The Traveling Salesman Problem on Grids with Forbidden Neighborhoods

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We introduce the Traveling Salesman Problem with forbidden neighborhoods (TSPFN). This is an extension of the Euclidean TSP in the plane where direct connections between points that are too close are forbidden. The TSPFN is motivated by an application in laser beam melting. In the production of a workpiece in several layers using this method one hopes to reduce the internal stresses of the workpiece by excluding the heating of positions that are too close. The points in this application are often arranged in some regular (grid) structure.

In this paper we study optimal solutions of TSPFN instances where the points in the Euclidean plane are the points of a regular grid. Indeed, we explicitly determine the optimal values for the TSPFN and its associated path version on rectangular regular grids for different minimal distances of the points visited consecutively. For establishing lower bounds on the optimal values we use combinatorial counting arguments depending on the parities of the grid dimensions. Furthermore we provide construction schemes for optimal TSPFN tours for the considered cases.

Key words. Shortest Hamiltonian Path Problem; Traveling Salesman Problem; Constrained Euclidean Traveling Salesman Problem; Grid; Explicit Solutions

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1 Introduction

In this paper we study an extension of the Traveling Salesman Problem (TSP), the so called *TSP with forbidden neighborhoods* (TSPFN): Given l points (also called nodes or vertices) in the Euclidean plane and a radius $r \in \mathbb{R}$, the task is to determine a shortest Hamiltonian cycle $(t_0, \dots, t_{l-1}, t_0)$ over the l points (also called *tour*) such that $\|t_i - t_{(i+1) \bmod l}\| > r$ is satisfied for all $i \in \{0, \dots, l-1\}$ (the feasible connections between the points can be interpreted as edges of a graph with l nodes). These inequalities ensure that points traversed successively have a distance larger than r in the plane. We will mainly consider the Euclidean norm for measuring distances between points in the objective function but we extend our results to the 1-norm, also called Manhattan distance, as well.

In general, the $\mathcal{NP}$-hard TSP is one of the most famous (combinatorial) optimization problems with high importance in both operations research and theoretical computer science. We refer to the books [1, 10, 16, 25] and the references therein for extensive material on the TSP, its variants

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and various applications, details on many heuristic and exact methods as well as on relevant theoretical results. An important special case is the metric TSP in which the symmetric costs between the nodes satisfy the triangle inequality. The algorithm of Christofides approximates the total cost of an optimal metric TSP tour within a factor of 1.5 [8]. In the Euclidean TSP, a special metric TSP, the nodes lie in $\mathbb{R}^2$ (or more generally, in $\mathbb{R}^d$ for some d) and distances are measured according to the ℓ_2-norm. Even this special case is $\mathcal{NP}$-hard [15, 23]. Arora et al. [6] proved that if $\mathcal{P} \neq \mathcal{NP}$, then the metric TSP and several other problems do not allow for a *Polynomial-Time Approximation Scheme* (PTAS). In contrast to this Arora [5] showed that there is a PTAS for the Euclidean TSP.

There exist many variants of the TSP motivated by various applications. The one most similar to the TSPFN is the maximum scatter TSP [3, 7, 17] where the goal is to maximize the length of a shortest edge of a tour. We can directly relate this problem to the TSPFN with $r \in \mathbb{R}_0^+$: The task of the maximum scatter TSP is to find a TSPFN tour such that the radius r of the forbidden neighborhood is maximized. Recently, Hoffmann et al. [17] studied the maximum scatter TSP on regular grids.

The TSPFN is motivated by an application in mechanical engineering, more precisely in laser beam melting. This technology is used for building complex workpieces in several layers, similar to 3D printing. For each layer new material has to be heated up at several points. The question is now how to choose the order of the points to be treated in each layer such that internal stresses are low. Furthermore, one is interested in low cycle times of the workpieces. One idea discussed in [20] was to look for short paths between the points without connecting points that are too close such that the heat quantity in each region is not too high within short periods. In particular the layers are rectangular non-regular grids in all instances provided to us by Richard Kordaß and Thomas Töppel from the Fraunhofer IWU, see also [20].

Motivated by the mentioned application we will concentrate on the TSPFN on regular grids in this paper: We are given mn points $V = \{p^0, p^1, \dots, p^{mn-1}\}$ $(m \leq n)$ on the $m \times n$ grid (where m represents the rows and n the columns) in the Euclidean plane and a radius $r \in \mathbb{R}_0^+$.[1] The task is to determine a shortest Hamiltonian cycle $(p^{t_0}, \dots, p^{t_{mn-1}}, p^{t_0})$ over the mn points such that the following inequalities are satisfied: $\|p^{t_i} - p^{t_{(i+1) \bmod mn}}\| > r,\ i \in \{0, \dots, mn-1\}$.

The complexity of the TSPFN on regular grids has not been studied before and is currently unknown: On the one hand the TSPFN is highly related to the $\mathcal{NP}$-hard Euclidean TSP and on the other hand the arrangement of the points is rather restricted by only allowing grids.

There is also some connection to the TSP on grid graphs, but there usually not all possible edges of the grid are present and the distances are normally either measured as the Manhattan distance or as the length of a shortest path between the points along the grid. Itai et al. [18] showed that the Hamiltonian cycle problem is $\mathcal{NP}$-complete on planar grid graphs. A solid planar grid graph is a planar grid graph without holes. Umans and Lenhart [27] give a polynomial time algorithm for determining a Hamiltonian cycle in a solid planar grid graph. Recently Arkin et al. [4] gave a systematic study of Hamiltonicity of grids. The complexity of solving the TSP on a solid planar grid graph is still open, see, e. g., Arkin et al. [2] or Problem 54 in [13].

In this paper we study the length and the structure of shortest Hamiltonian cycles and Hamiltonian paths, also denoted as *open* tours, for the smallest reasonable forbidden neighborhoods on the grid, namely $r \in \{1, \sqrt{2}\}$ as well as the case $r = 0$, see Figure 1 for an illustration. Furthermore we present schemes for the construction of optimal solutions. We have also implemented all algorithms developed and discussed in this paper in MATLAB [22]. A current version of our implementation can be downloaded from [19].

A linear integer programming (ILP) formulation of the TSPFN using binary edge variables $x_{jk} = x_{kj} \in \{0, 1\}$, $j, k \in \{0, 1, \dots, mn-1\}$, $j \neq k$ (we number the cells from 0 to $mn - 1$), is

[1] Note that we rotated the m-n-coordinate system of the Euclidean plane by 90 degrees in clockwise direction in order to work with the usual matrix numbering of the grid cells later on.

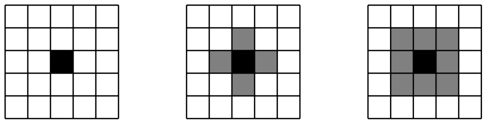


Figure 1: Illustration of forbidden neighborhoods for different values of the radius: We consider $r = 0$, $r = 1$ and $r = \sqrt{2}$ from left to right. The current grid cell is the black one in the middle, the forbidden cells are the gray ones around it and the cells we are allowed to visit next are white.

given by the following extension of the classical formulation of the TSP by Dantzig et al. [12]:

$$\min \sum_{j=0}^{mn-2} \sum_{k=j+1}^{mn-1} \|p^j - p^k\| x_{jk}$$

$$\sum_{\substack{k=0\\k\neq j}}^{mn-1} x_{kj} = 2, \qquad j \in \{0,1,\ldots,mn-1\}, \tag{1a}$$

$$\sum_{\substack{j,k\in S\\j<k}} x_{jk} \leq |S| - 1, \qquad \emptyset \neq S \subsetneq \{0,1,\ldots,mn-1\}, \tag{1b}$$

$$x_{jk} = 0, \qquad j,k \in \{0,1,\ldots,mn-1\}, j<k, \text{ with } \|p^j - p^k\| \leq r, \tag{1c}$$

$$x_{jk} \in \{0,1\}, \qquad j,k \in \{0,1,\ldots,mn-1\}, j<k. \tag{1d}$$

The degree constraints (1a), the subtour elimination constraints (1b) and the binarity conditions (1d) ensure that the selected edges correspond to a Hamiltonian cycle. Edges between nodes that are closer than r are forbidden via (1c).

When solving this ILP model with the help of a standard solver that just separates the subtour elimination constraints (1b) some instances with $m \geq 25$ were already quite time-consuming in our test. Reasons for this might be the high number of symmetries and depending on r the large number of additionally needed subtour elimination constraints forbidding small subtours.

The paper is structured as follows. In Section 2 we start with the case $r = 0$, i. e., without forbidden neighborhood, and present shortest Hamiltonian paths and cycles. We study the case $r = 1$ in Section 3. Dividing the grid cells in *inner* and *outer* vertices allows us to derive lower bounds on the lengths of Hamiltonian paths and cycles. Then we provide construction schemes depending on the parities of m and n that show that these lower bounds are tight. For thin grids where $m \in \{1, 2, 4\}$ we have to provide special constructions in each of these cases. In Section 4 we study the case $r = \sqrt{2}$. Here the shortest allowed edges have a length of two. By dividing the grid cells into four disjoint classes according to the parity of their coordinates we show that for $m \geq 4$ there exists an optimal tour with the following structure: First we visit all vertices of one class and then we go to the next one via a step of length $\sqrt{5}$ and so on. For $m = 3$ there does not exist an optimal tour with this structure, but six edges of length $\sqrt{5}$ are needed in addition to edges of length two. Throughout the paper we are also able to relatively easily extend our results to the TSPFN with the distances measured according to the 1-norm. We conclude the paper in Section 5 and give suggestions for future work.

