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A fresh CP look at mixed-binary QPs: New formulations and relaxations

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Abstract Triggered by Burer's seminal characterization from 2009, many copositive (CP) reformulations of mixed-binary QPs have been discussed by now. Most of them can be used as proper relaxations, if the intractable co(mpletely)positive cones are replaced by tractable approximations. While the widely used approximation hierarchies have the disadvantage to use positive-semidefinite (psd) matrices of orders which rapidly increase with the level of approximation, alternatives focus on the problem of keeping psd matrix orders small, with the aim to avoid memory problems in the interior point algorithms. This work is the first to treat the various variants from a common theoretical perspective, using concise arguments and discussing conic duality.

Keywords Copositivity · Completely positive · Conic optimization · Quadratic optimization · Reformulations · Nonlinear optimization · Nonconvex optimization

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1 Introduction

1.1 Motivation and basic ideas

Copositive optimization (or copositive programming coined in [6], abbreviated CP) is a special case of conic optimization, which consists of minimizing a linear function over a (convex) cone subject to additional (inhomogeneous) linear (inequality or equality) constraints. This problem class has a close connection to that of quadratic optimization, which represents the simplest class of hard problems in continuous optimization [17] – to minimize a (possibly indefinite) quadratic form over a polyhedron given in standard form:

$$\min\left\{\mathsf{x}^\top \mathsf{Q}\mathsf{x} : \mathsf{A}\mathsf{x} = \mathsf{b}\,,\ \mathsf{x}\in\mathbb{R}^n_+\right\}\ . \tag{1}$$

By bold-faced lower-case letters we denote vectors in n-dimensional Euclidean space $\mathbb{R}^n$ (e.g. the zero vector o), by bold-faced upper-case letters matrices (e.g. the zero matrix O), and by $^\top$ transposition. The nonnegative orthant is denoted by $\mathbb{R}^n_+ := \{\mathsf{x}\in\mathbb{R}^n : x_i \geq 0 \text{ for all } i\in[1\!:\!n]\}$. For two integers m and n with $m \leq n$ we abbreviate $[m\!:\!n]$ for the integer interval $\{m, m+1, \dots, n\}$.

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The basic lifting idea goes back to Shor [21] and linearizes the quadratic form $\mathsf{x}^\top \mathsf{Q}\mathsf{x} = \operatorname{trace}(\mathsf{x}^\top \mathsf{Q}\mathsf{x}) = \operatorname{trace}(\mathsf{Q}\mathsf{x}\mathsf{x}^\top) = \langle \mathsf{Q}, \mathsf{x}\mathsf{x}^\top \rangle$ by introducing the new symmetric matrix variable $\mathsf{X} = \mathsf{x}\mathsf{x}^\top$ and using the Frobenius inner product $\langle \mathsf{X}, \mathsf{Y} \rangle = \operatorname{trace}(\mathsf{X}\mathsf{Y})$. If $\mathsf{A}\mathsf{x} \in \mathbb{R}^m_+$ for all $\mathsf{x} \in \mathbb{R}^n_+$ and $\mathsf{b} \in \mathbb{R}^m_+$, then the linear constraints in (1) can be squared, to arrive in a similar way at linear constraints of the form $\langle \mathsf{A}_i, \mathsf{X} \rangle = b_i^2$, where $\mathsf{A}_i = \mathsf{a}_i \mathsf{a}_i^\top$ and $\mathsf{a}_i^\top$ is the i-th row of A.

Now the set of all these $\mathsf{X} = \mathsf{x}\mathsf{x}^\top$ generated by nonnegative x is non-convex since rank $(\mathsf{x}\mathsf{x}^\top) = 1$. The convex hull

$$\operatorname{conv}\left\{\mathsf{x}\mathsf{x}^\top : \mathsf{x} \in \mathbb{R}^n_+\right\} ,$$

results in a convex matrix cone called the cone of *completely positive matrices* (see e.g. [3]). Note that a similar construction dropping nonnegativity constraints leads to

$$\operatorname{conv}\left\{\mathsf{x}\mathsf{x}^\top : \mathsf{x} \in \mathbb{R}^n\right\} ,$$

the cone of positive-semidefinite matrices, the basic set in *Semidefinite Optimization (SDP)*, wherefrom above lifting idea was borrowed.

Both variants of lifting give rise to relaxations which are considerably tighter than previously used ones. But one can say more: Burer [10] showed in a seminal paper that the CP approach (for surveys see e.g. [4,12,13, 16]) is in fact a reformulation rather than a relaxation, of any mixed-binary QP, under mild conditions. Since then, many alternative CP reformulations of mixed-binary QPs have been discussed, and one of the aims of this article is a unified view of them. We will observe that although the primal CP problems are all identical, their relaxations and their duals (and the relaxation of the duals) may differ, which is important in practical implementations.

In this paper we will review the reformulation of Burer [10], along with some related reformulations inspired by [1,11,14] and some new reformulations introduced in this paper. In Section 2 we will consider these from a common theoretical perspective, allowing us to clearly see that they are all reformulations of each other. In Section 3 we will then consider the dual problems to these reformulations and look at how the reformulation used effects the duality gap. Finally we look at conditions for when there is no duality gap (i.e. strong duality).

Note that in an effort to reduce notation, we use min and max at places where we could write inf and sup if attainability is not (yet) guaranteed. For the readers' convenience, we repeat some of the short proofs which may be already found elsewhere scattered around in the existing literature. To the best of our knowledge, this unified view on the problem class considered here is novel.

1.2 Further notation and terminology

The set of all $d \times p$ matrices is denoted by $\mathbb{R}^{d\times p}$, the subset of those with no negative entries by $\mathbb{R}^{d\times p}_+$, and

$$\mathcal{S}^d := \left\{\mathsf{X} \in \mathbb{R}^{d\times d} : \mathsf{X} = \mathsf{X}^\top\right\} .$$

By $\mathsf{I}_n \in \mathcal{S}^n$ we denote the identity matrix with columns $\mathsf{e}_i \in \mathbb{R}^n$, $i \in [1:n]$.

As already mentioned, Frobenius inner product is denoted by $\langle \mathsf{S}, \mathsf{X} \rangle = \operatorname{trace}(\mathsf{S}\mathsf{X})$, where $\{\mathsf{S}, \mathsf{X}\} \subset \mathcal{S}^d$. With regard to any duality, for a given cone $\mathcal{A}$, we consider its dual cone

$$\mathcal{A}^* = \{\mathsf{b} : \langle \mathsf{a}, \mathsf{b} \rangle \geq 0 \text{ for all } \mathsf{a} \in \mathcal{A}\} .$$

We let $\mathcal{S}^d_+$ be the positive semidefinite cone, i.e. the set of matrices $X \in \mathcal{S}^d$ such that X is positive-semidefinite (psd). Furthermore, $\mathcal{N}^d := \left\{\mathsf{X} \in \mathcal{S}^d : X_{ij} \geq 0 \text{ for all } i,j\right\}$, while $\mathcal{COP}^d$ denotes the cone of all symmetric $d \times d$ copositive matrices:

$$\mathcal{COP}^d = \{\mathsf{M} \in \mathcal{S}^d : \mathsf{x}^\top \mathsf{M}\mathsf{x} \geq 0 \text{ for all } \mathsf{x} \in \mathbb{R}^d_+\},$$

and $\mathcal{CP}^d$ the cone of all completely positive matrices of order d:

$$\mathcal{CP}^d = \operatorname{conv}\left\{\mathsf{x}\mathsf{x}^\top : \mathsf{x} \in \mathbb{R}^d_+\right\} = \left\{\mathsf{X} \in \mathcal{S}^d : \mathsf{X} = \sum_{i=1}^{p(d)} \mathsf{f}_i \mathsf{f}_i^\top \text{ for some } \mathsf{f}_i \in \mathbb{R}^d_+\right\} .$$

Here the upper bound $p(d)$ on the necessary number of summands for X, the so-called *cp-rank* of X, can be set to $p(d) = \max\left\{\binom{d+1}{2} - 4, d\right\}$ which is asymptotically tight for large d [8,20].

1.3 Setup and new variants of CP formulation

In this paper we consider the following mixed-binary quadratic optimization problem:

$$\begin{aligned} \min\ & \mathsf{x}^\top \mathsf{Q}\mathsf{x} + 2\mathsf{c}^\top \mathsf{x} \\ \text{s.t. } & \mathsf{a}_i^\top \mathsf{x} = b_i \qquad \text{for } i\in[1:m] \\ & \mathsf{x} \in \mathbb{R}^n_+ \\ & x_j \in \{0,1\} \qquad \text{for } j \in B\,, \end{aligned} \tag{P}$$

where $B \subseteq [1:n]$; $\mathsf{Q} \in \mathcal{S}^n$; $\mathsf{b} \in \mathbb{R}^m$; and $\{\mathsf{c}, \mathsf{a}_1, \dots, \mathsf{a}_m\} \subset \mathbb{R}^n$, with $\mathsf{a}_1, \dots, \mathsf{a}_m$ being linearly independent.

We define the polyhedron $\mathcal{Z} := \{\mathsf{x} \in \mathbb{R}^n_+ : \mathsf{a}_i^\top \mathsf{x} = b_i,\ i\in[1:m]\}$, noting that the feasible set of (P), feas (P), is contained in $\mathcal{Z}$. We shall then assume there exists $\mathsf{x}_0 \in \mathcal{Z}$ such that $(\mathsf{x}_0)_j > 0$ for all $j\in[1:n]$. If this is not the case then either $\mathcal{Z} = \emptyset$ (and so (P) is infeasible) or there exists a $j\in[1:n]$ such that $x_j = 0$ for all $\mathsf{x} \in \mathcal{Z}$ (and thus also for all x feasible in (P)). In this second case we can discard the variable x_j from problem (P), reducing the size of the problem. The checks required in this paragraph can all be performed with simple linear optimization problems.

