Tight cycle relaxations for the cut polytope

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Abstract

We study the problem of optimizing an arbitrary weight function $w^T z$ over the metric polytope of a graph $G = (V, E)$, a well-known relaxation of the cut polytope. We define the signed graph (G, E^-), where E^- consists of the edges of G having negative weight. We characterize the sign patterns of the weight vector w such that all optimal vertices of the metric polytope are integral, and we show that they correspond to signed graphs with no odd-K_5 minor. Our result has significant implications for unconstrained zero-one quadratic programming problems. We relate the strength of the cycle relaxation of the boolean quadric polytope to the sign pattern of the objective function. In this case, the sign patterns such that all optimal vertices of the cycle relaxation are integral correspond to signed graphs with no odd-K_4 minor.

Keywords: cut polytope; metric polytope; boolean quadric polytope; cycle relaxation; signed graph; odd minor.

1 Introduction

For an undirected graph $G = (V, E)$ without loops and parallel edges, and for a node subset $S \subseteq V$, the cut $\delta(S)$ is the set of all of the edges of G having exactly one endpoint in S. The cut polytope of G, denoted by $\mathrm{CUT}(G)$, is defined as the convex hull of the incidence vectors of cuts in G. We consider the problem

$$\min\{w^T z : z \in \mathrm{CUT}(G)\}, \tag{CUT}$$

where $w : E \to \mathbb{R}$ is a weight function with arbitrary sign pattern. This problem reduces to the classical *maximum cut* problems when $w \leq 0$. The maximum cut problem, and hence problem (CUT), is NP-hard [20, 11].

The *cycle relaxation* of (CUT) is:

$$\begin{array}{ll} \min & w^T z \\ \text{s.t.} & z \in \mathrm{MET}(G), \end{array} \tag{MET}$$

where $\mathrm{MET}(G)$ is the *metric polytope*, i.e., the polytope defined by the vectors $z \in \mathbb{R}^E$ that satisfy the system of inequalities

$$z(D) - z(C \setminus D) \leq |D| - 1 \qquad C \text{ cycle of } G, D \subseteq C, |D| \text{ odd} \tag{1}$$

$$0 \leq z_e \leq 1 \qquad e \in E, \tag{2}$$

where, for $T \subseteq E$, $z(T) = \sum_{e \in T} z_e$. Barahona and Mahjoub [2] proved that CUT(G)= MET(G) if and only if G is not contractible to K_5. In other words, if G contains a K_5 minor, all optima to (MET) might be fractional. This is for example the case when G is a complete graph.

Note that the above condition is only based on the topology of G. We are interested in finding out whether the knowledge of some structural property of w can be exploited to relax the previous condition on the graph topology, in order to have that all the optimal vertices of (MET) are integral. Can we guarantee, even when $\mathrm{MET}(G)$ has fractional vertices, that (MET) is a tight relaxation of (CUT) for some classes of

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weight functions w? In particular, our goal is to relate the strength of (MET) as a relaxation of (CUT) to the sign pattern of w. Without loss of generality we assume $w_e \neq 0$ for all $e \in E$. We introduce the signed graph $(G, E^-(w))$, where the set of *negative edges* is defined as

$$E^-(w) = \{e \in E : w_e < 0\}. \tag{3}$$

We also define the set of *positive edges* as $E^+(w) = E \setminus E^-(w)$. We might write E^- and E^+ in place of $E^-(w)$ and $E^+(w)$, if w is clear from the context. Basic notions on signed graphs and minors are presented in the Section 2. We characterize the sign patterns of w such that all optimal vertices of (MET) are integral, i.e. they are incidence vectors of cuts in G. Our main results are the following.

Theorem 1. *If the signed graph $(G, E^-(w))$ does not contain an odd-K_5 minor, then any optimal vertex of* (MET) *is integral.*

Conversely, we show that for any signed graph (G, Σ) with an odd-K_5 minor, there exists a vector of weights w whose sign pattern is compatible with Σ, i.e., $E^-(w) = \Sigma$, and such that (MET) has an optimal fractional vertex.

Proposition 1. *Let $G = (V, E)$ and $\Sigma \subseteq E$. If (G, Σ) has an odd-K_5 minor, then there exist $w \in \mathbb{R}^E$ such that $E^-(w) = \Sigma$ and* (MET) *has a unique optimum that is fractional.*

Note that (MET) can be solved in polynomial time with the ellipsoid method, since there is a polynomial-time separation algorithm for inequalities (1)[14, 2, 8]. Thus, Theorem 1 provides a new class of polynomially-solvable instances for (CUT).