2 Results for $r = 0$

We start with the study of optimal tours on regular $m \times n$ grids without forbidden neighborhood, hence just the Euclidean TSP on a rectangular grid. We assign a pair of coordinates to each point, where the left upper point of the grid is related to the tuple $(1, 1)$ and the right lower point of the grid has coordinates (m, n), see Figure 2. In the following we also denote the grid as

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$

Figure 2: Visualization of the numbering of the grid cells for the 3×4 grid.

a graph where the points are the vertices and line segments between the points are the edges of this graph, i. e., each edge corresponds to a feasible connection between points visited successively. Let us further denote vertices as *odd (even) vertices* if their coordinates sum up to an odd (even) number.

Lemma 1. *Each optimal TSP tour on an $m \times n$ grid has the following length:*

a) $2(n-1)$ *for* $m = 1$*,*

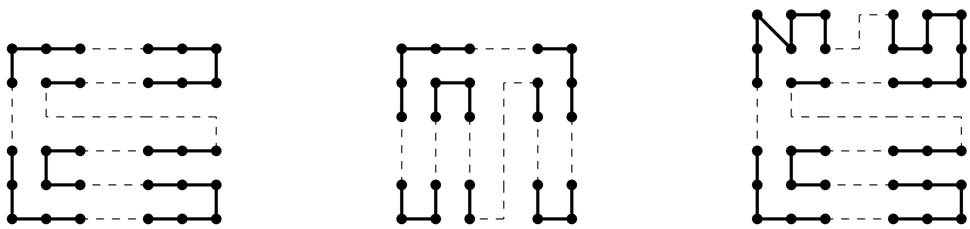
b) mn *for* m *or* n *even, see Figures 3a and 3b for illustrations of optimal tours,*

c) $mn - 1 + \sqrt{2}$ *for* m *and* n *odd,* $m \geq 3$*, see Figure 3c for an illustration of an optimal tour.*

Proof. **a):** The vertices $(1,1)$ and $(1,n)$ are $n-1$ away from each other and must both be visited.

b): There are mn vertices on the $m \times n$ grid, hence a tour contains mn edges. The shortest edges of the graph have length one. Hence mn is a lower bound on the length of an optimal tour. But a tour using only edges of length one can easily be constructed because m or n is even. Possible drawing patterns are illustrated in Figures 3a and 3b.

c): Edges of length one always link an odd and an even vertex but if m and n are both odd, the number of even vertices exceeds the number of odd vertices by one. Hence each tour must contain at least one edge connecting two even vertices. Such an edge has length at least $\sqrt{2}$. Hence $mn - 1 + \sqrt{2}$ is a lower bound on the length of an optimal tour. But a tour with this length always exists because the drawing pattern depicted in Figure 3c can be applied. □



(a) (b) (c)

Figure 3: Optimal TSP tours on $m \times n$ grids for m even, n even and both m and n odd, respectively.

We call a tour a *rook* tour (in the style of the term knight's tour, see, e. g., [26]) if and only if all of its edges connect odd and even vertices from adjacent cells. In the lemma above we implicitly proved that an $m \times n$ grid allows for a rook tour if and only if $m \geq 2$ and m or n is even.

The results of Lemma 1 stay basically the same if we use the Manhattan distance instead of the Euclidean norm. We only have to replace $mn - 1 + \sqrt{2}$ by $mn + 1$ in Lemma 1 c). A shortest Hamiltonian path on an $m \times n$ grid with $m \geq 2$ has length $mn - 1$ for both norms.

The situation gets more involved if we allow varying grid widths of the rows and columns, i. e., if we are given two sets M and N containing m respectively n (not necessarily consecutive) integers or reals. The coordinates of the mn vertices are obtained by the Cartesian product of M and N. In this case the use of edges that go diagonally or that connect non-adjacent cells might be necessary in order to get optimal solutions, see Figure 4 for two small examples. In both examples one can easily check that the best rook tour is longer than the optimal tour.

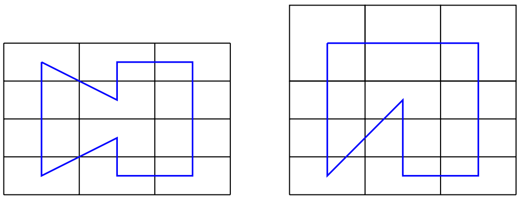


Figure 4: Optimal TSPFN tours on two 4×3 grids. In the example on the left the best rook tours have length 18, the shown optimal tour has length $12 + 2\sqrt{5} \approx 16.47$. On the right, the best rook tours have length 19, the optimal tours have length $15 + \sqrt{8} \approx 17.83$.

These two examples show that we cannot expect to find an easy construction rule for optimal tours on grids with varying row and column widths, even if we do not consider a forbidden neighborhood. For this reason, we will focus on obtaining first results for the TSPFN on grids with equidistant row and column widths in the remainder of the paper.

3 Results for $r = 1$

In this section we study the TSPFN on $m \times n$ grids with $r = 1$ that can also be interpreted as a TSP on the complement of special unit distance graphs. In unit distance graphs the vertices correspond to points in the plane and an edge between two vertices is present if the distance between the two points equals 1.

We will show that for $m \geq 5$ optimal tours can be obtained as a combination of shortest Hamiltonian paths on the odd and on the even vertices, respectively. To do so we first establish a lower bound on the length of Hamiltonian paths on the odd and on the even vertices for different parities of m and n. As we apply a forbidden neighborhood with radius one in this section, the shortest possible steps have length $\sqrt{2}$.

In the following lemma we examine how many steps of length larger than $\sqrt{2}$ are at least needed in each feasible tour for different dimensions of the grid. To simplify the corresponding counting argument we define the following sets: First, let us divide the vertices on the grid into *outer* and *inner* vertices. Outer and inner vertices are arranged in a kind of nested layers starting at the border of the rectangular grid with an outer layer and then alternating between the two types. Each layer only consists of the boundary of the corresponding rectangle that one derives by deleting all previous layers from the original grid. We refer to Figure 5 for depictions of the layers for various grid dimensions.

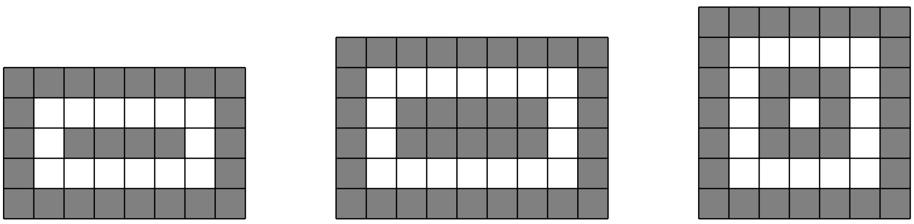


Figure 5: Division of the vertices into outer vertices (gray cells) and inner vertices (white cells) on the 5×8, 6×9 and 7×7 grid.

Furthermore, we introduce the notations *outer odd* and *outer even* vertex as well as *inner odd* and *inner even* vertex, for an illustration see Figure 6. Finally, we introduce two further terms to address special types of outer vertices. An outer vertex is called *friendly* vertex if it has an edge of length $\sqrt{2}$ to another outer vertex. A *corner* vertex lies in one of the four corners of the first outer layer. These are the four points $(1,1)$, $(1,n)$, $(m,1)$ and (m,n). We refer to Figure 7 for an illustration of both definitions.

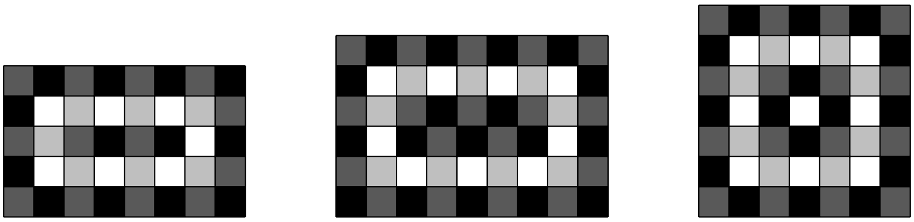


Figure 6: Division of the vertices into outer odd vertices (black cells), outer even vertices (dark gray cells), inner odd vertices (light gray cells) and inner even vertices (white cells) on the 5×8, 6×9 and 7×7 grid.

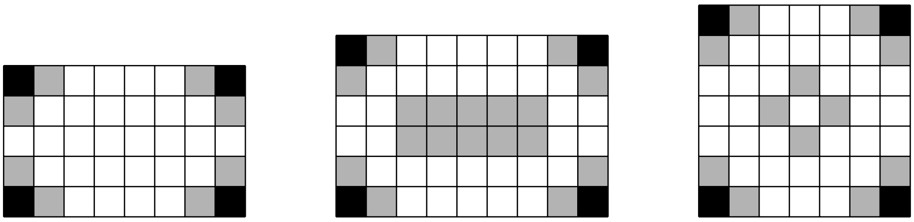


Figure 7: We highlighted the corner vertices (black cells) and the friendly vertices (gray cells) on the 5×8, 6×9 and 7×7 grid.

Lemma 2. *For an $m \times n$ grid with $m \geq 2$ the following numbers are lower bounds on the number of edges with length at least 2 in any Hamiltonian path on the odd and on the even vertices, respectively:*

a) $\frac{m}{2} - 1$ *for odd and even vertices, resp., with m even,*

b) $\frac{n}{2} - 1$ *for odd and even vertices, resp., with m odd and n even,*

c) $\frac{n+m}{2} - 1$ *for even vertices with m and n odd,*

d) $\max\{0, \frac{n-m}{2} - 1\}$ *for odd vertices with m and n odd.*

Proof. We use the following idea to establish lower bounds: Outer vertices, except for friendly vertices, can only be reached and left over edges with length $\sqrt{2}$ from inner vertices. The same holds true for paths of friendly vertices that are obtained by connecting all friendly vertices with common edges of length $\sqrt{2}$.

