^0 and some $\varepsilon_0 \ge \varepsilon > 0$. Set $i = 0$.
1. Run the fixed-point iteration $z^{i+1} = T_{\text{DRS}}(z^i)$ until $\|F^{i+1}\|_2 \le \varepsilon_0$.
2. Set $z^0 = z^{i+1}$ and $k = 0$, run Algorithm 2.
3. If $\|F^k\|_2 \le \varepsilon$, stop. Otherwise, if some local optimality condition is detected, set $\varepsilon_0 = \mu\|F^{i+1}\|_2$ and go back to step 1.

The following theorem shows that Algorithm 3 always converges to a global optimal solution.

THEOREM 5.4. *Assume that $\varepsilon = 0$. Then, one of the following two cases will happen for Algorithm 3:*

(i) A global optimal solution is obtained in step 2.

(ii) An infinite sequence $\{z^i\}$ is obtained and

$$\lim_{i \to \infty} \|F^i\|_2 = 0.$$

In any case, a global optimal solution to $F(z) = 0$ is obtained.

Proof. It is sufficient to show that the sequence $\{z^i\}$ generated by $z^{i+1} = T(z^i)$ converges to a global solution, when the LM iteration in step 2 always goes to a local solution to $\min_z \phi(z)$. The result follows immediately from the fact that T is non-expansive. □

Under some regular conditions, a simpler globalzied algorithm is introduced. In the first stage, we run the fixed-point iteration

$$z^{i+1} = T(z^i) \tag{5.7}$$

until the iterate point falls in a neighborhood of an optimal solution. Denote the terminate point of the first stage as $\bar{z}$. Then, starting from $z^0 = \bar{z}$, we perform

$$z^{k+1} = z^k + s^k, \tag{5.8}$$

where s^k is a LM step obtained by (approximately) solving

$$[J_k^T J_k + \|F^k\|_2 I]s = -J_k^T F^k, \tag{5.9}$$

which is a special case of (5.2) with $\lambda_k \equiv 1$. It will be shown that this special LM iteration converges to a global solution quadratically from a good initial point. The method is summarized in Algorithm 4.

Algorithm 4: *A Globalized method*
0. Choose z^0 and some $\varepsilon_0 \geq \varepsilon > 0$. Set $i = 0$ and $k = 0$.
1. Run fixed-point iteration (5.7) until $\|F(\bar{z})\|_2 \leq \varepsilon_0$.
2. Starting from $z^0 = \bar{z}$, run LM iteration (5.8) until $\|F^k\|_2 \leq \varepsilon$.

In step 2 of Algorithm 4, let $z^* \in Z^*$ be the solution satisfying

$$\|z^* - \bar{z}\|_2 = \mathrm{dist}(\bar{z}, Z^*).$$

Note that F is strongly semi-smooth. Hence, there exists some $\kappa > 0$ such that for any $s \in \mathbb{R}^n$ and any $J \in \partial_B F(z^* + s)$,

$$\|F(z^* + s) - F(z^*) - Js\|_2 \leq \kappa \|s\|_2^2, \quad \text{as } \|s\|_2 \to 0. \tag{5.10}$$

Suppose that BD-regular condition holds at point z^*. Thus, from [34, Corollary 4.1], there exist $0 < \sigma_0 \leq \sigma_1$ and $\delta_1 > 0$ such that for any $z \in \mathbb{R}^n$ with $\mathrm{dist}(z, Z^*) \leq \mathrm{dist}(\bar{z}, Z^*) \leq \delta_1$,

$$\sigma_0 \leq \sigma_{\min}(J_z) \leq \sigma_{\max}(J_z) \leq \sigma_1, \tag{5.11}$$

where J_z is an element of $\partial_B F(z)$, $\sigma_{\min}(J_z)$ and $\sigma_{\max}(J_z)$ are the minimum and maximum singular values of J_z. It is well known that BD regularity implies that error bound condition (EBN) holds, i.e., there exist $\delta_2 > 0$ and $\delta_0 > 0$ such that

$$\|F(z)\|_2 \geq \delta_0 \mathrm{dist}(z, Z^*), \quad \forall z \in \{z | \mathrm{dist}(z, Z^*) \leq \delta_2\}. \tag{5.12}$$

Next we state that the iteration in step 2 of Algorithm 4 converges to a global solution quadratically if ε_0 is small enough.