Let $w \in \mathbb{R}^E$ and $E^- = E^-(w)$. Let $\mathcal{C}(G, E^-)$ be the set of *odd cycles* of the signed graph (G, E^-), i.e. the cycles that contain an odd number of edges in E^-. Moreover, for an odd cycle $C \in \mathcal{C}(G, E^-)$, define $C^- = C \cap E^-$ and $C^+ = C \cap E^+$. We define a relaxation of the metric polytope obtained by keeping only the constraints (1) having $C \in \mathcal{C}(G, E^-)$ and $D = C^-$, the nonnegativity constraints associated to the positive edges, and the constraints $z_e \leq 1$ associated to the negative edges. Precisely, we denote by $\mathrm{MET}(G, E^-)$ the polyhedron defined by the vectors $z \in \mathbb{R}^E$ that satisfy the system of inequalities

$$z(C^-) - z(C^+) \leq |C^-| - 1 \qquad C \in \mathcal{C}(G, E^-) \tag{4}$$

$$z_e \geq 0 \qquad e \in E^+, \tag{5}$$

$$z_e \leq 1 \qquad e \in E^-. \tag{6}$$

Note that the above inequality system is derived from the inequality system defining $\mathrm{MET}(G)$ by including only the inequalities (1)(2) that can be written in the form $a^T z \geq b$ where a and w have the same sign pattern.

To prove Theorem 1 and Proposition 1, we will first prove the following result.

Theorem 2. *A vector $z^* \in \mathbb{R}^E$ is an optimum of* (MET) *with weights w if and only if it is an optimum of*

$$\begin{array}{ll} \min & w^T z \\ s.t. & z \in \mathrm{MET}(G, E^-). \end{array} \tag{MET$^-$}$$

Theorem 2 is an extension of a result of Poljak [25] in the case where $E = E^-$.

An interesting geometric consequence of Theorem 2 concerns the polyhedral fans of $\mathrm{MET}(G)$ and $\mathrm{MET}(G, E^-)$. For $T \subseteq E$, denote by $\mathcal{R}(T)$ the orthant $\{z \in \mathbb{R}^E : z_e \geq 0\ \forall e \in T, z_e \leq 0\ \forall e \in E \setminus T\}$. We will prove the following result, where we denote by $A + B$ the Minkowski sum of two subsets A and B of $\mathbb{R}^E$.

Proposition 2. *For each $E^- \subseteq E$, z is a vertex of $\mathrm{MET}(G, E^-)$ if and only if it is a vertex of $\mathrm{MET}(G)$ whose normal cone contains in its relative interior a vector in $\mathcal{R}(E^-)$. Moreover*

$$\mathrm{MET}(G, E^-) = \mathrm{MET}(G) + \mathcal{R}(E^+).$$

The above result implies that (4)–(6) is a facial description of $\text{MET}(G)+\mathcal{R}(E^+)$, and such facial description is derived from the facial description of $\text{MET}(G)$ by removing from (1)–(2) some specific inequalities. We remark that in general the Minkowski sum of a polytope and an orthant could have a facial structure that is far more complex than that of the polytope itself. One example is the perfect matching polytope of G: all its facets have coefficients in $\{0,-1,+1\}$, but the Minkowski sum of the perfect matching polytope and the nonnegative orthant has facets whose coefficients take on all the integer values between 0 and $|E|/2-2$ [6]. Moreover, since (MET^-) is unbounded if $w \notin \mathcal{R}(E^+)$, it also follows that the polyhedral fan of $\text{MET}(G,E^-)$ is a refinement of $\mathcal{R}(E^-)$.

Our results concerning the metric polytope also have significant implications for the problem

$$\begin{aligned} \max \quad & c^T x + x^T Q x \\ \text{s.t.} \quad & x \in \{0,1\}^n, \end{aligned} \tag{QP}$$

where $c \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$. Since $x_i^2 = x_i$ for all $x \in \{0,1\}^n$ and $i = 1,\dots,n$, we can assume wlog that $Q = (q_{ij})_{1\le i,j\le n}$ is an upper-triangular matrix with zero diagonal. The sparsity pattern of Q is captured by the undirected graph $G = (V,E)$ with $V = \{1,\dots,n\}$ and $E = \{ij : i,j \in V, i<j, q_{ij} \neq 0\}$.