Each vertex has to be visited exactly once and we can start and finish the Hamiltonian path at an outer vertex.[2] Hence if the number of outer vertices exceeds the number of inner vertices by k (when we count friendly outer vertices that can be connected as one vertex), we need at least $k - 1$ edges of length at least 2.

Now we present the number of outer and inner vertices for the different parities of the grid to obtain the lower bounds stated above. The results are summarized in Table 1 where $f_{oo}(n,m)$ denotes the number of outer odd vertices, $f_{oe}(n,m)$ denotes the number of outer even vertices, $f_{io}(n,m)$ denotes the number of inner odd vertices and $f_{ie}(n,m)$ denotes the number of inner even vertices. For m odd we further have to distinguish between the cases $m \bmod 4 = 1$ and $m \bmod 4 = 3$. In order to derive the lower bounds for the minimal length of Hamiltonian paths

[2]In slight abuse of notation we will often write in this paper that a path has a start (the first node mentioned) and an end vertex (the last node mentioned) although we are in the undirected case.

m **even**	
$f_{oo}(n,m) = f_{oe}(n,m)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-4) + \frac{m \bmod 4}{2}$
$f_{io}(n,m) = f_{ie}(n,m)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-6)$
$f_{oo}(n,m) - f_{io}(n,m) = f_{oe}(n,m) - f_{ie}(n,m)$	$\frac{m}{2}$

m **odd,** n **even**	$m \bmod 4 = 1$	$m \bmod 4 = 3$
$f_{oo}(n,m) = f_{oe}(n,m)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-4) + \frac{n-m+1}{2}$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-4) + n - m + 2$
$f_{io}(n,m) = f_{ie}(n,m)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-6)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-6) + \frac{n-m+1}{2}$
$f_{oo}(n,m) - f_{io}(n,m)$	$\frac{n}{2}$	$\frac{n}{2}$
$f_{oe}(n,m) - f_{ie}(n,m)$	$\frac{n}{2}$	$\frac{n}{2}$

m, n **odd**	$m \bmod 4 = 1$	$m \bmod 4 = 3$
$f_{oo}(n,m)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-6) + \frac{n-m}{2}$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-6) + \max\{1, n-m\}$
$f_{io}(n,m)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-6)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-6) + \frac{n-m}{2}$
$f_{oe}(n,m)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-2) + \frac{n-m}{2} + 1$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-2) + n - m + 4$
$f_{ie}(n,m)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-6)$	$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-6) + \frac{n-m}{2} + 1$
$f_{oo}(n,m) - f_{io}(n,m)$	$\frac{n-m}{2}$	$\max\{1, \frac{n-m}{2}\}$
$f_{oe}(n,m) - f_{ie}(n,m)$	$\frac{n+m}{2}$	$\frac{n+m}{2}$

Table 1: Numbers of the different kinds of vertices, where $f_{oo}(n,m)$ is the number of outer odd vertices, $f_{oe}(n,m)$ is the number of outer even vertices, $f_{io}(n,m)$ is the number of inner odd vertices and $f_{ie}(n,m)$ is the number of inner even vertices. We also included the differences of the odd and the even vertices, respectively.

stated in Lemma 2, we have to subtract 1 from the differences between the number of outer and inner vertices as the Hamiltonian paths might start and end at outer vertices.

We state the calculation of the number of outer even vertices for the case m even in detail. The other cases are rather similar as can be seen in Figure 8. First note that for m even (as well as for n even) we have a symmetric situation for odd and even vertices (simply rotate the grid by 180 degrees). Let us consider the number of outer even vertices in the case m even, where we count paths of connected friendly vertices as one vertex. Then we get

$$f_{oe}(n,m) = f_{oo}(n,m) = \sum_{i=0}^{\lfloor \frac{m}{4} \rfloor -1} (n+m-8i-4) + \frac{m \bmod 4}{2}.$$

This formula can be derived in the following way. First, let us neglect that friendly vertices which can be connected by edges of length $\sqrt{2}$ are counted as just one outer vertex. For each grid layer of size $(m-4i) \times (n-4i)$, $i \in \{0, 1, \ldots, \lfloor \frac{m}{4} \rfloor - 1\}$, there are in total exactly $n-4i$ even outer vertices when jointly considering the upper and lower border of the grid block. Similarly there are in total $m-4i$ even outer vertices when jointly considering the left and right border of the grid block. But now we counted the vertices with coordinates $(1,1)$ and $(1, m-4i)$ twice, hence we subtract 2 for each block. Additionally there are two paths of length 2 of friendly vertices for

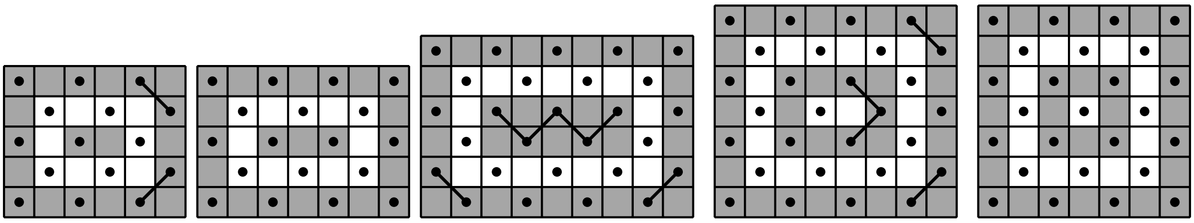


Figure 8: Visualization of the number of outer and inner even vertices for various grid dimensions representing the five cases considered in Table 1. The outer vertices have gray cells and the inner vertices white ones. Each path (single vertices are paths of length 0) represents one of the counted vertices.

each of the grid blocks, thus for counting them correctly we again subtract 2.

Finally the number of outer vertices depends on $m \bmod 4$. If $m \bmod 4 = 2$ we get additionally one large path of friendly vertices on the middle layer, else the vertices on this layer have already been counted. For an illustration we refer to the third grid from the left in Figure 8 and to the second grid from the right in the same figure if one rotates the whole grid by 90 degrees.

In a similar way one can derive $f_{io}(n,m) = f_{ie}(n,m)$. Then $f_{oe}(n,m) - f_{ie}(n,m) = f_{oo}(n,m) - f_{io}(n,m)$ simplifies to $\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor - 1} 2 + \frac{m \bmod 4}{2} = \frac{m}{2}$.

For m and n odd, the situation for odd and even vertices is not symmetric. For determining the number of even outer vertices note that there do not exist friendly vertices in this case. For an illustration we refer to the most right grid in Figure 8. □

Corollary 3. *For an $m \times n$ grid with $m \geq 2$ the following numbers are lower bounds on the length of a shortest TSPFN tour with $r = 1$:*

a) $m(n-1)\sqrt{2} + 2(m-2) + 2\sqrt{5}$ for m even,

b) $n(m-1)\sqrt{2} + 2(n-2) + 2\sqrt{5}$ for m odd and $m \neq n$,

c) $(m^2 - m - 1)\sqrt{2} + 2(m-1) + 2\sqrt{5}$ for $m = n$ odd.

Proof. A trivial lower bound on the length of any tour is $(mn-2)\sqrt{2} + 2\sqrt{5}$ because we have to switch between odd and even vertices at least twice in a tour and the shortest edges to do so have length $\sqrt{5}$. But we can improve this bound using Lemma 2.

a): Due to Lemma 2 a) a TSPFN tour with $r = 1$ must contain at least $m - 2$ edges of length at least 2 between pairs of odd as well as between pairs of even vertices. Hence we obtain $m(n-1)\sqrt{2} + 2(m-2) + 2\sqrt{5}$ as a lower bound on the length of any tour.

b): For this case we use the results from Lemma 2 b)–d) and additionally assume $m \neq n$. For both n even and n odd, resp., a TSPFN tour with $r = 1$ must contain at least $n - 2$ edges of length at least 2 between pairs of odd or pairs of even vertices, resp., as $2\left(\frac{n}{2} - 1\right) = \frac{n+m}{2} - 1 + \frac{n-m}{2} - 1 = n - 2$. Hence $n(m-1)\sqrt{2} + 2(n-2) + 2\sqrt{5}$ is a lower bound on the length of any tour.

c): For $m = n$ odd we can again apply Lemma 2 c) and d). In this case a TSPFN tour with $r = 1$ must contain at least $m - 1$ edges of length at least 2. Thus we obtain $(m^2 - m - 1)\sqrt{2} + 2(m-1) + 2\sqrt{5}$ as a lower bound on the length of any tour. □

Next we determine the length of optimal TSPFN tours with $r = 1$. We first consider grids with $m = 3$ or $m \geq 5$ and study the cases with $m \in \{1, 2, 4\}$ afterwards.