We shall also assume that the following key assumption holds for this problem (cf. the discussion in [7]):

$$\mathsf{v} \in \mathcal{Z} \quad \implies \quad v_j \le 1 \text{ for all } j \in B. \tag{2}$$

This assumption can always be made to hold by adding slack variables. Letting $\mathcal{Z}_\infty$ be the recession cone of $\mathcal{Z}$, i.e. $\mathcal{Z}_\infty := \{\mathsf{z} \in \mathbb{R}^n : \mathsf{x} + \lambda \mathsf{z} \in \mathcal{Z} \text{ for all } \mathsf{x} \in \mathcal{Z},\ \lambda \ge 0\} = \{\mathsf{z} \in \mathbb{R}^n_+ : \mathsf{a}_i^\top \mathsf{z} = 0,\ i\in[1:m]\}$, the key assumption implies the following property (by looking at $\mathsf{v} = \mathsf{x}_0 + t\mathsf{z}$ with $t \uparrow \infty$):

$$\mathsf{z} \in \mathcal{Z}_\infty \quad \implies \quad z_j = 0 \text{ for all } j \in B\,. \tag{3}$$

We shall let $\{\mathsf{a}_{m+1}, \dots, \mathsf{a}_n\} \subset \mathbb{R}^n$ be such that $\mathsf{a}_{m+1}, \dots, \mathsf{a}_n$ are linearly independent vectors with $\mathsf{a}_i^\top \mathsf{a}_{m+j} = 0$ for all $i\in[1:m]$ and all $j\in[1:n-m]$. Then letting $\mathsf{B} \in \mathbb{R}^{(n+1)\times(n+1-m)}$ be such that

$$\mathsf{B} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \mathsf{x}_0 & \mathsf{a}_{m+1} & \mathsf{a}_{m+2} & \cdots & \mathsf{a}_n \end{bmatrix},$$

we have

$$\left\{ \begin{pmatrix} \zeta \\ \mathsf{z} \end{pmatrix} \in \mathbb{R}^{n+1} : \mathsf{a}_i^\top \mathsf{z} = b_i \zeta \text{ for } i\in[1:m] \right\} = \left\{ \mathsf{B}\mathsf{y} : \mathsf{y} \in \mathbb{R}^{n+1-m} \right\}. \tag{4}$$

Note that linear independence of $\mathsf{a}_{m+1}, \dots, \mathsf{a}_n$ implies that B has full column rank, and thus for any $\mathsf{Y} \in \mathcal{S}^{n+1-m}$ we have

$$\mathsf{B}\mathsf{Y}\mathsf{B}^\top \in \mathcal{S}^{n+1}_+ \quad \iff \quad \mathsf{Y} \in \mathcal{S}^{n+1-m}_+\,, \quad \text{and} \tag{5}$$

$$\mathsf{B}\mathsf{Y}\mathsf{B}^\top = 0 \quad \iff \quad \mathsf{Y} = 0. \tag{6}$$

In [10], Burer showed that provided the key assumption (2) holds, then (P) has the same optimal value as the problem (CPP) below.

$$\begin{aligned} \min\ & \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} \\ \text{s.t. } & \mathsf{a}_i^\top \mathsf{x} = b_i \qquad \text{for } i\in[1:m] \\ & \langle \mathsf{a}_i \mathsf{a}_i^\top, \mathsf{X} \rangle = b_i^2 \qquad \text{for } i\in[1:m] \\ & X_{jj} = x_j \qquad \text{for } j \in B \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{CP}^{n+1}\,. \end{aligned} \tag{CPP}$$

In order to show this, he also proved the result

$$\left\{ \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} : (\mathsf{x}, \mathsf{X}) \in \text{feas (CPP)} \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 1 \\ \mathsf{x} \end{pmatrix} \begin{pmatrix} 1 \\ \mathsf{x} \end{pmatrix}^\top : \mathsf{x} \in \text{feas (P)} \right\} + \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ \mathsf{z} \end{pmatrix} \begin{pmatrix} 0 \\ \mathsf{z} \end{pmatrix}^\top : \mathsf{z} \in \mathcal{Z}_\infty \right\}.$$

In the sequel we shall study two alternatives which are equivalent to (CPP). By 'equivalent' we mean that their feasible sets are the same, and they have the same objective functions. Hence the following two problems present alternative CP reformulations of the problem (P). The first reformulation was put forward by [1]

$$\begin{aligned} \min\ & \langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} \\ \text{s. t. } & \textstyle\sum_{i=1}^m \left(\mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i - 2b_i\, \mathsf{a}_i^\top \mathsf{x} + b_i^2 x_0\right) = 0 \\ & \textstyle\sum_{j\in B}(X_{jj} - x_j) = 0 \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \in \mathcal{CP}^{n+1}, \end{aligned} \tag{7}$$

along with its doubly-nonnegative (DNN) relaxation,

$$\begin{aligned} \min\ & \langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} \\ \text{s. t. } & \textstyle\sum_{i=1}^m \left(\mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i - 2b_i\, \mathsf{a}_i^\top \mathsf{x} + b_i^2\right) = 0 \\ & \textstyle\sum_{j\in B}(X_{jj} - x_j) = 0 \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \in \mathcal{N}^{n+1} \cap \mathcal{S}_+^{n+1} . \end{aligned} \tag{8}$$

While the second reformulation is inspired by [11,14], it was apparently not yet put forward in the published literature:

$$\begin{aligned} \min\ & \langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} \\ \text{s. t. } & \textstyle\sum_{j\in B}(X_{jj} - x_j) = 0 \\ & \mathsf{B}\mathsf{Y}\mathsf{B}^\top = \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \in \mathcal{CP}^{n+1}\,,\ \mathsf{Y} \in \mathcal{S}^{n+1-m} . \end{aligned} \tag{9}$$

We will also consider the following new relaxation of this:

$$\begin{aligned} \min\ & \langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} \\ \text{s. t. } & \textstyle\sum_{j\in B}(X_{jj} - x_j) = 0 \\ & \mathsf{B}\mathsf{Y}\mathsf{B}^\top = \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \\ & \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \in \mathcal{N}^{n+1}\,,\ \mathsf{Y} \in \mathcal{S}_+^{n+1-m} . \end{aligned} \tag{10}$$

2 Reformulation

In this section we shall prove the equivalence of problems (CPP), (7) and (9) and similar results for the relaxations (8) and (10). We split this into several parts. We begin in Subsection 2.1 by considering sets related to the linear constraints of (P). Then in Subsection 2.2 we consider sets related to the binary constraints of (P). In Subsection 2.3 we combine these to look at the optimization problems explicitly. Finally, in Subsection 2.4, we will compare our new optimization problems with those that previously appeared in the literature. We will switch to a more convenient notation for ease of comparison and establishing the proofs.

2.1 Linear Constraints: aggregation and facial reduction

In this subsection we will look at sets related to the linear constraints of (P). These results are inspired by the work done in [1,11] and [14, Section 2.1]. We shall consider the following four linear subspaces in $\mathcal{S}^{n+1}$:

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \in \mathcal{S}^{n+1} : \exists \mathsf{Y} \in \mathcal{S}^{n+1-m} \text{ s.t. } \mathsf{B}\mathsf{Y}\mathsf{B}^\top = \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \right\} = \mathsf{B}\mathcal{S}^{n+1-m}\mathsf{B}^\top,$$

$$\mathcal{L}_2 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X}\end{pmatrix} \in \mathcal{S}^{n+1} : \begin{matrix} \mathsf{a}_i^\top \mathsf{x} = b_i x_0 & \text{for } i \in [1:m]\,, \\ \mathsf{a}_i^\top \mathsf{X} = b_i \mathsf{x}^\top & \text{for } i \in [1:m] \end{matrix} \right\},$$

$$\mathcal{L}_3 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : \begin{array}{ll} \mathsf{a}_i^\top \mathsf{x} = b_i x_0 & \text{for } i \in [1\!:\!m]\,, \\ \mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i = b_i^2 x_0 & \text{for } i \in [1\!:\!m] \end{array} \right\},$$

$$\mathcal{L}_4 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : \sum_{i=1}^m \left(\mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i - 2b_i\, \mathsf{a}_i^\top \mathsf{x} + b_i^2 x_0 \right) = 0 \right\}.$$

Note that from the definition of B, we have $(\mathsf{B}\mathsf{Y}\mathsf{B}^\top)_{00} = (\mathsf{Y})_{00}$, and thus for $\mathcal{L}_1$, an additional constraint of $x_0 = 1$ is equivalent to fixing $(\mathsf{Y})_{00} = 1$.

We will now show that when intersecting with the positive semidefinite cone these four subspaces coincide.

Theorem 1 *We have*

(a) $\mathcal{L}_1 = \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \mathcal{L}_4\,,$

(b) $\mathcal{L}_1 \cap \mathcal{S}_+^{n+1} = \mathcal{L}_2 \cap \mathcal{S}_+^{n+1} = \mathcal{L}_3 \cap \mathcal{S}_+^{n+1} = \mathcal{L}_4 \cap \mathcal{S}_+^{n+1}$ *and furthermore*

(c) $\mathcal{L}_1 \cap \mathcal{S}_+^{n+1} = \mathsf{B}\mathcal{S}_+^{n+1-m}\mathsf{B}^\top\,.$

Proof. From (4), applied to the columns of $\begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} = \mathsf{B}\mathsf{Y}\mathsf{B}^\top$, it is easy to see that $\mathcal{L}_1 \subseteq \mathcal{L}_2$ holds. The inclusions $\mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \mathcal{L}_4$ are trivial. From (5) we have $\mathcal{L}_1 \cap \mathcal{S}_+^{n+1} = \mathsf{B}\mathcal{S}_+^{n+1-m}\mathsf{B}^\top$. We are thus left to show that $\mathcal{L}_2 \subseteq \mathcal{L}_1$ for (a), and $\mathcal{L}_4 \cap \mathcal{S}_+^{n+1} \subseteq \mathcal{L}_1$ for (b).

First we shall show that $\mathcal{L}_2 \subseteq \mathcal{L}_1$. The zero matrix is trivially in both of these sets and we now consider an arbitrary nonzero $\mathsf{Z} = \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{L}_2$. As $\mathsf{Z} \in \mathcal{S}^{n+1}$, for $p = \operatorname{rank}(\mathsf{Z}) \in [1\!:\!n+1]$, there exists $\{\boldsymbol{\zeta}, \boldsymbol{\mu}\} \subset \mathbb{R}^p$ and $\mathsf{z}_1, \ldots, \mathsf{z}_p \in \mathbb{R}^n$ such that $\mu_k \neq 0$ for all k and $\begin{pmatrix} \zeta_1 \\ \mathsf{z}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \zeta_p \\ \mathsf{z}_p \end{pmatrix}$ are orthonormal and $\mathsf{Z} = \sum_{k=1}^p \mu_k \begin{pmatrix} \zeta_k \\ \mathsf{z}_k \end{pmatrix} \begin{pmatrix} \zeta_k \\ \mathsf{z}_k \end{pmatrix}^\top$. For all $i \in [1\!:\!m]$ and $l \in [1\!:\!p]$ we have

$$0 = \begin{pmatrix} \zeta_l \\ \mathsf{z}_l \end{pmatrix}^\top \begin{pmatrix} 0 \\ \mathsf{o} \end{pmatrix} = \begin{pmatrix} \zeta_l \\ \mathsf{z}_l \end{pmatrix}^\top \begin{pmatrix} \mathsf{a}_i^\top \mathsf{x} - b_i x_0 \\ \mathsf{X}\mathsf{a}_i - b_i \mathsf{x} \end{pmatrix} = \begin{pmatrix} \zeta_l \\ \mathsf{z}_l \end{pmatrix}^\top \mathsf{Z} \begin{pmatrix} -b_i \\ \mathsf{a}_i \end{pmatrix} = \mu_l \begin{pmatrix} \zeta_l \\ \mathsf{z}_l \end{pmatrix}^\top \begin{pmatrix} -b_i \\ \mathsf{a}_i \end{pmatrix} = \mu_l (\mathsf{a}_i^\top \mathsf{z}_l - b_i \zeta_l).$$

By (4) this implies that for all $l \in [1\!:\!p]$ there exists $\mathsf{y}_l \in \mathbb{R}^{n+1-m}$ such that $\begin{pmatrix} \zeta_l \\ \mathsf{z}_l \end{pmatrix} = \mathsf{B}\mathsf{y}_l$. Letting $\mathsf{Y} = \sum_{k=1}^p \mu_k \mathsf{y}_k \mathsf{y}_k^\top \in \mathcal{S}^{n+1-m}$, we get $\mathsf{Z} = \mathsf{B}\mathsf{Y}\mathsf{B}^\top \in \mathcal{L}_1$.

