On Approximation Algorithms for Concave Mixed-Integer Quadratic Programming

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July 20, 2017

Abstract

Concave Mixed-Integer Quadratic Programming is the problem of minimizing a concave quadratic polynomial over the mixed-integer points in a polyhedral region. In this work we describe an algorithm that finds an ϵ-approximate solution to a Concave Mixed-Integer Quadratic Programming problem. The running time of the proposed algorithm is polynomial in the size of the problem and in $1/\epsilon$, provided that the number of integer variables and the number of negative eigenvalues of the objective function are fixed. The running time of the proposed algorithm is expected unless $\mathcal{P} = \mathcal{NP}$.

Key words: mixed-integer quadratic programming; approximation algorithms

1 Introduction

Mixed-Integer Quadratic Programming (MIQP) problems are optimization problems in which the objective function is a quadratic polynomial, the constraints are linear inequalities, and some of the variables are required to be integers:

$$
\begin{aligned}
\text{minimize} \quad & x^\top H x + h^\top x \\
\text{subject to} \quad & W x \geq w \\
& x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}.
\end{aligned}
\tag{1}
$$

In this formulation, H is symmetric, and all the data is rational. *Concave MIQP* is the special case of MIQP when the objective is concave, which occurs when H is negative semidefinite. Concave quadratic cost functions are frequently encountered in real-world problems concerning economies of scale, which corresponds to the economic phenomenon of "decreasing marginal cost" (see [13, 28]). Concave MIQP is strongly $\mathcal{NP}$-complete [29, 14, 12] and it is still $\mathcal{NP}$-hard in very restricted settings such as the problem to minimize $\sum_{i=1}^n (w_i^\top x)^2$ over $x \in \{0,1\}^n$ [24], or when the concave quadratic objective has only one negative eigenvalue [27].

If we assume that the dimension n is fixed, then Concave MIQP is polynomially solvable. Del Pia, Dey, and Molinaro [12] showed that in fixed dimension we can decide in polynomial time if a MIQP problem is bounded or not. If the problem is bounded, they show how we can replace the original polyhedron $Q := \{x \in \mathbb{R}^n : Wx \geq w\}$ with a polytope Q' that contains at least one optimum solution. Cook, Hartmann, Kannan, and McDiarmid [5, 16] showed that in fixed dimension we can find a description of the integer hull $\text{conv}\{x \in \mathbb{Z}^n : x \in Q'\}$ of Q' in polynomial time, and this result can be extended to the mixed-integer hull $Q'_I := \text{conv}\{x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} : x \in Q'\}$ by discretization [6, 17]. Since the minimum of a Concave MIQP on a polytope Q' is always achieved at one of the vertices of Q'_I, Concave MIQP can now be solved in fixed dimension by evaluating all vertices of Q'_I and by selecting one with lowest objective value. In this work, we will not assume that the dimension n of the problem is fixed.

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1.1 Our contribution

In order to state our approximation result, we first give the definition of ϵ-approximation. Consider an instance of MIQP, and let $g(x)$ denote the objective function. Let x^* be an optimum solution of the problem, and let $g_{\max}$ be the maximal value of $g(x)$ on the feasible region. For $\epsilon \in [0,1]$, we say that a feasible point $x^\diamond$ is an ϵ-*approximate solution* if

$$|g(x^\diamond) - g(x^*)| \le \epsilon \cdot |g_{\max} - g(x^*)|.$$

In the case that the problem is unbounded or infeasible, the definition fails, and we expect our algorithm to return an indicator that the problem is unbounded or infeasible. If the objective function has no upper bound on the feasible region, our definition loses its value because any feasible point is an ϵ-approximation for any $\epsilon > 0$. The definition of ϵ-approximation has some useful invariance properties. For instance, it is preserved under dilation and translation of the objective function, and it is insensitive to affine transformations of the objective function and of the feasible region. Our definition of approximation has been used in earlier works, and we refer to [23, 33, 3, 9] for more details.

In this paper we describe an algorithm that finds an ϵ-approximate solution to Concave MIQP. The key idea used in the algorithm consists in decomposing the original feasible set into a number of polyhedral subregions with a special property. We then obtain separately an ϵ-approximate solution for each of the subregions, and the best one will be an ϵ-approximate solution for the original Concave MIQP problem. In order to obtain an ϵ-approximate solution in a subregion, we adapt classic algorithms in the continuous Quadratic Programming (QP) setting based on mesh partition and linear underestimators. These algorithms were introduced by Pardalos and Rosen [26], and later perfected by Vavasis [33]. Parts of our analysis are based on the one given by Vavasis. These algorithms are not applicable to the mixed-integer case as they rely on the convexity of the feasible region. Our contribution is to adapt such algorithmic techniques to the mixed-integer case: the geometry of the mixed-integer points in each subregion in fact allows us to relax the convexity requirement. In our algorithm we need to solve a number of Mixed-Integer Linear Programming (MILP) problems with a fixed number of integer variables. These problems can be solved in polynomial time by Lenstra [20].