Theorem 4. *An optimal TSPFN tour with $r = 1$ on an $m \times n$ grid has:*

a) the length $m(n-1)\sqrt{2} + 2(m-2) + 2\sqrt{5}$ for $m \geq 6$ even, see Figure 9 for illustrations of optimal tours,

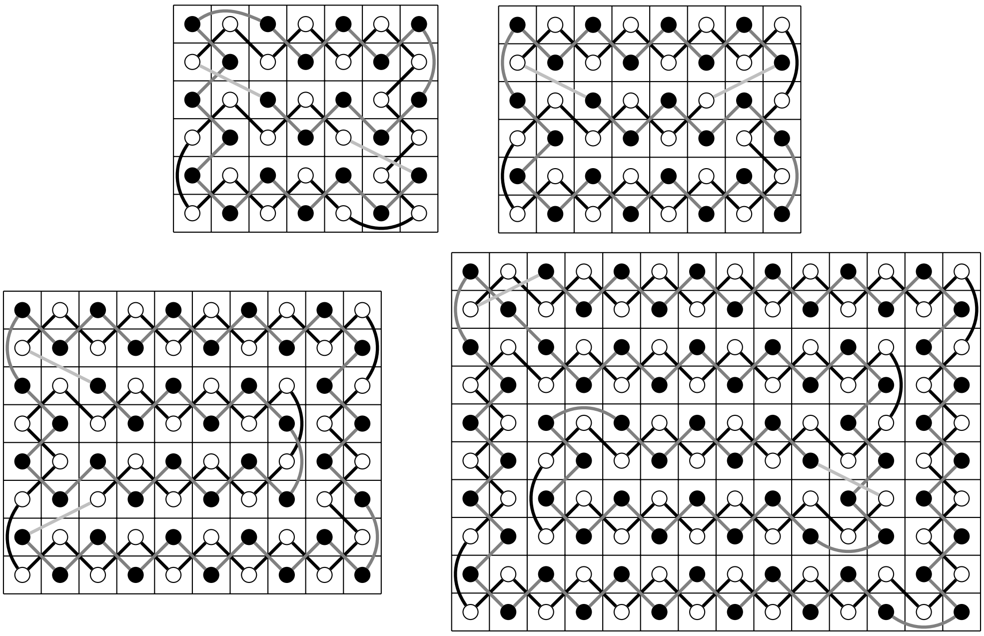


Figure 9: Illustration of optimal TSPFN tours with $r = 1$ on $m \times n$ grids with m or n even. In particular we consider the following grid dimensions: 6×7, 6×8, 8×10 and 10×14.

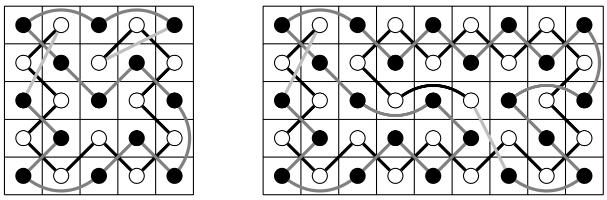


Figure 10: Visualization of optimal TSPFN tours with $r = 1$ on $m \times n$ grids with $m \bmod 4 = 1$ and $n \geq 5$ odd. In particular we display the 5×5 grid and the 5×9 grid.

b) the length $n(m-1)\sqrt{2} + 2(n-2) + 2\sqrt{5}$ for $m \geq 5$ odd and $m \neq n$, see Figures 10 and 11 for depictions of optimal tours,

c) the length $(m^2 - m - 1)\sqrt{2} + 2(m-1) + 2\sqrt{5}$ for $m \geq 5$ odd and $m = n$, see Figures 10, 11 and 12 for illustrations of optimal tours,

d) the length $2n\sqrt{2} + 2(n-2) + 2\sqrt{5}$ for $m = 3$ and $n \geq 4$ and the length $5\sqrt{2} + 4 + 2\sqrt{5}$ for $m = n = 3$, see Figure 13 for depictions of optimal tours.

Proof. Due to Corollary 3 the numbers stated in Theorem 4 a)–d) are lower bounds on the length of an optimal TSPFN tour with $r = 1$ and the respective dimension. Hence suggesting construction schemes for tours having these particular lengths is sufficient to complete the proof.

a) and b) with m or n even: In the following we assume, w. l. o. g., that m is even. Otherwise we can rotate the grid in the construction below by 90 degrees to switch the roles of m and n. Note in this context that the roles of m and n in the length formulas in a) and b) are simply exchanged.

A shortest Hamiltonian path on the odd vertices is easily constructed by starting at vertex $(2, 1)$, then visiting in clockwise direction all outer vertices on the two outside layers of the $m \times n$ grid. This can be done using only edges of length $\sqrt{2}$ except for two edges of length 2 for leaving corner vertices. The remaining vertices, i. e., the odd vertices on the remaining layers forming an $(m-4) \times (n-4)$ grid, are handled as follows: Visit consecutively all odd vertices in the rows $(3, 4)$,$(5, 6)$,. . .,$(m-3, m-2)$. Vertices located in the same pair of rows can be visited using only

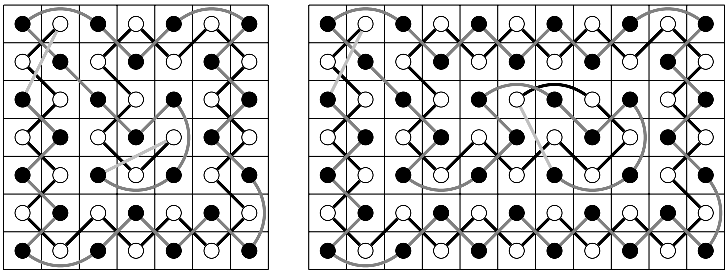


Figure 11: Illustration of optimal TSPFN tours with $r = 1$ on $m \times n$ grids with $m \bmod 4 = 3$ and $n \geq m$ odd. In particular we consider the 7×7 grid and the 7×11 grid.

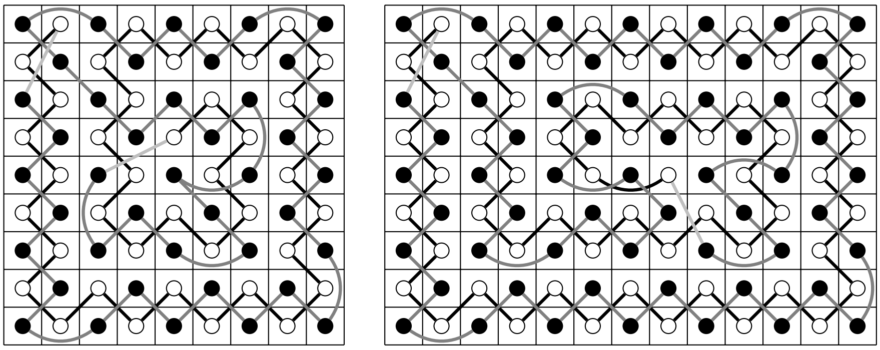


Figure 12: Visualization of optimal TSPFN tours with $r = 1$ on $m \times n$ grids with $m \bmod 4 = 1$ and $n \geq m$ odd. In particular we display the 9×9 grid and the 9×13 grid.

edges of length $\sqrt{2}$, changing the row pairs requires edges of length 2. Hence we need $\frac{m}{2} - 3$ edges of length 2 for changing between the pairs of rows and in the entire Hamiltonian path we need $\frac{m}{2} - 1$ edges of length 2 and otherwise only edges of length $\sqrt{2}$. Hence the total length of the suggested Hamiltonian path complies with the lower bound established in Lemma 2 a).

For the even vertices we distinguish several cases. If $m \bmod 4 = 0$ a Hamiltonian path of optimal length can be obtained by mirroring the Hamiltonian path for the odd vertices after row $\frac{m}{2}$. In the case $m \bmod 4 = 2, m \geq 6$ and n odd we rotate the Hamiltonian path for the odd vertices by 180 degrees. For $m = 6$ and n even a path of optimal length can be obtained by mirroring the Hamiltonian path for the odd vertices after column $\frac{n}{2}$.

It remains the case $m \bmod 4 = 2, m \geq 10$ and n even. Let us recall that the path on the odd vertices goes from $(2, 1)$ to $(m - 3, n - 2)$. Hence we start in $(1, 3)$ and zigzag until $(3, 1)$ in clockwise direction. From there we go over $(1, 1), (2, 2)$ to $(3, 3)$ and then visit all even vertices in each pair of rows using a zigzag path as long as we are at position $(m - 6, n - 2)$. From there we go down to $(m - 2, n - 2)$ and $(m - 2, n - 4)$. Finally, we visit all remaining even vertices in rows $m - 2, m - 3$ and then in $m - 4, m - 5$ such that the path ends in $(m - 4, n - 4)$.

In all cases the two Hamiltonian paths can be connected by edges of length $\sqrt{5}$, thus we obtain a tour of total length $(nm - m)\sqrt{2} + 2(m - 2) + 2\sqrt{5}$ meeting the lower bound from Corollary 3 a) and b). We refer to Figure 9 for a visualization of the constructions just described.

b), c) and d) with m **and** n **odd:** A shortest Hamiltonian path on the odd vertices can be constructed by starting at vertex $(1, 2)$, then visiting in counter-clockwise direction all outer vertices on the border of the $m \times n$ grid and then doing the same on the grids of smaller dimension as long as it is not possible to use a further step of length $\sqrt{2}$. One can easily check that all the $\max\{0, \frac{n-m}{2} - 1\}$ missing odd vertices lie in exactly one row. We connect these vertices by edges of length 2 to the other odd vertices. The position of the last vertex equals:

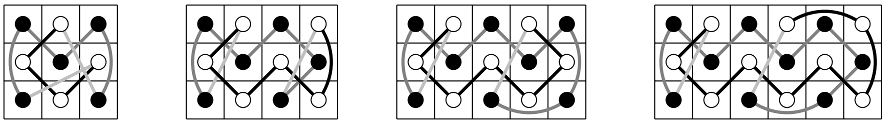


Figure 13: Illustration of optimal TSPFN tours with $r = 1$ on $3 \times n$ grids with $n \geq 3$.

conditions on m and n	position
$m = n = 3$	$(2,3)$
$m = 3, n \geq 5$	$(1,4)$
$m = n \geq 5$ and $m \bmod 4 = 1$	$(\frac{m-1}{2}, \frac{m+1}{2})$
$m \neq n, m \geq 5$ and $m \bmod 4 = 1$	$(\frac{m+1}{2}, n - \frac{m+1}{2})$
$m = n \geq 7$ and $m \bmod 4 = 3$	$(\frac{m+1}{2}, \frac{m+1}{2} + 1)$
$m \neq n, m \geq 7$ and $m \bmod 4 = 3$	$(\frac{m-1}{2}, \frac{m-1}{2} + 3)$

Hence the total length of the suggested Hamiltonian path on the odd vertices coincides with the lower bound established in Lemma 2 d).