We shall now show that $\mathcal{L}_4 \cap \mathcal{S}_+^{n+1} \subseteq \mathcal{L}_1$ using a similar but different proof. We consider an arbitrary $\begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{L}_4 \cap \mathcal{S}_+^{n+1}$. As $\begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}_+^{n+1}$, there exists $\{\boldsymbol{\zeta}, \mathsf{z}_1, \ldots, \mathsf{z}_n\} \subset \mathbb{R}^n$ such that $\begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} = \sum_{k=1}^n \begin{pmatrix} \zeta_k \\ \mathsf{z}_k \end{pmatrix} \begin{pmatrix} \zeta_k \\ \mathsf{z}_k \end{pmatrix}^\top$. We then have

$$0 = \sum_{i=1}^m \left(\mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i - 2b_i\, \mathsf{a}_i^\top \mathsf{x} + b_i^2 x_0 \right) = \sum_{i=1}^m \sum_{k=1}^n (\mathsf{a}_i^\top \mathsf{z}_k - b_i \zeta_k)^2 \geq 0$$

Therefore $\mathsf{a}_i^\top \mathsf{z}_k = b_i \zeta_k$ for all i, k, and thus, from (4), for all k there exists $\mathsf{y}_k \in \mathbb{R}^{n+1-m}$ such that $\begin{pmatrix} \zeta_k \\ \mathsf{z}_k \end{pmatrix} = \mathsf{B}\mathsf{y}_k$. Now letting $\mathsf{Y} = \sum_{k=1}^n \mathsf{y}_k \mathsf{y}_k^\top \in \mathcal{S}^{n+1-m}$, we get $\mathsf{Z} = \mathsf{B}\mathsf{Y}\mathsf{B}^\top \in \mathcal{L}_1$. □

In spite of these equivalences, we still consider these as separate sets, even when intersected with the positive semidefinite cone. The difference in descriptions is especially important when considering optimization problems and their duals later in the paper.

2.2 Binary constraints: aggregation

In this subsection we will consider the following subspaces, related to the binary constraints of (P):

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : X_{jj} = x_j \quad \text{for } j \in B \right\},$$

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : \sum_{j \in B} (X_{jj} - x_j) = 0 \right\}.$$

Similarly to the previous section, we will see that when intersected with certain sets, the resultant intersection is independent of the choice of $\mathcal{B}_1$ or $\mathcal{B}_2$. To show this we first need the following two results:

Lemma 1 *Let $j \in [1:n]$ and $\mu \in \mathbb{R}$ such that $x_j \leq \mu$ for all $\mathsf{x} \in \mathcal{Z}$. Then for $\begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{L}_2 \cap \mathcal{N}^{n+1}$ we have $X_{jk} \leq \mu x_k$ for all $k \in [1:n]$.*

Proof. Consider an arbitrary $k \in [1:n]$ and let $\mathsf{Z} = \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix}$. From $\mathsf{Z} \in \mathcal{N}^{n+1}$ we have $x_k \geq 0$ and we now consider two cases:

1. If $x_k = 0$ then let $\widehat{\mathsf{x}}$ be the k^{th} column of X. From $\mathsf{Z} \in \mathcal{N}^{n+1}$ we have $\widehat{\mathsf{x}} \in \mathbb{R}^n_+$, and from $\mathsf{Z} \in \mathcal{L}_2$, for all $i \in [1:m]$ we have $\mathsf{a}_i^\top \widehat{\mathsf{x}} = (\mathsf{a}_i^\top \mathsf{X})_k = (b_i \mathsf{x}^\top)_k = b_i x_k = 0$. Therefore $\widehat{\mathsf{x}} \in \mathcal{Z}_\infty$ and by looking at $\mathsf{v} = \mathsf{x}_0 + t\widehat{\mathsf{x}}$ with $t \uparrow \infty$, this implies that $0 = \widehat{x}_j = X_{jk}$.
2. If $x_k > 0$ then let $\widehat{\mathsf{x}}$ be the k^{th} column of $\frac{1}{x_k}\mathsf{X}$. From $\mathsf{Z} \in \mathcal{N}^{n+1}$ we have $\widehat{\mathsf{x}} \in \mathbb{R}^n_+$, and from $\mathsf{Z} \in \mathcal{L}_2$, for all $i \in [1:m]$ we have $\mathsf{a}_i^\top \widehat{\mathsf{x}} = \frac{1}{x_k}(\mathsf{a}_i^\top \mathsf{X})_k = \frac{1}{x_k}(b_i \mathsf{x}^\top)_k = b_i$. Therefore $\widehat{\mathsf{x}} \in \mathcal{Z}$ and thus $\mu \geq \widehat{x}_j = \frac{1}{x_k} X_{jk}$. □

Lemma 2 *We have $\mathcal{B}_1 \cap \mathcal{L}_2 \cap \mathcal{N}^{n+1} = \mathcal{B}_2 \cap \mathcal{L}_2 \cap \mathcal{N}^{n+1}$.*

Proof. We trivially have $\mathcal{B}_1 \subseteq \mathcal{B}_2$, and the proof will be completed if we can show $\mathcal{B}_2 \cap \mathcal{L}_2 \cap \mathcal{N}^{n+1} \subseteq \mathcal{B}_1$.

Considering an arbitrary $\begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_2 \cap \mathcal{L}_2 \cap \mathcal{N}^{n+1}$, from Lemma 1 and (2), for all $j \in B$ we have $X_{jj} - x_j \leq 0$ and thus $0 = \sum_{k \in B}(X_{kk} - x_k) \leq X_{jj} - x_j \leq 0$. Therefore $\begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_1$. □

We are now ready to present the main result of this subsection:

Theorem 2 *For all $i, j \in [1:4]$ and $k, l \in [1:2]$ we have*

$$\begin{aligned} \mathcal{B}_1 &\subseteq \mathcal{B}_2, \\ \mathcal{B}_k \cap \mathcal{L}_i \cap \mathcal{CP}^{n+1} &= \mathcal{B}_l \cap \mathcal{L}_j \cap \mathcal{CP}^{n+1}, \\ \mathcal{B}_k \cap \mathcal{L}_i \cap \mathcal{S}^{n+1}_+ \cap \mathcal{N}^{n+1} &= \mathcal{B}_l \cap \mathcal{L}_j \cap \mathcal{S}^{n+1}_+ \cap \mathcal{N}^{n+1}. \end{aligned}$$

Proof. We trivially have $\mathcal{B}_1 \subseteq \mathcal{B}_2$. Noting that $\mathcal{CP}^{n+1} \subseteq \mathcal{S}^{n+1}_+ \cap \mathcal{N}^{n+1}$ and using Theorem 1, without loss of generality we assume $i = j = 2$. The results then follow directly from Lemma 2. □

2.3 Optimization problems: new CP formulations and relaxations

For $k \in [1:2]$, $l \in [1:4]$ we will now consider the problem

$$\min \left\{ \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_k \cap \mathcal{L}_l \cap \mathcal{CP}^{n+1} \right\}. \tag{11}$$

For $(k, l) = (1, 3)$, problem (11) expressed explicitly is in fact problem (CPP), which was shown in [10] by Burer to have the same optimal value as (P). From Theorem 2 we have that problems (11) are equivalent for all choices of k and l, therefore they all have the same optimal value as (P). In particular, considering (11) expressed explicitly for $(k, l) = (2, 4)$ and $(k, l) = (2, 1)$ we get problems (7) and (9) respectively.

Unfortunately, optimizing over the completely positive cone is a very difficult problem (in fact an NP-hard problem [6, 15, 17]), and thus we wish to consider approximations of this. A natural relaxation of problem (11) is

$$\min \left\{ \langle \mathsf{Q}, \mathsf{X} \rangle + 2\mathsf{c}^\top \mathsf{x} : \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_k \cap \mathcal{L}_l \cap \mathcal{S}^{n+1}_+ \cap \mathcal{N}^{n+1} \right\}, \tag{12}$$

and again from Theorem 2 we have that problems (12) are equivalent for all choices of k and l.

For $(k,l) = (1,3)$ this becomes the following natural relaxation to (CPP):

$$\begin{aligned}
\min\ & \langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} \\
\text{s.t. } & \mathsf{a}_i^\top \mathsf{x} = b_i \quad \text{for } i\in[1:m] \\
& \langle \mathsf{a}_i \mathsf{a}_i^\top, \mathsf{X}\rangle = b_i^2 \quad \text{for } i\in[1:m] \\
& X_{jj} = x_j \text{ for all } j \in B \\
& \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1}.
\end{aligned} \tag{13}$$

Computationally this is still a fairly difficult problem, mainly due to the fact that we have large positive semidefinite constraints and we never have an interior point for the problem. Indeed, for any feasible $(1, \mathsf{x}, \mathsf{X})$ and any i we have

$$\begin{pmatrix} -b_i \\ \mathsf{a}_i \end{pmatrix}^\top \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \begin{pmatrix} -b_i \\ \mathsf{a}_i \end{pmatrix} = \mathsf{a}_i^\top \mathsf{X} \mathsf{a}_i - 2b_i\, \mathsf{a}_i^\top \mathsf{x} + b_i^2 = 0,$$

which implies that $\begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix}$ is on the boundary of the positive semidefinite cone.

Both of these difficulties can be solved by considering the equivalent problem of (12) with $(k,l) = (1,1)$, which is expressed explicitly as

$$\begin{aligned}
\min\ & \langle \mathsf{Q}, \mathsf{X}\rangle + 2\mathsf{c}^\top \mathsf{x} \\
\text{s.t. } & X_{jj} = x_j \text{ for all } j \in B \\
& \mathsf{B}\mathsf{Y}\mathsf{B}^\top = \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \\
& \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{N}^{n+1},\ \mathsf{Y} \in \mathcal{S}_+^{n+1-m}.
\end{aligned} \tag{14}$$

If there are no binary constraints, then for $\varepsilon > 0$ small enough the point $Y = \varepsilon \mathsf{I} + (1-\varepsilon)\mathsf{e}_0\mathsf{e}_0^\top$ is strictly feasible for this problem. If there are binary constraints then unfortunately we are unable to guarantee strict feasibility (or even feasibility).

Computationally (14) still has the problem that if $|B|$ is large then we have a large number of constraints. This can be solved by considering the equivalent problem of (12) with $(k,l) = (2,1)$ which is expressed explicitly as problem (10). Further support for this choice will be provided when we consider the dual problems in Subsection 3.2.

2.4 Comparison to previous results

We have been considering a (nonconvex) binary quadratic problem (P) with m linear equality constraints and n variables, of which $|B|$ of the variables are further restricted to be binary.

