Computation of Graphical Derivative for a Class of Normal Cone Mappings under a Very Weak Condition

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Abstract. Let $\Gamma := \{x \in \mathbb{R}^n \,|\, q(x) \in \Theta\}$, where $q : \mathbb{R}^n \to \mathbb{R}^m$ is a twice continuously differentiable mapping, and Θ is a nonempty polyhedral convex set in $\mathbb{R}^m$. In this paper, we first establish a formula for exactly computing the graphical derivative of the normal cone mapping $N_\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $x \mapsto N_\Gamma(x)$, under the condition that $M_q(x) := q(x) - \Theta$ is metrically subregular at the reference point. Then, based on this formula, we exhibit formulae for computing the graphical derivative of solution mappings and present characterizations of the isolated calmness for a broad class of generalized equations. Finally, applying to optimization, we get a new result on the isolated calmness of stationary point mappings.

Key Words. Computation, Graphical Derivative, Normal Cone Mapping, Generalized Equation, Isolated Calmness

1 Introduction

Graphical derivative of a set-valued mapping at a point in its graph is the set-valued mapping whose graph is the (Bouligand-Severi) tangent/contingent cone to the graph of the given set-valued mapping at the reference point. This concept also known as the outer graphical derivative [24] was introduced by Aubin [1] who called it the contingent derivative. The terminology "graphical derivative" was used by Rockafellar and Wets [30], Dontchev and Rockafellar [10], and many other people.

The graphical derivative is a powerful tool in variational analysis and its applications [2, 10, 30]. One can use it to investigate the stability and sensitivity of constraint and variational systems, and more general, generalized equations [2, 10, 20, 21, 23, 24, 30], or even, characterize some nice properties of set-valued mappings, such as the metric regularity, the Aubin/Lipschitz-like property [2, 10, 11], the isolated calmness/the local upper Lipschitz property, the strong metric subregularity [10, 21, 23]. Also, its graph can play a mediate role in computing the dual derivative-like constructions [8, 15, 19]. Although it is the key in dealing with some important issues in variational analysis and its applications, unfortunately, computation of the graphical derivative of a set-valued mapping is generally a challenging task. The problem has studied by many researchers for a long time, and many interesting results in the direction have been established; see [2, 10, 17, 19, 21, 23, 24, 30, 31] and the references therein.

In this paper, we concern the computation and application of the graphical derivative DN_Γ of the normal cone mapping $N_\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $x \mapsto N_\Gamma(x)$, where $\Gamma := \{x \,|\, q(x) \in \Theta\}$ with $q : \mathbb{R}^n \to \mathbb{R}^m$

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being a twice continuously differentiable mapping, and Θ being a nonempty polyhedral convex set in $\mathbb{R}^m$. Note that normal cones keep certain first-order information on the underlying sets, the derivative as well as the coderivative of the normal cone mappings are therefore objects of second-order analysis. To our best knowledge, the first result closely related to the research conducted in this paper was due to Dontchev and Rockafellar [8], where the authors gave an exact description of the graph of DN_Γ under the condition that Γ is a polyhedral convex set, and then used it to compute the generalized Hessian/the limiting second-order subdifferential [26] of the indicator function of Γ. The latter was a crucial step in characterizing the strong regularity for variational inequalities over polyhedral convex sets in [8]. Henrion et al. [17, Remark 3.1] indicated that a formula for computing the DN_Γ can be obtained by [30, Corollary 13.43 & Exercise 13.17], provided $M_q(x) := q(x)-\Theta$ is metrically regular around the reference point. Moreover, if $\Theta := \mathbb{R}^m_-$ and both the Magasarian-Fromovitz and the constant rank constraint qualifications are satisfied, then this formula can be much more simplified [17, Theorems 3.1 & 3.2]. Gfrerer and Outrata [14, Theorem 1] proved the formula holds if $\Theta := \mathbb{R}^\ell_-$ and the metric regularity is replaced by the metric subregularity at the reference point plus a uniform metric regularity around this point. Recently, in the case $\Theta := \{0_{\mathbb{R}^{m_1}}\}\times\mathbb{R}^{m-m_1}_-$ and under the metric subregularity, Gfrerer and Mordukhovich [15, Theorem 5.1] showed that the result is still valid if the local uniform metric regularity is relaxed to the so-called bounded extreme point property. For more information on recent results in this direction as well as related issues, we refer the reader to [10, 13, 14, 15, 17, 19, 20, 27, 28, 29] and the references therein. Among other things, the motivation of these studies came from the applications to investigating isolated calmness for generalized equations [14, 17] and tilt stability in optimization [15].

The aim of this paper is to extend and unify the above mentioned results on computation of the graphical derivative DN_Γ, and thereby obtain some new results on applications of the graphical derivative. More precisely, we first establish a formula for exactly computing the graphical derivative of the normal cone mapping $N_\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $x \mapsto N_\Gamma(x)$, under the condition that $M_q(x) := q(x) - \Theta$ is metrically subregular at the reference point. Then, based on this formula, we exhibit formulae for computing the graphical derivative of solution mappings and present characterizations of the isolated calmness for a broad class of generalized equations. Finally, applying to optimization, we get a new result on the isolated calmness of stationary point mappings.

The structure of the paper is as follows. After recalling some basic notions and needed properties from variational analysis in the next section, we present the formula for computing the graphical derivative of the normal cone mappings in Section 3. Then, in Section 4, we show how to use this formula to compute the graphical derivative of solution mappings as well as derive the new results on the isolated calmness for generalized equations and stationary point mappings. Finally, we conclude the paper in Section 5 where we give some remarks on the perspective of the obtained results.

2 Preliminaries

In this paper, all spaces are assumed to be Euclidean spaces with scalar product $\langle\cdot,\cdot\rangle$ and Euclidean norm $\|\cdot\|$. For a given set $C \subset \mathbb{R}^n$, the distance from $x \in \mathbb{R}^n$ to C is denoted by $d_C(x)$, that is, $d_C(x) := \inf\big\{\|x-u\| \,|\, u \in C\big\}$, and $\mathrm{pos}C := \big\{\sum_{i=1}^{k} \lambda_i c_i \,|\, \lambda_i \geq 0,\, c_i \in C \cup \{0\},\, i = 1,\ldots k,\, k \in \mathbb{N}\big\}$, $C^\perp := \big\{u \in \mathbb{R}^n \,|\, \langle u, x\rangle = 0 \text{ for all } x \in C\big\}$, and $C^0 := \big\{u \in \mathbb{R}^n \,|\, \langle u, x\rangle \leq 0 \text{ for all } x \in C\big\}$. The closed ball with centre $\bar{x}$ and radius $r > 0$ is denoted by $\mathbb{B}_r(\bar{x})$, and put $\mathbb{B} := \mathbb{B}_1(0)$. As usual, we use the little-o notation as a Landau symbol, that is, for $x \to \bar{x}$, one has $\alpha = o(\beta)$ if and only if $\lim\limits_{x\to\bar{x}} \alpha(x) = 0$, $\lim\limits_{x\to\bar{x}} \beta(x) = 0$, and $\lim\limits_{x\to\bar{x}} \frac{\alpha(x)}{\beta(x)} = 0$. For $\alpha \in \mathbb{R}$, one puts $[\alpha]_+ := \max\{\alpha, 0\}$.

Below are basic notions and facts from variational analysis, which are frequently used in the sequel; see [26, 30] for more details.

Definition 2.1. ([30, Chapter 6]). Let Ω be a nonempty subset of $\mathbb{R}^n$.

(*i*) The (Bouligand-Severi) *tangent/contingent cone* to the set Ω at $\bar{x} \in \Omega$ is defined by

$$T_\Omega(\bar{x}) := \big\{v \in \mathbb{R}^n \,|\, \text{there exist } t_k \downarrow 0,\ v_k \to v \ \text{ with } \ \bar{x} + t_k v_k \in \Omega \text{ for all } k \in \mathbb{N}\big\}.$$

(*ii*) The (Fréchet) *regular normal cone* to Ω at $\bar{x} \in \Omega$ is the set $\widehat{N}_C(\bar{x})$ given by

$$\widehat{N}_\Omega(\bar{x}) := \Big\{v \in \mathbb{R}^n \,|\, \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le 0\Big\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$.

(*iii*) The (Mordukhovich) *limiting/basic normal cone* to Ω at $\bar{x} \in \Omega$ is the set $N_\Omega(\bar{x})$ defined by

$$N_\Omega(\bar{x}) = \big\{v \in \mathbb{R}^n \,|\, \text{there exist } x_k \to \bar{x},\ v_k \in \widehat{N}_\Omega(x_k) \text{ with } v_k \to v\big\}.$$

If $\bar{x} \notin \Omega$, put $N_\Omega(\bar{x}) = \widehat{N}_\Omega(\bar{x}) := \emptyset$ by convention.

It is known [26, Theorem 1.10] that the regular normal cone is the dual of the tangent cone:

$$\widehat{N}_\Omega(\bar{x}) = \{v \in \mathbb{R}^n \,|\, \langle v, u\rangle \le 0 \ \text{ for all } u \in T_\Omega(\bar{x})\big\}. \tag{2.1}$$

If Ω is convex, the above tangent cone and normal cones reduce to the tangent cone and normal cone in the sense of convex analysis.

Definition 2.2. (see [26, 30]). Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with its graph $\mathrm{gph}\Phi := \big\{(x, y) \,|\, y \in \Phi(x)\big\}$ and its domain $\mathrm{Dom}\Phi := \big\{x \,|\, \Phi(x) \neq \emptyset\big\}$.

(*i*) Given a point $\bar{x} \in \mathrm{Dom}\Phi$, the *graphical derivative* of Φ at $\bar{x}$ for $\bar{y} \in \Phi(\bar{x})$ is the set-valued mapping $D\Phi(\bar{x}|\bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$D\Phi(\bar{x}|\bar{y})(v) := \big\{w \in \mathbb{R}^m \,|\, (v, w) \in T_{\mathrm{gph}\Phi}(\bar{x}, \bar{y})\big\} \ \text{ for all } v \in \mathbb{R}^n,$$

that is, $\mathrm{gph}D\Phi(\bar{x}|\bar{y}) := T_{\mathrm{gph}\Phi}(\bar{x}, \bar{y})$.

(*ii*) The *regular coderivative* of Φ at a given point $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ is the set-valued mapping $\widehat{D}^*\Phi(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$\widehat{D}^*\Phi(\bar{x}, \bar{y})(y^*) := \big\{x^* \in \mathbb{R}^n \,|\, (x^*, -y^*) \in \widehat{N}_{\mathrm{gph}\Phi}(\bar{x}, \bar{y})\big\} \ \text{ for all } y^* \in \mathbb{R}^m.$$

In the case $\Phi(\bar{x}) = \{\bar{y}\}$, one writes $D\Phi(\bar{x})$ and $\widehat{D}^*\Phi(\bar{x})$ for $D\Phi(\bar{x}|\bar{y})$ and $\widehat{D}^*\Phi(\bar{x}, \bar{y})$, respectively.

We note that if $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a single-valued mapping that is differentiable at $\bar{x}$, then $D\Phi(\bar{x}) = \nabla\Phi(\bar{x})$ and $\widehat{D}^*\Phi(\bar{x}) = \nabla\Phi(\bar{x})^*$.

Definition 2.3. ([26, 30]). Let $\varphi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in \mathbb{R}^n$ with $\bar{y} := \varphi(\bar{x})$ finite.

(*i*) The *regular subdifferential* (known also as the presubdifferential and as the Fréchet/viscosity subdifferential) of φ at $\bar{x}$ is defined by

$$\widehat{\partial}\varphi(\bar{x}) := \big\{x^* \in \mathbb{R}^n \,|\, (x^*, -1) \in \widehat{N}_{\mathrm{epi}\varphi}(\bar{x}, \bar{y})\big\},$$

where $\mathrm{epi}\varphi := \big\{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \,|\, \alpha \ge \varphi(x)\big\}$ is the *epigraph* of φ.