Problem (QP) is NP-hard in general. A classic approach is to linearize (QP) by introducing new variables $y_{ij} = x_i x_j$ for all $ij \in E$ [9, 17]. Padberg [24] defined the *boolean quadric polytope* of G as

$$\text{BQP}(G) = \text{conv}\{(x,y) \in \{0,1\}^{V\cup E} : y_{ij} = x_i x_j \ \forall\, ij \in E\}$$

and restated (QP) as the equivalent problem

$$\begin{aligned} \max \quad & c^T x + q^T y \\ \text{s.t.} \quad & (x,y) \in \text{BQP}(G). \end{aligned} \tag{BQP}$$

where $q = (q_{ij})_{ij\in E}$. We assume that G contains no isolated node. The *edge relaxation* of (BQP) is:

$$\begin{aligned} \max \quad & c^T x + q^T y \\ \text{s.t.} \quad & (x,y) \in \text{BQP}_2(G), \end{aligned} \tag{BQP2}$$

where the polyhedron $\text{BQP}_2(G)$ is defined by the vectors $(x,y) \in \mathbb{R}^{V\cup E}$ that satisfy the system of inequalities

$$\begin{aligned} & y_{ij} \le x_i,\ y_{ij} \le x_j && ij \in E \\ & y_{ij} \ge 0,\ y_{ij} \ge x_i + x_j - 1 && ij \in E. \end{aligned}$$

Padberg [24] proved that $\text{BQP}_2(G)$ coincides with $\text{BQP}(G)$ if and only if G is acyclic. A better approximation of $\text{BQP}(G)$ is given by the polyhedron $\text{BQP}_3(G)$, defined by the vectors $(x,y) \in \text{BQP}_2(G)$ that also satisfy the inequalities

$$x(V^2_{C,D}) - x(V^0_{C,D}) + y(C\setminus D) - y(D) \le \tfrac{|D|-1}{2}$$

for each cycle C of G, and $D \subseteq C$ with $|D|$ odd, where $V_C \subseteq V$ are the nodes of the cycle C, $V^2_{C,D} = \{j \in V_C : |\{ij \in D\}| = 2\}$ and $V^0_{C,D} = \{j \in V_C : |\{ij \in D\}| = 0\}$. Correspondingly, the cycle relaxation of (BQP) is:

$$\begin{aligned} \max \quad & c^T x + q^T y \\ \text{s.t.} \quad & (x,y) \in \text{BQP}_3(G). \end{aligned} \tag{BQP3}$$

It was shown by Padberg that if G is series-parallel, then $\text{BQP}(G) = \text{BQP}_3(G)$ [24]. De Simone later proved that the converse also holds [7].

The results of Padberg and De Simone [7, 24] indicate that the complexity of the inequality description of $\text{BQP}(G)$ depends on the sparsity of G: $\text{BQP}(G) = \text{BQP}_2(G)$ if and only if G has no K_3 minor and $\text{BQP}(G) = \text{BQP}_3(G)$ if and only if G has no K_4 minor.

Our goal is to relate the strength of the cycle relaxation (BQP3) to the sign pattern of q. We associate to q the signed graph defined by the pair $(G, E^-(q))$, where the set of negative edges is $E^-(q) = \{ij \in E : q_{ij} < 0\}$ and the set of positive edges is $E^+(q) = E \setminus E^-(q)$. For the edge relaxation, it has been shown that any optimal vertex of $\text{BQP}_2(G)$ is integral if the signed graph $(G, E^-(q))$ is balanced, i.e., if it contains no odd-K_3 minor [18, 5]. For the cycle relaxation of (BQP), we characterize the sign patterns of q such that all optimal vertices of (BQP3) are integral.

Theorem 3. *If $(G, E^-(q))$ has no odd-K_4 minor, then any optimal vertex of* (BQP3) *is integral.*

Conversely, we show that for any signed graph (G, Σ) with an odd-K_4 minor, there exists a vector q whose sign pattern is compatible with Σ, i.e., $E^-(q) = \Sigma$, and such that (BQP3) has an optimal fractional vertex.

Proposition 3. *Let $G = (V, E)$ and $\Sigma \subseteq E$. If (G, Σ) has an odd-K_4 minor, then there exist $c \in \mathbb{R}^V$ and $q \in \mathbb{R}^E$ such that $E^-(q) = \Sigma$ and* (BQP3) *has a unique optimum that is fractional.*

We summarize related work and our contribution in Table 1.