In [10], Burer showed that this has the same optimal value as the completely positive problem (CPP) with a completely positive constraint of order $n+1$ and $2m+|B|$ linear equality constraints. We explicitly exclude the *normalization constraint* that the upper left corner entry of the matrix $\begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix}$ be one, which can be treated as fixing a variable rather than a constraint.

In this section we have looked at how we can deal with some of the linear constraints by aggregating them. This results in the equivalent completely positive problem (7), which has two homogeneous linear equality constraints and a completely positive constraint of order $n+1$.

In [1], the authors introduced an equivalent copositive problem with only one equality constraint, however in general this can have a larger order completely positive constraint, possibly even as high as order $2n+1$.

Burer's original completely positive reformulation has the natural relaxation (13), which is a positive semidefinite problem with $2m+|B|$ linear equality constraints, $\frac{1}{2}n(n-1) - |B|$ inequality constraints and a positive semidefinite constraint of order $n+1$. Indeed, we have $\frac{1}{2}n(n-1)$ inequality constraints for $\mathcal{N}^n$, rather than the expected $\frac{1}{2}n(n+1)$ inequality constraints as the on-diagonal entries are already restricted to be nonnegative by the positive semidefinite constraint; the same reasoning, using $x_j = X_{jj}$ applies to all $j \in B$, reducing the number of inequality constraints further. This problem has the difficulty that it never has an interior point.

In [11], Burer showed how we can deal with the linear constraints which would give us the relaxation (14). This is a positive semidefinite problem with $|B|$ linear equality constraints, $\frac{1}{2}n(n-1)-|B|$ inequality constraints and a positive semidefinite constraint of order $n+1-m$.

In this section we have shown that this is also equivalent to problem (10), which is a positive semidefinite problem with one linear equality constraint, $\frac{1}{2}n(n-1)-|B|$ inequality constraints and a positive semidefinite constraint of order $n+1-m$.

3 Duality

We will now investigate the dual problems to the reformulations introduced in the first half of this paper. In particular we will look at how the duality gap is effected by the choice of reformulation. We will also consider when we can guarantee there is a zero duality gap through Slater's condition holding in the dual.

3.1 Conic optimization

We begin by recalling basic results on a more general class of conic optimization problems. For ease of reference the proofs of these results will be provided here although some may be found scattered in the literature.

We will primarily consider conic optimization problems of the following form:

$$\begin{aligned} \min_{\mathsf{x}} \quad & \langle \mathsf{q}, \mathsf{x}\rangle \\ \text{s. t.} \quad & \langle \mathsf{d}_0, \mathsf{x}\rangle = 1\,, \\ & \mathsf{x} \in \mathcal{L}\cap\mathcal{K}, \end{aligned} \tag{15}$$

where $\mathcal{K} \subseteq \mathbb{R}^n$ is a closed convex pointed cone, $\mathcal{L} = \{\mathsf{d}_1, \ldots, \mathsf{d}_m\}^\perp = \{\mathsf{x} \in \mathbb{R}^n : \langle \mathsf{d}_i, \mathsf{x}\rangle = 0, \text{ for } i \in \in [1\!:\!m]\}$ is a linear subspace, and $\{\mathsf{q}, \mathsf{d}_0, \mathsf{d}_1, \ldots, \mathsf{d}_m\} \subset \mathbb{R}^n$ are such that $\langle \mathsf{d}_0, \mathsf{x}\rangle \geq 0$ for all $\mathsf{x} \in \mathcal{L}\cap\mathcal{K}$, i.e. $\mathsf{d}_0 \in (\mathcal{L}\cap\mathcal{K})^*$. In Example 2 we will see how this connects to general conic optimization problems.

In order to consider the Lagrangian dual problem to this, we could consider $\mathcal{L}\cap\mathcal{K}$ to be a single closed convex cone and use the Lagrangian function

$$\Lambda(\mathsf{x}; y, \mathsf{z}) \;=\; \langle \mathsf{q}, \mathsf{x}\rangle + y(1 - \langle \mathsf{d}_0, \mathsf{x}\rangle) - \langle \mathsf{z}, \mathsf{x}\rangle \;=\; y + \langle \mathsf{q} - y\mathsf{d}_0 - \mathsf{z}, \mathsf{x}\rangle.$$

Letting opt (15) be the optimal value to problem (15), we get

$$\operatorname{opt}(15) \;=\; \min_{\mathsf{x}\in\mathbb{R}^n} \max_{\substack{y\in\mathbb{R},\\ \mathsf{z}\in(\mathcal{L}\cap\mathcal{K})^*}} \Lambda(\mathsf{x}; y, \mathsf{z}) \;\geq\; \max_{\substack{y\in\mathbb{R},\\ \mathsf{z}\in(\mathcal{L}\cap\mathcal{K})^*}} \min_{\mathsf{x}\in\mathbb{R}^n} \Lambda(\mathsf{x}; y, \mathsf{z}) \;=\; \max_{\substack{y\in\mathbb{R},\\ \mathsf{z}\in(\mathcal{L}\cap\mathcal{K})^*}} \{y : \mathsf{q} - y\mathsf{d}_0 = \mathsf{z}\}$$

and thus the dual problem is

$$\begin{aligned} \max_{y} \quad & y \\ \text{s. t.} \quad & \mathsf{q} - y\mathsf{d}_0 \in (\mathcal{L}\cap\mathcal{K})^*\,. \end{aligned} \tag{16}$$

An advantage of this approach is given in the lemma below:

Lemma 3 opt (15) = opt (16) *provided at least one of problems* (15) *and* (16) *are feasible, even if Slater's condition is violated for both problems.*

Proof. We have that opt (15) $\geq$ opt (16) and thus if opt (16) $= \infty$ then the result is trivial. From now on we suppose that opt (16) $< \infty$ and consider an arbitrary $\lambda >$ opt (16). We shall show that $\lambda >$ opt (15), and thus opt (15) $\leq$ opt (16), completing the proof.

As $\lambda >$ opt (16) we have $\mathsf{q} - \lambda\mathsf{d}_0 \notin (\mathcal{L}\cap\mathcal{K})^*$, and thus there exists $\mathsf{z} \in \mathcal{L}\cap\mathcal{K}$ such that $\langle \mathsf{z}, \mathsf{q} - \lambda\mathsf{d}_0\rangle < 0$ and from the assumption on d_0 we have $\langle \mathsf{z}, \mathsf{d}_0\rangle \geq 0$. We now consider two cases:

1. If $\langle \mathsf{z}, \mathsf{d}_0\rangle = 0$ then $\langle \mathsf{z}, \mathsf{q} - y\mathsf{d}_0\rangle = \langle \mathsf{z}, \mathsf{q}\rangle = \langle \mathsf{z}, \mathsf{q} - \lambda\mathsf{d}_0\rangle < 0$ for all $y \in \mathbb{R}$ and thus (16) is infeasible. As we assume at least one of the problems is feasible, there exists an x which is feasible for (15). Therefore for all $\mu \geq 0$ we have that $\mathsf{x} + \mu\mathsf{z}$ is feasible for (15), and considering $\mu \to \infty$ we get opt (15) $= -\infty < \lambda$.
2. If $\langle \mathsf{z}, \mathsf{d}_0\rangle > 0$ then without loss of generality $\langle \mathsf{z}, \mathsf{d}_0\rangle = 1$. We then have that z is feasible for (15) and $\lambda > \langle \mathsf{z}, \mathsf{q} - \lambda\mathsf{d}_0\rangle + \lambda = \langle \mathsf{z}, \mathsf{q}\rangle \geq$ opt (15). ☐

The main disadvantage of this approach is that although we have $(\mathcal{L}\cap\mathcal{K})^* = \operatorname{cl}\left(\mathcal{L}^\perp+\mathcal{K}^*\right)$, characterizing this closure is often a nontrivial problem. This means we are unable to use this dual in practice. Instead we often consider separating the conic and linear constraints, and then taking the resulting (possibly different) Lagrangian dual. By this we mean first noting that

$$\operatorname{opt}(15) = \min_{\mathsf{u},\mathsf{x}} \left\{ \langle \mathsf{q},\mathsf{x}\rangle : \begin{array}{ll} 1=\langle \mathsf{d}_0,\mathsf{x}\rangle, & \mathsf{x}\in\mathcal{L}, \\ \mathsf{u}-\mathsf{x}=0, & \mathsf{u}\in\mathcal{K} \end{array}\right\}.$$

Then considering the corresponding Lagrangian function

$$\begin{aligned} \Lambda(\mathsf{u},\mathsf{x};y,\mathsf{v},\mathsf{w},\mathsf{z}) &= \langle \mathsf{q},\mathsf{x}\rangle + y(1-\langle \mathsf{d}_0,\mathsf{x}\rangle) + \langle \mathsf{v},\mathsf{u}-\mathsf{x}\rangle - \langle \mathsf{w},\mathsf{x}\rangle - \langle \mathsf{z},\mathsf{u}\rangle \\ &= y + \langle \mathsf{x}, \mathsf{q}-y\mathsf{d}_0-\mathsf{v}-\mathsf{w}\rangle + \langle \mathsf{u},\mathsf{v}-\mathsf{z}\rangle, \end{aligned}$$

we get

$$\begin{aligned} \operatorname{opt}(15) &= \min_{\mathsf{u},\mathsf{x}\in\mathbb{R}^n} \max_{\substack{y\in\mathbb{R},\\ \mathsf{v}\in\mathbb{R}^n,\\ \mathsf{w}\in\mathcal{L}^\perp,\\ \mathsf{z}\in\mathcal{K}^*}} \Lambda(\mathsf{u},\mathsf{x};y,\mathsf{v},\mathsf{w},\mathsf{z}) \\ &\geq \max_{\substack{y\in\mathbb{R},\\ \mathsf{v}\in\mathbb{R}^n,\\ \mathsf{w}\in\mathcal{L}^\perp,\\ \mathsf{z}\in\mathcal{K}^*}} \min_{\mathsf{u},\mathsf{x}\in\mathbb{R}^n} \Lambda(\mathsf{u},\mathsf{x};y,\mathsf{v},\mathsf{w},\mathsf{z}) \quad = \quad \max_{\substack{y\in\mathbb{R},\\ \mathsf{v}\in\mathbb{R}^n,\\ \mathsf{w}\in\mathcal{L}^\perp,\\ \mathsf{z}\in\mathcal{K}^*}} \left\{ y : \mathsf{q}-y\mathsf{d}_0 = \mathsf{w}+\mathsf{v},\ \mathsf{v}=\mathsf{z}\right\}, \end{aligned}$$

and thus the dual problem is

$$\begin{aligned} \max_{y} \quad & y \\ \text{s.t.} \quad & \mathsf{q}-y\mathsf{d}_0 \in \mathcal{L}^\perp+\mathcal{K}^*. \end{aligned} \tag{17}$$

This dual problem could have alternatively been obtained as a relaxation of our original dual problem (16) by ignoring the closure operation in $(\mathcal{L}\cap\mathcal{K})^* = \operatorname{cl}\left(\mathcal{L}^\perp+\mathcal{K}^*\right)$.