(*ii*) The *limiting subdifferential* (known also as the Mordukhovich/basic subdifferntial) of φ at $\bar{x}$ is defined by

$$\partial\varphi(\bar{x}) := \big\{x^* \in \mathbb{R}^n \,|\, (x^*, -1) \in N_{\mathrm{epi}\varphi}(\bar{x}, \bar{y})\big\}.$$

If $|\varphi(\bar{x})| = \infty$ then put $\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x}) := \emptyset$ by convention.

Note that $\widehat{\partial}\varphi(\bar{x}) \subset \partial\varphi(\bar{x})$ and if φ is a convex function then both $\widehat{\partial}\varphi(\bar{x})$ and $\partial\varphi(\bar{x})$ coincide with the subdifferential in the sense of convex analysis:

$$\widehat{\partial}\varphi(\bar{x}) = \partial\varphi(\bar{x}) = \big\{x \in \mathbb{R}^n \,|\, \langle x^*, x-\bar{x}\rangle \le \varphi(x) - \varphi(\bar{x}) \text{ for all } x \in \mathbb{R}^n\big\}.$$

In addition, it is not difficult to see that for $\bar{x} \in \mathbb{R}^n$ with $\varphi(\bar{x}) \in \mathbb{R}$, one has $x^* \in \widehat{\partial}\varphi(\bar{x})$ if and only if

$$\liminf_{x\to\bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \ge 0.$$

Recall [10, Section 3.8] that a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be metrically subregular at $\bar{x}$ for $\bar{y}$ if $(\bar{x}, \bar{y}) \in \mathrm{gph}\Phi$ and there exist $\kappa, r > 0$ such that

$$d\big(x, \Phi^{-1}(\bar{y})\big) \le \kappa d\big(\bar{y}, \Phi(x)\big) \text{ for all } x \in \mathbb{B}_r(\bar{x}).$$

Using this property, Gfrerer and Mordukhovich [15] introduced the following constraint qualification in the nonlinear programming setting.

Definition 2.4. ([15, Definition 3.2]). Consider the constraint set

$$\Gamma := \{x \in \mathbb{R}^n \,|\, q(x) \in \Theta\},$$

where $q : \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable mapping and Θ is a nonempty closed set in $\mathbb{R}^m$. One says that the *metric subregularity constraint qualification* (MSCQ) holds at $\bar{x} \in \Gamma$ if $M_q(x) := q(x) - \Theta$ is metrically subregular at $\bar{x}$ for 0.

In fact, if Γ is the constraint set of a nonlinear programming, then the validity of MSCQ amounts to the existence of a local error bound [18], which is weaker than most known constraint qualifications [3, 4, 5, 16, 22, 25]. Furthermore, from definition we see that MSCQ is fulfilled at every $x \in \Gamma$ near $\bar{x}$ whenever it holds at $\bar{x} \in \Gamma$.

Combining [12, Proposition 3.4] and [26, Corollary 1.15], we are able to get the following formula for computing the normal cones to inverse sets.

Lemma 2.5. *Let $q : \mathbb{R}^n \to \mathbb{R}^m$ be a twice continuously differentiable mapping, and let Θ be a nonempty closed convex set in $\mathbb{R}^m$ and $\bar{x} \in \Gamma := \{x \,|\, q(x) \in \Theta\}$. Suppose that MSCQ holds at $\bar{x}$. Then, there exists $\delta > 0$ such that*

$$N_\Gamma(x) = \widehat{N}_\Gamma(x) = \nabla q(x)^* N_\Theta(y) \ \text{ for all } x \in \Gamma \cap \mathbb{B}_\delta(\bar{x}) \text{ and } y := q(x).$$

3 Graphical Derivative of Normal Cone Mapping

From now on, we assume that $\Gamma := \{x \,|\, q(x) \in \Theta\}$, where $q : \mathbb{R}^n \to \mathbb{R}^m$ is a twice continuously differentiable mapping and $\Theta := \{y \in \mathbb{R}^m \,|\, \langle b_i, y\rangle \le \alpha_i,\ i = 1, 2, ..., \ell\}$, with $b_i \in \mathbb{R}^m\backslash\{0\}$ and $\alpha_i \in \mathbb{R}$. Fix $\bar{x} \in \Gamma$ and $\bar{x}^* \in N_\Gamma(\bar{x})$, and put $\overline{\Lambda} := \{\lambda \in N_\Theta(\bar{y}) \,|\, \nabla q(\bar{x})^T\lambda = \bar{x}^*\}$, $\bar{y} := q(\bar{x})$, and $I_q(\bar{x}) := \{i = 1, 2, \dots, \ell \,|\, \langle b_i, \bar{y}\rangle = \alpha_i\}$, and $\overline{K} := T_\Gamma(\bar{x}) \cap \{\bar{x}^*\}^\perp$.

Since Θ is a nonempty polyhedral convex set, by Lemma 2.5, if MSCQ holds at $\bar{x}$, then $\overline{\Lambda}$ is a nonempty polyhedral convex set. In this case, for each $v \in \overline{K}$, the problem LP(v)

$$\begin{array}{ll} \min & -v^T\nabla^2\big(\lambda^T q\big)(\bar{x})v \\ \text{subject to} & \lambda \in \overline{\Lambda} \end{array}$$

is a linear programming with its dual program DP(v)

$$\begin{array}{ll} \max & \langle \bar{x}^*, z\rangle \\ \text{subject to} & \nabla q(\bar{x})z + v^T\nabla^2 q(\bar{x})v \in T_\Theta(\bar{y}), \end{array}$$

where $T_\Theta(\bar y)$ can be computed by $T_\Theta(\bar y) = \{w \in \mathbb{R}^m \,|\, \langle b_i, w\rangle \le 0 \text{ for all } i \in I_q(\bar x)\}$; see [6, p.126].

To proceed, we need the following auxiliary result.

Lemma 3.1. *Suppose that MSCQ is valid at $\bar x$ and $\bar y := q(\bar x)$. Then, for each $v \in \overline{K}$ and $\lambda \in \overline{\Lambda}$, one has*

$$\widehat{N}_{\overline{K}}(v) = \big\{\nabla q(\bar x)^T \mu \,|\, \mu^T \nabla q(\bar x) v = 0,\ \mu \in T_{N_\Theta(\bar y)}(\lambda)\big\}, \tag{3.1}$$

where $N_\Theta(\bar y) = \mathrm{pos}\{b_i \,|\, i \in I_q(\bar x)\}$, $I_q(\bar x) := \big\{i = 1, 2, \ldots, \ell \,|\, \langle b_i, \bar y\rangle = \alpha_i\big\}$, and

$$T_{N_\Theta(\bar y)}(\lambda) = \mathrm{pos}\{b_i \,|\, i \in I_q(\bar x)\} - \mathbb{R}_+\lambda.$$

Consequently, for $v \in \overline{K}$, one has

$$\widehat{N}_{\overline{K}}(v) = \left\{ \sum_{i \in I_q(\bar x)} t_i b_i^T \nabla q(\bar x) - t_0 \bar x^* \,|\, t_0, t_i \in \mathbb{R}_+,\ i \in I_q(\bar x) \right\} \cap \{v\}^\perp. \tag{3.2}$$

Proof. We first prove the following inclusion holds:

$$\widehat{N}_{\overline{K}}(v) \subset \big\{\nabla q(\bar x)^T \mu \,|\, \mu^T \nabla q(\bar x) v = 0,\ \mu \in T_{N_\Theta(\bar y)}(\lambda)\big\}. \tag{3.3}$$

Take any $v^* \in \widehat{N}_{\overline{K}}(v)$. Since $\widehat{N}_{\overline{K}}(v) = \overline{K}^0 \cap \{v\}^\perp = \big(\nabla q(\bar x)^T N_\Theta(\bar y) + \mathbb{R}\bar x^*\big) \cap \{v\}^\perp$, there exist $\tilde\mu \in N_\Theta(\bar y)$ and $\alpha \in \mathbb{R}$ such that $v^* = \nabla q(\bar x)^T(\tilde\mu) + \alpha \bar x^* = \nabla q(\bar x)^T(\tilde\mu + \alpha\lambda)$. For $\mu := \tilde\mu + \alpha\lambda$, noting that $\lambda, \tilde\mu \in N_\Theta\big(\bar y\big)$, we have $\lambda + k^{-1}\mu = (1 + k^{-1}\alpha)\lambda + k^{-1}\tilde\mu \in N_\Theta(\bar y)$ for all $k \in \mathbb{N}$. The latter implies $\mu \in T_{N_\Theta(\bar y)}(\lambda)$. This together with $v^* = \nabla q(\bar x)^T \mu$ and $\mu^T \nabla q(\bar x) v = \langle v^*, v\rangle = 0$ shows that (3.3) holds. We next justify the converse inclusion. Take an arbitrary $\mu \in T_{N_\Theta(\bar y)}(\lambda)$ with $\mu^T \nabla q(\bar x) v = 0$. Then, we have $\lambda + t\mu \in N_\Theta(\bar y)$ for some $t > 0$. So, by Lemma 2.5, it holds $\nabla q(\bar x)^T(\lambda + t\mu) \in N_\Gamma(\bar x) = T_\Gamma(\bar x)^0$. Hence,

$$0 \ge \langle \nabla q(\bar x)^T(\lambda + t\mu), w\rangle = \langle \bar x^*, w\rangle + t\mu^T \nabla q(\bar x) w = t\mu^T \nabla q(\bar x) w,$$

for all $w \in \overline{K}$, which guarantees $\nabla q(\bar x)^T \mu \in \overline{K}^0$. This together with $\mu^T \nabla q(\bar x) v = 0$ implies $\nabla q(\bar x)^T \mu \in \widehat{N}_{\overline{K}}(v)$. So, equality (3.1) has been justified. Finally, it is not difficult to see that (3.2) is a straightforward consequence of (3.1). □

We now arrive at the main result of this section.

Theorem 3.2. *Let MSCQ be satisfied at $\bar x \in \Gamma$, and $\bar x^* \in N_\Gamma(\bar x)$. Then, one has*

$$T_{\mathrm{gph} N_\Gamma}(\bar x, \bar x^*) = \big\{(v, v^*) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \exists \lambda \in \overline{\Lambda}(v) \,:\, v^* \in \nabla^2\big(\lambda^T q\big)(\bar x) v + \widehat{N}_{\overline{K}}(v)\big\}. \tag{3.4}$$

In other words, the graphical derivative of the normal cone mapping $x \mapsto N_\Gamma(x)$ is given by

$$D N_\Gamma(\bar x | \bar x^*)(v) = \big\{\nabla^2\big(\lambda^T q\big)(\bar x) v \,|\, \lambda \in \overline{\Lambda}(v)\big\} + \widehat{N}_{\overline{K}}(v). \tag{3.5}$$

Here $\overline{\Lambda}(v)$ is the optimal solution set of the linear programming LP(v), *and the cone $\widehat{N}_{\overline{K}}(v)$ can be computed by* (3.2).