The running time of the proposed algorithm is polynomial in the size of the problem and in $1/\epsilon$, provided that the number p of integer variables, and the number k of negative eigenvalues of H are fixed. We now state the main result of this paper.

Theorem 1. *There is an algorithm to find an ϵ-approximate solution to Concave MIQP* (1) *in time*

$$O\left(2^p \left\lceil \sqrt{\frac{k}{\epsilon}} \right\rceil^k \tau \right).$$

In the above formula, k denotes the number of negative eigenvalues of H, and τ denotes the time to solve a MILP problem with p integer variables and of the same size as (1).

We recall that the size of a problem is the sum of the sizes of the data defining it (see [30, 4] for more details). Thus the size of MIQP problem (1) is the sum of the sizes of H, h, W, w. The time to solve a MILP problem with p integer variables and of size φ is $\tau = \text{poly}(\varphi) 2^{\text{poly}(p)}$ (see [20, 7] for more details). In particular, we have that $\tau \ge \varphi$, and thus the size of the problem does not appear explicitly in the running time of our algorithm.

The proposed algorithm has running time polynomial in the size of the problem and in $1/\epsilon$, and exponential in both k and p. We now explain why this running time is expected unless $\mathcal{P} = \mathcal{NP}$. First, we consider the dependence on ϵ. Suppose we had a similar approximation algorithm, but with running time polynomial in $|\log \epsilon|$. Vavasis [33] showed that, for $k = 1$ and $p = 0$, such an algorithm could solve Concave QP with one concave direction in polynomial time, and the latter problem is $\mathcal{NP}$-complete [27]. Suppose now that we had an approximation algorithm with running time polynomial in $1/\epsilon$ and in k. Vavasis [33] showed that, for $p = 0$, such algorithm could solve 3SAT in polynomial time, again implying $\mathcal{P} = \mathcal{NP}$. Finally, the existence of an approximation algorithm with running time polynomial in p, even in the MILP case $k = 0$ and for ϵ fixed, would allow us to solve MAX3SAT in polynomial time, again implying $\mathcal{P} = \mathcal{NP}$.

We remark that some of the procedures that we describe yield an alternative algorithm to solve Concave MIQP in polynomial time when the dimension n is fixed. This alternative algorithm, in contrast with the one described at the beginning of Section 1, does not use the method given in [12] to bound the feasible region.

1.2 Related approximation algorithms

In the IPCO version of this paper [11] we presented a different "flatness-based" algorithm to find an ϵ-approximate solution to Concave MIQP. The key decomposition idea used in [11] is similar to the one that we use in the "parity-based" algorithm presented in this paper. However, the obtained decomposition of the original feasible set into subregions is radically different. In the flatness-based algorithm, we start by obtaining just one subregion inside the original polyhedron. We then show that the remaining parts of the original polyhedron are "flat", and we subdivide each of them, using Lenstra's algorithm, into a number of polyhedra that live in a space with one less integer variable. By applying recursively the above step to the lower-dimensional polyhedra, we obtain the desired decomposition. In the parity-based algorithm no recursion is needed to obtain the subregions. We first partition the mixed-integer lattice $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ into 2^p translated mixed-integer lattices, based on the parity of the vectors in $\mathbb{Z}^p$. For each translated lattice, we then obtain one subregion directly from the original polyhedron. The main benefits of this different approach are:

(i) We do not need to apply Lenstra's algorithm for the decomposition. (We still need Lenstra's algorithm as a subroutine to solve MILPs with a fixed number of integer variables.)

(ii) We do not need to bound the feasible region, or to reduce the problem to a full-dimensional one. In fact, those steps are necessary in the flatness-based algorithm only to be able to decompose the outer region of the polyhedron using Lenstra's algorithm.

(iii) The total number of subregions that we obtain is 2^p instead of (at most) $(2k \cdot 4p(p+1)2^{p(p-1)/4})^{p+1}$.

(iv) The parity-based algorithm lends itself better to parallel implementation. This is because the 2^p subregions are obtained from the original polyhedron in a non-recursive fashion.