The advantage of this approach is that we have $\mathcal{L}^\perp = \operatorname{span}\{\mathsf{d}_i : i\in[1\!:\!m]\}$ and we generally know a characterization of the set $\mathcal{K}^*$. Due to this advantage it shall be this dual that we consider from now on in the paper.

The disadvantage is that due to the absence of the closure, there could be a duality gap, i.e. we may have $\operatorname{opt}(15) > \operatorname{opt}(17)$.

Example 1 ([9, Example 3.3]) Consider problem (15) and its duals (16) and (17) for

$$\mathsf{Q} = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}, \qquad \mathsf{D}_0 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \qquad \mathcal{L} = \left\{\mathsf{X}\in\mathcal{S}^3 : x_{11}=0\right\}, \qquad \mathcal{K}=\mathcal{CP}^3$$

Considering the primal problem, we have

$$\mathcal{L}\cap\mathcal{K} = \{\mathsf{X}\in\mathcal{CP}^3 : x_{11}=0\} = \left\{\mathsf{X}\in\mathcal{S}^3 : x_{1i}=0 \text{ for } i\in[1\!:\!3],\ \begin{pmatrix} x_{22} & x_{23}\\ x_{23} & x_{33}\end{pmatrix} \in\mathcal{CP}^2\right\},$$

$$\operatorname{feas}(15) = \left\{\mathsf{X}\in\mathcal{S}^3 : x_{1i}=0 \text{ for } i\in[1\!:\!3],\ x_{33}=1,\ x_{23}\geq 0,\ x_{23}^2\leq x_{22}\right\}, \qquad \operatorname{opt}(15)=1.$$

For the dual problems, from basic results on copositivity (see e.g. [14, Theorem 1.1]) we have

$$\mathcal{N}^3 \;\subseteq\; \mathcal{L}^\perp+\mathcal{K}^* \;=\; \left\{\lambda\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} : \lambda\in\mathbb{R}\right\} + \mathcal{COP}^3 \;\subseteq\; \left\{\mathsf{Z}\in\mathcal{S}^3 : \begin{array}{c} \begin{pmatrix} z_{22} & z_{23}\\ z_{23} & z_{33}\end{pmatrix}\in\mathcal{COP}^2, \\ z_{22}=0 \;\Rightarrow\; z_{12}\geq 0 \end{array}\right\},$$

and thus $\operatorname{feas}(17) = \{y\in\mathbb{R} : y\leq 0\}$ and $\operatorname{opt}(17) = 0 < \operatorname{opt}(15)$.

Considering an arbitrary $\mathsf{Z}\in\mathcal{S}^3$ with $\begin{pmatrix} z_{22} & z_{23}\\ z_{23} & z_{33}\end{pmatrix}\in\mathcal{COP}^2$, from [5, Lemma 4.1] we get that $\mathsf{Z}+\varepsilon\mathsf{I}_3\in\mathcal{L}^\perp+\mathcal{K}^*$ for all $\varepsilon>0$. Combining this observation with the inclusion relations above, we get

$$\operatorname{cl}(\mathcal{L}^\perp+\mathcal{K}^*) = \left\{\mathsf{Z}\in\mathcal{S}^3 : \begin{pmatrix} z_{22} & z_{23}\\ z_{23} & z_{33}\end{pmatrix}\in\mathcal{COP}^2\right\} = (\mathcal{L}\cap\mathcal{K})^*,$$

and thus $\operatorname{feas}(16) = \{y\in\mathbb{R} : y\leq 1\}$ and $\operatorname{opt}(16) = 1 = \operatorname{opt}(15)$.

In the following example we will briefly look at how the results so far presented in this subsection connect to general conic optimization problems.

Example 2 Consider the following general conic optimization problem:

$$\begin{aligned} \min_{\mathsf{x}} \quad & \langle \widetilde{\mathsf{q}}, \mathsf{z}\rangle \\ \text{s.t.} \quad & \langle \widetilde{\mathsf{d}}_i, \mathsf{z}\rangle = f_i \qquad \text{for } i\in[1:m], \\ & \mathsf{z}\in\mathcal{C}, \end{aligned} \tag{18}$$

where $\left\{\widetilde{\mathsf{q}}, \widetilde{\mathsf{d}}_1, \ldots, \widetilde{\mathsf{d}}_m\right\} \subset \mathbb{R}^{n-1}$, and $\mathcal{C} \subseteq \mathbb{R}^{n-1}$ is a closed convex pointed cone. This problem is equivalent to problem (15) with $\mathcal{K} = \mathcal{C} \times \mathbb{R}_+ \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$, $\mathsf{q} = (\widetilde{\mathsf{q}}, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $\mathsf{d}_0 = (\mathsf{o}, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\mathsf{d}_i = (\widetilde{\mathsf{d}}_i, -f_i) \in \mathbb{R}^{n-1} \times \mathbb{R}$ for $i\in[1:m]$. The dual problem of the form (16) for this is then

$$\begin{aligned} \max_{y_0} \quad & y_0 \\ \text{s.t.} \quad & (\widetilde{\mathsf{q}}, -y_0) \in \operatorname{cl}\Big((\mathcal{C}^* \times \mathbb{R}_+) + \operatorname{span}\{(\widetilde{\mathsf{d}}_i, -f_i) : i\in[1:m]\}\Big) . \end{aligned} \tag{19}$$

Removing the closure to give problem (17) and then simplifying gives the problem

$$\begin{aligned} \max_{\mathsf{y}} \quad & \mathsf{f}^\top \mathsf{y} \\ \text{s.t.} \quad & \widetilde{\mathsf{q}} - \sum_{i=1}^m y_i \widetilde{\mathsf{d}}_i \in \mathcal{C}^*. \end{aligned}$$

This is the standard dual for a general conic optimization problem, and thus we see that, in general conic optimization, all nontrivial dual gaps (i.e., when at least one of the primal or dual problems is feasible) is caused by the removal of the closure operation.

In order to guarantee equality of the optimal values we will use Slater's condition. This requires the existence of a so called strictly feasible point, and we know from the discussion in Subsection 2.3 that in general our primal problems will not contain such a point. For this reason we will focus on the dual side. We will use the following well known theorem, and for the sake of completeness we include the proof.

Theorem 3 *Consider problems* (15) *and* (17) *along with their assumptions. We have*

$$\operatorname{int}\left(\mathcal{L}^\perp + \mathcal{K}^*\right) = \mathcal{L}^\perp + \operatorname{int}\left(\mathcal{K}^*\right) \neq \emptyset,$$

and we say that Slater's condition holds for (17) *if*

1. *there exists* $\widehat{y} \in \mathbb{R}$ *such that* $\mathsf{q} - \widehat{y}\mathsf{d}_0 \in \operatorname{int}\left(\mathcal{L}^\perp + \mathcal{K}^*\right)$.

An equivalent condition to this is:

2. $\langle \mathsf{q}, \mathsf{u}\rangle > 0$ *for all* $\mathsf{u} \in \mathcal{L} \cap \mathcal{K} \setminus \{\mathsf{o}\}$ *such that* $\langle \mathsf{d}_0, \mathsf{u}\rangle = 0$.

If Slater's condition holds for (17) *then we have* $\operatorname{opt}(15) = \operatorname{opt}(17)$.
If $\operatorname{feas}(15) \neq \emptyset$ *then another two equivalent conditions to Slater's condition for* (17) *are:*

3. *there exists* $\mu \in \mathbb{R} \cup \{\infty\}$ *such that the set* $\{\mathsf{x} \in \operatorname{feas}(15) : \langle \mathsf{q}, \mathsf{x}\rangle \leq \mu\}$ *is nonempty and compact,*
4. *for all* $\mu \in \mathbb{R}$ *we have that the set* $\{\mathsf{x} \in \operatorname{feas}(15) : \langle \mathsf{q}, \mathsf{x}\rangle \leq \mu\}$ *is compact.*

Therefore, if $\operatorname{feas}(15) \neq \emptyset$ *and Slater's condition holds for* (17) *then the optimal value to* (15) *is attained.*

Proof. The characterizations of $\operatorname{int}\left(\mathcal{L}^\perp + \mathcal{K}^*\right)$ follow trivially from the definition of the interior and from $\mathcal{K}^*$ being a convex cone with nonempty interior (see e.g. [2, Theorem 2.3]).

If Slater's condition, *1*, holds then we can approach the optimal value to (16) from the interior, and thus the closure is not necessary (i.e., for all $y \in \operatorname{feas}(16)$ and all $\varepsilon \in]0, 1]$ we have $\mathsf{q} - (y + \varepsilon(\widehat{y} - y))\mathsf{d}_0 \in \operatorname{int}\left(\mathcal{L}^\perp + \mathcal{K}^*\right)$, and thus $y + \varepsilon(\widehat{y} - y) \in \operatorname{feas}(17)$). This implies that $\operatorname{opt}(17) = \operatorname{opt}(16) = \operatorname{opt}(15)$, with the last equality following from Lemma 3.

We are thus left to prove the equivalence relations. In doing this we will repeatedly use the result that for a closed convex pointed cone $\mathcal{C} \subseteq \mathbb{R}^n$ we have $\operatorname{int}\mathcal{C}^* = \{\mathsf{z} \in \mathbb{R}^n : \langle \mathsf{x}, \mathsf{z}\rangle > 0 \text{ for all } \mathsf{x} \in \mathcal{C} \setminus \{\mathsf{o}\}\}$ (see e.g. [2, Equation (2.1)]). We split the proof of the equivalence relations into four parts:

1⇔2: This follows directly from the observation

$$\begin{aligned}\{\mathsf{q}\in\mathbb{R}^n : \langle\mathsf{q},\mathsf{u}\rangle>0 \text{ for all } \mathsf{u}\in\mathcal{L}\cap\mathcal{K}\setminus\{\mathsf{o}\} \text{ s.t. } \langle\mathsf{d}_0,\mathsf{u}\rangle=0\}\\ &= \operatorname{int}(\{\mathsf{u}\in\mathcal{L}\cap\mathcal{K} : \langle\mathsf{d}_0,\mathsf{u}\rangle=0\}^*)\\ &= \operatorname{int}(\operatorname{cl}(\operatorname{span}\{\mathsf{d}_0\}+\mathcal{L}^\perp+\mathcal{K}^*))\\ &= \operatorname{int}(\operatorname{span}\{\mathsf{d}_0\}+(\mathcal{L}^\perp+\mathcal{K}^*))\\ &= \operatorname{span}\{\mathsf{d}_0\}+\operatorname{int}(\mathcal{L}^\perp+\mathcal{K}^*)\\ &= \{\mathsf{q}\in\mathbb{R}^n : \exists y\in\mathbb{R} \text{ s.t. } \mathsf{q}-y\mathsf{d}_0\in\operatorname{int}(\mathcal{L}^\perp+\mathcal{K}^*)\}.\end{aligned}$$