Table 1: Related work and main results

	Conditions on $G = (V, E)$	Conditions on (G, Σ)
	each vertex is integral	each optimal vertex is integral
(BQP2)	$\Leftrightarrow \nexists\ K_3$ minor[24]	$\Leftrightarrow \nexists$ odd-K_3 minor, $\Sigma = E^-(q)$[18, 5]
(BQP3)	$\Leftrightarrow \nexists\ K_4$ minor[24, 7]	$\Leftrightarrow \nexists$ odd-K_4 minor, $\Sigma = E^-(q)$
(MET)	$\Leftrightarrow \nexists\ K_5$ minor[2]	$\Leftrightarrow \nexists$ odd-K_5 minor, $\Sigma = E^-(w)$

The problem of recognizing if a signed graph (G, Σ) contains an odd-K_k minor can be solved in polynomial-time for fixed k [19, 21]. Moreover, in the case where $\Sigma = E$, the problem of recognizing if (G, Σ) contains an odd-K_4 minor reduces to checking if the underlying graph has an *odd K_4-subdivision*, i.e. a subdivision of K_4 such that each triangle has become an odd cycle, see [26].

Corollary 1. *If $E^-(q) = E$ and G has no odd K_4-subdivision, then any optimal vertex of* (BQP3) *is integral.*

An older result by Gerards [12] describes a recursive polynomial-time algorithm to check if a graph contains an odd K_4-subdivision. A similar algorithm was later given by the same author to recognize if a signed graph contains an odd-K_4 minor [13].

We briefly mention some more related work. Galluccio et al. [10] and McCormick et al. [22] also investigated classes of polynomially-solvable instances of the maximum cut problem that are characterized by properties of the objective function, but their approach is not polyhedral. As mentioned earlier, Theorem 2 was proven by Poljak [25] for the case where $E^- = E$. The proof of Poljak is based on a duality argument: the author shows that there always exists an optimal dual solution to (MET) where a constraint of the form (1) has positive dual variable only if $C = D$. Finally, Boros et al. [3] used tools from pseudo-boolean optimization to associate the optimal value of (BQP3) to the problem of balancing a weighted signed graph.

The paper is organized as follows. Section 2 introduces some required preliminary notions on signed graphs and odd minors and Section 3 presents illustrative examples of our main results. In Section 4, we focus on the metric polytope and we prove Theorem 2, Theorem 1 and Proposition 1. In Section 5, we study the cycle relaxation of the boolean quadric polytope and we prove Theorem 3 and Proposition 3.

2 Preliminaries on signed graphs

A *signed graph* is a pair (G, Σ), where G is a graph and Σ, called the *signature*, is a subset of edges of G. The edges of Σ are called the *negative edges*, and the edges not in Σ are called the *positive edges* . A subset of edges of G is called *odd* if it contains an odd number of negative edges, otherwise it is called *even*. Two signatures are *equivalent* if their symmetric difference is a cut. The operation of changing to an equivalent signature is called *signature-exchange*. If Σ_1 and Σ_2 are equivalent signatures, then (G, Σ_1) and (G, Σ_2) have the same set of odd cycles. The *deletion* of an edge consists in the removal of the edge from the graph. The *contraction* of an edge consists in possibly doing a signature-exchange, so that the edge becomes positive, and then in removing the edge and identifying its endnodes.

A *minor* of (G, Σ) is a signed graph that can be obtained by any of the following operations: signature-exchanges, edge deletions, and edge contractions. An *odd-K_t* is a signed complete graph on t nodes where

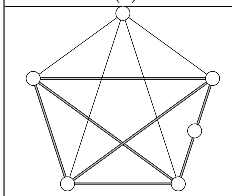
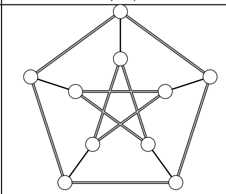
	(i)	(ii)
(G,Σ)		
$\exists K_5$ minor	Yes	Yes
$\mathrm{MET}(G)$ is integral	No	No
$\exists$ odd-K_5 minor	No	Yes
$\forall w \in \mathbb{R}^E : E^-(w) = \Sigma$, each optimal vertex of (MET) is integral	Yes	No

Figure 1: The signed graphs of Example 1

all edges are negative. A signed graph is called *balanced* if it has no odd cycles. Equivalently: (i) a balanced graph has no odd-K_3 minor; (ii) a balanced graph admits a signature exchange that resigns all edges to be negative. Signed graphs with no odd-K_4 minor coincide with the class of *strongly bipartite* graphs [27], and signed graphs with no odd-K_5 minor coincide with the class of *weakly bipartite* graphs [15]. We refer to [26, 4] for more notions on signed graphs.