Proof. We first justify the inclusion

$$\big\{(v, v^*) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \exists \lambda \in \overline{\Lambda}(v) \,:\, v^* \in \nabla^2\big(\lambda^T q\big)(\bar x) v + \widehat{N}_{\overline{K}}(v)\big\} \subset T_{\mathrm{gphN}_\Gamma}(\bar x, \bar x^*). \tag{3.6}$$

Take any $(v, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$ with $v^* \in \nabla^2\big(\lambda^T q\big)(\bar x) v + \widehat{N}_{\overline{K}}(v)$ for some $\lambda \in \overline{\Lambda}(v)$. Let $\alpha > 0$ satisfy $\alpha \|\nabla^2\big(\lambda^T q\big)(\bar x)\| < 2^{-1}$. Then $2I + \nabla^2\big((\alpha\lambda)^T q\big)(\bar x)$ is positively definite. Since Θ is a polyhedral convex set, by [6, Theorem 3.70], there exists $r > 0$ such that $\bar{x}$ is the unique global solution of the problem (P):

$$\min \|\bar{x} + \alpha\bar{x}^* - x\|^2 \text{ subject to } x \in \Gamma \cap \mathbb{B}_r(\bar{x}).$$

Since λ is an optimal solution to $LP(v)$, the optimal solution set to $DP(v)$ is nonempty. Moreover, by [6, Proposition 2.191], for any optimal solution z to $DP(v)$, we have

$$\big\langle \lambda, \nabla q(\bar{x})z + v^T\nabla^2 q(\bar{x})v\big\rangle = 0.$$

From $v^* \in \nabla^2\big(\lambda^T q\big)(\bar{x})v + \widehat{N}_{\overline{K}}(v)$, by Lemma 3.1, it follows that $v^* = \nabla^2\big(\lambda^T q\big)(\bar{x})v + \nabla q(\bar{x})^T\mu$, for some $\mu \in T_{N_\Theta(\bar{y})}(\lambda) \cap \big\{\nabla q(\bar{x})v\big\}^\perp$, where $\bar{y} := q(\bar{x})$. Thus

$$\langle v^*, v\rangle = v^T\nabla^2\big(\lambda^T q\big)(\bar{x})v + \mu^T\nabla q(\bar{x})v = v^T\nabla^2\big(\lambda^T q\big)(\bar{x})v.$$

Since Θ is a polyhedral convex set, $\lambda \in N_\Theta(\bar{y})$, and $\mu \in T_{N_\Theta(\bar{y})}(\lambda)$, there exists $\bar{t} > 0$ satisfying $\lambda + t\mu \in N_\Theta(\bar{y})$ for all $t \in (0, \bar{t})$. Furthermore, the $\bar{t}$ can be chosen such that

$$\big\langle q(\bar{x} + tv + \frac{1}{2}t^2 z), b_i\big\rangle < \alpha_i \text{ for all } i \notin I_q(\bar{x}). \tag{3.7}$$

For each $t \in (0, \bar{t})$, pick an arbitrary global solution x_t to the problem (P_t):

$$\min \|\bar{x} + tv + \frac{1}{2}t^2 z + \alpha(\bar{x}^* + tv^*) - x\|^2 \text{ subject to } x \in \Gamma \cap \mathbb{B}_r(\bar{x}).$$

Then $x_t \to \bar{x}$ as $t \downarrow 0$. Indeed, if this is not true, there exist $t_k \downarrow 0$ and $\big\{x_{t_k}\big\} \subset \Gamma \cap \mathbb{B}_r(\bar{x})$ converging to some $x_0 \in \Gamma \cap \mathbb{B}_r(\bar{x})$ with $x_0 \neq \bar{x}$. Since x_{t_k} is a global solution to (P_{t_k}), it holds

$$\|\bar{x} + t_k v + \frac{1}{2}t_k^2 z + \alpha(\bar{x}^* + t_k v^*) - x\|^2 \geq \|\bar{x} + t_k v + \frac{1}{2}t_k^2 z + \alpha(\bar{x}^* + t_k v^*) - x_{t_k}\|^2,$$

for all $x \in \Gamma \cap \mathbb{B}_r(\bar{x})$. Letting $k \to \infty$, we get

$$\|\bar{x} + \alpha\bar{x}^* - x\|^2 \geq \|\bar{x} + \alpha\bar{x}^* - x_0\|^2 \text{ for all } x \in \Gamma \cap \mathbb{B}_r(\bar{x}),$$

which shows that x_0 is a global solution to (P). This contradicts the global solution uniqueness of the problem (P). So $\lim\limits_{t\downarrow 0} x_t = \bar{x}$ and hence we can consider $x_t \in \text{int}\mathbb{B}_r(\bar{x})$ for all $t \in (0, \bar{t})$. According to the first order optimality condition [30, Theorem 10.1],

$$\alpha(\bar{x}^* + tv^*) + t\big(\frac{1}{t}(\bar{x} + tv - x_t) + \frac{1}{2}tz\big) \in N_\Gamma(x_t) \text{ for all } t \in (0, \bar{t}).$$

The latter implies

$$\lim_{t\downarrow 0} \frac{d_{N_\Gamma(x_t)}(\bar{x} + tv^*)}{t} = 0, \tag{3.8}$$

provided

$$\lim_{t\downarrow 0} \frac{1}{t}(\bar{x} + tv - x_t) = 0. \tag{3.9}$$

Moreover, (3.8) and (3.9) together guarantee $(v, v^*) \in T_{\text{gph}N_\Gamma}(\bar{x}, \bar{x}^*)$. So, we next prove (3.9) holds. Since $v \in \overline{K}$ and MSCQ is fulfilled at $\bar{x}$, we have $\nabla q(\bar{x})v \in T_\Theta(\bar{y})$, that is,

$$\big\langle \nabla q(\bar{x})v, b_i\big\rangle \leq 0 \text{ for all } i \in I_q(\bar{x}). \tag{3.10}$$

On the other hand, since z is feasible for the dual problem $DP(v)$,

$$\langle \nabla q(\bar{x})z, b_i\rangle + \langle v^T\nabla^2 q(\bar{x})v, b_i\rangle \le 0 \text{ for all } i \in I_q(\bar{x}). \tag{3.11}$$

Thus, from (3.10) and (3.11) it follows that

$$\begin{aligned}\langle q(\bar{x}+tv+\tfrac{1}{2}t^2 z), b_i\rangle &= \langle q(\bar{x}), b_i\rangle + t\langle \nabla q(\bar{x})(v), b_i\rangle \\ &\quad +\tfrac{1}{2}t^2\big(\langle \nabla q(\bar{x})z, b_i\rangle + \langle v^T\nabla^2 q(\bar{x})v, b_i\rangle\big) + o(t^2) \qquad (3.12)\\ &\le \alpha_i + o(t^2),\end{aligned}$$

for all $i \in I_q(\bar{x})$ and $t \in (0,\bar{t})$. Since M_q is metrically subregular at $\bar{x}$ for 0, we can assume, for some $\kappa_1 > 0$,

$$d_\Gamma(\bar{x}+tv+\frac{1}{2}t^2 z) \le \kappa_1 d_\Theta\big(q(\bar{x}+tv+\frac{1}{2}t^2 z)\big) \text{ for all } t \in (0,\bar{t}).$$

Note that $\Theta := \{y \in \mathbb{R}^m \mid \langle b_i, y\rangle \le \alpha_i,\ i = 1,2,...,\ell\}$. By Hoffman's lemma [6, Theorem 2.200], there exists $\kappa_2 > 0$ such that

$$d_\Theta\big(q(\bar{x}+tv+\frac{1}{2}t^2 z)\big) \le \kappa_2 \sum_{i=1}^{\ell} \Big[\big\langle b_i, q(\bar{x}+tv+\frac{1}{2}t^2 z)\big\rangle - \alpha_i\Big]_+ \text{ for all } t \in (0,\bar{t}). \tag{3.13}$$

Combining (3.7) and (3.12)-(3.13), we get

$$d\big(\bar{x}+tv+\frac{1}{2}t^2 z, \Gamma\big) \le o(t^2) \text{ for all } t \in (0,\bar{t}).$$

This implies that, for each $t \in (0,\bar{t})$, there exists $\tilde{x}_t \in \Gamma$ such that

$$\|\bar{x}+tv+\frac{1}{2}t^2 z - \tilde{x}_t\| = o(t^2), \tag{3.14}$$

which ensures that $\lim_{t\downarrow 0}\tilde{x}_t = \bar{x}$. Choosing the $\bar{t}$ smaller if necessary, we assume $\tilde{x}_t \in \Gamma \cap \mathbb{B}_r(\bar{x})$ for all $t \in (0,\bar{t})$. Since x_t is a global solution to (P_t), we have

$$\|\bar{x}+tv+\frac{1}{2}t^2 z + \alpha\big(\bar{x}^*+tv^*\big) - x_t\|^2 \le \|\bar{x}+tv+\frac{1}{2}t^2 z + \alpha\big(\bar{x}^*+tv^*\big) - \tilde{x}_t\|^2,$$

and consequently,

$$\begin{aligned}&\|\bar{x}+tv+\tfrac{1}{2}t^2 z - x_t\|^2 + 2\alpha\langle \bar{x}^*+tv^*, \bar{x}+tv+\tfrac{1}{2}t^2 z - x_t\rangle\\ &\le \|\bar{x}+tv+\tfrac{1}{2}t^2 z - \tilde{x}_t\|^2 + 2\alpha\langle \bar{x}^*+tv^*, \bar{x}+tv+\tfrac{1}{2}t^2 z - \tilde{x}_t\rangle\\ &\le \|\bar{x}+tv+\tfrac{1}{2}t^2 z - \tilde{x}_t\|^2 + 2\alpha\|\bar{x}^*+tv^*\|\|\bar{x}+tv+\tfrac{1}{2}t^2 z - \tilde{x}_t\|.\end{aligned}$$

Combining this with (3.14), we arrive at

$$\|\bar{x}+tv+\frac{1}{2}t^2 z - x_t\|^2 + 2\alpha\langle \bar{x}^*+tv^*, \bar{x}+tv+\frac{1}{2}t^2 z - x_t\rangle \le o(t^2). \tag{3.15}$$

Since $\langle b_i, q(x_t) - q(\bar{x})\rangle \le 0$ for all $i \in I_q(\bar{x})$, it holds $q(x_t) - q(\bar{x}) \in T_\Theta(\bar{y})$. So, taking into account that $\lambda + t\mu \in N_\Theta(\bar{y})$ and $\nabla q(\bar{x})^T\mu = v^* - \nabla^2(\lambda^T q)(\bar{x})v$, we have

$$\begin{aligned} 0 &\ge \big\langle \lambda + t\mu, q(x_t) - q(\bar{x})\big\rangle \\ &= \big\langle \lambda + t\mu, \nabla q(\bar{x})(x_t - \bar{x}) + \tfrac{1}{2}(x_t - \bar{x})^T \nabla^2 q(\bar{x})(x_t - \bar{x})\big\rangle + o(\|x_t - \bar{x}\|^2) \\ &= \big\langle \nabla q(\bar{x})^T\lambda + t\nabla q(\bar{x})^T\mu, x_t - \bar{x}\big\rangle \\ &\quad + \tfrac{1}{2}(x_t - \bar{x})^T \nabla^2\big(\lambda^T q\big)(\bar{x})(x_t - \bar{x}) + o(\|x_t - \bar{x}\|^2) \\ &= \langle \bar{x}^* + tv^*, x_t - \bar{x}\rangle - t\langle \nabla^2(\lambda^T q)(\bar{x})v, x_t - \bar{x}\rangle \\ &\quad + \tfrac{1}{2}(x_t - \bar{x})^T \nabla^2\big(\lambda^T q\big)(\bar{x})(x_t - \bar{x}) + o(\|x_t - \bar{x}\|^2), \end{aligned}$$

showing that

$$-\langle \bar{x}^* + tv^*, x_t - \bar{x}\rangle \ge -t\langle \nabla^2(\lambda^T q)(\bar{x})v, x_t - \bar{x}\rangle + \frac{1}{2}(x_t - \bar{x})^T \nabla^2\big(\lambda^T q\big)(\bar{x})(x_t - \bar{x}) + o(\|x_t - \bar{x}\|^2).$$