2⇒4: Suppose for the sake of contradiction that statement *2* is true but there exists $\mu\in\mathbb{R}$ such that $\{\mathsf{x}\in\operatorname{feas}(15) : \langle\mathsf{q},\mathsf{x}\rangle\le\mu\}$ is not compact. This set is the intersection of closed sets and thus is itself closed. Therefore it must be unbounded, i.e. for all $i\in\mathbb{N}$ there exists $\mathsf{x}_i\in\operatorname{feas}(15)$ such that $\|\mathsf{x}_i\|_2>i$ and $\langle\mathsf{q},\mathsf{x}_i\rangle\le\mu$.
For all i, letting $\mathsf{u}_i=\frac{1}{\|\mathsf{x}_i\|_2}\mathsf{x}_i$ we have $\|\mathsf{u}_i\|_2=1$ and thus the sequence $\{\mathsf{u}_i : i\in\mathbb{N}\}$ is limited to a compact set. Therefore there is a limiting subsequence tending towards a limiting point u.
For all i we have $\mathsf{u}_i\in\mathcal{L}\cap\mathcal{K}$ and $\|\mathsf{u}_i\|_2=1$, and as $\mathcal{L}\cap\mathcal{K}$ is closed this implies that $\mathsf{u}\in\mathcal{L}\cap\mathcal{K}\setminus\{\mathsf{o}\}$. We also have $\langle\mathsf{d}_0,\mathsf{u}\rangle=\lim_{i\to\infty}\frac{1}{\|\mathsf{x}_i\|_2}\langle\mathsf{d}_0,\mathsf{x}_i\rangle=0$ and $\langle\mathsf{q},\mathsf{u}\rangle\le\lim_{i\to\infty}\frac{\mu}{\|\mathsf{x}_i\|_2}=0$. This gives the contradiction that statement *2* does not hold.
4$\wedge$ feas (15) $\neq\emptyset\Rightarrow$*3:* This follows from considering an arbitrary $\mathsf{x}\in\operatorname{feas}(15)$ and letting $\mu=\langle\mathsf{q},\mathsf{x}\rangle$.
3⇒2: Suppose there exists $\mu\in\mathbb{R}\cup\{\infty\}$ such that the set $\{\mathsf{x}\in\operatorname{feas}(15) : \langle\mathsf{q},\mathsf{x}\rangle\le\mu\}$ is nonempty and compact, and additionally suppose for the sake of contradiction that there exists a $\mathsf{u}\in\mathcal{L}\cap\mathcal{K}\setminus\{\mathsf{o}\}$ such that $\langle\mathsf{q},\mathsf{u}\rangle\le 0=\langle\mathsf{d}_0,\mathsf{u}\rangle$. Then there exists $\mathsf{v}\in\operatorname{feas}(15)$ such that $\langle\mathsf{q},\mathsf{v}\rangle\le\mu$, and for all $\lambda\ge0$ we have $\mathsf{v}+\lambda\mathsf{u}\in\operatorname{feas}(15)$ and $\langle\mathsf{q},\mathsf{v}+\lambda\mathsf{u}\rangle\le\langle\mathsf{q},\mathsf{v}\rangle\le\mu$, which is a contradiction. □

Corollary 1 *If* feas (15) *is nonempty and bounded then Slater's condition holds for* (17).

Remark 1 Applying Theorem 3 to Example 2, we have that Slater's condition holding for the dual problem (i.e., problem (19) with the closure removed) is equivalent to the following:

1. there exists $\mathsf{y}\in\mathbb{R}^m$ such that $\widetilde{\mathsf{q}}-\sum_{i=1}^m y_i\widetilde{\mathsf{d}}_i\in\operatorname{int}\mathcal{C}^*$,
2. $\langle\widetilde{\mathsf{q}},\mathsf{z}\rangle>0$ for all $\mathsf{z}\in\mathcal{C}\setminus\{\mathsf{o}\}$ such that $\langle\widetilde{\mathsf{d}}_i,\mathsf{z}\rangle=0$ for $i\in[1:m]$.

When feas (18) $\neq\emptyset$ then a further two equivalent conditions are:

3. there exists $\mu\in\mathbb{R}\cup\{\infty\}$ such that the set $\{\mathsf{z}\in\operatorname{feas}(18) : \langle\widetilde{\mathsf{q}},\mathsf{z}\rangle\le\mu\}$ is nonempty and compact,
4. for all $\mu\in\mathbb{R}$ the set $\{\mathsf{z}\in\operatorname{feas}(18) : \langle\widetilde{\mathsf{q}},\mathsf{z}\rangle\le\mu\}$ is compact.

3.2 Our problems

All of the results of the previous subsection naturally extend to other Euclidean spaces, including the spaces $\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n$ and $\mathcal{S}^{n+1-m}$ which shall be considered in this subsection.

For $k\in[1:2]$ and $l\in[1:4]$, we will consider the dual problems to

$$\begin{aligned}&\min\ \langle\mathsf{Q},\mathsf{X}\rangle+2\mathsf{c}^\top\mathsf{x}\\ &\text{s.t. } x_0=1\\ &\quad\begin{pmatrix}x_0 & \mathsf{x}^\top\\ \mathsf{x} & \mathsf{X}\end{pmatrix}\in\mathcal{B}_k\cap\mathcal{L}_l\cap\mathcal{CP}^{n+1},\end{aligned}\tag{20}$$

and the duals of their relaxations

$$\begin{aligned}&\min\ \langle\mathsf{Q},\mathsf{X}\rangle+2\mathsf{c}^\top\mathsf{x}\\ &\text{s.t. } x_0=1\\ &\quad\begin{pmatrix}x_0 & \mathsf{x}^\top\\ \mathsf{x} & \mathsf{X}\end{pmatrix}\in\mathcal{B}_k\cap\mathcal{L}_l\cap\mathcal{S}_+^{n+1}\cap\mathcal{N}^{n+1}.\end{aligned}\tag{21}$$

For $l=1$ this relaxation is equivalently to the following reduced problem, which we will also consider the dual of:

$$\begin{aligned}
\min_{\mathsf{Y}} \quad & \left\langle \mathsf{B}^\top \begin{pmatrix} 0 & \mathsf{c}^\top \\ \mathsf{c} & \mathsf{Q} \end{pmatrix} \mathsf{B}, \mathsf{Y} \right\rangle \\
\text{s.t.} \quad & \left\langle \mathsf{B}^\top \begin{pmatrix} 1 & \mathsf{o}^\top \\ \mathsf{o} & \mathsf{O} \end{pmatrix} \mathsf{B}, \mathsf{Y} \right\rangle = 1 \\
& \mathsf{BYB}^\top \in \mathcal{B}_k \\
& \mathsf{BYB}^\top \in \mathcal{N}^{n+1} \\
& \mathsf{Y} \in \mathcal{S}_+^{n+1-m}.
\end{aligned} \tag{22}$$

We begin by considering problems (20) and (21) (for $k\in[1:2]$ and $l\in[1:4]$). In order to apply the results from the previous subsection we note that $\mathcal{B}_k \cap \mathcal{L}_l$ is a linear subspace with $(\mathcal{B}_k \cap \mathcal{L}_l)^\perp = \mathcal{B}_k^\perp + \mathcal{L}_l^\perp$, where

$$\begin{aligned}
\mathcal{L}_1^\perp &= \left\{ \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{S}^{n+1} : \mathsf{B}^T \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \mathsf{B} = \mathsf{O} \right\}, \\
\mathcal{L}_2^\perp &= \operatorname{span}\left\{ \begin{pmatrix} 2b_i & -\mathsf{a}_i^\top \\ -\mathsf{a}_i & \mathsf{O} \end{pmatrix} : i\in[1:m] \right\} + \operatorname{span}\left\{ \begin{pmatrix} 0 & -b_i \mathsf{e}_j^\top \\ -b_i \mathsf{e}_j & \mathsf{a}_i \mathsf{e}_j^\top + \mathsf{e}_j \mathsf{a}_i^\top \end{pmatrix} : i\in[1:m],\ j\in[1:n] \right\}, \\
\mathcal{L}_3^\perp &= \operatorname{span}\left\{ \begin{pmatrix} 2b_i & -\mathsf{a}_i^\top \\ -\mathsf{a}_i & \mathsf{O} \end{pmatrix} : i\in[1:m] \right\} + \operatorname{span}\left\{ \begin{pmatrix} b_i^2 & \mathsf{o}^\top \\ \mathsf{o} & -\mathsf{a}_i \mathsf{a}_i^\top \end{pmatrix} : i\in[1:m] \right\}, \\
\mathcal{L}_4^\perp &= \operatorname{span}\left\{ \sum_{i=1}^m \begin{pmatrix} b_i^2 & -b_i \mathsf{a}_i^\top \\ -b_i \mathsf{a}_i & \mathsf{a}_i \mathsf{a}_i^\top \end{pmatrix} \right\}, \\
\mathcal{B}_1^\perp &= \operatorname{span}\left\{ \begin{pmatrix} 0 & -\mathsf{e}_j^\top \\ -\mathsf{e}_j & 2\mathsf{e}_j \mathsf{e}_j^\top \end{pmatrix} : j \in \mathcal{B} \right\}, \qquad \mathcal{B}_2^\perp = \operatorname{span}\left\{ \sum_{j\in\mathcal{B}} \begin{pmatrix} 0 & -\mathsf{e}_j^\top \\ -\mathsf{e}_j & 2\mathsf{e}_j \mathsf{e}_j^\top \end{pmatrix} \right\}.
\end{aligned}$$

Recalling that $(\mathcal{CP}^{n+1})^* = \mathcal{COP}^{n+1}$ and $(\mathcal{S}_+^{n+1} \cap \mathcal{N}^{n+1})^* = \mathcal{S}_+^{n+1} + \mathcal{N}^{n+1}$ (see e.g. [14, Theorem 1.32]), and using the results from the previous subsection, we then get that the dual problems to (20) and (21) for $k\in[1:2]$ and $l\in[1:4]$ are respectively

$$\begin{aligned}
\max_y \quad & y \\
\text{s.t.} \quad & \begin{pmatrix} -y & \mathsf{c}^\top \\ \mathsf{c} & \mathsf{Q} \end{pmatrix} \in \mathcal{B}_k^\perp + \mathcal{L}_l^\perp + \mathcal{COP}^{n+1},
\end{aligned} \tag{23}$$

$$\begin{aligned}
\max_y \quad & y \\
\text{s.t.} \quad & \begin{pmatrix} -y & \mathsf{c}^\top \\ \mathsf{c} & \mathsf{Q} \end{pmatrix} \in \mathcal{B}_k^\perp + \mathcal{L}_l^\perp + \mathcal{S}_+^{n+1} + \mathcal{N}^{n+1}.
\end{aligned} \tag{24}$$

We now consider the dual to problem (22) for $k\in[1:2]$. The set $\{\mathsf{Y} \in \mathcal{S}^{n+1-m} : \mathsf{BYB}^\top \in \mathcal{B}_k\}$ is a linear subspace with $\{\mathsf{Y} \in \mathcal{S}^{n+1-m} : \mathsf{BYB}^\top \in \mathcal{B}_k\}^\perp = \mathsf{B}^\top \mathcal{B}_k^\perp \mathsf{B}$. We now let $\mathcal{K} = \{\mathsf{Y} \in \mathcal{S}_+^{n+1-m} : \mathsf{BYB}^\top \in \mathcal{N}^{n+1}\}$ and consider the following result:

Lemma 4 *Letting $\mathcal{K} = \{\mathsf{Y} \in \mathcal{S}_+^{n+1-m} : \mathsf{BYB}^\top \in \mathcal{N}^{n+1}\}$, we have that $\mathcal{K}$ is a proper cone with $\mathcal{K}^* = \mathcal{S}_+^{n+1-m} + \mathsf{B}^\top \mathcal{N}^{n+1} \mathsf{B}$.*

Proof. It is trivial to see that $\mathcal{K}$ is a closed convex pointed cone. For $\varepsilon \in]0,1]$, letting $\mathsf{Y} = \varepsilon \mathsf{I} + (1-\varepsilon)\mathsf{e}_0 \mathsf{e}_0^\top$, we have $\mathsf{Y} \in \operatorname{int} \mathcal{S}_+^{n+1-m}$ and $\mathsf{BYB}^\top = \varepsilon \mathsf{B}^\top \mathsf{B} + (1-\varepsilon) \begin{pmatrix} 1 \\ \mathsf{x}_0 \end{pmatrix} \begin{pmatrix} 1 \\ \mathsf{x}_0 \end{pmatrix}^\top$ which is strictly positive for $\varepsilon > 0$ small enough. Therefore $\mathsf{Y} \in \operatorname{int} \mathcal{K}$ for $\varepsilon > 0$ small enough, completing the proof that $\mathcal{K}$ is a proper cone. The remainder of the proof then follows from the following equalities:

$$(\mathcal{S}_+^{n+1-m})^* = \mathcal{S}_+^{n+1-m}, \tag{25}$$

$$\begin{aligned}(\mathsf{B}^\top\mathcal{N}^{n+1}\mathsf{B})^* &= \{\mathsf{Y}\in\mathcal{S}^{n+1-m} : \langle \mathsf{Y}, \mathsf{B}^\top\mathsf{X}\mathsf{B}\rangle \geq 0 \text{ for all } \mathsf{X}\in\mathcal{N}^{n+1}\}\\ &= \{\mathsf{Y}\in\mathcal{S}^{n+1-m} : \langle \mathsf{B}\mathsf{Y}\mathsf{B}^\top, \mathsf{X}\rangle \geq 0 \text{ for all } \mathsf{X}\in\mathcal{N}^{n+1}\}\\ &= \{\mathsf{Y}\in\mathcal{S}^{n+1-m} : \mathsf{B}\mathsf{Y}\mathsf{B}^\top\in\mathcal{N}^{n+1}\}, \end{aligned} \tag{26}$$

$$\{\mathsf{Y}\in\mathcal{S}^{n+1-m} : \mathsf{B}\mathsf{Y}\mathsf{B}^\top\in\mathcal{N}^{n+1}\}^* = (\mathsf{B}^\top\mathcal{N}^{n+1}\mathsf{B})^{**} = \mathrm{cl}(\mathsf{B}^\top\mathcal{N}^{n+1}\mathsf{B}) = \mathsf{B}^\top\mathcal{N}^{n+1}\mathsf{B}, \tag{27}$$

$$\begin{aligned}\mathcal{K}^* &= \mathrm{cl}\left((\mathcal{S}_+^{n+1-m})^* + \{\mathsf{Y}\in\mathcal{S}^{n+1-m} : \mathsf{B}\mathsf{Y}\mathsf{B}^\top\in\mathcal{N}^{n+1}\}^*\right) &(28)\\ &= \mathrm{cl}\left(\mathcal{S}_+^{n+1-m} + \mathsf{B}^\top\mathcal{N}^{n+1}\mathsf{B}\right)\\ &= \mathcal{S}_+^{n+1-m} + \mathsf{B}^\top\mathcal{N}^{n+1}\mathsf{B}. &(29)\end{aligned}$$

The equalities (25) and (26) follow from the well known result that both $\mathcal{S}_+^{n+1-m}$ and $\mathcal{N}^{n+1}$ are self-dual (see e.g. [14, Theorem 1.32]). The equalities in (27) follow from (26), [2, Corollary 2.1] and $\mathcal{N}^{n+1}$ being a polyhedron implying that $\mathsf{B}^\top\mathcal{N}^{n+1}\mathsf{B}$ is a polyhedron. Finally, the equality (28) follows from [2, Corollary 2.2] and equality (29) follows from [14, Corollary 1.12]. □

Therefore the dual problem to (22) for $k\in[1:2]$ is

$$\begin{aligned}\max_{y}\quad & y\\ \text{s.t.}\quad & \mathsf{B}^\top\begin{pmatrix}0 & \mathsf{c}^\top\\ \mathsf{c} & \mathsf{Q}\end{pmatrix}\mathsf{B} - y\mathsf{B}^\top\begin{pmatrix}1 & \mathsf{o}^\top\\ \mathsf{o} & \mathsf{O}\end{pmatrix}\mathsf{B}\in \mathsf{B}^\top\mathcal{B}_k^\perp\mathsf{B} + \mathsf{B}^\top\mathcal{N}^{n+1}\mathsf{B} + \mathcal{S}_+^{n+1-m},\end{aligned}$$

or equivalently

$$\begin{aligned}\max_{y,\mathsf{Z}}\quad & y\\ \text{s.t.}\quad & \begin{pmatrix}-y & \mathsf{c}\\ \mathsf{c}^\top & \mathsf{Q}\end{pmatrix} - Z\in\mathcal{B}_k^\perp + \mathcal{N}^{n+1}\\ & \mathsf{B}^\top\mathsf{Z}\mathsf{B}\in\mathcal{S}_+^{n+1-m}.\end{aligned} \tag{30}$$

We can use the following lemma to compare the dual problems with each other:

Lemma 5 *We have*

$$\begin{aligned}\mathcal{B}_2^\perp &\subseteq \mathcal{B}_1^\perp,\\ \mathcal{L}_4^\perp \subseteq \mathcal{L}_3^\perp &\subseteq \mathcal{L}_2^\perp = \mathcal{L}_1^\perp\\ \mathcal{L}_1^\perp + \mathcal{S}_+^{n+1} &= \{\mathsf{Z}\in\mathcal{S}^{n+1} : \mathsf{B}^\top\mathsf{Z}\mathsf{B}\in\mathcal{S}_+^{n+1-m}\},\\ \mathcal{B}_1^\perp + \mathcal{L}_2^\perp + \mathcal{COP}^{n+1} &= \mathcal{B}_2^\perp + \mathcal{L}_2^\perp + \mathcal{COP}^{n+1},\\ \mathcal{B}_1^\perp + \mathcal{L}_2^\perp + \mathcal{S}_+^{n+1} + \mathcal{N}^{n+1} &= \mathcal{B}_2^\perp + \mathcal{L}_2^\perp + \mathcal{S}_+^{n+1} + \mathcal{N}^{n+1}.\end{aligned}$$

Proof. It is trivial to see that $\mathcal{B}_2^\perp\subseteq\mathcal{B}_1^\perp$ and from Theorem 1 we have $\mathcal{L}_4^\perp\subseteq\mathcal{L}_3^\perp\subseteq\mathcal{L}_2^\perp=\mathcal{L}_1^\perp$. From Theorem 1 we also have

$$\mathrm{cl}(\mathcal{L}_1^\perp + \mathcal{S}_+^{n+1}) = (\mathcal{L}_1\cap\mathcal{S}_+^{n+1})^* = (\mathsf{B}\mathcal{S}_+^{n+1-m}\mathsf{B}^\top)^* = \{\mathsf{Z}\in\mathcal{S}^{n+1} : \mathsf{B}^\top\mathsf{Z}\mathsf{B}\in\mathcal{S}_+^{n+1-m}\},$$

and by considering the faces of the positive semidefinite cone it can be shown that $(\mathcal{L}_1^\perp + \mathcal{S}_+^{n+1})$ is in fact closed (see e.g. [18,19]).

To prove the final two equalities, it is sufficient to show that $\mathcal{B}_1^\perp + \mathcal{L}_2^\perp + \mathcal{N}^{n+1} = \mathcal{B}_2^\perp + \mathcal{L}_2^\perp + \mathcal{N}^{n+1}$. From Lemma 2 we have that $\mathcal{B}_1\cap\mathcal{L}_2\cap\mathcal{N}^{n+1} = \mathcal{B}_2\cap\mathcal{L}_2\cap\mathcal{N}^{n+1}$. Therefore

$$\mathrm{cl}(\mathcal{B}_1^\perp + \mathcal{L}_2^\perp + \mathcal{N}^{n+1}) = (\mathcal{B}_1\cap\mathcal{L}_2\cap\mathcal{N}^{n+1})^* = (\mathcal{B}_2\cap\mathcal{L}_2\cap\mathcal{N}^{n+1})^* = \mathrm{cl}(\mathcal{B}_2^\perp + \mathcal{L}_2^\perp + \mathcal{N}^{n+1}).$$

The proof is then completed by noting that as the sets $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{L}_2$ and $\mathcal{N}^{n+1}$ are all polyhedra, then so are their Minkowski sums, and in particular this means that these sums are closed. □

Remark 2 From Lemma 5, we see that as l increases the feasible sets of the dual problems shrink (or at least remain the same). This means that the duality gap is monotonically increasing with l. Therefore picking $l = 1$ for problem (20) (and its relaxation (22)) not only gives the smallest problems in terms of number of variables and constraints (and order of positive semidefinite constraints for the relaxation), but also gives the smallest duality gap. For both of these problems, with $l = 1$, the feasible sets of their primal and dual problems are independent of k. However picking $k = 2$ will reduce the number of constraints in the primal problems, and the number of variables in the dual problems (when compared to $k = 1$).

From the previous remark, we have that for the completely positive problem, we should always use (20) with $k = 2$ and $l = 1$, whilst for the relaxation, we should always use (22) with $k = 2$. This will provide the smallest problems and duality gaps. As an example of a duality gap occurring see below:

Example 3 Consider the problem

$$\min_{\mathsf{x}} \left\{ 2x_2 - 2x_1x_2 : x_1 = 1,\ \mathsf{x} \in \mathbb{R}^2_+ \right\}, \tag{31}$$

i.e., $n = 2$, $m = b_1 = 1$, $\mathsf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathsf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathsf{Q} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and opt (31) = 0. For $l = 3$, the completely positive reformulation (20) and its dual (23) are repectively

$$\min_{\mathsf{x},\mathsf{X}} \left\{ 2x_2 - 2x_{12} : x_1 = 1,\ x_{11} = 1,\ \begin{pmatrix} 1 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{CP}^3 \right\}, \tag{32}$$

$$\max_{u,v,y} \left\{ y : \begin{pmatrix} -y & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} + u \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + v \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{COP}^3 \right\}. \tag{33}$$

It can then be shown that

$$\text{feas}\,(32) = \left\{ (\mathsf{x}, \mathsf{X}) \in \mathbb{R}^2_+ \times \mathcal{S}^2 : x_1 = 1,\ x_{11} = 1,\ x_{12} = x_2,\ x_2^2 \le x_{22} \right\} \quad \text{and} \quad \text{feas}\,(33) = \emptyset.$$

Therefore opt $(33) = -\infty < 0 =$ opt $(32) =$ opt (31).