LEMMA 5.5. *Suppose that BD-regular condition holds at a solution z^* which satisfies $\|\bar{z} - z^*\|_2 = \mathrm{dist}(\bar{z}, Z^*)$. Let $\bar{z}$ be close enough to Z^* such that*

$$\|F(\bar{z})\|_2 \leq \varepsilon_0 := \delta_0 \min\{1/c, \delta_1, \delta_2\}, \tag{5.13}$$

where $c = c_0 + 1$ and c_0 is a constant defined by

$$c_0 := \frac{8 + \sigma_1 \kappa}{\sigma_0^2}. \tag{5.14}$$

Then, the iteration in step 2 of Algorithm 4 converges quadratically to z^*, i.e.,

$$\|z^{k+1} - z^*\|_2 \leq c_0 \|z^k - z^*\|_2^2. \tag{5.15}$$

Proof. Denote $\Psi_k^\mu = \Psi_k + \mu_k I$, $\Psi_k = J_k^T J_k$ and $\mu_k = \|F^k\|_2$. Then the LM subproblem (5.9) is rewritten as

$$\Psi_k^\mu s = -J_k^T F^k.$$

Let $h^k = z^k - z^*$. Then we obtain

$$\begin{aligned} h^{k+1} = z^{k+1} - z^* &= h^k - (\Psi_k^\mu)^{-1} J_k^T F^k \\ &= (\Psi_k^\mu)^{-1} (\Psi_k + \mu_k I) h^k - (\Psi_k^\mu)^{-1} J_k^T F^k \\ &= \mu_k (\Psi_k^\mu)^{-1} h^k + (\Psi_k^\mu)^{-1} J_k^T (J_k h^k - F^k) \\ &= I_1^k + I_2^k. \end{aligned} \tag{5.16}$$

We first estimate I_1^k. Since $T = I - F$ is non-expansive, we have

$$\mu_k = \|F^k - F(z^*)\|_2 = \|(z^k - z^*) - (T(z^k) - T(z^*))\|_2 \le 2\|h^k\|_2.$$

This together with the fact that $\|(\Psi_k^\mu)^{-1}\| \le 1/(\sigma_0^2 + \mu_k)$ implies that

$$\|I_1^k\|_2 \le \frac{\mu_k}{\sigma_0^2 + \mu_k}\|h^k\|_2 \le \frac{2}{\sigma_0^2}\|h^k\|_2^2. \tag{5.17}$$

We now show that $\|h^k\|_2$ converges to zero. For $k = 0$, the inequalities (5.12) and (5.13) imply that

$$\|h^0\|_2 = \mathrm{dist}(\bar{z}, Z^*) \le \|F(\bar{z})\|_2/\delta_0 \le 1/c \le \sigma_0^2/8.$$

Hence, (5.17) yields $\|I_1^0\|_2 \le \|h^0\|_2/4$. For any $k \ge 0$, by noticing that $\|(\Psi_k^\mu)^{-1}J_k^T\| \le \sigma_1/\sigma_0^2$ and F is semi-smooth, we have

$$\|I_2^k\|_2 \le \|(\Psi_k^\mu)^{-1}J_k^T\| \cdot \|F^k - F(z^*) - J_k h^k\|_2 \le \frac{1}{4}\|h^k\|_2$$

for a sufficiently small $\|h^k\|_2$. Therefore, by induction one has that $\|h^{k+1}\|_2 \le \|h^k\|_2/2$ for all $k \ge 0$, that is $\|h^k\|_2 \to 0$, which also indicates that $\{z^k\}$ converges to z^*.

From (5.10) and $\|(\Psi_k^\mu)^{-1}J_k^T\| \le \sigma_1/\sigma_0^2$, we obtain an estimate of I_2^k as

$$\|I_2^k\|_2 \le \|(\Psi_k^\mu)^{-1}J_k^T\| \cdot \|J_k h^k - F^k\|_2 \le \frac{\sigma_1\kappa}{\sigma_0^2}\|h^k\|_2^2. \tag{5.18}$$

Combining (5.16), (5.17) and (5.18), one has

$$\|z^{k+1} - z^*\|_2 \le c_0\|z^k - z^*\|_2^2,$$

where c_0 is a positive constant defined by (5.14). Therefore, if $\mathrm{dist}(\bar{z}, Z^*) < 1/c$ (where $c = c_0 + 1$), the iteration in step 2 converges to z^* quadratically. This condition holds true automatically when ε_0 is chosen as in (5.13), since

$$\mathrm{dist}(\bar{z}, Z^*) \le \frac{1}{\delta_0}\|F(\bar{z})\|_2 \le \frac{1}{\delta_0}\varepsilon_0 \le \frac{1}{c}.$$