Hence, using the fact that $\langle \lambda, \nabla q(\bar{x})z + v^T\nabla^2 q(\bar{x})v\rangle = 0$, $\langle v^*, v\rangle = v^T\nabla^2(\lambda^T q)(\bar{x})v$ and $\langle \bar{x}^*, v\rangle = 0$, we get the estimate:

$$\begin{aligned} &\langle \bar{x}^* + tv^*, \bar{x} + tv + \tfrac{t^2}{2}z - x_t\rangle = -\langle \bar{x}^* + tv^*, x_t - \bar{x}\rangle + \tfrac{t^2}{2}\langle \bar{x}^*, z\rangle + t^2\langle v^*, v\rangle + o(t^2) \\ &\ge -t\langle \nabla^2(\lambda^T q)(\bar{x})v, x_t - \bar{x}\rangle + \tfrac{1}{2}(x_t - \bar{x})^T \nabla^2\big(\lambda^T q\big)(\bar{x})(x_t - \bar{x}) + \tfrac{1}{2}t^2\langle \nabla q(\bar{x})^T\lambda, z\rangle \\ &\quad + t^2 v^T\nabla^2(\lambda^T q)(\bar{x})v + o(t^2) + o(\|x_t - \bar{x}\|^2) \\ &= -t\langle \nabla^2(\lambda^T q)(\bar{x})v, x_t - \bar{x}\rangle + \tfrac{1}{2}(x_t - \bar{x})^T \nabla^2\big(\lambda^T q\big)(\bar{x})(x_t - \bar{x}) \\ &\quad - \tfrac{1}{2}t^2 v^T\nabla^2(\lambda^T q)(\bar{x})v + o(t^2) + o(\|x_t - \bar{x}\|^2) \\ &= -\tfrac{1}{2}(\bar{x} + tv - x_t)^T \nabla^2(\lambda^T q)(\bar{x})(\bar{x} + tv - x_t) + o(t^2) + o(\|x_t - \bar{x}\|^2), \end{aligned}$$

which implies that

$$\begin{aligned} \alpha\langle \bar{x}^* + tv^*, \bar{x} + tv + \tfrac{t^2}{2}z - x_t\rangle &\ge -\tfrac{1}{2}\alpha(\bar{x} + tv - x_t)^T \nabla^2(\lambda^T q)(\bar{x})(\bar{x} + tv - x_t) \\ &\qquad + o(t^2) + o(\|x_t - \bar{x}\|^2) \\ &\ge -\tfrac{1}{2}\alpha\|\nabla^2(\lambda^T q)(\bar{x})\|\|\bar{x} + tv - x_t\|^2 + o(t^2) + o(\|x_t - \bar{x}\|^2) \\ &\ge -\tfrac{1}{4}\|\bar{x} + tv - x_t\|^2 + o(t^2) + o(\|x_t - \bar{x}\|^2). \end{aligned}$$

The latter is due to $\alpha\|\nabla^2\big(\lambda^T q\big)(\bar{x})\| < 2^{-1}$. Consequently, by (3.15), we have

$$\begin{aligned} &\tfrac{1}{2}\|\bar{x} + tv - x_t\|^2 + \tfrac{1}{4}t^4\|z\|^2 + t^3\langle z, v\rangle + t^2\langle z, \bar{x} - x_t\rangle \\ &= \|\bar{x} + tv + \tfrac{1}{2}t^2 z - x_t\|^2 - \tfrac{1}{2}\|\bar{x} + tv - x_t\|^2 \le \|\bar{x} + tv + \tfrac{1}{2}t^2 z - x_t\|^2 \\ &\qquad + 2\alpha\langle \bar{x}^* + tv^*, \bar{x} + tv + \tfrac{t^2}{2}z - x_t\rangle + o(t^2) + o(\|x_t - \bar{x}\|^2) \\ &\le o(t^2) + o(\|x_t - \bar{x}\|^2). \end{aligned}$$

Hence, the following estimate holds

$$\|\bar{x} + tv - x_t\|^2 \le o(t^2) + o(\|x_t - \bar{x}\|^2). \tag{3.16}$$

This allows us to choose $\tilde{t} > 0$ such that, for each $t \in (0, \tilde{t})$,

$$\|\bar{x} + tv - x_t\|^2 \le \frac{1}{2}(t^2 + \|x_t - \bar{x}\|^2),$$

or equivalently,

$$\|\bar{x} - x_t\|^2 + 2t^2\|v\|^2 + 4t\langle v, \bar{x} - x_t\rangle \le t^2.$$

The latter guarantees the validity of the following estimate

$$\begin{aligned}\|\bar{x} - x_t\| - 2t\|v\| &\le \|\bar{x} + 2tv - x_t\| \\ &= \sqrt{\|\bar{x} - x_t\|^2 + 2t^2\|v\|^2 + 4t\langle v, \bar{x} - x_t\rangle + 2t^2\|v\|^2} \le t\sqrt{1 + 2\|v\|^2},\end{aligned}$$

for all $t \in (0, \tilde{t})$. Thus, we have

$$\|\bar{x} - x_t\| \le t(2\|v\| + \sqrt{1 + 2\|v\|^2}) \ \text{ for all } t \in (0, \tilde{t}).$$

Combining this with (3.16) yields

$$\|\bar{x} + tv - x_t\| = o(t),$$

that is, (3.9) is valid. By (3.8) and (3.9), we get $(v, v^*) \in T_{\mathrm{gph}N_\Gamma}(\bar{x}, \bar{x}^*)$, and hence inclusion (3.6) has been justified.

We next justify the converse inclusion:

$$T_{\mathrm{gphN}_\Gamma}(\bar{x}, \bar{x}^*) \subset \big\{(v, v^*) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \exists \lambda \in \overline{\Lambda}(v) \,:\, v^* \in \nabla^2\big(\lambda^T q\big)(\bar{x})v + \widehat{N}_{\overline{K}}(v)\big\}. \tag{3.17}$$

Take any $(v, v^*) \in T_{\mathrm{gph}N_\Gamma}(\bar{x}, \bar{x}^*)$. By definition, there exist $t_k \downarrow 0$, $v_k \to v$, and $v_k^* \to v^*$ such that $x_k^* := \bar{x}^* + t_k v_k^* \in N_\Gamma(x_k)$ with $x_k := \bar{x} + t_k v_k$. Let $I_q(x_k) := \{i = 1, 2, \ldots, \ell \,|\, \langle b_i, q(x_k)\rangle = \alpha_i\}$. Using a subsequence if necessary, we can assume $I_q(x_k) = \tilde{I}$ for all k, with some fixed index set $\tilde{I} \subset I_q(\bar{x})$. So, it holds

$$\langle b_i, q(\bar{x})\rangle + t_k\langle b_i, \nabla q(\bar{x})v_k\rangle + o(t_k) = \langle b_i, q(x_k)\rangle \begin{cases} = \alpha_i & \text{if } i \in \tilde{I}, \\ < \alpha_i & \text{if } i \in I_q(\bar{x})\backslash\tilde{I}, \end{cases}$$

for all k. Thus

$$t_k\langle b_i, \nabla q(\bar{x})v_k\rangle + o(t_k) \ \begin{cases} = 0 & \text{if } i \in \tilde{I}, \\ < 0 & \text{if } i \in I_q(\bar{x})\backslash\tilde{I}, \end{cases}$$

for all k. Dividing by t_k and then letting $k \to \infty$, we get

$$\langle b_i, \nabla q(\bar{x})v\rangle \ \begin{cases} = 0 & \text{if } i \in \tilde{I}, \\ \le 0 & \text{if } i \in I_q(\bar{x})\backslash\tilde{I}. \end{cases} \tag{3.18}$$

To proceed, we next prove that there exist $\delta > 0, \kappa > 0$ such that for all $x \in \Gamma \cap \mathbb{B}_\delta(\bar{x})$ and $x^* \in \mathbb{R}^n$, one has

$$\Lambda(x, x^*) \cap \kappa\|x^*\|\mathbb{B}_{\mathbb{R}^m} \neq \emptyset \ \text{ whenever } \Lambda(x, x^*) \neq \emptyset. \tag{3.19}$$

Indeed, let $\delta > 0$ be such that MSCQ holds at every $x \in \Gamma \cap \mathbb{B}_\delta(\bar{x})$ with some constant $\gamma > 0$. Take any $x^* \in \mathbb{R}^n$ such that $\Lambda(x, x^*) \neq \emptyset$. If $x^* = 0$ then $0 \in \Lambda(x, x^*) \cap \kappa\|x^*\|\mathbb{B}_{\mathbb{R}^m}$ for any $\kappa > 0$. Suppose now that $x^* \neq 0$. Since $N_\Gamma(x) = \widehat{N}_\Gamma(x) = \nabla q(x)^T N_\Theta(y)$ with $y := q(x)$, and $\Lambda(x, x^*) \neq \emptyset$, it holds $\|x^*\|^{-1}x^* \in \widehat{N}_\Gamma(x) \cap \mathbb{B}_{\mathbb{R}^n}$. By [26, Corollary 1.96], $\widehat{N}_\Gamma(x) \cap \mathbb{B}_{\mathbb{R}^n} = \widehat{\partial} d_\Gamma(x)$. Hence, one has

$$\liminf_{u \to x} \frac{d_\Gamma(u) - d_\Gamma(x) - \langle \|x^*\|^{-1}x^*, u - x\rangle}{\|u - x\|} \geq 0.$$

Note that $d_\Gamma(x) = d_\Theta\big(q(x)\big) = 0$ and $d_\Gamma(u) \leq \gamma d_\Theta\big(q(u)\big)$ for all u near x. We have

$$\liminf_{u \to x} \frac{d_\Theta\big(q(u)\big) - d_\Theta\big(q(x)\big) - \langle (\gamma\|x^*\|)^{-1}x^*, u - x\rangle}{\|u - x\|} \geq 0.$$

This means $(\gamma\|x^*\|)^{-1}x^* \in \widehat{\partial}(d_\Theta \circ q)(x) \subset \partial(d_\Theta \circ q)(x)$. So, it follows from [26, Corollary 3.43] and [26, Corollary 1.96] that

$$(\gamma\|x^*\|)^{-1}x^* \in \partial(d_\Theta \circ q)(x) \subset \nabla q(x)^T \partial d_\Theta(y) = \nabla q(x)^T \big(N_\Theta(y) \cap \mathbb{B}_{\mathbb{R}^m}\big),$$

that is, $(\gamma\|x^*\|)^{-1}x^* = \nabla q(x)^T \tilde{\lambda}$ for some $\tilde{\lambda} \in N_\Theta(y) \cap \mathbb{B}_{\mathbb{R}^m}$. Put $\lambda := \gamma\|x^*\|\tilde{\lambda}$ and $\kappa := \gamma$. We have $\lambda \in \Lambda(x, x^*) \cap \kappa\|x^*\|\mathbb{B}_{\mathbb{R}^m}$. Thus (3.19) holds.

Since $x_k^* \in N_\Gamma(x_k)$, $(x_k, x_k^*) \to (\bar{x}, \bar{x}^*)$ and $MSCQ$ is fulfilled at $\bar{x}$, we have $\Lambda(x_k, x_k^*) \neq \emptyset$ for all k sufficiently large. Note that $\{\|x_k^*\|\}$ is bounded. By (3.19), there exist $\kappa, c_1 > 0$ and $\lambda^k \in \Lambda(x_k, x_k^*)$ such that

$$\|\lambda_k\| \leq \kappa\|x_k^*\| \leq c_1,$$

for all k sufficiently large. Hence, by [3, Lemma 1] and using a subsequence if necessary, we can assume $\lambda^k \to \tilde{\lambda} \in \overline{\Lambda}$ and $\lambda^k \in \tilde{\Theta} := \operatorname{pos}\{b_i \,|\, i \in I_0\}$, where $I_0 \subset \tilde{I}$ and $\{b_i\}_{i \in I_0}$ is linearly independent. For each $x^* \in \mathbb{R}^n$, put

$$\Psi_{I_0}(x^*) := \big\{\lambda \in \mathbb{R}^m \,|\, \nabla q(\bar{x})^T \lambda = x^*,\ \lambda = \sum_{i \in I_0} \lambda_i b_i,\ \lambda_i \geq 0\big\}.$$

Since $\lambda^k \to \tilde{\lambda}$, $\lambda^k \in \tilde{\Theta}$, and $\tilde{\Theta}$ is closed, it holds $\tilde{\lambda} \in \tilde{\Theta}$. So, $\tilde{\lambda} \in \Psi_{I_0}(\bar{x}^*)$ and thus $\Psi_{I_0}(\bar{x}^*) \neq \emptyset$. On the other hand, since $\tilde{\Theta}$ is a polyhedral convex cone in $\mathbb{R}^m$, there exist $a_i \in \mathbb{R}^m$, $i = 1, 2, ..., p$, such that $\tilde{\Theta} = \{\lambda \in \mathbb{R}^m \,|\, \langle a_i, \lambda\rangle \leq 0,\ i = 1, 2, ..., p\}$. Thus, we have

$$\Psi_{I_0}(x^*) := \big\{\lambda \in \mathbb{R}^m \,|\, \nabla q(\bar{x})^T \lambda = x^*,\ \langle a_i, \lambda\rangle \leq 0,\ i = 1, 2, ..., p\big\}.$$