Concave QP is the continuous version of Concave MIQP, and can be obtained by setting $p = 0$ in (1). Concave QP is also $\mathcal{NP}$-complete [29, 31], even when the concave quadratic objective has only one concave direction [27]. In [33], Vavasis gives an algorithm to find an ϵ-approximate solution to Concave QP whose running time is

$$O\left(\left\lceil \sqrt{\frac{k}{\epsilon}} \right\rceil^k \ell\right),$$

where ℓ denotes the time to solve a Linear Programming (LP) problem of the same size as the original QP problem. Vavasis' algorithm is polynomial in the size of the problem and in $1/\epsilon$, provided that the number of negative eigenvalues of H is fixed. Moreover, this running time is expected unless $\mathcal{P} = \mathcal{NP}$. Our algorithm reduces to Vavasis' when $p = 0$, thus it can be seen as an extension of Vavasis' algorithm to the mixed-integer case. This shows that the computational effort needed to find an ϵ-approximate solution to a Concave QP or to a Concave MIQP are essentially the same, as long as the number of integer variables is small. This is not the first time that the same type of problem can be solved to the same extent and with the same complexity in the continuous case, and in the mixed-integer case with a fixed number of integer variables. In fact, celebrated results by Khachiyan [19] and by Lenstra [20] show that this is true also for linear problems: both Linear Programming, and Mixed-Integer Linear Programming with a fixed number of integer variables can be solved in polynomial time.

In [32, 34], Vavasis obtains approximation algorithms for general QP problems, provided that the feasible region is bounded. One of the main ingredients in these results is the use of weak Löwner-John pairs for the polytope describing the feasible points.

In [10], De Loera, Hemmecke, Köppe, and Weismantel present an algorithm to find an ϵ-approximate solution for the problem of minimizing a polynomial function over mixed-integer points in a polytope. The running time is polynomial in the maximum total degree of the polynomial, the input size, and $1/\epsilon$, provided

that the dimension is fixed. Using a stronger notion of approximation, they also show the existence of a fully polynomial-time approximation scheme for the problem of maximizing a non-negative polynomial over mixed-integer points in a polytope, when the number of variables is fixed. Both algorithms are based on Barvinok's theory for encoding all lattice points of a polyhedron in terms of short rational functions [1, 2].

In [18], Hildebrand, Weismantel, and Zemmer give a fully polynomial-time approximation scheme for MIQP problems, provided that the dimension is fixed, and the objective is homogeneous with at most one positive or negative eigenvalue. Their approach links the subdivision strategy of Papadimitriou and Yannakakis [25] with real algebraic certificates for the positivity of polynomials.

2 Description of the algorithm

In this section we describe our algorithm to find an ϵ-approximate solution to a Concave MIQP problem. The analysis of its running time provides a proof to Theorem 1.

Our starting point is a Concave MIQP problem of the form (1) in which H is negative semidefinite with rank k.

2.1 Diagonalization

The first task of our algorithm is to construct a rational linear change of variables $(y, z) = L^\top x$ that transforms the objective function of (1) in separable form, where the negative-definite part of the problem is confined to k variables. This transformation maps (1) into an equivalent problem of the form:

$$
\begin{aligned}
\text{minimize} \quad & y^\top D y + c^\top y + f^\top z \\
\text{subject to} \quad & (y, z) \in P \\
& (y, z) \in \mathcal{L},
\end{aligned}
\tag{2}
$$

where $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{n-k}$, D is diagonal and negative definite, P is a rational polyhedron in $\mathbb{R}^n$, and $\mathcal{L}$ is a subset of $\mathbb{R}^n$ isomorphic to $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

In order to obtain this change of variables, we first compute a rational decomposition of H of the form $H = L\tilde{D}L^\top$, where the matrix $\tilde{D}$ is diagonal, whose first k diagonal entries are negative and the rest are zero, and where L is non-singular. Such a decomposition can be obtained in $O(n^3)$ time via symmetric Gaussian elimination [8] or $LDL^\top$ decomposition (see, e.g., [15]). The importance of these decomposition techniques is that, in contrast to other factorizations like the Eigenvalue decomposition or the Cholesky decomposition, they are rational decompositions; i.e., if the matrix H is rational then all numbers that appear in the decomposition are rational and polynomially sized.

We now perform the change of basis $(y, z) = L^\top x$, and we end up with an equivalent problem of the form (2), where y is a k-dimensional vector, z is a $(n-k)$-dimensional vector, D is diagonal and negative definite, and

$$
\mathcal{L} := \{(y, z) \in \mathbb{R}^n : (y, z) = L^\top x, \text{ for } x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}\}.
$$

Note that $\mathcal{L}$ is a mixed-integer lattice. In fact, if we denote by $\ell^1, \ldots, \ell^n$ the columns of $L^\top$, we can write the set $\mathcal{L}$ in the form

$$
\mathcal{L} = \{(y, z) = \sum_{i=1}^n x_i \ell^i : x_1, \ldots, x_p \in \mathbb{Z}, \ x_{p+1}, \ldots, x_n \in \mathbb{R}\}.
$$

2.2 Boundedness of MIQP

The next task of our algorithm is to detect if a Concave MIQP problem of the form (2) is unbounded or not. As we describe below, this can be done by solving $2k+1$ MILP problems with p integer variables and of the same size as (2). We remark that detecting if a general MIQP problem is unbounded is $\mathcal{NP}$-hard even in the pure continuous case (see [22, 12]).