The proof is completed. □

REMARK 5.6. *We claim that the quadratical convergence rate in Lemma 5.5 is preserved if the subproblem in step 2 of Algorithm 4 is approximately solved. To be specific, the step s^k is obtained by approximately solving system (5.9) such that s^k satisfies*

$$\|r^k\|_2 \le \min\{\bar{\eta}, \|F^k\|_2^2\},$$

where $r^k = [J_k^T J_k + \|F^k\|_2 I]s^k + J_k^T F^k$ and $\bar{\eta} > 0$ is some constant. The claim can be proved by the same lines as Lemma 5.5. The only difference is that in this case (5.16) is revised to

$$z^{k+1} - z^* = I_1 + I_2 + (\Psi_k^\mu)^{-1} r^k$$

and the additional term satisfies

$$\|(\Psi_k^\mu)^{-1} r^k\|_2 \le \frac{1}{\sigma_0^2}\|F^k\|_2^2 \le \frac{4}{\sigma_0^2}\|h^k\|_2^2.$$

From Proposition 2.8, we have

$$\|F^i\|_2^2 = \|T(z^i) - z^i\|_2^2 \leq \frac{4\|z^0 - z^*\|_2^2}{i+1},$$

which implies that the total number of fixed-point iterations in step 1 is $O(1/\varepsilon_0^2)$ such that $\|F^{i+1}\|_2 \leq \varepsilon_0$. In step 2, Lemma 5.5 indicates that $O(\log(1/\varepsilon))$ LM iterations are required to generate $\|F^k\|_2 \leq \varepsilon$. In summary, an iteration complexity of Algorithm 4 is provided in the following theorem.

THEOREM 5.7. *Suppose that BD-regular condition holds over the solution set Z^*. Let ε_0 be given as (5.13). Then, Algorithm 4 takes at most $O(1/\varepsilon_0^2)$ fixed-point iterations and $O(\log(1/\varepsilon))$ LM iterations to generate a point z^k such that $\|F^k\|_2 \leq \varepsilon$.*

6. Numerical Results. In this section, we conduct proof-of-concept numerical experiments on our proposed schemes for the fixed-point mappings induced from the FBS and DRS methods by applying them to a variety of problems, including compressive sensing, logistic regression, linear programming, quadratic programming and image denoising. All numerical experiments are performed in MATLAB on a MacBook Pro computer with a Intel Core i7 (2.5 GHZ) CPU and 16GB memory.

6.1. Applications to the FBS Method. Consider the system of nonlinear equations corresponding to the FBS method as:

$$F(x) = x - \mathbf{prox}_{tf}(x - t\nabla h(x)) = 0.$$

The generalized Jacobian matrix of $F(x)$ is

$$J(x) = I - M(x)(I - t\partial^2 h(x)), \tag{6.1}$$

where $M(x) \in \partial\mathbf{prox}_{tf}(x - t\nabla h(x))$ and $\partial^2 h(x)$ is the generalized Hessian matrix of $h(x)$.

6.1.1. LASSO Regression. The Lasso regression problem is to find x such that

$$\min \frac{1}{2}\|Ax - b\|_2^2 \text{ subject to } \|x\|_1 \leq \lambda, \tag{6.2}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $\lambda \geq 0$ are given. Let $h(x) = \frac{1}{2}\|Ax - b\|_2^2$ and $f(x)$ be the indicator function $1_\Omega(x)$ on the set $\Omega = \{x \mid \|x\|_1 \leq \lambda\}$. Then the proximal mapping corresponding to $f(x)$ is the projection onto the ℓ_1-norm ball. For a given $z \in \mathbb{R}^n$, let $|z_{[1]}| \geq |z_{[2]}| \geq \ldots \geq |z_{[n]}|$, α be the largest value of $\left(\sum_{i=1}^k |z_{[i]}| - \lambda\right)/k$, $k = 1, \ldots, n$, and p be the corresponding k of α. It can be verified that