For the Hamiltonian path on the even vertices we distinguish four cases: $m = 3$, $m = 5$ as well as $m \geq 7$ and $m \bmod 4 = 1$ or $m \bmod 4 = 3$. Each constructed Hamiltonian path on the even vertices starts at vertex $(3,1)$.

Let $m = 3$ and $n \geq m$ odd. Then our path goes from $(3,1)$ to $(1,1)$ by an edge of length 2 and continues zigzagging to $(1,n)$ by edges of length $\sqrt{2}$. The remaining $\frac{n-1}{2}$ vertices, all lying in row 3, are reached via edges of length 2. The last vertex of our path is $(3,3)$, see Figure 13.

Next we consider the case $m = 5$ and $n \geq m$ odd. A solution satisfying the desired length requirement for $m = n = 5$ is shown in Figure 10. So, let $n > m = 5$ in the following. We start the path on the even vertices at $(3,1)$, go in counter-clockwise direction along the border until position $(4, n-3)$, i. e., directly below the end of the path on the odd vertices. From there we go to $(3, n-4)$ and then visit all $\frac{n-7}{2}$ missing even vertices in this row using steps of length 2. From $(3,3)$ we go over $(2,2)$ to $(1,1)$ and $(1,3)$ and then continue zigzagging until $(1,n)$. After that we go to $(3,n)$, $(3,n-2)$, $(4,n-1)$, $(5,n)$ and end the path at $(5,n-2)$. In total we used exactly $\frac{n+3}{2}$ edges of length 2. The construction is visualized in Figure 10.

Now let $7 \leq m \leq n$, m,n odd and $m \bmod 4 = 3$. First we consider the case $m < n$. The path on the even vertices starts at $(3,1)$ and mainly[3] zigzags in counter-clockwise direction until position $(\frac{m-1}{2} + 1, \frac{m-1}{2} + 3)$, i. e., directly below the end of the path on the odd vertices. After that we go to $(\frac{m-1}{2}, \frac{m-1}{2} + 2)$ and with an edge of length 2 to $(\frac{m-1}{2}, \frac{m-1}{2} + 4)$. All missing even vertices in rows $\frac{m-1}{2}$ and $\frac{m-1}{2} + 1$ are visited via a zigzag path. Finally, we connect all missing vertices in row $\frac{m-1}{2} + 2$ via edges of length 2, ending in $(\frac{m-1}{2} + 2, \frac{m-1}{2} + 4)$. The construction for $m = n$ is similar. We start again at $(3,1)$ and zigzag in the same way as above until we reach $(\frac{m+1}{2}, \frac{m+1}{2})$. The three missing vertices are visited in clockwise direction, ending in $(\frac{m+1}{2} + 1, \frac{m+1}{2} - 1)$. In Figure 11 we visualize the constructions just described.

It remains to consider the case $9 \leq m \leq n$, m,n odd, $m \bmod 4 = 1$. As above we distinguish between $m = n$ and $m \neq n$. If $m = n$, starting at $(3,1)$ we zigzag in counter-clockwise direction with the same special treatment of the upper left corner until position $(\frac{m-1}{2}, \frac{m-1}{2})$, i. e., exactly left to the end vertex of the odd path. Then using a zigzag subpath we go to $(\frac{m-1}{2} - 1, \frac{m-1}{2} + 3)$. After that we go down two units, and then two units to the left. We continue with two single steps down right and reach all missing vertices by a zigzag subpath, ending in $(\frac{m-1}{2} + 1, \frac{m-1}{2} - 1)$, see also Figure 12 for an illustration of the construction.

If $m \neq n$ we proceed similarly, starting at $(3,1)$ we go until position $(\frac{m+1}{2} + 1, n - \frac{m+1}{2})$, i. e., directly below the end of the odd path. We continue by connecting all nodes in row $\frac{m+1}{2}$ left to this position with edges of length 2. Then we use again a zigzag subpath until we reach

[3] The only exception is in the upper left corner, when we go from $(1,3)$ to $(1,1)$ and then over $(2,2)$, to $(3,3)$ and $(4,4)$.

$(\frac{m+1}{2}, n - \frac{m+1}{2} + 3)$. For connecting the missing four vertices we go two cells to the left and then diagonally down right. Finally, we reach $(\frac{m+1}{2} + 2, n - \frac{m+1}{2} + 1)$ by a further edge of length 2. The construction is visualized in Figure 12.

In all variants the Hamiltonian paths on the even vertices use in total $\frac{n+m}{2} - 1$ edges of length 2 and all other edges have length $\sqrt{2}$. Thus the lengths coincide with the lower bound established in Lemma 2 c). The two Hamiltonian paths can be connected by edges of length $\sqrt{5}$, thus we obtain tours of total length $(nm-n)\sqrt{2}+2(n-2)+2\sqrt{5}$ for $n > m$ and $(m^2-m-1)\sqrt{2}+2(m-1)+2\sqrt{5}$ otherwise, both meeting the lower bounds from Corollary 3 b) and c). In particular, except for the case $m = n = 3$, we always connect the start vertex of the odd path at $(1, 2)$ with the start vertex of the even path at $(3, 1)$.

d) with n even: For n even and $m = 3$ we use a construction similar to the one for the odd vertices. The Hamiltonian path on the odd vertices starts at $(1, 2)$ and zigzags in counter-clockwise direction until $(3, n)$. All remaining odd vertices in row one are visited using edges of length 2, ending in $(1, 4)$. For the path on the even vertices we go from $(3, 1)$ to $(1, 1)$ and then zigzag in clockwise direction until $(3, n - 1)$. The missing vertices in row three are connected via edges of length 2, ending in $(3, 3)$. To close the tour we connect the start as well as the end vertices of the two Hamiltonian paths on the odd and even vertices. For an illustration we refer to Figure 13. □

The lengths of the associated shortest Hamiltonian paths directly follow from the last theorem and its proof.

Corollary 5. *An optimal open TSPFN tour with $r = 1$ on an $m \times n$ grid with $m \geq 6$ even or $m \geq 3$ odd is exactly $\sqrt{5}$ shorter than the optimal TSPFN tour in this case. For the exact lengths we refer the reader to Theorem 4.*

A Hamiltonian path visiting all odd (or even) vertices is called a bishop's path (in the style of the term knight's tour, see, e. g., [26]) if and only if all vertices are connected by edges of length $\sqrt{2}$. The following result on the existence of bishop's paths is another direct consequence of Theorem 4.

Corollary 6. *An $m \times n$ grid allows for a bishop's path on the odd vertices if and only $m = 2$ or $m \geq 3$ odd and $m \in \{n, n-2\}$. A bishop's path on the even vertices exists if and only if $m = 2$.*

To complete our analysis of optimal TSPFN tours with $r = 1$ we now consider the grid sizes that have not been covered in Theorem 4 as their lengths are not equal to the lower bounds established in Corollary 3. In particular we study the $1 \times n$ grids for $n \geq 5$ (for $n \leq 4$ there does not exist a feasible tour), the $2 \times n$ grids for $n \geq 3$ (for $n = 2$ there does not exist a feasible tour) and the $4 \times n$ grids.

Lemma 7. *An optimal TSPFN tour with $r = 1$ on an $m \times n$ grid has the following length:*

a) 12 *for* $m = 1$ *and* $n = 5$*;* 16 *for* $m = 1$ *and* $n = 6$*,*

b) $2n + 6$ *for* $m = 1$ *and* $n \geq 7$*,*

c) $4\sqrt{2} + 2\sqrt{5}$ *for* $m = 2$ *and* $n = 3$*;* $4\sqrt{2} + 4 + 2\sqrt{5}$ *for* $m = 2$ *and* $n = 4$*;* $4\sqrt{2} + 8 + 2\sqrt{5}$ *for* $m = 2$ *and* $n = 5$*,*

d) $(2n - 6)\sqrt{2} + 4 \cdot 2 + 2\sqrt{5}$ *for* $m = 2$ *and* $n \geq 6$*,*

e) $(4n - 5)\sqrt{2} + 3 \cdot 2 + 2\sqrt{5}$ *for* $m = 4$*.*

Proof. **a):** These values were determined by solving our ILP stated in (1) for the respective grid sizes. For a depiction of optimal tours we refer to Figure 14.

b): We checked the instances with $n \leq 11$ explicitly. The tours are shown in Figure 14, too. To establish $2n + 6$ as a lower bound for $m = 1$ and $n \geq 12$ we consider nodes $(1, 1)$ to $(1, 6)$ and

1	4	2	5	3

1	5	2	4	6	3

1	5	2	7	3	6	4

1	7	2	8	4	6	3	5

1	8	2	9	7	5	3	6	4

1	9	2	10	8	3	5	7	4	6

1	10	2	11	9	3	8	6	4	7	5

Figure 14: Depiction of optimal TSPFN tours with $r = 1$ on $1 \times n$ grids with $5 \leq n \leq 11$.

nodes $(1, n-5)$ to $(1, n)$ explicitly. By symmetry it suffices to consider the first six nodes and to show that for any feasible tour this part contributes at least 3 units more than just using edges of length 2.

For connecting $(1, 1)$ and $(1, 2)$ each with two other vertices we need at least one edge of length at least 3 in both cases. If $(1, 1)$ is visited in the tour using at least one edge of length at least 4, then our lower bound is valid. Hence let us assume that $(1, 1)$ is visited using edges of length 2 and length 3. If $(1, 2)$ is visited using at least one edge of length at least 4, then our lower bound is valid. But if we as well connect $(1, 2)$ by edges of length 2 and 3, one of the two edges connecting $(1, 3)$ to the rest of the tour has length at least 3 as otherwise a subtour would be implied. Hence in summary $2n + 6$ is a lower bound for $m = 1$ and $n \geq 12$.