We first define two functions. The first one is the part of the objective function that depends on y:

$$
q(y) := y^\top D y + c^\top y.
$$

The second one is a function that associates to each $\bar{y} \in \mathbb{R}^k$, the optimal value of the MILP problem obtained as the restriction of (2) to the set of points $(\bar{y}, z)$. Formally,

$$\phi(\bar{y}) := \min\{f^\top z : (\bar{y}, z) \in P \cap \mathcal{L}\},$$

for all $\bar{y}$ for which the minimum exists. If, for a fixed $\bar{y}$, the MILP on the right-hand side is infeasible, we write $\phi(\bar{y}) := \infty$. Similarly, if, for a fixed $\bar{y}$, the MILP on the right-hand side is unbounded, we write $\phi(\bar{y}) := -\infty$. Our MIQP problem (2) is now equivalent to the unconstrained problem

$$\text{minimize} \quad q(y) + \phi(y)$$
$$\text{subject to} \quad y \in \mathbb{R}^k.$$

Given a set $S \subseteq \mathbb{R}^n$, we denote by

$$\pi(S) = \{y \in \mathbb{R}^k : \exists z \in \mathbb{R}^{n-k} \text{ with } (y, z) \in S\}$$

the projection of S onto the space of the y variables. The next proposition characterizes when our MIQP is unbounded.

Proposition 1. *A Concave MIQP problem of the form* (2) *is unbounded if and only if*

(i) *For every* $y \in \pi(P \cap \mathcal{L})$, $\phi(y) = -\infty$, *or*

(ii) *Region* $\pi(P \cap \mathcal{L})$ *is unbounded.*

Proof. In this proof we denote the feasible set of (2) by $\mathcal{F}$, i.e., $\mathcal{F} := P \cap \mathcal{L}$. Condition (i) trivially implies that (2) is unbounded. In fact, the existence of a single $\bar{y}$ such that $\phi(\bar{y}) = -\infty$ implies that (2) is unbounded because $\phi(\bar{y})$ is a restriction of (2).

To prove the sufficiency of condition (ii), we now assume that $\pi(\mathcal{F})$ is unbounded. As $\pi(\mathcal{F}) \subseteq \pi(\text{conv}\,\mathcal{F})$, we have that $\pi(\text{conv}\,\mathcal{F})$ is unbounded as well. By Meyer's theorem [21] we have that $\text{conv}\,\mathcal{F}$ is a rational polyhedron, and Fourier's method implies that $\pi(\text{conv}\,\mathcal{F})$ is a rational polyhedron as well. Let y^r be a nonzero rational vector in the recession cone of $\pi(\text{conv}\,\mathcal{F})$. It follows that there exists a rational vector z^r such that (y^r, z^r) is in the recession cone of $\text{conv}\,\mathcal{F}$. Let $(\bar{y}, \bar{z}) \in \mathcal{F}$, and consider the ray $\{(\bar{y}, \bar{z}) + t(y^r, z^r) : t \geq 0\}$. The objective function $y^\top D y + c^\top y + f^\top z$ evaluated on the ray is $t^2(y^{r\top} D y^r) + O(t)$. Since the leading term is negative, the objective function tends to $-\infty$ along the ray. It follows that (2) is unbounded because, for every $\bar{t} \in \mathbb{R}$, the ray $\{(\bar{y}, \bar{z}) + t(y^r, z^r) : t \geq \bar{t}\}$ contains (infinitely many) points in $\mathcal{F}$.

To prove necessity of the conditions, we now assume that (2) is unbounded, and show that at least one of the two conditions holds. Consider the relaxation of (2) obtained by dropping the constraint $(y, z) \in \mathcal{L}$. Also the latter continuous problem is unbounded, and in this case Vavasis [31] proved that there exists a rational ray $\{(\tilde{y}, \tilde{z}) + t(y^r, z^r) : t \geq 0\} \subseteq P$ along which the objective function of (2) tends to $-\infty$. At least one among y^r and z^r must be nonzero.