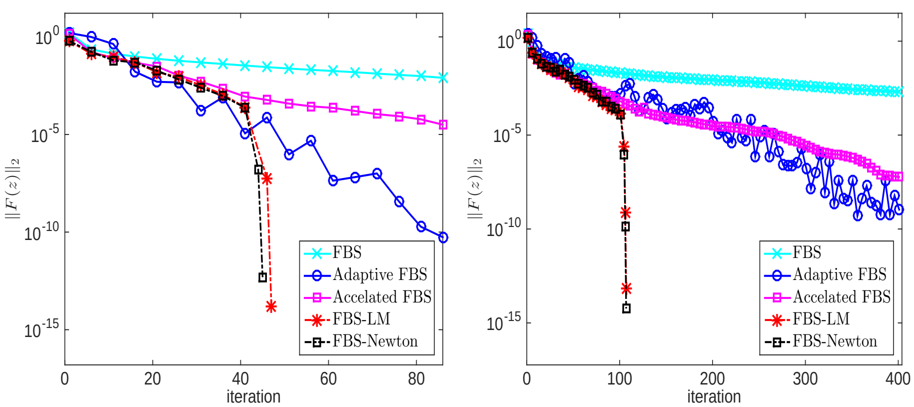
$$\left(\mathbf{prox}_{tf}(z)\right)_i = \begin{cases} z_i & \text{if } \alpha < 0, \\ 0, & \text{if } |z_i| < \alpha \text{ and } \alpha > 0, \\ z_i - \alpha\text{sign}(z_i), & \text{otherwise.} \end{cases}$$

Consequently, all possible nonzero elements of the Jacobian matrix $M(z)$ are

$$M(z)_{ij} = \begin{cases} 1 & \text{if } \alpha < 0, j = i \\ 1 - \alpha\text{sign}(z_i)\text{sign}(z_j)/p, & \text{if } |z_i| \geq \alpha \text{ and } \alpha > 0, j = [1], \ldots, [p]. \end{cases}$$

We compare the performance of FBS-LM (Algorithm 3) and FBS-Newton (Algorithm 1) with the basic FBS and the adaptive FBS and accelerated FBS methods in [17] on solving

(6.2). The test matrix $A \in \mathbb{R}^{m\times n}$ is set to the Gaussian matrix whose elements are generated independent and identically distributed from the normal distribution. A sparse signal $x^* \in \mathbb{R}^n$ is generated with k nonzero components whose values are equal to 1. Then the right hand side b is constructed from $b = Ax^* + 0.05\epsilon$, where ϵ are normally distributed noises. The parameter λ is set to $0.9\|x^*\|_1$. In FBS-LM and FBS-Newton, we first perform a limited-memory quasi-Newton method (FBS-QN), an approach similar to [47] in which the quasi-Newton matrix is replaced by the limited-memory matrix, until the fixed-point residual $\|F(z)\|_2$ is less than 10^{-4}. Then the second-order type methods are activated. FBS-QN is used rather than FBS itself in the initial stage because FBS-QN is faster than FBS in these examples. Figure 6.1 shows convergence history of $\|F(z)\|_2$ on $n = 1000$ and $m = 500$ with respect to $k = 50$ and 150. Both FBS-LM and FBS-Newton show quadratic convergence rates when the fixed-point residual is small enough.



(a) $k = 50$

(b) $k = 150$

FIG. 6.1. *residual history of LASSO* (6.11) *on* $n = 1000$, $m = 500$ *and* $\mu = 0.9\|x\|_1$

6.1.2. Logistic Regression. Suppose that the outcome of random Bernoulli trials is $b_i \in \{0, 1\}, i = 1, \dots, m$. The success probability of the ith trial is $P(b_i = 1|x) = e^{A_i x}/(1 + e^{A_i x}$. The so-called sparse logistic regression problem is

$$\min \ \mu\|x\|_1 + h(x), \tag{6.3}$$

where $h(x) = \sum_{i=1}^m \log(e^{A_i x} + 1) - b_i A_i x$. The proximal mapping corresponding to $f(x) = \mu\|x\|_1$ is the so-called shrinkage operator defined as

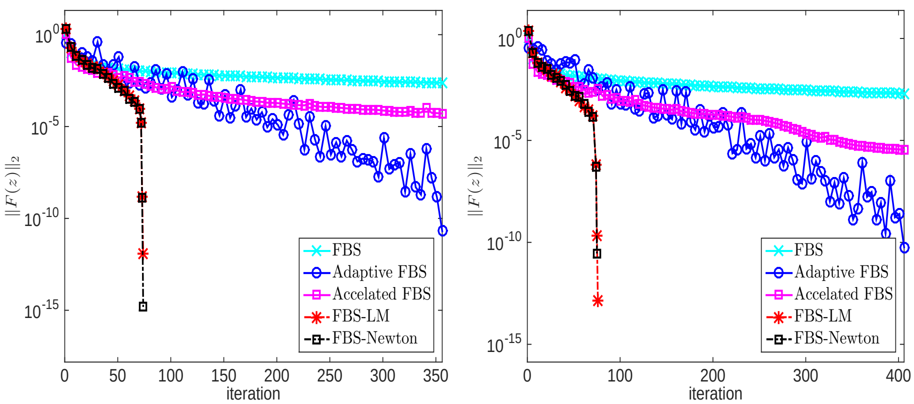
$$\left(\mathbf{prox}_{tf}(z)\right)_i = \operatorname{sign}(z_i)\max(|z_i| - \mu t, 0).$$