By Hoffman's lemma [6, Theorem 2.200], there exists $\beta > 0$ such that

$$d_{\Psi_{I_0}(\bar{x}^*)}(\lambda^k) \leq \beta\big(\|\nabla q(\bar{x})^T \lambda^k - \bar{x}^*\| + \sum_{i=1}^{p} \big[\langle a_i, \lambda^k\rangle\big]_+\big) = \beta\|\nabla q(\bar{x})^T \lambda^k - \bar{x}^*\|, \tag{3.20}$$

for all k. The above equality is due to $\lambda^k \in \tilde{\Theta}$ for all k. Since $q \in \mathcal{C}^2$, for some $c_2 > 0$, it holds

$$\|\nabla q(\bar{x}) - \nabla q(x_k)\| \leq c_2\|\bar{x} - x_k\| = c_2 t_k\|v_k\|,$$

for all k sufficiently large. Hence, we get the estimate:

$$\begin{aligned}
\|\nabla q(\bar{x})^T \lambda^k - \bar{x}^*\| &= \|t_k v_k^* + \big(\nabla q(\bar{x}) - \nabla q(x_k)\big)^T \lambda^k\| \\
&\leq t_k\|v_k^*\| + \|\nabla q(\bar{x}) - \nabla q(x_k)\|\|\lambda^k\| \\
&\leq t_k(\|v_k^*\| + c_1 c_2\|v_k\|),
\end{aligned}$$

for all k sufficiently large. Combining this with (3.20) yields that

$$d_{\Psi_{I_0}(\bar{x}^*)}(\lambda^k) \le \beta t_k(\|v_k^*\| + c_1c_2\|v_k\|),$$

for all k sufficiently large. Thus there exists $\tilde{\lambda}^k \in \Psi_{I_0}(\bar{x}^*)$ such that

$$\|\lambda^k - \tilde{\lambda}^k\| \le \beta t_k(\|v_k^*\| + c_1c_2\|v_k\|)$$

for all k sufficiently large. Put $\mu^k := \frac{\lambda^k - \tilde{\lambda}^k}{t_k}$. Due to the boundedness of $\{\|\mu^k\|\}$, we can assume that $\{\mu^k\}$ converges to some $\mu \in \mathbb{R}^m$. Since $\tilde{\lambda}, \lambda^k, \tilde{\lambda}^k \in \tilde{\Theta}$, one has $\tilde{\lambda} = \sum_{i\in I_0} \tilde{\lambda}_i b_i$, $\lambda^k = \sum_{i\in I_0} \lambda_i^k b_i$ and $\tilde{\lambda}^k = \sum_{i\in I_0} \tilde{\lambda}_i^k b_i$ for some $\tilde{\lambda}_i, \lambda_i^k, \tilde{\lambda}_i^k \ge 0$. By (3.18), we get

$$(\mu^k)^T \nabla q(\bar{x})v = \frac{1}{t_k}\big(\lambda^k - \tilde{\lambda}^k\big)^T \nabla q(\bar{x})v = \sum_{i\in I_0} \frac{1}{t_k}\big(\lambda_i^k - \tilde{\lambda}_i^k\big)\big\langle b_i^T, \nabla q(\bar{x})v\big\rangle = 0.$$

Letting $k \to \infty$, we have

$$\mu^T \nabla q(\bar{x})v = 0. \tag{3.21}$$

Note that, by (3.18),

$$\langle \bar{x}^*, v\rangle = \langle \nabla q(\bar{x})^T \tilde{\lambda}^k, v\rangle = \sum_{i\in I_0} \tilde{\lambda}_i^k \big\langle b_i, \nabla q(\bar{x})v\big\rangle = 0.$$

Combining this with (3.18) allows us to conclude

$$v \in \overline{K}. \tag{3.22}$$

Moreover, for each $\lambda \in \overline{\Lambda}$, it holds

$$(\tilde{\lambda}^k - \lambda)^T q(x_k) = (\tilde{\lambda}^k - \lambda)^T \big(q(\bar{x}) + t_k \nabla q(\bar{x})v_k\big) + \frac{1}{2}t_k^2(\tilde{\lambda}^k - \lambda)^T v_k^T \nabla^2 q(\bar{x})v_k + o(t_k^2). \tag{3.23}$$

Since $\lambda \in N_\Theta(\bar{y})$, we have $\lambda = \sum_{i\in I_q(\bar{x})} \lambda_i b_i$ for some $\lambda_i \ge 0$. Thus,

$$\begin{aligned}
(\tilde{\lambda}^k - \lambda)^T \big(q(\bar{x}) + t_k \nabla q(\bar{x})v_k\big) &= (\tilde{\lambda}^k - \lambda)^T q(\bar{x}) + t_k(\tilde{\lambda}^k - \lambda)^T \nabla q(\bar{x})v_k \\
&= (\tilde{\lambda}^k - \lambda)^T q(\bar{x}) = \sum_{i\in I_q(\bar{x})} (\tilde{\lambda}_i^k - \lambda_i)\langle b_i, q(\bar{x})\rangle \\
&= \sum_{i\in I_0} (\tilde{\lambda}_i^k - \lambda_i)\alpha_i - \sum_{i\in I_q(\bar{x})\setminus I_0} \lambda_i \alpha_i \\
&\le \sum_{i\in I_0} (\tilde{\lambda}_i^k - \lambda_i)\langle b_i, q(x_k)\rangle - \sum_{i\in I_q(\bar{x})\setminus I_0} \lambda_i \langle b_i, q(x_k)\rangle \\
&= (\tilde{\lambda}^k - \lambda)^T q(x_k).
\end{aligned}$$

This together with (3.23) guarantees $\frac{1}{2}{t_k}^2(\tilde{\lambda}^k - \lambda)^T v_k^T \nabla^2 q(\bar{x})v_k + o(t_k^2) \ge 0$. Dividing the latter by $\frac{1}{2}{t_k}^2$ and then letting $k \to \infty$, we have $(\tilde{\lambda} - \lambda)^T v^T \nabla^2 q(\bar{x})v \ge 0$ for all $\lambda \in \overline{\Lambda}$, that is, $\tilde{\lambda} \in \overline{\Lambda}(v)$.

Moreover, noting that $\nabla q(x_k)^T\lambda^k = x_k^* = \nabla q(\bar{x})^T\tilde{\lambda}^k + t_k v_k^*$, it holds

$$\begin{aligned} v^* &= \lim_{k\to\infty} v_k^* = \lim_{k\to\infty} \frac{\nabla q(x_k)^T\lambda^k - \nabla q(\bar{x})^T\tilde{\lambda}^k}{t_k} \\ &= \lim_{k\to\infty} \frac{\big(\nabla q(x_k) - \nabla q(\bar{x})\big)^T\tilde{\lambda}^k + \nabla q(x_k)^T(\lambda^k - \tilde{\lambda}^k)}{t_k} \\ &= \lim_{k\to\infty} \frac{\big(t_k\nabla^2 q(\bar{x})v_k + o(t_k)\big)^T\tilde{\lambda}^k}{t_k} + \lim_{k\to\infty} \frac{\nabla q(x_k)^T(\lambda^k - \tilde{\lambda}^k)}{t_k} \\ &= \nabla^2(\tilde{\lambda}^T q)(\bar{x})v + \nabla q(\bar{x})^T\mu. \end{aligned} \tag{3.24}$$

Case 1: $\mu \in T_{N_\Theta(\bar{y})}(\tilde{\lambda})$. By Lemma 3.1, from (3.21)-(3.22) and the fact $\tilde{\lambda} \in \overline{\Lambda}$ it follows that $\nabla q(\bar{x})^T\mu \in \widehat{N}_{\overline{K}}(v)$. Thus, by (3.24), $v^* \in \nabla^2(\tilde{\lambda}^T q)(\bar{x})v + \widehat{N}_{\overline{K}}(v)$ with $\tilde{\lambda} \in \overline{\Lambda}(v)$. This shows $(v, v^*) \in T(\bar{x}, \bar{x}^*)$.

Case 2: $\mu \notin T_{N_\Theta(\bar{y})}(\tilde{\lambda})$. Put

$$J := \{i \in I_0 \,|\, \tilde{\lambda}_i = 0,\, \mu_i < 0\}.$$

We will show that $J \neq \emptyset$. Suppose to the contrary that $J = \emptyset$. Let

$$\tilde{I}_0 := \{i \in I_0 \,|\, \tilde{\lambda}_i = 0,\, \mu_i = 0\}$$

and

$${\mu'}^k := \sum_{i\in I_0} {\mu'}_i^k b_i,$$

where

$${\mu'}_i^k := \begin{cases} \mu_i^k & \text{if } i \in I_0\backslash\tilde{I}_0 \\ 0 & \text{if } i \in \tilde{I}_0 \end{cases} \qquad \text{and } \ \mu_i^k := \frac{\lambda_i^k - \tilde{\lambda}_i^k}{t_k}.$$

Taking into account the linear independence of $\{b_i\}_{i\in I_0}$, we get $\lim\limits_{k\to\infty} {\mu'}^k = \mu$. Hence, by $J = \emptyset$, $\tilde{\lambda} + t_k{\mu'}^k = \sum\limits_{i\in I_0}(\tilde{\lambda}_i + t_k{\mu'}_i^k)b_i \in N_\Theta(\bar{y})$, for all k sufficiently large. This implies $\mu \in T_{N_\Theta(\bar{y})}(\tilde{\lambda})$, which is a contradiction. Thus $J \neq \emptyset$.

For each $i \in J$, since

$$\begin{cases} \mu_i < 0 \\ \lim\limits_{k\to\infty} \frac{\lambda_i^k - \tilde{\lambda}_i^k}{t_k} = \mu_i, \end{cases}$$

there exists $\bar{k} \in \mathbb{N}$ such that

$$\frac{\lambda_i^{\bar{k}} - \tilde{\lambda}_i^{\bar{k}}}{t_{\bar{k}}} \le \frac{\mu_i}{2} < 0.$$

Put

$$\tilde{\mu} := \mu + 2\frac{\tilde{\lambda}^{\bar{k}} - \tilde{\lambda}}{t_{\bar{k}}} = \sum_{i\in I_0} \big(\mu_i + 2\frac{\tilde{\lambda}_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}}\big)b_i.$$

We next prove

$$\tilde{\mu} \in T_{N_\Theta(\bar{y})}(\tilde{\lambda}).$$

Note that if $\tilde{\lambda}_i = 0$ then

$$\mu_i + 2\frac{\tilde{\lambda}_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}} \geq \mu_i,$$

and if $i \in J$ then

$$\mu_i + 2\frac{\tilde{\lambda}_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}} \geq \mu_i \geq \mu_i + 2\frac{\tilde{\lambda}_i^{\bar{k}} - \tilde{\lambda}_i^{\bar{k}}}{t_{\bar{k}}} \geq \mu_i - 2\frac{\mu_i}{2} = 0.$$

Thus, we have

$$J' := \big\{i \in I_0 \,|\, \tilde{\lambda}_i = 0,\, \mu_i + 2\frac{\tilde{\lambda}_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}} < 0\big\} = \emptyset.$$

Put

$$\tilde{\mu}^k := \sum_{i \in I_0 \backslash J_0} \big(\frac{\lambda_i^k - \tilde{\lambda}_i^k}{t_k} + 2\frac{\lambda_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}}\big) b_i,$$

where

$$J_0 := \big\{i \in I_0 \,|\, \tilde{\lambda}_i = 0,\, \mu_i + 2\frac{\tilde{\lambda}_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}} = 0\big\}.$$