Suppose y^r is nonzero, and let $(\bar{y}, \bar{z}) \in \mathcal{F}$. The ray $\{(\bar{y}, \bar{z}) + t(y^r, z^r) : t \geq 0\}$ is contained in P. Moreover, for every $\bar{t} \in \mathbb{R}$, the ray $\{(\bar{y}, \bar{z}) + t(y^r, z^r) : t \geq \bar{t}\}$ contains points in $\mathcal{F}$. This implies that the ray $\{\bar{y} + ty^r : t \geq \bar{t}\}$ contains points in $\pi(\mathcal{F})$ for every $\bar{t} \in \mathbb{R}$, and hence $\pi(\mathcal{F})$ is unbounded.

The other case is that $y^r = 0$, in which case z^r is nonzero. Our ray $\{(\tilde{y}, \tilde{z}) + t(y^r, z^r) : t \geq 0\}$ can then be written as $\{(\tilde{y}, \tilde{z} + tz^r) : t \geq 0\}$. Also, $f^\top z^r < 0$ because the objective function decreases along the ray by assumption. Let $\bar{y} \in \pi(\mathcal{F})$, let $\bar{z}$ be a vector such that $(\bar{y}, \bar{z}) \in \mathcal{F}$, and consider the ray $\{(\bar{y}, \bar{z} + tz^r) : t \geq 0\}$. This new ray is contained in the polyhedron $\{(y, z) \in P : y = \bar{y}\}$. Moreover, since (y^r, z^r) is rational, the ray $\{(\bar{y}, \bar{z} + tz^r) : t \geq \bar{t}\}$ contains points in $\mathcal{F}$ for every $\bar{t} \in \mathbb{R}$. Finally, $f^\top(\bar{z} + tz^r) = t(f^\top z^r) + O(1)$ tends to $-\infty$ along the ray, implying $\phi(\bar{y}) = -\infty$. $\square$

The characterization given in Proposition 1 allows us to determine whether problem (2) is unbounded or not. For every $j = 1, \ldots, k$, solve

$$\min\{y_j : (y, z) \in P \cap \mathcal{L}\},$$
$$\max\{y_j : (y, z) \in P \cap \mathcal{L}\}.$$

Each problem is a MILP with p integer variables and of the same size as (2). If any of these MILPs is unbounded, then MIQP (2) is unbounded by Proposition 1. Otherwise, let $(\bar{y}, \bar{z})$ be a point in $P \cap \mathcal{L}$, which can be, for example, an optimum solution of one of the $2k$ MILPs just solved. We can now compute $\phi(\bar{y})$ by solving another MILP with p integer variables and of the same size as (2). As the point $\bar{y}$ is in $\pi(P \cap \mathcal{L})$, Proposition 1 implies that $\phi(\bar{y}) = -\infty$ if and only if problem (2) is unbounded.

We note that we could have checked if $\pi(P \cap \mathcal{L})$ is unbounded by solving only $k + 1$ MILP problems instead of $2k$. However, checking it as described above allows us later, in (3), to avoid solving $2k$ similar MILP problems, effectively rendering this check free from a computational perspective. We now assume that MIQP problem (2) is bounded.

We remark that the results presented so far in Section 2 provide an alternative algorithm to solve Concave MIQP in polynomial time when the dimension n is fixed. In fact, using the above technique in fixed dimension, we can decide in polynomial time if a Concave MIQP problem of the form (1) is bounded or not. Assume now that the problem is bounded. Then, a minimum of the Concave MIQP on the polyhedron $Q := \{x \in \mathbb{R}^n : Wx \geq w\}$ is always achieved at all feasible points in one of the minimal faces of the mixed-integer hull $Q_I = \text{conv}\{x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} : Wx \geq w\}$ of Q. As described in Section 1, in fixed dimension a description of Q_I can be found in polynomial time. Concave MIQP can now be solved by evaluating a feasible point in each minimal face of Q_I and by selecting one with lowest objective value. In particular, this algorithm does not use the method given in [12] to bound the feasible region.

2.3   Decomposition

In this section we show how to decompose the MIQP problem (2) into 2^p MIQP problems restricted to a smaller feasible region. To construct this decomposition we will solve $2^p \cdot 2k$ MILP problems with p integer variables.