Hence, the Jacobian matrix $M(z)$ is diagonal matrix whose diagonal entries are

$$(M(z))_{ii} = \begin{cases} 1, & \text{if } |z_i| > \mu t, \\ 0, & \text{otherwise.} \end{cases}$$

We first generate a data matrix A and a solution x^* with k nonzero elements in the same fashion as Lasso regression in section 6.1.1. Then the vector b is constructed based on the probability $P(b_i = 1|x^*)$. In FBS-LM and FBS-Newton, we first use FBS-QN until the fixed-point residual $\|F(z)\|_2$ is less than 10^{-4}, then the second-order type methods are performed.

The residual history $\|F(z)\|_2$ on $n = 2000$, $m = 1000$ and $\mu = 1$ with respect to $k = 200$ and 600 is shown in Figure 6.2. Both FBS-LM and FBS-Newton show quadratic convergence rates when the fixed-point residual is small enough.



(a) $k = 200$

(b) $k = 600$

FIG. 6.2. *residual history of the logistic regression problem* (6.3) *on* $n = 2000$, $m = 1000$ *and* $\mu = 1$

6.1.3. General Quadratic Programming. Consider the general quadratic programming

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x \text{ subject to } Ax \leq b, \tag{6.4}$$

where $Q \in \mathbb{R}^{n \times n}$ is symmetric positive definite, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The dual problem is

$$\max_{y \geq 0} \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x + y^T (Ax - b),$$

which is equivalent to

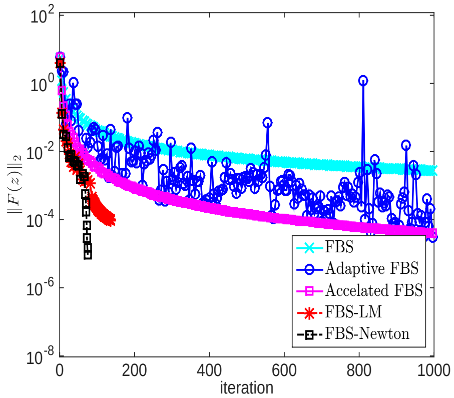
$$\min_{y \geq 0} \frac{1}{2} y^T (AQ^{-1}A^T) y + (AQ^{-1}c + b)^T y. \tag{6.5}$$

The primal solution can be recovered by $x = -Q^{-1}(A^T y + c)$.

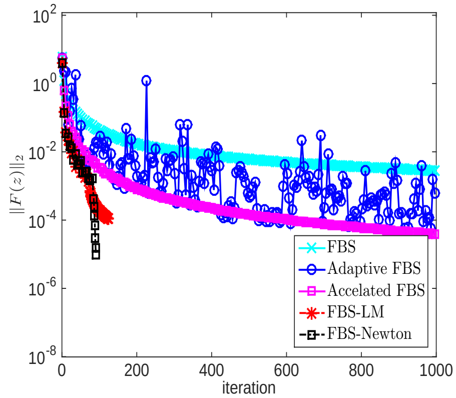
We apply all methods to the dual problem (6.5) on two problems “LISWET1” and “LISWET2” from the test set [26]. The matrix dimensions of A are $m = 10000$ and $n = 10002$, respectively. In particular, the matrix Q is the identity and A is a tri-diagonal matrix. We switch to the second-order type methods when $\|F(z)\|_2$ is less than 10^{-3}. The residual history $\|F(z)\|_2$ of all methods is shown in Figure 6.3. Although quadratic convergence is not achieved by FBS-LM and FBS-Newton, they are significantly faster than other types of FBS in terms of the number of the iterations.

6.1.4. Total Variation Denoising. Given a noisy image u, the total variation denoising problem is

$$\min_x \mu |\nabla x| + \frac{1}{2} \|x - u\|_2^2, \tag{6.6}$$



(a) LISWET1



(b) LISWET2

FIG. 6.3. *residual history of quadratic programming*

where μ is a constant and the total variation is $|\nabla x| = \sum_{ij} \sqrt{(x_{i+1,j} - x_{ij})^2 + (x_{i,j+1} - x_{ij})^2}$. Note that $|\nabla x| = \max_{\|y\|_\infty \leq 1} \langle y, \nabla x \rangle$. Consequently, we obtain

$$\min_x \mu |\nabla x| + \frac{1}{2}\|x - u\|_2^2 = \max_{\|y\|_\infty \leq 1} \min_x \mu \langle y, \nabla x \rangle + \frac{1}{2}\|x - u\|_2^2.$$

Hence, the dual problem is

$$\min_{\|y\|_\infty \leq 1} \frac{1}{2} \left\| \nabla \cdot y - \frac{1}{\mu} u \right\|_2^2. \tag{6.7}$$

The primal solution x can be recovered by $x = u - \mu \nabla \cdot x$.