Now let us suggest a TSPFN tour attaining this bound: All vertices are connected using edges of length 2 except for the six vertices most left and most right on the grid, respectively, that are connected as depicted in Figure 15 using one edge of length 3 and one edge of length 4.

1	$n-1$	2	n	3	$n-2$

$\frac{n}{2}-2$	$\frac{n}{2}+3$	$\frac{n}{2}$	$\frac{n}{2}+2$	$\frac{n}{2}-1$	$\frac{n}{2}+1$

$\frac{n+7}{2}$	$\frac{n-3}{2}$	$\frac{n+3}{2}$	$\frac{n-1}{2}$	$\frac{n+5}{2}$	$\frac{n+1}{2}$

Figure 15: The numbers in the cells indicate the order in which the left six and right six vertices on the $1 \times n$ grid with $n \geq 12$ can be visited in order to attain a TSPFN tour of length $2n + 6$. The upper right grid depicts the optimal order of the last six vertices if n is even and the lower right grid deals with the case n odd.

c): We determined these values by solving our ILP stated in (1) for the respective grid sizes. For a depiction of optimal tours we refer to Figure 16.

1	5	3
4	2	6

1	6	8	3
5	2	4	7

1	7	10	3	9
6	2	5	8	4

1	10	3	9	4	7
11	2	12	6	8	5

1	12	3	11	7	9	6
13	2	14	4	10	5	8

1	14	16	12	4	8	6	9
15	2	13	3	11	5	10	7

Figure 16: Illustration of optimal TSPFN tours with $r = 1$ on $2 \times n$ grid with $3 \leq n \leq 8$.

d): The cases with $n \in \{6, 7, 8\}$ were checked explicitly, see also Figure 16. A TSPFN tour with $r = 1$ on the $2 \times n$ grid with $n \geq 9$ has to contain two edges of length at least $\sqrt{5}$ because we have to switch from odd to even vertices. Next let us consider the four corner vertices $(1, 1)$, $(2, 1)$, $(1, n)$ and $(2, n)$. We will show that on the left and on the right side an optimal tour has a certain structure, implying the lower bound $(2n - 6)\sqrt{2} + 4 \cdot 2 + 2\sqrt{5}$.

There are a few relevant possibilities to visit the two corner vertices $(1, 1)$ and $(2, 1)$:

1. If $(1, 1)$ and $(2, 1)$ are visited by their two shortest incident edges of length $\sqrt{2}$ and 2, then none of the vertices in the second column can be visited by an edge of length $\sqrt{2}$ as otherwise a subtour is implied.
2. If $(1, 1)$ and $(2, 1)$ are visited by edges of length $\sqrt{2}$ and $\sqrt{5}$, then either $(1, 2)$ or $(2, 2)$ has to be visited by an edge of length at least 2 as otherwise a subtour is implied.

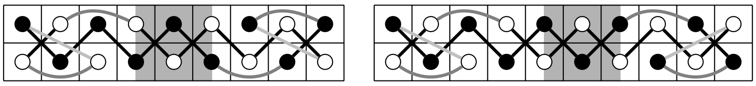


Figure 17: Visualization of drawing patterns for optimal TSPFN tours with $r = 1$ on $2 \times n$ grids. The left depiction shows the odd case and the right one the even case. Note that tours for longer grids can easily be derived by extending the gray part in the middle.

3. If, w. l. o. g., we use edges of length $\sqrt{2}$ and $\sqrt{5}$ for $(1, 1)$ and edges of length $\sqrt{2}$ and 2 for vertex $(2, 1)$, then another edge of length at least 2 is needed for visiting $(1, 2)$ without creating a subtour.
4. Another possibility is to use edges of length $\sqrt{2}$ and $\sqrt{5}$ for vertex $(1, 1)$ and edges of length $\sqrt{2}$ and $\sqrt{10}$ for vertex $(2, 1)$, but $\sqrt{2} + \sqrt{10} > 4$.

All other feasible constructions use edges that are in sum too long to yield a further improvement. For this also note that $(1, 1), (2, 1)$ and $(1, n), (2, n)$ have distance at least $n - 1 \geq 8$. In summary we need at least two additional edges of length at least two both on the left and the right side of the grid and $(2n - 6)\sqrt{2} + 8 + 2\sqrt{5}$ is a valid lower bound for the length of TSPFN tours with $r = 1$ on the $2 \times n$ grid with $n \geq 9$.

Tours attaining this lower bound can be constructed according to the pattern visualized in Figure 17. For larger grids tours can be derived by enlarging the middle part, preserving the subpaths on the left and the right end of the grid.

e): Lemma 2 a) ensures that for the $4 \times n$ grid the Hamiltonian paths on the odd and on the even vertices, resp., have to contain at least one step of length larger than $\sqrt{2}$. Corresponding Hamiltonian paths with length $(2n - 2)\sqrt{2} + 2$ can easily be constructed. But these paths have to start and end at a corner vertex and an adjacent outer vertex with distance 2 to the corner vertex because traversing a corner vertex needs one edge of length at least 2. One can easily check that it is not possible to connect the start and end vertices of the two Hamiltonian paths of minimal length (on the odd and on the even vertices) with edges of length $\sqrt{5}$. Furthermore using at least one edge of length longer than $\sqrt{5}$ for connecting odd and even vertices gives tours that are longer than $(4n - 5)\sqrt{2} + 6 + 2\sqrt{5}$, which is therefore a valid lower bound.

A tour with exactly this length can easily be constructed as follows: On the even vertices we construct a Hamiltonian path of length $(2n - 2)\sqrt{2} + 2$ starting at vertex $(1, 1)$ and going in a zigzag path in counter-clockwise direction until position $(1, 3)$. On the odd vertices we construct a Hamiltonian path of length $(2n - 3)\sqrt{2} + 4$ starting at vertex $(2, 1)$ and going in clockwise direction using a zigzag path to $(4, 3)$. From there we go to $(4, 1)$ and end at vertex $(3, 2)$. Finally, we connect these two paths by two edges of length $\sqrt{5}$, see Figure 18. □

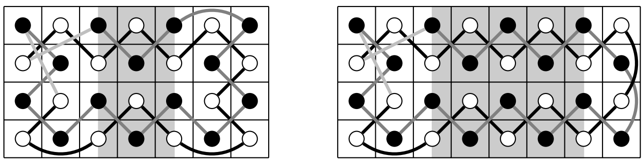


Figure 18: Illustration of optimal TSPFN tours with $r = 1$ on the $4 \times n$ grid for the cases n odd on the left and n even on the right. For larger grids the gray part has to be extended.

The results and corresponding proofs presented in this section can be simplified if we consider the Manhattan distance.

Corollary 8. *For $1 \times n$ grids with $n \geq 5$ the optimal TSPFN tours with $r = 1$ suggested in Lemma 7 a) and b) are also optimal with respect to the Manhattan distance. For the $m \times n$ grid with $m \geq 2$ and $n \geq 3$ TSPFN tours with $r = 1$ and minimal Manhattan distance always have the length $2(mn - 2) + 6$.*

Proof. When considering the Manhattan distance the shortest possible edges have length 2 and the shortest edges for connecting odd and even vertices have length 3. Hence $2(nm-2)+6$ is a lower bound on the length of an optimal tour. But the constructions of optimal tours for the various grid sizes in Theorem 4 and Lemma 7 all attain this lower bound. ☐

4 Results for $r = \sqrt{2}$

In this section we examine optimal TSPFN tours on $m \times n$ grids with $r = \sqrt{2}$. First let us extend our definition of odd and even vertices: We now denote vertices according to the parity of their coordinates, e. g., (o, e)-vertices if their first coordinate is odd and their second coordinate is even. We will show that for $m \geq 4$ optimal tours can be obtained as a combination of shortest Hamiltonian paths on the (o, e)-vertices, (o, o)-vertices, (e, o)-vertices and (e, e)-vertices.

Lemma 9. *The value $2(mn-4)+4\sqrt{5}$ is a lower bound on the length of a shortest TSPFN tour with $r = \sqrt{2}$ on the $m \times n$ grid with $m \geq 2$.*

Proof. All feasible edges have length at least 2. To switch between (o, e)-vertices, (o, o)-vertices, (e, o)-vertices and (e, e)-vertices we have to use edges of length at least $\sqrt{5}$. Hence in total $2(mn-4)+4\sqrt{5}$ is a valid lower bound. ☐

Theorem 10. *An optimal TSPFN tour with $r = \sqrt{2}$ on the $m \times n$ grid with $m \geq 4$ has length $2(mn-4)+4\sqrt{5}$.*

Proof. The claimed length is a valid lower bound for any feasible tour due to Lemma 9. Now let us suggest 8 different drawing patterns for shortest Hamiltonian paths on the (o, o)-vertices, (o, e)-vertices, (e, o)-vertices and (e, e)-vertices. We subdivide the drawing patterns with respect to their start and end vertices, where we have to take into account the different parities of the rows and columns of the grid. The drawing patterns 1, 2 and 7 start at the upper left vertex, i. e., $(1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 2)$, respectively. For the other drawing patterns the start vertex is shifted by one unit to the right or downwards.