Consider the mixed-integer lattice

$$\bar{\mathcal{L}}^0 := \{(y, z) \in \mathbb{R}^n : (y, z) = L^\top x, \text{ for } x \in 2\mathbb{Z}^p \times \mathbb{R}^{n-p}\},$$

and the cosets of $\bar{\mathcal{L}}^0$ in $\mathcal{L}$ defined by

$$\bar{\mathcal{L}}^v := \bar{\mathcal{L}}^0 + L^\top(v, 0),$$

for every binary vector $v \in \{0, 1\}^p$. Equivalently,

$$\bar{\mathcal{L}}^v = \{(y, z) = \sum_{i=1}^n x_i \ell^i : x_i \text{ even if } i \in \{1, \dots, p\}, \ v_i = 0,$$
$$x_i \text{ odd if } i \in \{1, \dots, p\}, \ v_i = 1,$$
$$x_{p+1}, \dots, x_n \in \mathbb{R}\}.$$

Note that the family $\bar{\mathcal{L}}^v$, for $v \in \{0, 1\}^p$, forms a partition of $\mathcal{L}$. We now observe a key property of cosets which will play a key role in our approximation algorithm.

Claim 1. *Let $v \in \{0, 1\}^p$, and let (y^0, z^0) and (y^1, z^1) be vectors in $\bar{\mathcal{L}}^v$. Then the midpoint of the segment joining (y^0, z^0) and (y^1, z^1) is in $\mathcal{L}$.*

Proof of claim. Let $(y^\bullet, z^\bullet)$ be the midpoint of the segment joining (y^0, z^0) and (y^1, z^1), i.e.,

$$(y^\bullet, z^\bullet) := \frac{1}{2}(y^0, z^0) + \frac{1}{2}(y^1, z^1).$$

Each (y^β, z^β), for $\beta = 0, 1$, can be written as $(\tilde{y}^\beta, \tilde{z}^\beta) + L^\top(v, 0)$, for some $(\tilde{y}^\beta, \tilde{z}^\beta) \in \bar{\mathcal{L}}^0$. Moreover, we can write each $(\tilde{y}^\beta, \tilde{z}^\beta)$ as:

$$(\tilde{y}^\beta, \tilde{z}^\beta) = L^\top x^\beta,$$

6

for $x^\beta \in 2\mathbb{Z}^p \times \mathbb{R}^{n-p}$. We obtain

$$(y^\bullet, z^\bullet) = \frac{1}{2}(L^\top x^0 + L^\top(v,0)) + \frac{1}{2}(L^\top x^1 + L^\top(v,0)) = L^\top \frac{x^0 + x^1}{2} + L^\top(v,0).$$

For every $i = 1, \ldots, p$, scalars x_i^0 and x_i^1 are both even, therefore $x_i^0 + x_i^1$ is even and $(x_i^0 + x_i^1)/2$ is an integer. The vector $(y^\bullet, z^\bullet)$ is then in $\mathcal{L}$ because it is the sum of two vectors in $\mathcal{L}$. $\diamond$

Next, we compute for each $v \in \{0,1\}^p$ and each $j = 1, \ldots, k$ the 2 MILP problems

$$\begin{aligned} l_j^v &:= \min\{y_j : (y,z) \in P \cap \bar{\mathcal{L}}^v\}, \\ u_j^v &:= \max\{y_j : (y,z) \in P \cap \bar{\mathcal{L}}^v\}. \end{aligned} \tag{3}$$

Each of these MILPs is bounded, since the same problems over the larger feasible region $P \cap \mathcal{L}$ are bounded by Proposition 1. For each $v \in \{0,1\}^p$, we define the polytope $\bar{P}^v$ as

$$\bar{P}^v := \{(y,z) \in P : l_j^v \le y_j \le u_j^v, \ j = 1, \ldots, k\},$$

and we consider the MIQP problem (2) restricted to feasible region $\bar{P}^v \cap \mathcal{L}$

$$\begin{aligned} \text{minimize} \quad & y^\top D y + c^\top y + f^\top z \\ \text{subject to} \quad & (y,z) \in \bar{P}^v \\ & (y,z) \in \mathcal{L}. \end{aligned} \tag{4}$$

In Section 2.4 we will show how to construct an ϵ-approximate solution (y^v, z^v) to each Concave MIQP problem (4). We now show that the vector (y^Δ, z^Δ) that achieves the minimum objective function value among all the 2^p vectors (y^v, z^v), for $v \in \{0,1\}^p$, is an ϵ-approximate solution to the Concave MIQP problem (2).

Since $\mathcal{L}^v$, for $v \in \{0,1\}^p$, forms a partition of $\mathcal{L}$, there exists $\bar{v} \in \{0,1\}^p$ such that $\mathcal{L}^{\bar{v}}$ contains a global optimum solution of (2). Then the ϵ-approximate solution $(y^{\bar{v}}, z^{\bar{v}})$ to the corresponding Concave MIQP (4) is also an ϵ-approximate solution to the Concave MIQP (2). As the objective value of (y^Δ, z^Δ) is at most that of $(y^{\bar{v}}, z^{\bar{v}})$, we have that the vector (y^Δ, z^Δ) is an ϵ-approximate solution to the Concave MIQP (2). The ϵ-approximate solution to the original Concave MIQP (1) is then the vector $x^\Delta := L^{-\top}(y^\Delta, z^\Delta)$.