We solve the dual problem (6.7) on two images "phantom" and "cameraman". We compare the performance of FBS-QN and FBS-Newton (Algorithm 1) with the basic FBS and the adaptive FBS and accelerated FBS methods. Figure 6.4 shows convergence history of $\|F(z)\|_2$. FBS-Newton (as well as FBS-LM) stagnates after a few iterations. The reason might be that the iteration is away from the contraction region. On the other hand, FBS-QN is still faster than the accelerated FBS on reaching an accuracy of 10^{-2} in terms of the number of the iterations. Finally, the original, noisy and recovered images of "phantom" are shown in Figure 6.5.

6.2. Applications to the DRS Method. Consider optimization problems with linear constraints

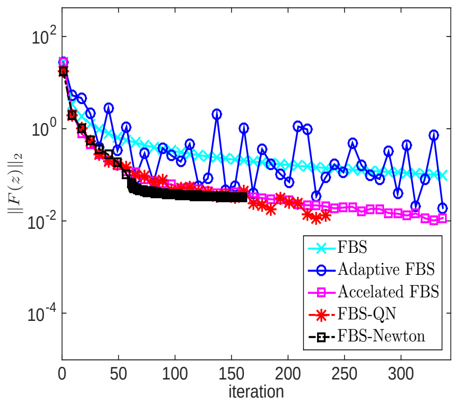
$$\min \ f(x) \quad \text{subject to } Ax = b, \tag{6.8}$$

where $A \in \mathbb{R}^{m \times n}$ is of full row rank and $b \in \mathbb{R}^m$. Let $h(x)$ be the indicator function $1_\Omega(x)$ on the set $\Omega = \{x \mid Ax = b\}$. It follows from [37, Proposition 2.4.3] that the proximal mapping with respect to $h(x)$ is

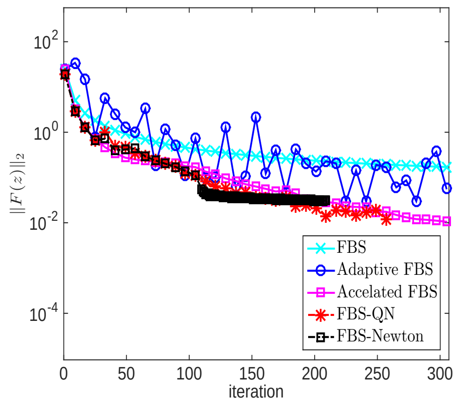
$$\mathbf{prox}_{th}(x) = \mathcal{P}_\Omega(x) = (I - \mathcal{P}_{A^T})x + (A^T(AA^T)^{-1})b,$$

where $\mathcal{P}_{A^T} = A^T(AA^T)^{-1}A$ is the orthogonal projection onto the range space of A^T. Hence, the system of nonlinear equations (4.1) corresponding to the DRS fixed-point mapping (2.6) reduces to

$$F(z) = Dz + \beta - \mathbf{prox}_{tf}((2D - I)z + 2\beta) = 0, \tag{6.9}$$

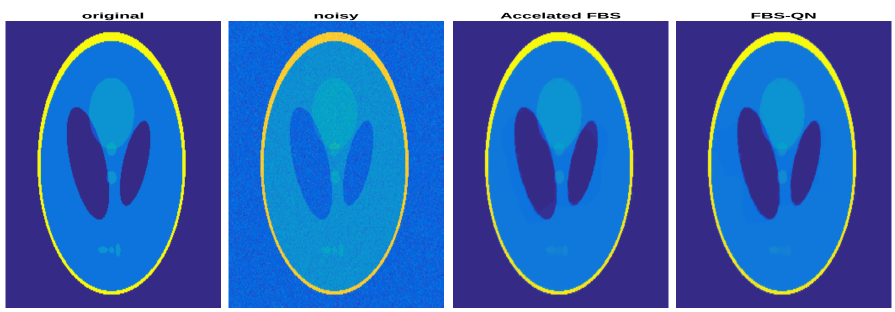


(a) phantom



(b) cameraman

FIG. 6.4. *residual history of total variation denoising*



(a) phantom

FIG. 6.5. *illustration of total variation denoising*

where

$$D = I - \mathcal{P}_{A^T} \quad \text{and} \quad \beta = (A^T(AA^T)^{-1})b.$$

The generalized Jacobian matrix of $F(z)$ is in the form of

$$J(z) = D - M(z)(2D - I) = \Phi(z)\mathcal{P}_{A^T} - \Psi(z), \tag{6.10}$$

where $M(z) \in \partial \mathbf{prox}_{tf}((2D-I)z + 2\beta)$, $\Psi(z) = M(z) - I$ and $\Phi(z) = 2M(z) - I$.