Then $\lim_{k\to\infty} \tilde{\mu}^k = \tilde{\mu}$, and

$$\tilde{\lambda} + t_k \tilde{\mu}^k = \sum_{i \in J_0} \tilde{\lambda}_i b_i + \sum_{i \in I_0 \backslash J_0} \big(\tilde{\lambda}_i + \lambda_i^k - \tilde{\lambda}_i^k + 2\frac{\lambda_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}} t_k\big) b_i.$$

Take any $i \in I_0 \backslash J_0$. If $\tilde{\lambda}_i > 0$ then $\tilde{\lambda}_i + \lambda_i^k - \tilde{\lambda}_i^k + 2\frac{\lambda_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}} t_k > 0$ for all k sufficiently large. If $\tilde{\lambda}_i = 0$ then, by $J' = \emptyset$, $\mu_i + 2\frac{\tilde{\lambda}_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}} > 0$. Note that $\lim_{k\to\infty} \frac{\lambda_i^k - \tilde{\lambda}_i^k}{t_k} = \mu_i$. We have

$$\tilde{\lambda}_i + \lambda_i^k - \tilde{\lambda}_i^k + 2\frac{\lambda_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}} t_k = \tilde{\lambda}_i + t_k\big(\frac{\lambda_i^k - \tilde{\lambda}_i^k}{t_k} + 2\frac{\lambda_i^{\bar{k}} - \tilde{\lambda}_i}{t_{\bar{k}}}\big) > 0,$$

for all k sufficiently large. Hence, $\tilde{\lambda} + t_k \tilde{\mu}^k \in N_\Theta(\bar{y})$ for all k sufficiently large. Due to the fact $\lim_{k\to\infty} \tilde{\mu}^k = \tilde{\mu}$, the latter guarantees $\tilde{\mu} \in T_{N_\Theta(\bar{y})}(\tilde{\lambda})$. In addition, it holds

$$\begin{aligned} \nabla q(\bar{x})^T \tilde{\mu} &= \nabla q(\bar{x})^T \big(\mu + 2\tfrac{\tilde{\lambda}^{\bar{k}} - \tilde{\lambda}}{t_{\bar{k}}}\big) \\ &= \nabla q(\bar{x})^T \mu + \tfrac{2}{t_{\bar{k}}} \nabla q(\bar{x})^T (\tilde{\lambda}^{\bar{k}} - \tilde{\lambda}) \\ &= \nabla q(\bar{x})^T \mu. \end{aligned}$$

By Lemma 3.1, we have $\nabla q(\bar{x})^T \tilde{\mu} \in \widehat{N}_{\overline{K}}(v)$ and thus $v^* \in \nabla^2(\tilde{\lambda}^T q)(\bar{x})v + \widehat{N}_{\overline{K}}(v)$ with $\tilde{\lambda} \in \overline{\Lambda}(v)$. This shows that (3.17) holds and (3.4) has been justified.

Finally, we see that (3.4) is equivalent to (3.5), due to the definition of the graphical derivative. The proof is complete. □

Remark 3.3. In the above proof, we have combined the development of the approach and techniques of Gfrerer and Outrata [14] with some ideas from Ioffe and Outrata [12].

The following example exhibits the situation where the mapping M_q is metrically subregular, but not metrically regular, while (3.4) and (3.5) hold.

Example 3.4. Let $\Theta := \{(y_1, y_2) \,|\, -y_1 \le 0,\, -y_2 \le 0,\, y_1 + y_2 - 1 \le 0\}$, $q(x) := (x_2 + x_1^2, 0)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, and $\bar{x} := (0, 0,)$. Then, one has

$$\Gamma := \{x \in \mathbb{R}^2 \,|\, q(x) \in \Theta\} = \{x \in \mathbb{R}^2 \,|\, -x_1^2 \le x_2 \le 1 - x_1^2\} \text{ and } \nabla q(\bar{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, since $(0, 0) \notin \operatorname{int}\big(q(\bar{x}) + \nabla q(\bar{x})\mathbb{R}^2 - \Theta\big)$, by the Robinson-Ursescu theorem, M_q is not metrically regular at $\bar{x}$ for 0. However, M_q is metrically subregular at $\bar{x}$ for 0, as $d_\Gamma(x) \le d_\Theta\big(q(x)\big)$ for all x near $\bar{x}$. Hence, Theorem 3.2 can be applicable for this situation. By a direct verification, for all $x \in \mathbb{R}^2$ near $\bar{x}$, it holds $N_\Theta(0, 0) = \mathbb{R}^2_-$, $T_\Gamma(0, 0) = \mathbb{R} \times \mathbb{R}_+$, $\nabla^2\big(\lambda^T q\big)(\bar{x}) = \begin{pmatrix} 2\lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$, and

$$N_\Gamma(x) = \begin{cases} \{(0,0)\} & \text{if } -x_1^2 < x_2 \\ \begin{pmatrix} 2x_1 \\ 1 \end{pmatrix} \mathbb{R}_- & \text{if } -x_1^2 = x_2 \\ \emptyset & \text{otherwise.} \end{cases}$$

Case 1: $\bar{x}^* := (0, 0) \in N_\Gamma(\bar{x})$. Then $\overline{K} = T_\Gamma(0, 0) = \mathbb{R} \times \mathbb{R}_+$, $\overline{\Lambda} = \{0\} \times \mathbb{R}_-$, $\overline{\Lambda}(v) = \{0\} \times \mathbb{R}_-$, and

$$T_{\mathrm{gph} N_\Gamma}(0, 0, 0, 0) = \big(\mathbb{R} \times (0, +\infty) \times \{(0, 0)\}\big) \cup \big(\mathbb{R} \times \{(0, 0)\} \times \mathbb{R}_-\big).$$

Hence,

$$\begin{aligned} &\{(v, v^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, \exists \lambda \in \overline{\Lambda}(v) \,:\, v^* \in \nabla^2\big(\lambda^T q\big)(\bar{x})v + \widehat{N}_{\overline{K}}(v)\} \\ &= \left\{(v, v^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, \exists \lambda \in \{0\} \times \mathbb{R}_- \,:\, v^* \in \begin{pmatrix} 2\lambda_1 & 0 \\ 0 & 0 \end{pmatrix} v + \widehat{N}_{\mathbb{R} \times \mathbb{R}_+}(v)\right\} \\ &= \big(\mathbb{R} \times (0, +\infty) \times \{(0, 0)\}\big) \cup \big(\mathbb{R} \times \{(0, 0)\} \times \mathbb{R}_-\big) \\ &= T_{\mathrm{gph} N_\Gamma}(0, 0, 0, 0). \end{aligned}$$

Case 2: $\bar{x}^* := (0, -1) \in N_\Gamma(\bar{x})$. Then $\overline{K} = T_\Gamma(0, 0) \cap \{\bar{x}^*\}^\perp = \mathbb{R} \times \{0\}$, $\overline{\Lambda} = \{-1\} \times \mathbb{R}_-$, $\overline{\Lambda}(v) = \{-1\} \times \mathbb{R}_-$, and

$$T_{\mathrm{gph} N_\Gamma}(0, 0, 0, -1) = \{(v_1, v_2, v_1^*, v_2^*) \in \mathbb{R}^4 \,|\, v_1^* = -2v_1,\ v_2 = 0\}.$$

Hence,

$$\begin{aligned} &\{(v, v^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, \exists \lambda \in \overline{\Lambda}(v) \,:\, v^* \in \nabla^2\big(\lambda^T q\big)(\bar{x})v + \widehat{N}_{\overline{K}}(v)\} \\ &= \left\{(v, v^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, \exists \lambda \in \{-1\} \times \mathbb{R}_- \,:\, v^* \in \begin{pmatrix} 2\lambda_1 & 0 \\ 0 & 0 \end{pmatrix} v + \widehat{N}_{\mathbb{R} \times \{0\}}(v)\right\} \\ &= \{(v_1, v_2, v_1^*, v_2^*) \in \mathbb{R}^4 \,|\, v_1^* = -2v_1,\ v_2 = 0\} \\ &= T_{\mathrm{gph} N_\Gamma}(0, 0, 0, -1). \end{aligned}$$

The above computation shows that (3.4) and (3.5) are valid.

For the case where Γ is a feasible set of a nonlinear programming, it may happen that the assumption of Theorem 3.2 is fulfilled, while the bounded extreme point property in the sense of [15, Definition 3.3] is invalid.

Example 3.5. Let $q(x) := (-x_1, x_1 - x_1^2 x_2^2)$, $\Theta := \{(0,0)\}$, $\Gamma := \big\{x \in \mathbb{R}^2 \,|\, q(x) \in \Theta\big\} = \{0\} \times \mathbb{R}$, and $\bar{x} := (0,0)$. We see that

$$N_\Theta\big(q(\bar{x})\big) = \mathbb{R}^2,\ T_\Gamma(\bar{x}) = \{0\} \times \mathbb{R},\ \nabla q(\bar{x}) = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix},\ \nabla^2(\lambda^T q)(\bar{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ N_\Gamma(x) = \mathbb{R} \times \{0\},$$

for all $x \in \Gamma$. Furthermore, since the gradients of the equality constraints at $\bar{x}$ are linearly dependent, the bounded extreme point property in the sense of [15, Definition 3.3] is invalid at $\bar{x}$. Obviously, M_q is metrically subregular at $\bar{x}$ for 0, and so Theorem 3.2 can be applicable for this situation. We next directly justify this claim. Let $\bar{x}^* := (0,0) \in N_\Gamma(\bar{x})$. By simple computation, we have $\overline{K} = \{0\} \times \mathbb{R}$, $\overline{\Lambda}(v) = \overline{\Lambda} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathbb{R}$, and $T_{\mathrm{gph} N_\Gamma}(0,0,0,0) = \mathbb{R}^3 \times \{0\}$. Hence,

$$\begin{aligned}
&\big\{(v, v^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, \exists \lambda \in \overline{\Lambda}(v) \,:\, v^* \in \nabla^2\big(\lambda^T q\big)(\bar{x})v + \widehat{N}_{\overline{K}}(v)\big\} \\
&= \big\{(v, v^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, v^* \in \widehat{N}_{\{0\}\times\mathbb{R}}(v)\big\} \\
&= \mathbb{R}^3 \times \{0\} = T_{\mathrm{gph} N_\Gamma}(0,0,0,0).
\end{aligned}$$

This shows that (3.4) and (3.5) hold.

The next result gives us a formula for computing the regular coderivative of the normal cone mapping, which is a direct consequence of Theorem 3.2.

Corollary 3.6. *Under the assumption of Theorem 3.2, one has*

$$\widehat{D}^* N_\Gamma(\bar{x}, \bar{x}^*)(u^*) = \big\{u \,|\, \langle u, v\rangle - \big\langle u^*, \nabla^2\big(\lambda^T q\big)(\bar{x})v\big\rangle \le 0 \ \text{for all } v \in \overline{K},\ \lambda \in \overline{\Lambda}(v),\ -u^* \in T_{\overline{K}}(v)\big\}.$$

Proof. By definition of coderivative, $u \in \widehat{D}^* N_\Gamma(\bar{x}, \bar{x}^*)(u^*)$ if and only if $(u, -u^*) \in \widehat{N}_{\mathrm{gph} N_\Gamma}(\bar{x}, \bar{x}^*)$. Furthermore, by (2.1), the latter amounts to $\langle u, v\rangle - \langle u^*, v^*\rangle \le 0$ for all $(v, v^*) \in T_{\mathrm{gph} N_\Gamma}(\bar{x}, \bar{x}^*)$. So, by Theorem 3.2, the necessary and sufficient condition for $u \in \widehat{D}^* N_\Gamma(\bar{x}, \bar{x}^*)(u^*)$ is

$$\langle u, v\rangle - \big\langle u^*, \nabla^2\big(\lambda^T q\big)(\bar{x})v + w\big\rangle \le 0 \ \text{ for all } \lambda \in \overline{\Lambda}(v) \text{ and } w \in \widehat{N}_{\overline{K}}(v),$$

or equivalently,

$$\langle u, v\rangle - \big\langle u^*, \nabla^2\big(\lambda^T q\big)(\bar{x})v\big\rangle \le 0 \ \text{ for all } \lambda \in \overline{\Lambda}(v),\ v \in \overline{K},\ -u^* \in \big(\widehat{N}_{\overline{K}}(v)\big)^0.$$

Hence, noting that $\overline{K}$ is convex, we arrive at the conclusion. □

4 Application to Generalized Equation

In this section, we first consider the parametric generalized equation of the form:

$$0 \in F(x, y) + N_\Gamma(x), \tag{4.25}$$

where $F : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$ is a continuous differentiable mapping, x is a variable, y is a parameter, and $\Gamma := \{x \in \mathbb{R}^n \,|\, q(x) \in \Theta\}$, $\Theta \subset \mathbb{R}^m$ is a nonempty polyhedral set in $\mathbb{R}^m$, and $q : \mathbb{R}^n \to \mathbb{R}^m$ is a twice continuously differentiable mapping. Denote by S the solution mapping to (4.25) given by

$$S(y) := \big\{x \in \mathbb{R}^n \,|\, 0 \in F(x, y) + N_\Gamma(x)\big\},$$

which assigns the corresponding set of equilibria to each value of the parameter y.