2.4 Approximation in one subregion

In this section we present an algorithm to find an ϵ-approximate solution (y^v, z^v) of MIQP problem (4). This ϵ-approximate solution can be found by solving $\lceil \sqrt{k/\epsilon} \rceil^k$ MILP problems with p integer variables.

We now fix a vector $v \in \{0,1\}^p$. For ease of notation we drop the v superscript, i.e., we write $\bar{P}$, $\bar{\mathcal{L}}$, l_j, u_j instead of $\bar{P}^v$, $\bar{\mathcal{L}}^v$, l_j^v, u_j^v, respectively. We also assume that $l_j < u_j$ for all $j = 1, \ldots, k$, because otherwise y_j is uniquely determined and can be dropped from the problem. In order to simplify some computations, we assume that the coordinates of the vector y are translated and rescaled so that

$$[l_1, u_1] \times \cdots \times [l_k, u_k] = [0,1]^k.$$

Note that this affine transformation depends on $\bar{v}$ and changes the data defining (4). In particular: (i) the negative entries of D can change, but still remain negative, (ii) the set $\mathcal{L}$ can become a *translated* mixed-integer lattice, and (iii) the vector c, the polytope $\bar{P}$, and the function $q(y)$ can change.

We now place an $(m+1) \times \cdots \times (m+1)$ grid of points in the cube $[0,1]^k$. The value of m is the ceiling of $\sqrt{k/\epsilon}$, and the reason behind this choice will be explained later. The coordinates of the points of the grids have the form $(i_1/m, i_2/m, \ldots, i_k/m)$, where $i_1, \ldots, i_k \in \{0, 1, \ldots, m\}$. The grid partitions $[0,1]^k$ into m^k subcubes. Next, for each subcube C, we construct an affine underestimator of the restriction of $q(y)$ to C. In what follows, we denote by γ the absolute value of the smallest diagonal entry of D.

Claim 2. *For each subcube C, we can construct an affine function $\mu(y)$ such that for every $y \in C$ we have*

$$\mu(y) \le q(y) \le \mu(y) + \frac{\gamma k}{4m^2}.$$

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Proof of claim. Let C be a particular subcube, say $C = [r_1, s_1] \times \cdots \times [r_k, s_k]$, where $s_j - r_j = 1/m$ for every j. For each $j = 1, \ldots, k$, the affine univariate function

$$\lambda_j(y_j) := d_{jj}(r_j + s_j)y_j + c_j y_j - d_{jj} r_j s_j \tag{5}$$

satisfies $\lambda_j(r_j) = d_{jj} r_j^2 + c_j r_j$, and $\lambda_j(s_j) = d_{jj} s_j^2 + c_j s_j$. We define the affine function from $\mathbb{R}^k$ to $\mathbb{R}$ given by

$$\mu(y) := \sum_{j=1}^{k} \lambda_j(y_j).$$

The separability of q implies that $\mu(y)$ and $q(y)$ attain the same values at all vertices of C. As q is concave, this in particular implies that $\mu(y) \leq q(y)$.

We now show that $q(y) - \mu(y) \leq \gamma k/(4m^2)$. From the separability of q, we obtain

$$q(y) - \mu(y) = \sum_{j=1}^{k} (d_{jj} y_j^2 + c_j y_j - \lambda_j(y_j)).$$

Using the explicit formula for λ_j given in (5), it can be derived that:

$$d_{jj} y_j^2 + c_j y_j - \lambda_j(y_j) = -d_{jj}(y_j - r_j)(s_j - y_j).$$

The univariate function on the right-hand side is concave, and its maximum is achieved at $y_j = (r_j + s_j)/2$. This maximum value is $-d_{jj}/(4m^2)$. Therefore, as $-d_{jj} \leq \gamma$, for $j = 1, \ldots, k$, we establish that $q(y) - \mu(y) \leq \gamma k/(4m^2)$. ◇

For each subcube C_t, $t = 1, \ldots, m^k$, our algorithm now constructs the corresponding affine function $\mu_t(y)$ described in Claim 2. Then, we minimize $\mu_t(y) + f^\top z$ on each subcube C_t. This can be done by solving the following MILP problem with p integer variables:

$$\begin{aligned} \text{minimize} \quad & \mu_t(y) + f^\top z \\ \text{subject to} \quad & (y, z) \in \bar{P} \\ & y \in C_t \\ & (y, z) \in \mathcal{L}. \end{aligned}$$

Finally, our algorithm returns the best solution $(y^\diamond, z^\diamond)$ among all the m^k optimum solutions just obtained.