6.2.1. Basis Pursuit. A special case of (6.8) is the ℓ_1 minimization problem:

$$\min_{x \in \mathbb{R}^n} \quad \|x\|_1 \quad \text{subject to} \quad Ax = b. \tag{6.11}$$

Let $f(x) = \|x\|_1$. The proximal mapping with respect to $f(x)$ is

$$\mathbf{prox}_{tf}(x) = \text{sign}(x) \cdot \max\{0, |x| - t\}.$$

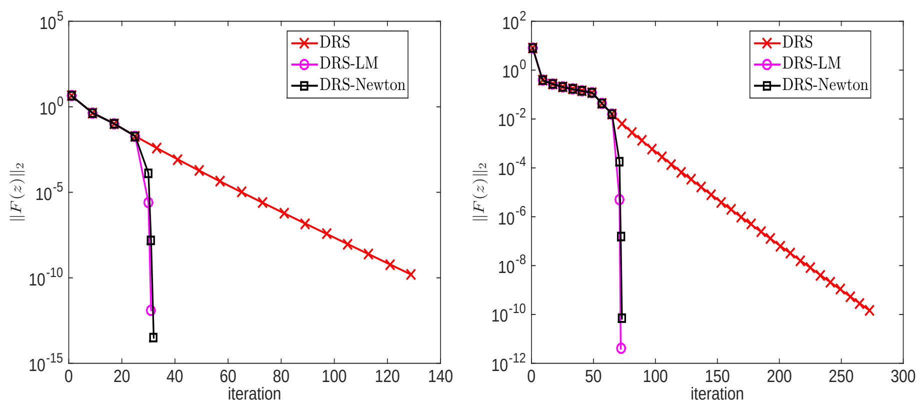
Then the generalized Jacobian matrix $M(z)$ is a diagonal matrix with diagonal entries

$$M_{ii}(z) \begin{cases} = 1, & |((2D-I)z + 2\beta)_i| > t, \\ = 0, & |((2D-I)z + 2\beta)_i| < t, \\ \in [0,1], & |((2D-I)z + 2\beta)_i| = t. \end{cases}$$

In particular, we can choose $M(z)$ such that $M_{ii}(z) = 1$ when $|((2D-I)z+2\beta)_i| = t$. Consequently, $\Phi(z)$ and $\Psi(z)$ are diagonal matrices whose diagonal entries are

$$\begin{cases} \Psi_{ii}(z) = 0, \quad \Phi_{ii}(z) = 1, & |((2D-I)z+2\beta)_i| \geq t, \\ \Psi_{ii}(z) = -1, \quad \Phi_{ii}(z) = -1, & |((2D-I)z+2\beta)_i| < t. \end{cases} \tag{6.12}$$

We compare the performance of DRS, DRS-LM (Algorithm 3) and DRS-Newton (Algorithm 1) on solving (6.11). The test matrix $A \in \mathbb{R}^{m\times n}$ is set to the Gaussian matrix whose elements are generated independent and identically distributed from the normal distribution. A sparse signal $x^* \in \mathbb{R}^n$ is generated with k nonzero components whose values are normally distributed. Then the right hand side b is constructed from $b = Ax^*$. In DRS-LM and DRS-Newton, we first perform the original DRS until the fixed-point residual $\|F(z)\|_2$ is less than 10^{-2}. Then the second-order type methods are activated. Figure 6.6 shows convergence history of $\|F(z)\|_2$ on $n = 1000$ and $m = 500$ with respect to $k = 50$ and 150. Both DRS-LM and DRS-Newton show quadratic or superlinear convergence rates as long as the initial point is good enough.