Based on the result of the preceding section, we next provide a formula for computing the graphical derivative of the solution mapping S.

Theorem 4.1. *Let $(\bar{y}, \bar{x}) \in \text{gph}S$ and let M_q be metrically subregular at $\bar{x}$ for 0. Then, one has*

$$DS(\bar{y}|\bar{x})(z) \subset \Big\{ v \,|\, -\nabla_y F(\bar{x}, \bar{y})z \in \nabla_x F(\bar{x}, \bar{y})v + \big\{ \nabla^2 \big(\lambda^T q\big)(\bar{x})v \,|\, \lambda \in \overline{\Lambda}(v) \big\} + \widehat{N}_{\overline{K}}(v) \Big\}, \quad (4.26)$$

for all $z \in \mathbb{R}^s$. Inclusion (4.26) holds as equality if assume further that $\nabla_y F(\bar{x}, \bar{y})$ is surjective. Here $\overline{K} := T_\Gamma(\bar{x}) \cap \{\bar{x}^*\}^\perp$ with $\bar{x}^* := -F(\bar{x}, \bar{y})$, and $\overline{\Lambda}(v)$ is the optimal solution set of LP(v).

Proof. Let $\varphi : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be the mapping given by

$$\varphi(y, x) := \big(x, -F(x, y)\big) \text{ for all } (y, x) \in \mathbb{R}^s \times \mathbb{R}^n.$$

Then gphS $= \varphi^{-1}\big(\text{gph}N_\Gamma\big)$ and $\nabla\varphi(\bar{y}, \bar{x})(z, v) = \big(v, -\nabla_x F(\bar{x}, \bar{y})v - \nabla_y F(\bar{x}, \bar{y})z\big)$ for all (z, v). Clearly, $\nabla_y F(\bar{x}, \bar{y})$ is surjective if and only if $\nabla\varphi(\bar{y}, \bar{x})$ is surjective. So, by [30, Theorem 6.31], we have

$$T_{\text{gph}S}(\bar{y}, \bar{x}) \subset \big\{ (z, v) \,|\, \big(v, -\nabla_x F(\bar{x}, \bar{y})v - \nabla_y F(\bar{x}, \bar{y})z\big) \in T_{\text{gph}N_\Gamma}\big((\bar{x}, -F(\bar{x}, \bar{y}))\big) \big\},$$

and the inclusion becomes equality if in addition $\nabla_y F(\bar{x}, \bar{y})$ is surjective. By definition of graphical derivative, we get the desired conclusion. □

Remark 4.2. The surjectivity of $\nabla_y F(\bar{x}, \bar{y})$ guarantees the parameterization (4.25) is ample at $\bar{x}$ in the sense of Dontchev and Rockafellar [10, p. 95]. This condition has been extensively used in variational analysis [10, 26, 30].

If q is an affine mapping, then $\{\nabla^2\big(\lambda^T q\big)(\bar{x})v \,|\, \lambda \in \overline{\Lambda}(v)\} = \{0\}$ and M_q is automatically metrically subregular. Hence, in this case, formula (4.26) can be much more simplified.

Corollary 4.3. ([9, Theorem 7.1]). *Consider the generalized equation (4.25) with $q : \mathbb{R}^n \to \mathbb{R}^m$ being an affine mapping. For any $(\bar{y}, \bar{x}) \in \text{gph}S$ and $\bar{x}^* := -F(\bar{x}, \bar{y})$, one has*

$$DS(\bar{y}|\bar{x})(z) \subset \Big\{ v \,|\, -\nabla_y F(\bar{x}, \bar{y})z \in \nabla_x F(\bar{x}, \bar{y})v + \widehat{N}_{\overline{K}}(v) \Big\} \text{ for all } z \in \mathbb{R}^s. \quad (4.27)$$

Inclusion (4.27) holds as equality if in addition $\nabla_y F(\bar{x}, \bar{y})$ is surjective.

If Γ is the constraint set of a nonlinear programming satisfying the constant rank constraint qualification (CRCQ) (see, e.g., [15]) at the reference point, then (4.26) can be also simplified. This is due to the facts that under CRCQ the mapping $\lambda \mapsto v^T\nabla^2\big(\lambda^T q\big)(\bar{x})v$ is constant on $\overline{\Lambda}$ for every $v \in \overline{K}$, and further $\nabla^2\big(\lambda_1^T q\big)(\bar{x})v - \nabla^2\big(\lambda_2^T q\big)(\bar{x})v \in \widehat{N}_{\overline{K}}(v)$ for all $\lambda^1, \lambda^2 \in \overline{\Lambda}$ and $v \in \overline{K}$; see [14, Lemma 4], [15, Proposition 5.3], and [17, Corollary 3.2].

Corollary 4.4. *Consider (4.25) with $\Theta := \{0_{\mathbb{R}^{m_1}}\} \times \mathbb{R}_-^{m-m_1}$ and $(\bar{y}, \bar{x}) \in \text{gph}S$. Assume that CRCQ is fulfilled at $\bar{x}$. Then, one has*

$$DS(\bar{y}|\bar{x})(z) \subset \Big\{ v \,|\, -\nabla_y F(\bar{x}, \bar{y})z \in \nabla_x F(\bar{x}, \bar{y})v + \nabla^2\big(\lambda^T q\big)(\bar{x})v + \widehat{N}_{\overline{K}}(v) \Big\}, \quad (4.28)$$

for all $z \in \mathbb{R}^s$ and $\lambda \in \overline{\Lambda}$. Inclusion (4.28) holds as equality if in addition $\nabla_y F(\bar{x}, \bar{y})$ is surjective.

Proof. Suppose that CRCQ is fulfilled at $\bar{x}$. Then M_q is metrically subregular at $\bar{x}$ for 0. So, by [15, Proposition 4.3], for each $\lambda \in \overline{\Gamma}$, we have

$$\overline{K} = \left\{ v \in \mathbb{R}^n \mid \langle \nabla q_i(\bar{x}), v \rangle \begin{cases} = 0 & \text{if } i \in E \cup I^+(\lambda) \\ \leq 0 & \text{if } i \in I(\bar{x}) \backslash I^+(\lambda) \end{cases} \right\}, \tag{4.29}$$

where $E := \{1, \ldots, m_1\}$, $I(\bar{x}) := \{i = m_1 + 1, \ldots, m \mid q_i(\bar{x}) = 0\}$, and $I^+(\lambda) := \{i \in I(\bar{x}) \mid \lambda_i > 0\}$. On the other hand, by [15, Proposition 5.3 (i)], for each $v \in \mathbb{R}^n$ satisfying

$$\langle \nabla q_i(\bar{x}), v \rangle = 0 \text{ whenever } i \in E \cup I^+ \text{ with } I^+ := \bigcup_{\lambda \in \overline{\Lambda}} I^+(\lambda)$$

the mapping $\lambda \mapsto v^T \nabla^2 \big(\lambda^T q\big)(\bar{x}) v$ is constant on $\overline{\Lambda}$. Combining this with (4.29) shows that the mapping $\lambda \mapsto v^T \nabla^2 \big(\lambda^T q\big)(\bar{x}) v$ is constant on $\overline{\Lambda}$ for each $v \in \overline{K}$. Hence $\overline{\Lambda}(v) = \overline{\Lambda}$ for all $v \in \overline{K}$. Next, following the proof scheme of [14, Lemma 4], we show that

$$\nabla^2 \big(\lambda_1^T q\big)(\bar{x}) v - \nabla^2 \big(\lambda_2^T q\big)(\bar{x}) v \in \widehat{N}_{\overline{K}}(v), \tag{4.30}$$

for all $\lambda^1, \lambda^2 \in \overline{\Lambda}$. Suppose to the contrary that there exist $v \in \overline{K}$ and $\lambda^1, \lambda^2 \in \overline{\Lambda}$ such that

$$\nabla^2 \big(\lambda_1^T q\big)(\bar{x}) v - \nabla^2 \big(\lambda_2^T q\big)(\bar{x}) v \notin \widehat{N}_{\overline{K}}(v).$$

Thus, by the convex separation theorem, one can find some w satisfying the following condition

$$\langle w, \nabla^2 \big(\lambda_1^T q\big)(\bar{x}) v - \nabla^2 \big(\lambda_2^T q\big)(\bar{x}) v \rangle < \langle w, u \rangle \text{ for all } u \in \widehat{N}_{\overline{K}}(v).$$

From (3.2) it follows that

$$\widehat{N}_{\overline{K}}(v) = \left\{ \sum_{i \in E \cup I(\bar{x})} t_i \nabla q_i(\bar{x}) - t_0 \bar{x}^* \mid t_i \in \mathbb{R}, i \in E, \; t_0, t_j \geq 0, j \in I(\bar{x}) \right\} \cap \{v\}^\perp.$$

Note that $\bar{x}^* \in \operatorname{pos}\big\{\nabla q_i(\bar{x}) \,|, i \in I(\bar{x})\big\}$ and $\nabla q_i(\bar{x}) v = 0$ for all $i \in E \cup I^+$. Hence, $\nabla q_i(\bar{x}) w = 0$ for all $i \in E \cup I^+$ and $\langle w, \nabla^2 \big(\lambda_1^T q\big)(\bar{x}) v - \nabla^2 \big(\lambda_2^T q\big)(\bar{x}) v \rangle < 0$. By [15, Proposition 5.3 (ii)], there exist $\tilde{\lambda} \in \overline{\Lambda}$ with $I^+(\tilde{\lambda}) = I^+$ and some $\tilde{v} \in \overline{K}$ such that

$$\langle \nabla q_i(\bar{x}), \tilde{v} \rangle \begin{cases} = 0 & \text{if } i \in E \cup I^+(\tilde{\lambda}) \\ < 0 & \text{if } i \in I(\bar{x}) \backslash I^+(\tilde{\lambda}). \end{cases}$$

Consequently, $\langle w + t\tilde{v}, \nabla^2 \big(\lambda_1^T q\big)(\bar{x}) v - \nabla^2 \big(\lambda_2^T q\big)(\bar{x}) v \rangle < 0$ and

$$\langle \nabla q_i(\bar{x}), w + t\tilde{v} \rangle \begin{cases} = 0 & \text{if } i \in E \cup I^+(\tilde{\lambda}) \\ < 0 & \text{if } i \in I(\bar{x}) \backslash I^+(\tilde{\lambda}), \end{cases}$$

for all $t > 0$ sufficiently large. This allows us to choose $\alpha, \beta > 0$ such that $\tilde{w} := w + \alpha \tilde{v} \in \overline{K}$, $v + \beta \tilde{w} \in \overline{K}$, $\langle \tilde{w}, \nabla^2 \big(\lambda_1^T q\big)(\bar{x}) v - \nabla^2 \big(\lambda_2^T q\big)(\bar{x}) v \rangle < 0$, and hence,

$$\big\langle v + \beta \tilde{w}, \nabla^2 \big(\lambda_1^T q\big)(\bar{x})(v + \beta \tilde{w}) - \nabla^2 \big(\lambda_2^T q\big)(\bar{x})(v + \beta \tilde{w}) \big\rangle < 0.$$

This contradicts the fact that the mapping $\lambda \mapsto v^T \nabla^2 \big(\lambda^T q\big)(\bar{x}) v$ is constant on $\overline{\Lambda}$ for each $v \in \overline{K}$. So (4.30) holds. Noting that $\overline{\Lambda}(v) = \overline{\Lambda}$ for all $v \in \overline{K}$, by Theorem 4.1, the conclusion follows. □

Next, we consider the so-called isolated calmness of S. This property introduced by Dontchev [7] possesses a great of variationally attractive characteristics, such as stability with respect to approximation, having infinitesimal characterizations, or closely related to some other important properties in variational analysis. For more information on the isolated calmness, we refer the reader to the monograph [10].