So when can we guarantee that there will be no duality gap? One simple case when this can be guaranteed is covered in the following theorem.

Theorem 4 *If the polyhedron $\mathcal{Z} = \{\mathsf{x} \in \mathbb{R}^n_+ : \mathsf{a}_i^\top \mathsf{x} = b_i,\ i \in [1:m]\}$ is bounded then Slater's condition holds for problems* (23)*,* (24) *and* (30) *(for all $k \in [1:2]$, $l \in [1:4]$).*

Proof. If $\mathcal{Z}$ is bounded then for $B = \emptyset$, the feasible sets of (20), (21) and (22) are bounded and nonempty. From Corollary (1) we then get that, for $B = \emptyset$, Slater's condition holds for problems (23), (24) and (30). For the case when $B \neq \emptyset$, we are simply increasing the size of the feasible sets for (23), (24) and (30), and thus Slater's condition continues to hold. □

If $\mathcal{Z}$ is unbounded then checking Slater's condition is somewhat more complicated:

Theorem 5 *Assume that the polyhedron $\mathcal{Z} = \{\mathsf{x} \in \mathbb{R}^n_+ : \mathsf{a}_i^\top \mathsf{x} = b_i,\ i \in [1:m]\}$ is unbounded, and let J be the unbounded indices, i.e.*

$$\emptyset \neq J := \{i \in [1:n] : \sup\{x_i : \mathsf{x} \in \mathcal{Z}\} = \infty\} \subseteq [1:n] \setminus B\,.$$

Let $\widetilde{\mathsf{Q}}$ be the principal submatrix of Q corresponding to the indices J and let $\widetilde{\mathsf{a}}_i$ be the subvector of a_i corresponding to the indices J.

Then Slater's condition holds for problem (23) *(for all $k \in [1:2]$, $l \in [1:4]$) if and only if $\mathsf{z}^\top \widetilde{\mathsf{Q}} \mathsf{z} > 0$ for all $\mathsf{z} \in \mathbb{R}^{|J|}_+ \setminus \{\mathsf{o}\}$ with $\widetilde{\mathsf{a}}_i^\top \mathsf{z} = 0$ for all $i \in [1:m]$.*

Slater's condition holds for problems (24) *and* (30) *(for all $k \in [1:2]$, $l \in [1:4]$) if and only if*

$$\widetilde{\mathsf{Q}} \in \text{int}\left(\mathcal{N}^{|J|} + \mathcal{S}^{|J|}_+ + \text{span}\{\widetilde{\mathsf{a}}_i \widetilde{\mathsf{a}}_i^\top : i \in [1:m]\} \right).$$

Proof. First we consider problem (23) for arbitrary $k\in[1:2]$, $l\in[1:4]$. From Theorem (3) we have that Slater's condition holds if and only if

$$\left\langle \begin{pmatrix} 0 & \mathsf{c}^\top \\ \mathsf{c} & \mathsf{Q} \end{pmatrix}, \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \right\rangle > 0 \qquad \text{for all } \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_k \cap \mathcal{L}_l \cap \mathcal{CP}^{n+1} \setminus \{\mathsf{O}\} \text{ such that } x_0 = 0.$$

or equivalently

$$\langle \mathsf{Q}, \mathsf{X} \rangle > 0 \qquad \text{for all } \mathsf{X} \in \mathcal{S}^n \setminus \{\mathsf{O}\} \text{ such that } \begin{pmatrix} 0 & \mathsf{o}^\top \\ \mathsf{o} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_k \cap \mathcal{L}_l \cap \mathcal{CP}^{n+1}.$$

From Theorem 2, this condition is independent of k and l, and from Lemma 1 it can be shown that $\begin{pmatrix} 0 & \mathsf{o}^\top \\ \mathsf{o} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_k \cap \mathcal{L}_l \cap \mathcal{CP}^{n+1}$ if and only if

$$\begin{aligned} X_{jk} &= 0 \text{ for all } j\in[1:n] \setminus J,\ k\in[1:n], \\ \widetilde{\mathsf{X}} &\in \operatorname{conv}\{\mathsf{v}\mathsf{v}^\top : \mathsf{v} \in \mathbb{R}^{|J|}_+,\ \widetilde{\mathsf{a}}_i^\top \mathsf{v} = 0 \text{ for all } i\in[1:m]\}, \end{aligned}$$

where $\widetilde{\mathsf{X}}$ is the principal submatrix of X corresponding to the indices J. This completes the proof for problem (23).

We next consider problem (24) for arbitrary $k\in[1:2]$, $l\in[1:4]$. Similarly to before, from Theorem 3 we have that Slater's condition holds if and only if

$$\langle \mathsf{Q}, \mathsf{X} \rangle > 0 \qquad \text{for all } \mathsf{X} \in \mathcal{S}^n \setminus \{\mathsf{O}\} \text{ such that } \begin{pmatrix} 0 & \mathsf{o}^\top \\ \mathsf{o} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_k \cap \mathcal{L}_l \cap \mathcal{S}^{n+1}_+ \cap \mathcal{N}^{n+1}.$$

From Theorem 2, this condition is independent of k and l, and from Lemma 1 it can be shown that $\begin{pmatrix} 0 & \mathsf{o}^\top \\ \mathsf{o} & \mathsf{X} \end{pmatrix} \in \mathcal{L}_2 \cap \mathcal{N}^{n+1}$ if and only if

$$\begin{aligned} X_{jk} &= 0 \text{ for all } j\in[1:n] \setminus J,\ k\in[1:n], \\ \widetilde{\mathsf{X}} &\in \mathcal{N}^{|J|} \\ \widetilde{\mathsf{X}}\widetilde{\mathsf{a}}_i &= 0 \text{ for all } i\in[1:m], \end{aligned}$$

and we then additionally have $\begin{pmatrix} 0 & \mathsf{o}^\top \\ \mathsf{o} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_k$. Therefore $\begin{pmatrix} 0 & \mathsf{o}^\top \\ \mathsf{o} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_k \cap \mathcal{L}_l \cap \mathcal{S}^{n+1}_+ \cap \mathcal{N}^{n+1}$ if and only if

$$\begin{aligned} X_{jk} &= 0 \text{ for all } j\in[1:n] \setminus J,\ k\in[1:n], \\ \langle \widetilde{\mathsf{a}}_i \widetilde{\mathsf{a}}_i^\top, \widetilde{\mathsf{X}} \rangle &= 0 \text{ for all } i\in[1:m], \\ \widetilde{\mathsf{X}} &\in \mathcal{S}^{|J|}_+ \cap \mathcal{N}^{|J|}, \end{aligned}$$

which completes the proof for problem (24).

Finally we consider problem (30) for all $k\in[1:2]$. From Theorem (3) we have that Slater's condition holds if and only if

$$\begin{aligned} \left\langle \mathsf{B}^\top \begin{pmatrix} 0 & \mathsf{c}^\top \\ \mathsf{c} & \mathsf{Q} \end{pmatrix} \mathsf{B}, \mathsf{Y} \right\rangle > 0 \text{ for all } \mathsf{Y} \in \mathcal{S}^{n+1-m}_+ \setminus \{\mathsf{O}\} \text{ s.t. } & \mathsf{B}\mathsf{Y}\mathsf{B}^\top \in \mathcal{B}_k,\ \mathsf{B}\mathsf{Y}\mathsf{B}^\top \in \mathcal{N}^{n+1}, \\ & \left\langle \mathsf{B}^\top \begin{pmatrix} 1 & \mathsf{o}^\top \\ \mathsf{o} & \mathsf{O} \end{pmatrix} \mathsf{B}, \mathsf{Y} \right\rangle = 0, \end{aligned}$$

or equivalently

$$\left\langle \begin{pmatrix} 0 & \mathsf{c}^\top \\ \mathsf{c} & \mathsf{Q} \end{pmatrix}, \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \right\rangle > 0 \text{ for all } \begin{pmatrix} x_0 & \mathsf{x}^\top \\ \mathsf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{B}_k \cap (\mathsf{B}\mathcal{S}^{n+1-m}_+\mathsf{B}^\top) \cap \mathcal{N}^{n+1} \setminus \{\mathsf{O}\} \text{ s.t. } x_0 = 0.$$

From Theorem 1, this is equivalent to the previous case for $l = 1$. □

Note that Slater's condition holding is independent of the set of binary variables, B. We also note that a simple sufficient condition for Slater's condition holding is that $\widetilde{\mathsf{Q}} \in \operatorname{int}(\mathcal{N}^{|J|} + \mathcal{S}^{|J|}_+)$.

The primal problems not being strictly feasible means that the optimal values for the dual problems may not be attained, and we now consider a simple example of this happening:

Example 4 Consider the problem

$$\min_{x} \left\{ x^2 : x = 1,\ x \geq 0 \right\}, \tag{34}$$

i.e., $1 = n = \mathsf{Q} = m = \mathsf{a}_1 = b_1 = \operatorname{opt}(34)$ and $\mathsf{c} = 0$. For $l = 4$, the completely positive reformulation to this (20) and its dual (23) are respectively

$$\min_{v,x} \left\{ v : v - 2x + 1 = 0,\ \begin{pmatrix} 1 & x \\ x & v \end{pmatrix} \in \mathcal{CP}^2 \right\}, \tag{35}$$

$$\max_{y,z} \left\{ y : \begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in \mathcal{COP}^2 \right\}. \tag{36}$$

It can then be shown that $\operatorname{feas}(35) = \{(1,1)\}$ and

$$\operatorname{feas}(36) = \left\{ (y,z) \in \mathbb{R}^2 : -1 \leq z \leq 0,\ y \leq z \right\} \cup \left\{ (y,z) \in \mathbb{R}^2 : z \geq 0,\ y \leq \frac{z}{1+z} \right\}.$$

Therefore $1 = \operatorname{opt}(34) = \operatorname{opt}(35) = \operatorname{opt}(36)$, but the optimal solution to (36) is not attained.

4 Conclusion and outlook

In this note, we have tried to unify all existing alternative CP reformulations and relaxations of mixed-binary QPs, complementing them with several novel approaches. We discussed in detail the duality requirements which ensure attainability and/or zero duality gap (strong duality) for these reformulations and relaxations, offering a concise overview of results either novel or scattered in the literature.

The following research questions could be interesting projects for the future:

1. Can we relax the nonnegative constraints by aggregating them (e.g. instead of requiring $X_{ij} \geq 0$ for all i, j, we let $\mathcal{I}_1, \ldots, \mathcal{I}_p \subseteq [1:n]^2$ and require $0 \leq \sum_{(i,j)\in\mathcal{I}_k} X_{ij}$ for all k)?
2. If we use approximation hierarchies like that of Parrilo, can we use the linear constraints in the original problem to reduce the order of the PSD matrices in this approximation (in a similar manner to what we did for the zero level approximation where $\mathcal{CP}^d$ is approximated by $\mathcal{S}_+^d \cap \mathcal{N}^d$)?

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