- Drawing patterns 1 to 6 can be derived by deleting exactly one edge from the patterns visualized in Figure 3a and 3b. They can be applied if m or n is even. Pattern 1 finishes at the next vertex to the right of the start vertex, i. e., at $(1, 3)$, $(1, 4)$, $(2, 3)$ and $(2, 4)$, respectively. Pattern 2 finishes at the next vertex below the start vertex, i. e., at $(3, 1)$, $(3, 2)$, $(4, 1)$ and $(4, 2)$, respectively. For pattern 3 we just interchange the role of the start and the end vertex in drawing pattern 2. Similarly, pattern 4 can be derived by interchanging the role of the start and the end vertex in pattern 1. Finally, drawing pattern 5 starts at $(3, 1), (3, 2), (4, 1)$ and $(4, 2)$, respectively, and finishes at $(5, 1)$, $(5, 2)$, $(6, 1)$ and $(6, 2)$, respectively. For drawing pattern 6 we mirror the start and end positions of pattern 5 at the diagonal from $(1, 1)$ to (m, m). Hence it starts at $(1, 3), (1, 4), (2, 3)$ and $(2, 4)$, respectively, and finishes at the vertices $(1, 5)$, $(1, 6)$, $(2, 5)$ and $(2, 6)$, respectively.
- Drawing pattern 7 finishes diagonally below the upper right vertex of the grid, i. e., at $(3, n-3), (3, n-2), (4, n-3)$ and $(4, n-2)$, respectively. It can be realized if m is odd or n is even, for an illustration see Figure 19. We only use this pattern for the constructions of type **10** below. Note, we can also rotate the pattern by 90, 180 and 270 degrees and then apply it.
- For drawing pattern 8 we delete the diagonal edge from the pattern visualized in Figure 3c. It starts at the vertices $(3, 3)$, $(3, 4)$, $(4, 3)$ and $(4, 4)$, respectively, and ends in $(1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 2)$ respectively. It can be realized if m and n are odd.

Table 2 lists all possible combinations of the parities of the dimension of the four grids corresponding to the (o, o)-, (o, e)-, (e, o)- and (e, e)-vertices. We present constructions of optimal

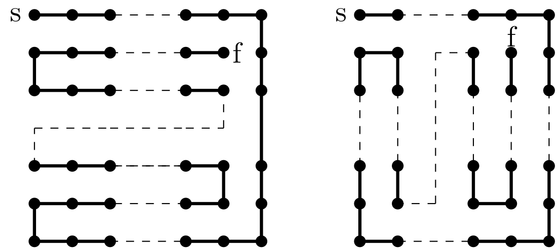


Figure 19: Drawing pattern 7 can be applied if m is odd or n is even. We say the path starts at s and ends in f. We highlighted only the nodes of one of the vertex classes of the original $m \times n$ grid.

TSPFN tours with $r = \sqrt{2}$ for all $m \times n$ grids with $m \geq 4$. Note that drawing patterns 1–4 are applicable for grids with $m \geq 4$, where the subgrids for our different vertex types have at least 2 rows and columns. For the other drawing patterns $m \geq 6$ is required, i. e., the subgrids must have at least 3 rows and columns.

	c	Parities column dimensions											
r		*e*	*e*	type	*e*	*o*	type	*o*	*e*	type	*o*	*o*	type
Parities row dim.	*e*	*ee*	*ee*	**1**	*ee*	*eo*	**1**	*eo*	*ee*	**1**	*eo*	*eo*	**1**
	e	*ee*	*ee*		*ee*	*eo*		*eo*	*ee*		*eo*	*eo*	
	e	*ee*	*ee*	**1**	*ee*	*eo*	**2**	*eo*	*ee*	**3**	*eo*	*eo*	**4**
	o	*oe*	*oe*		*oe*	*oo*		*oo*	*oe*		*oo*	*oo*	
	o	*oe*	*oe*	**1**	*oe*	*oo*	**5**	*oo*	*oe*	**6**	*oo*	*oo*	**7**
	e	*ee*	*ee*		*ee*	*eo*		*eo*	*ee*		*eo*	*eo*	
	o	*oe*	*oe*	**1**	*oe*	*oo*	**8**	*oo*	*oe*	**9**	*oo*	*oo*	**10**
	o	*oe*	*oe*		*oe*	*oo*		*oo*	*oe*		*oo*	*oo*	

Table 2: This table lists all possible combinations of dimensions of the four grids corresponding to the (o, o)-, (o, e)-, (e, o)- and (e, e)-vertices.

Optimal solutions can be derived by the schemes shown in Figure 20. Only for parity type **10** we explicitly describe the construction:

We start at vertex $(2, 2)$. Then we visit all (e, e)-vertices using drawing pattern 7. By using an edge of length $\sqrt{5}$ we reach vertex $(2, n-1)$ from $(4, n-2)$. After this we visit all (e, o)-vertices using drawing pattern 7 rotated by 90 degrees to the right and go with an edge of length $\sqrt{5}$ from $(n-2, n-3)$ to $(n-1, n-1)$. From there we visit all (o, o)-vertices using drawing pattern 7 rotated by 180 degrees and continue by an edge of length $\sqrt{5}$ from $(n-3, 3)$ to $(n-1, 2)$. Finally, we visit all (o, e)-vertices using drawing pattern 7 rotated by 90 degrees to the left and go back from $(3, 4)$ to $(2, 2)$ by an edge of length $\sqrt{5}$.

In general, all construction schemes can easily by applied if $m \geq 6$ because then all subgrids contain at least three rows and columns. If $m = 4$, then independent of the size of n we have parity type **1** that only uses drawing patterns 1 and 2. The case $m = 5$, $n \geq 8$ and $n \bmod 4 = 0$ leads again to type **1**. For $m = 5$, $n \geq 5$ and $n \bmod 4 \neq 0$, we are in one of the parity types **5** to **7**. In this case drawing patterns 3 and 4 can easily be applied and also drawing pattern 8 is feasible as we only use it if the smaller dimension of the subgrid is at least dimension 3. □

The length of shortest Hamiltonian paths is an immediate consequence of the theorem above.

Corollary 11. *An optimal open TSPFN tour with* $r = \sqrt{2}$ *on the* $m \times n$ *grid with* $m \geq 4$ *has length* $2(mn - 4) + 3\sqrt{5}$.

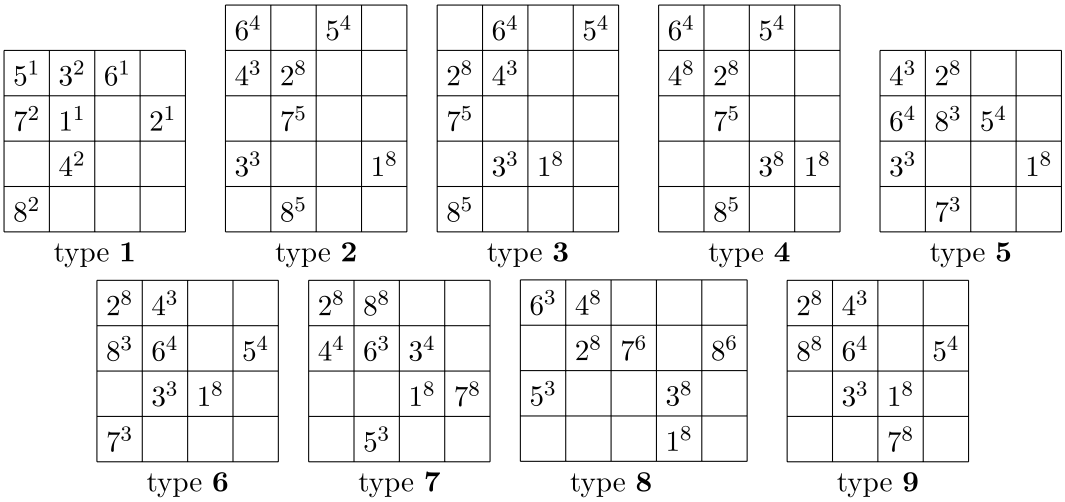


Figure 20: Visualization of the constructions of optimal TSPFN tours with $r = \sqrt{2}$ for the parity types **1**–**9**. We only show the decisive upper left part of the grid. The large numbers denote the start vertex 1 and by 2 to 8 the next important vertices where we jump from one subgrid to the next and accordingly change the parity type. The small number in the upper right corner of the vertices specifies which of the drawing patterns is used to connect the vertices of the same parity. The illustrations show that for changing the parity type always edges of length $\sqrt{5}$ can be used.

To complete our analysis of optimal TSPFN tours with $r = \sqrt{2}$ we consider the grid sizes with $1 \leq m \leq 3$ that have not been covered by Theorem 10 as their lengths are not equal to the lower bound established in Lemma 9. Clearly the neighborhoods $r = 1$ and $r = \sqrt{2}$ are equivalent on $1 \times n$ grids for $n \geq 5$ (for $n \leq 4$ there does not exist a feasible tour). Hence for this case we refer to Lemma 7 a) and b). For $2 \times n$ grids we were not able to discover an explicit formula for the lengths of the optimal tours and determined the exact values only for small n. For the length of optimal open tours we conjecture the following in this case:

Conjecture 12. *An optimal open TSPFN tour with $r = \sqrt{2}$ on an $2 \times n$ grid with $n \geq 20$ has length $4n + 1 + 2\sqrt{17} + \sqrt{37}$ for $n \geq 20$. In particular it consists of one step of length $\sqrt{37}$, two steps of length $\sqrt{17}$, two steps of length 5, one step of length 4, one step of length 3 and $2n - 8$ steps of length 2.*

Finally we consider $3 \times n$ grids with $n \geq 4$. Note that there does not exist a feasible tour on the 3×3 grid.

Lemma 13. *On the $3 \times n$ grid with $n \geq 4$ a lower bound for the length of any TSPFN tour with $r = \sqrt{2}$ is $2(3n - 6) + 6\sqrt{5}$ for Euclidean and $6n + 6$ for Manhattan distance.*

Proof. Let us first argue that there does not exist a tour using four edges of length $\sqrt{5}$ and otherwise only edges of length 2: In contrast to grids with $m \geq 4$ there only exist edges of length $\sqrt{5}$ between (e, e)- and (e, o)-vertices, between (e, o)- and (o, o)-vertices as well as between (o, o)- and (o, e)-vertices. Note, edges connecting vertices of types where there does not exist an edge of length $\sqrt{5}$ have length at least 3.