Recall [10] that a given set-valued mapping $\Phi : \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ is said to be isolated calm at $\bar{y} \in \mathbb{R}^s$ for $\bar{x} \in \Phi(\bar{y})$ if there exist $\kappa, r > 0$ such that $\Phi(y) \cap \mathbb{B}_r(\bar{x}) \subset \{\bar{x}\} + \kappa\|y - \bar{y}\|\mathbb{B}_{\mathbb{R}^n}$ for all $y \in \mathbb{B}_r(\bar{y})$.

The isolated calmness is equivalent to the so-called strong metric subregularity of the inverse [10, Theorem 3I.3]. It can be characterized by the graphical derivative. More concretely, we have the following result whose necessity part was given in [21, Proposition 2.1] while the other part was established in [23, Proposition 4.1].

Lemma 4.5. ([10, Theorem 4E.1]). *Let $\Phi : \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ and let $(\bar{y}, \bar{x}) \in \mathrm{gph}\Phi$. Then, Φ is isolated calm at $\bar{y}$ for $\bar{x}$ if and only if $D\Phi(\bar{y}|\bar{x})(0) = \{0\}$.*

Now, combining Lemma 4.5 with Theorem 4.1, we are able to derive a characterization of the isolated calmness.

Theorem 4.6. *Let $(\bar{y}, \bar{x}) \in \mathrm{gph}S$ and let M_q be metrically subregular at $\bar{x}$ for 0. If the implication*

$$\left.\begin{array}{l} 0 \in \nabla_x \mathcal{L}(\bar{x}, \bar{y}, \lambda)v + \widehat{N}_{\overline{K}}(v) \\ \lambda \in \overline{\Lambda}(v),\ v \in \mathbb{R}^n \end{array}\right\} \Rightarrow v = 0 \tag{4.31}$$

is valid, then S is isolated calm at $\bar{y}$ for $\bar{x}$. The inverse also holds if assume further that $\nabla_y F(\bar{x}, \bar{y})$ is surjective. Here $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^m \to \mathbb{R}^n$ is given by $\mathcal{L}(x, y, \lambda) := F(x, y) + \nabla q(x)^T \lambda$.

Proof. According to Theorem 4.1, we have

$$\begin{aligned} DS(\bar{y}|\bar{x})(0) &\subset \big\{v \mid \exists \lambda \in \overline{\Lambda}(v) \,:\, 0 \in \nabla_x F(\bar{x}, \bar{y})v + \nabla^2\big(\lambda^T q\big)(\bar{x})v + \widehat{N}_{\overline{K}}(v)\big\} \\ &= \big\{v \mid \exists \lambda \in \overline{\Lambda}(v) \,:\, 0 \in \nabla_x \mathcal{L}(\bar{x}, \bar{y}, \lambda)v + \widehat{N}_{\overline{K}}(v)\big\}, \end{aligned}$$

and the inclusion becomes equality whenever $\nabla_y F(\bar{x}, \bar{y})$ is surjective. Thus, by Lemma 4.5, we get the desired conclusion. □

Compared with [14, Theorem 7], we obtain here the same conclusion without the additional assumption of the so-called uniformly metric regularity. Moreover, in the polyhedral convex case, our result reduces to the one presented in the recent book of Dontchev and Rockafellar [10].

Corollary 4.7. ([10, Theorem 4G.1]). *Consider the generalized equation (4.25) with $\Gamma := \Theta$, $n = m$, and $q := I_n$ the identity mapping in $\mathbb{R}^n$. Let $(\bar{y}, \bar{x}) \in \mathrm{gph}S$ and $\bar{x}^* := -F(\bar{x}, \bar{y})$. Then, if*

$$\big(\nabla_x F(\bar{x}, \bar{y}) + N_{\overline{K}}\big)^{-1}(0) = \{0\} \tag{4.32}$$

then S is isolated calm at $\bar{y}$ for $\bar{x}$. Moreover, if in addition $\mathrm{rank}\nabla_y F(\bar{x}, \bar{y}) = n$, then property (4.32) is necessary and sufficient for S to have the isolated calmness at $\bar{y}$ for $\bar{x}$.

Proof. Under the given assumption, it holds that the mapping M_q is metrically subregular at $\bar{x}$ for 0, $N_{\overline{K}}(v) = \widehat{N}_{\overline{K}}(v)$, $\overline{\Lambda}(v) = \overline{\Lambda}(\bar{x}, \bar{x}^*) = \{\bar{x}^*\}$, and $\nabla_x \mathcal{L}(\bar{x}, \bar{y}, \lambda)v = \nabla_x F(\bar{x}, \bar{y})v$. Thus (4.32) is equivalent to (4.31). Moreover, $\mathrm{rank}\nabla_y F(\bar{x}, \bar{y}) = n$ if and only if $\nabla_y F(\bar{x}, \bar{y}) : \mathbb{R}^s \to \mathbb{R}^n$ is surjective. So, by Theorem 4.6, the conclusion follows. □

We now consider the parametric generalized equation:

$$w \in F(x, y) + N_\Gamma(x), \tag{4.33}$$

where $F : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$ is a continuous differentiable mapping, x is the variable, and $p := (y, w)$ represents the parameter, and $\Gamma := \{x \in \mathbb{R}^n \,|\, q(x) \in \Theta\}$ with $\Theta \subset \mathbb{R}^m$ being a polyhedron and $q : \mathbb{R}^n \to \mathbb{R}^m$ being a twice continuously differentiable mapping. Let $S : \mathbb{R}^s \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be the solution mapping of (4.33), that is,

$$S(p) := \big\{x \in \mathbb{R}^n \,|\, w \in F(x, y) + N_\Gamma(x)\big\} \text{ for all } p := (y, w) \in \mathbb{R}^s \times \mathbb{R}^n. \tag{4.34}$$

The following result gives us a characterization of the isolated calmness of the mapping $S(p)$.

Theorem 4.8. *Let $(\bar{p}, \bar{x}) \in \mathrm{gph}S$ and let M_q be metrically subregular at $(\bar{x}, 0)$. Then, the following assertions are equivalent:*

(i) The implication

$$\left.\begin{array}{l} 0 \in \nabla_x \mathcal{L}(\bar{x}, \bar{p}, \lambda)v + \widehat{N}_{\overline{K}}(v) \\ \lambda \in \overline{\Lambda}(v),\ v \in \mathbb{R}^n \end{array}\right\} \Rightarrow v = 0$$

is valid.

(ii) The solution mapping $S(p)$ is isolated calm at $\bar{p}$ for $\bar{x}$. Here $\mathcal{L} : \mathbb{R}^n \times \big(\mathbb{R}^s \times \mathbb{R}^n\big) \times \mathbb{R}^m \to \mathbb{R}^n$ is defined by $\mathcal{L}(x, p, \lambda) := F(x, y) - w + \nabla q(x)^T \lambda$ with $p := (y, w)$.

Proof. For $G(x, p) := F(x, y) - w$, we have

$$S(p) = \big\{x \in \mathbb{R}^n \,|\, 0 \in G(x, p) + N_\Gamma(x)\big\},$$

and $\nabla G_p(\bar{x}, \bar{p})(x, y, w) = \nabla_x F(\bar{x}, \bar{y})x + \nabla_y F(\bar{x}, \bar{y})y - w$ for all $(x, y, w) \in \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^n$. The latter shows that the partial derivative mapping $\nabla_p G(\bar{x}, \bar{p}) : \mathbb{R}^n \times (\mathbb{R}^s \times \mathbb{R}^n) \to \mathbb{R}^n$ is surjective. Thus, by Theorem 4.6, we get the desired conclusion. $\square$

Finally, we consider the parametric optimization problem:

$$\begin{array}{ll} \text{minimize} & g(x, y) - \langle w, x\rangle \\ \text{subject to} & x \in \Gamma, \end{array} \tag{4.35}$$

where $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ is twice continuously differentiable, $\Gamma := \{x \in \mathbb{R}^n \,|\, q(x) \in \Theta\}$, Θ is a nonempty polyhedral convex set in $\mathbb{R}^m$, $q : \mathbb{R}^n \to \mathbb{R}^m$ is twice continuously differentiable, x is a variable, and $y \in \mathbb{R}^s$ and $w \in \mathbb{R}^n$ are parameters.

Recall that the set-valued mapping $X_{KKT} : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$X_{KKT}(p) := \big\{x \in \mathbb{R}^n \,|\, 0 \in \nabla_x g(x, y) - w + N_\Gamma(x)\big\},\ p := (y, w) \in \mathbb{R}^s \times \mathbb{R}^n,$$

is called the stationary point mapping of (4.35).

Obviously, the stationary point mapping $X_{KKT}(p)$ is a special case of the set-valued mapping $S(p)$ given by (4.34). So, by Theorem 4.8, we get the corresponding characterization of isolated calmness of the stationary point mapping of (4.35).

Corollary 4.9. *Let $(\bar{p}, \bar{x}) \in \mathrm{gph}X_{KKT}$ and let M_q be metrically subregular at $(\bar{x}, 0)$. Then, the following assertions are equivalent:*

(i) The implication

$$\left.\begin{array}{l} 0 \in \nabla_x \mathcal{L}(\bar{x}, \bar{p}, \lambda)v + \widehat{N}_{\overline{K}}(v) \\ \lambda \in \overline{\Lambda}(v),\ v \in \mathbb{R}^n \end{array}\right\} \Rightarrow v = 0$$

is valid.

(ii) The mapping $X_{KKT}(p)$ is isolated calm at $\bar{p}$ for $\bar{x}$. Here $\mathcal{L} : \mathbb{R}^n \times \big(\mathbb{R}^s \times \mathbb{R}^n\big) \times \mathbb{R}^m \to \mathbb{R}^n$ is defined by $\mathcal{L}(x, p, \lambda) := \nabla_x g(x, y) - w + \nabla q(x)^T \lambda$ with $p := (y, w)$.

If in addition Γ is a convex set and $g(x, y)$ is a convex function in x, then the stationary point mapping $X_{KKT}(p)$ coincides with the optimal solution mapping of (4.35). Hence, in this case, Corollary 4.9 shows us a characterization of isolated calmness for the optimal solution mapping of the considered problem.

5 Concluding Remarks

The main results of this paper exhibit formulae for computing the graphical derivative of the normal cone mappings and the solution mappings of generalized equations, under a very weak condition. We also present characterizations of isolated calmness for a broad class of generalized equations. This class covers the generalized equation reformulation of variational inequalities over polyhedral convex sets, and especially, the stationary point mappings of parametric nonlinear programmings. In our view, these results can be applied to the study of the stability and sensitivity for more structural problems in which the specific features may help us to get better information on the considered issues. In addition, we think that it would be very interesting if we could use the obtained results on computation of graphical derivative to investigate the tilt-sability for general optimization problems, and the metric regularity for the class of related set-valued mappings. We will pay our attention to these topics in the coming time. Note that some nice results in this direction were recently reported in the literature; see, e.g., [14, 15, 27].

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