We now show that $(y^\diamond, z^\diamond)$ is an ϵ-approximate solution to the MIQP problem (4). In order to do so, in the next two claims we will obtain two different bounds. To simplify the notation in the next arguments, from now on we denote the objective function of MIQP (4) by

$$g(y, z) := q(y) + f^\top z = y^\top D y + c^\top y + f^\top z.$$

The first bound is an upper bound on the gap between the objective value at $(y^\diamond, z^\diamond)$ and the objective value at an optimum solution (y^*, z^*) of (4).

Claim 3. $g(y^\diamond, z^\diamond) - g(y^*, z^*) \leq \gamma k/(4m^2)$.

Proof of claim. Let $t^\diamond$ such that $(y^\diamond, z^\diamond) \in C_{t^\diamond}$ and t^* such that $(y^*, z^*) \in C_{t^*}$. We have

$$\begin{aligned} g(y^\diamond, z^\diamond) &\leq \mu_{t^\diamond}(y^\diamond) + f^\top z^\diamond + \frac{\gamma k}{4m^2} \\ &\leq \mu_{t^*}(y^*) + f^\top z^* + \frac{\gamma k}{4m^2} \\ &\leq g(y^*, z^*) + \frac{\gamma k}{4m^2}. \end{aligned}$$

The first inequality follow because, by Claim 2, we have $q(y^\diamond) \leq \mu_{t^\diamond}(y^\diamond) + \gamma k/(4m^2)$. The second inequality holds by definition of $(y^\diamond, z^\diamond)$. The third inequality follow because, by Claim 2, we have $\mu_{t^*}(y^*) \leq q(y^*)$. ◇

The second bound is a lower bound on the gap between the maximum and the minimum objective function values of the points in $\bar{P} \cap \mathcal{L}$. Without loss of generality, we now assume that the smallest diagonal entry of D, the one with absolute value γ, is d_{11}. By construction of $\bar{P}$ there exists a point $(y^0, z^0) \in \bar{P} \cap \bar{\mathcal{L}}$ such that $y_1^0 = 0$. Similarly, there is a point $(y^1, z^1) \in \bar{P} \cap \bar{\mathcal{L}}$ such that $y_1^1 = 1$. We define the midpoint of the segment joining (y^0, z^0) and (y^1, z^1) as

$$(y^\bullet, z^\bullet) := \frac{1}{2}(y^0, z^0) + \frac{1}{2}(y^1, z^1).$$

The vector $(y^\bullet, z^\bullet)$ is in $\mathcal{L}$ by Claim 1, and is in $\bar{P}$ since it is a convex combination of two points in $\bar{P}$. We are now ready to derive our lower bound.

Claim 4. $g(y^\bullet, z^\bullet) - g(y^*, z^*) \geq \gamma/4$.

Proof of claim. The claim follows from the chain of inequalities below.

$$\begin{aligned}
g(y^\bullet, z^\bullet) &= \frac{1}{2}\big(g(y^0, z^0) + g(y^1, z^1)\big) - \frac{1}{4}(y^0 - y^1)^\top D(y^0 - y^1) \\
&\geq g(y^*, z^*) - \frac{1}{4}(y^0 - y^1)^\top D(y^0 - y^1) \\
&= g(y^*, z^*) + \frac{1}{4}\sum_{i=1}^{k}(y_j^0 - y_j^1)^2(-d_{jj}) \\
&\geq g(y^*, z^*) + \frac{1}{4}(y_1^0 - y_1^1)^2(-d_{11}) \\
&= g(y^*, z^*) + \frac{\gamma}{4}.
\end{aligned}$$

The first inequality holds because $g(y^0, z^0) \geq g(y^*, z^*)$ and $g(y^1, z^1) \geq g(y^*, z^*)$. In order to obtain the second inequality, note that all the terms of the summation are nonnegative, thus a lower bound is given by the first term. In the last equation, we have used $-d_{11} = \gamma$, and $(y_1^0 - y_1^1)^2 = 1$ by choice of y^0 and y^1. $\diamond$

Using Claim 3, Claim 4, and the definition of m we obtain

$$\begin{aligned}
|g(y^\diamond, z^\diamond) - g(y^*, z^*)| &\leq \frac{\gamma k}{4m^2} \\
&\leq \frac{k}{m^2} \cdot |g(y^\bullet, z^\bullet) - g(y^*, z^*)| \\
&\leq \epsilon \cdot |g(y^\bullet, z^\bullet) - g(y^*, z^*)|.
\end{aligned}$$

Therefore, $(y^\diamond, z^\diamond)$ is an ϵ-approximate solution to the MIQP problem (4).

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