Building on the arguments above we now show that $2(nm - 6) + 6\sqrt{5}$ is a lower bound for the length of any TSPFN tour with $r = \sqrt{2}$ on the $3 \times n$ grid with $n \geq 4$: The shortest feasible options to connect the different vertex types such that we are able to visit all vertices are either

1. use one edge of length 4 and one of length 3 to construct a tour in which we visit all vertices of each parity type at once or

2. we do not visit all vertices of one type at once and hence use at least two additional edges of length (at least) $\sqrt{5}$.

For constructing a tour according to the second option above we can, e. g., first visit all (o, e)-vertices, then visit a part of the (o, o)-vertices, after this visit a part of the (e, o)-vertices, then visit all (e, e)-vertices, after this visit the rest of the (e, o)-vertices and finally visit the rest of the (o, o)-vertices. As $3\sqrt{5} + 4 + 3 + 2 \approx 15.71 > 13.42 \approx 6\sqrt{5}$ the second option is the shorter one when applying the Euclidean norm. For the Manhattan distance both options yield the same lower bound $6n + 6$. □

Next we show that the lower bound deduced above is in fact the optimal length of TSPFN tours with $r = \sqrt{2}$ on the $3 \times n$ grid with $n \geq 4$.

Theorem 14. *An optimal TSPFN tour with $r = \sqrt{2}$ on the $3 \times n$ grid with $n \geq 4$ has length $2(3n - 6) + 6\sqrt{5}$.*

Proof. The claimed length is a lower bound for the length of any TSPFN tour with $r = \sqrt{2}$ due to Lemma 13. A TSPFN tour with $r = \sqrt{2}$ of the claimed length is given for $4 \leq n \leq 6$ in Figure 21.

3	5	11	9
1	8	2	7
4	6	12	10

5	7	4	11	13
1	10	2	9	3
6	8	15	12	14

5	7	4	13	16	14
1	12	2	11	3	10
6	8	18	9	17	15

3	9	6	10
1	11	2	12
4	8	5	7

13	6	12	9	11
1	4	2	5	3
14	7	15	8	10

14	12	15	7	16	8
1	4	2	5	3	6
13	11	18	10	17	9

Figure 21: Optimal TSPFN tours with $r = \sqrt{2}$ on $3 \times n$ grids with $4 \leq n \leq 6$ minimizing the Euclidean distance on the left and the Manhattan distance on the right.

Furthermore an optimal tour can be obtained for all $3 \times n$ grids with $n \geq 7$ by following the construction visualized in Figure 22:

1. Visit all (e, o)-vertices starting in $(2, 1)$. Use an edge of length $\sqrt{5}$ to go from $(2, n-1)$ to $(1, n-3)$ if n is even and from $(2, n)$ to $(1, n-2)$ if n is odd.
2. Visit the (o, o)-vertices going in counter-clockwise direction until you reach $(3, 1)$. Use an edge of length $\sqrt{5}$ to go to $(1, 2)$.
3. Visit the (o, e)-vertices in counter-clockwise direction until reaching $(3, n-2)$ if n is even and $(3, n-3)$ if n is odd. Use an edge of length $\sqrt{5}$ to reach $(2, n)$ if n is even and $(2, n-1)$ if n is odd.
4. Visit all (e, e)-vertices. Use an edge of length $\sqrt{5}$ to go to $(1, 4)$.
5. Visit the remaining (o, e)-vertices in clockwise direction reaching $(3, n)$ if n is even and $(3, n-1)$ if n is odd. Use an edge of length $\sqrt{5}$ to reach $(1, n-1)$ if n is even and $(1, n)$ if n is odd.
6. Visit the remaining (o, o)-vertices in clockwise direction. Use an edge of length $\sqrt{5}$ to reach the start vertex $(2, 1)$. □

n	$n+2$	$n-1$	$2n+1$
1	$2n$	2	$2n-1$
$n+1$	$n+3$	$3n$	$n+4$

$\frac{5n-5}{2}$	$\frac{n+3}{2}$	$\frac{5n-3}{2}$	$\frac{5n+1}{2}$
$\frac{3n+5}{2}$	$\frac{n-1}{2}$	$\frac{3n+3}{2}$	$\frac{n+1}{2}$
$\frac{3n+1}{2}$	$\frac{5n+5}{2}$	$\frac{5n-1}{2}$	$\frac{5n+3}{2}$

Figure 22: Construction of an optimal TSPFN tour with $r = \sqrt{2}$ on the $3 \times n$ grid with $n \geq 7$ having length $2(3n - 4) + 6\sqrt{5}$. In the illustration we assume, w. l. o. g., that n is odd.

Corollary 15. *An optimal open TSPFN tour with $r = \sqrt{2}$ on the $3 \times n$ grid with $n \geq 4$ has length $2(3n-4) + 3\sqrt{5}$.*

Proof. The given length is a lower bound as we always need an edge of length $\sqrt{5}$ to connect Hamiltonian paths on the (o,o)-vertices, (e,e)-vertices, (e,o)-vertices or (o,e)-vertices. An open TSPFN tour of the claimed length can be obtained for all $3 \times n$ grids with $n \geq 4$ by following the construction illustrated in Figure 23:

1. Visit all (e,o)-vertices starting in $(2,1)$ using edges of length 2. Use an edge of length $\sqrt{5}$ to go to $(1, n-3)$ if n is even and $(1, n-2)$ if n is odd.
2. Visit all (o,o)-vertices going in counter-clockwise direction ending in $(1, n-1)$ if n is even and in $(1, n)$ if n is odd. Use an edge of length $\sqrt{5}$ to go to $(3, n)$ if n is even and to $(3, n-1)$ if n is odd.
3. Visit all (o,e)-vertices in counter-clockwise direction. Use an edge of length $\sqrt{5}$ to go from $(3, n-2)$ to $(2, n)$ if n is even and from $(3, n-3)$ to $(2, n-1)$ if n is odd.
4. Visit all (e,e)-vertices using edges of length 2 ending in $(2,2)$. □

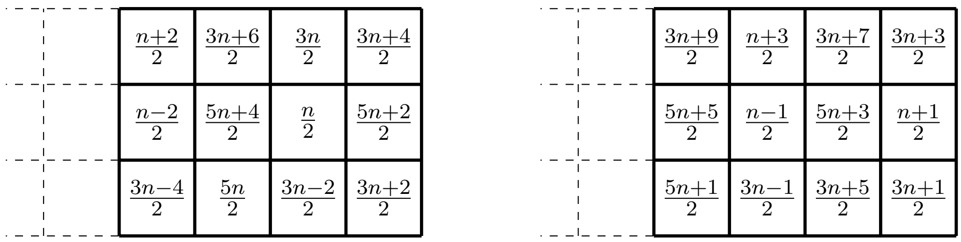


Figure 23: Visualization of an optimal open TSPFN tour with $r = \sqrt{2}$ on the $3 \times n$ grid with $n \geq 4$ having length $2(3n-4) + 3\sqrt{5}$. On the left we show the case n even and on the right the case n odd.

Let us extend the above results to the Manhattan distance. We can see in Figure 21 that optimal tours minimizing Euclidean and Manhattan distance are different for $3 \times n$ grids with $4 \leq n \leq 6$. In the following corollary we show that these are the only cases in which optimal tours do not coincide.

Corollary 16. *An optimal TSPFN tour with $r = \sqrt{2}$ minimizing the Manhattan distance has length $6n+6$ on the $3 \times n$ grid with $n \geq 7$. An optimal open TSPFN tour with $r = \sqrt{2}$ minimizing Manhattan distance has length $6n+1$ on the $3 \times n$ grid with $n \geq 4$.*

Proof. The statement on optimal open TSPFN tours follows immediately from Corollary 15 as the shortest edges to connect Hamiltonian paths on the (o,o)-vertices, (e,e)-vertices, (e,o)-vertices or (o,e)-vertices have Manhattan distance 3.

The statement on optimal TSPFN tours also directly follows from the results above as the length of the tours suggested in Theorem 14 is $6n+6$ in Manhattan distance, which is equal to the lower bound deduced in Lemma 13. Note that when applying the Manhattan distance the tours following construction option 1 in Lemma 13 lead to optimal solutions as well. Such tours can, e. g., be obtained by switching the last two vertices in the open tour suggested in Corollary 15 and then connecting start vertex $(2,1)$ and end vertex $(2,4)$. □

5 Conclusion and Future Work

In this paper we introduced a new variant of the traveling salesman problem (TSP). More precisely, we considered Euclidean TSP instances where direct connections between two points that are too close are forbidden, leading to the TSP with forbidden neighborhoods (TSPFN). After presenting a straightforward ILP formulation we studied the structure and the value of optimal solutions for instances on regular $m \times n$ grids for the smallest reasonable forbidden neighborhoods. In particular, we started without neighborhood, afterwards edges of length 1 and then additionally edges of length $\sqrt{2}$ were forbidden. We provided exact formulas for the lengths of optimal tours as well as construction schemes for tours that attain this value for almost all combinations of m and n.

It remains for future work to study larger neighborhoods. A natural next step would be to forbid all edges with length at most 2. Then the shortest possible edges would have length $\sqrt{5}$. Such connections correspond to so called *knight's moves* in chess. Knight's tours that only use such moves are well studied and their existence on regular $m \times n$ grids is well understood, see [14, 26]. There are construction schemes for such tours if they exist at all [9, 11, 21, 24]. An immediate question is how optimal solutions of the TSPFN look like if there does not exist a knight's tour on the respective grid.

Another interesting extension of our results would be the consideration of non-regular grids where not all grid points are present or where the row and column widths vary. It also remains for future work to determine the complexity of the Euclidean TSP on grids with varying row and column widths and the approximability of the general TSPFN.

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