(a) $k = 50$ (b) $k = 150$

FIG. 6.6. *residual history of the ℓ_1-minimization problem* (6.11) *on $n = 1000$ and $m = 500$*

6.2.2. Linear Programming. Another case of (6.8) is the classic linear programming problem

$$\min_{x\in\mathbb{R}^n} \quad c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0. \tag{6.13}$$

Let $f(x) = c^T x + 1_K(x)$ where $K := \{x \mid x \geq 0\}$. The corresponding proximal mapping is

$$\mathbf{prox}_{tf}(y) = \mathcal{P}_K(y - c).$$

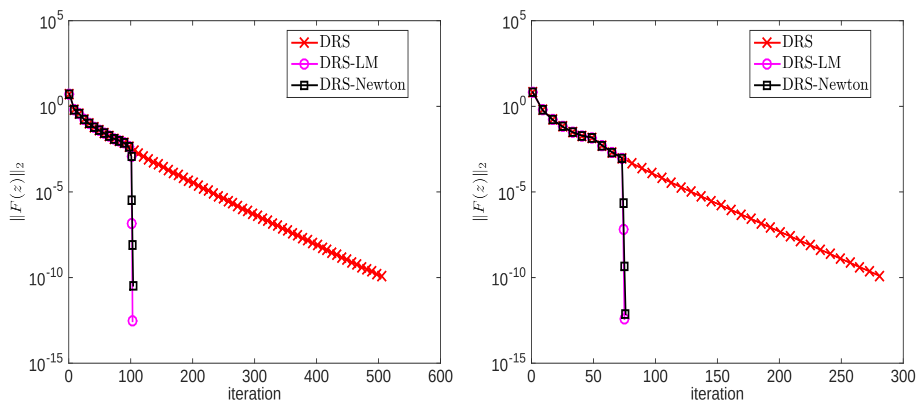
Every element of the generalized Jacobian $\partial\mathcal{P}_K$ at $(2D-I)z+\beta$ is a diagonal matrix with diagonal entries

$$M_{ii}(z) \begin{cases} = 1, & ((2D-I)z+\beta)_i > 0, \\ = 0, & ((2D-I)z+\beta)_i < 0, \\ \in [0,1], & ((2D-I)z+\beta)_i = 0. \end{cases}$$

In particular, we can choose $M(z)$ such that $M_{ii}(z) = 1$ when $((2D - I)z + \beta)_i = 0$. Consequently, we have

$$\begin{cases} \Psi_{ii}(z) = 0, \quad \Phi_{ii}(z) = 1, & ((2D - I)z + \beta)_i \geq 0, \\ \Psi_{ii}(z) = -1, \quad \Phi_{ii}(z) = -1, & ((2D - I)z + \beta)_i < 0. \end{cases} \tag{6.14}$$

We next compare the performance of DRS, DRS-LM and DRS-Newton on solving (6.13). The test matrix $A \in \mathbb{R}^{m \times n}$ is set to a random matrix whose elements are uniformly distributed in $(0, 1)$. The primal optimal solution $x^* \in \mathbb{R}^n$ is generated with $0.1n$ nonzero components whose values are normally distributed. Then the right hand side b is constructed from $b = Ax^*$ and the coefficient c is also randomly generated. In DRS-LM and DRS-Newton, we first perform the original DRS until the fixed-point residual $\|F(z)\|_2$ is less than 10^{-3}. Then the second-order type methods are activated. Figure 6.6 shows convergence history of $\|F(z)\|_2$ on $n = 1000$ with respect to $m = 300$ and 400. Again, both DRS-LM and DRS-Newton show quadratic or superlinear convergence rates as long as the initial point is good enough.



(a) $m = 300$ (b) $m = 400$

FIG. 6.7. *residual history of the LP problem* (6.13) *on* $n = 1000$

7. Conclusion. The purpose of this paper is to study second-order type methods for solving composite convex programs based on fixed-point mappings induced from many operator splitting approaches such as the FBS and DRS methods. The semi-smooth Newton method is theoretically guaranteed to converge to a global solution from an arbitrary initial point. The semi-smooth LM method converges quadratically as long as the initial point is close to the solution set enough. Our proposed algorithms are suitable to constrained convex programs when a fixed-point mapping is well-defined. Hence, this mechanism is a general paradigm. It may be able to bridge the gap between first-order and second-order type methods. They are indeed promising from our preliminary numerical experiments on a number of applications. In particular, quadratic or superlinear convergence is observable in some examples of Lasso regression, logistic regression, basis pursuit and linear programming.

There are a number of future directions worth pursuing from this point on, including theoretical analysis and a comprehensive implementation of these second-order algorithms. Since solving the corresponding system of linear equations is computationally dominant, it is important to explore the structure of the linear system and design certain suitable preconditioners. The performance on matrix optimization including semidefinite programming and low-rank optimization is also interesting. Although our current analysis is limited to the Euclidean space, the proposed algorithms can undoubtedly be generalized to the Hilbert space.

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