3 Examples

The following example illustrates Theorem 1 and Proposition 1

Example 1. *Let $G = K_5$. By the result of Barahona and Mahjoub [2],* $\mathrm{MET}(G)$ *has fractional vertices. Let (G,Σ) be the signed graph in Fig. 1(i), where the thicker lines correspond to the negative edges. Since this signed graph has no odd-K_5 minor, by Theorem 1, for each $w \in \mathbb{R}^E$ with $E^-(w) = \Sigma$ all the optimal vertices of* (MET) *are integral.*

Now let (G,Σ) be the signed graph in Fig. 1(ii). Since (G,Σ) contains an odd-K_5 minor, by Proposition 1, there exists a weight vector $w \in \mathbb{R}^E$ with $E^-(w) = \Sigma$, such that the unique optimum to (MET) *is fractional.*

The next example illustrates Theorem 3 and Proposition 3

Example 2. *Let $G = K_5$. By the results of Padberg and De Simone [24, 7], we have that both* $\mathrm{BQP}_2(G)$ *and* $\mathrm{BQP}_3(G)$ *have fractional vertices. Let (G,Σ) be the signed graph in Fig. 2(i), where the thicker lines correspond to the negative edges. By the results in [18, 5], since this signed graph has no odd-K_3 minor, for each $q \in \mathbb{R}^E$ with $E^-(q) = \Sigma$ all the optimal vertices of* (BQP2) *(and of* (BQP3)*) are integral.*

Now let (G,Σ) be the signed graph in Fig. 2(ii). Since (G,Σ) contains an odd-K_3 minor, the results in [18, 5] imply that there is a vector $q \in \mathbb{R}^E$ with $E^-(q) = \Sigma$ such that the unique optimum to (BQP2) *is fractional. However, since (G,Σ) has no odd-K_4 minor, by Theorem 3 for each $q \in \mathbb{R}^E$ with $E^-(q) = \Sigma$ all the optimal vertices of* (BQP3) *are integral.*

4 Optimizing over the metric polytope

In the following we consider a vector of weights $w \in \mathbb{R}^E$ and we let $E^- = E^-(w)$. Consider the signed graph (G, E^-) and let $E^+ = E \setminus E^-$. For any edge subset $L \subseteq E$ we denote by $L^- = L \cap E^-$ and $L^+ = L \cap E^+$ the negative and positive edges of L, respectively. For an odd cycle $C \in \mathcal{C}(G, E^-)$ and a vector $z \in \mathrm{MET}(G, E^-)$, we say that C is tight at z if the corresponding cycle constraint is tight at z, i.e. $z(C^-) - z(C^+) = |C^-| - 1$. Finally, for any $e \in E$ we define $\mathcal{C}_e = \{C \in \mathcal{C}(G, E^-) : e \in C\}$.

We first prove that, to solve (MET), one can equivalently solve (MET^-).

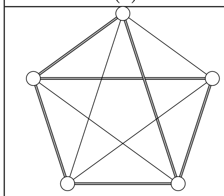
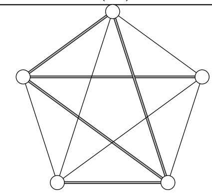
	(i)	(ii)
(G, Σ)		
$\exists K_3$ minor	Yes	Yes
$\mathrm{BQP}_2(G)$ is integral	No	No
$\exists$ odd-K_3 minor	No	Yes
$\forall q \in \mathbb{R}^E : E^-(q) = \Sigma$, each optimal vertex of (BQP2) is integral	Yes	No
$\exists K_4$ minor	Yes	Yes
$\mathrm{BQP}_3(G)$ integral	No	No
$\exists$ odd-K_4 minor	No	No
$\forall q \in \mathbb{R}^E : E^-(q) = \Sigma$, each optimal vertex of (BQP3) is integral	Yes	Yes

Figure 2: The signed graphs of Example 2

Proof of Theorem 2. We first prove the "if" direction, i.e., we assume that $z^* \in \mathbb{R}^E$ is an optimum of (MET$^-$), and we show that z^* is an optimum of (MET). To this purpose, it is sufficient to prove that $z^* \in \mathrm{MET}(G)$, since $\mathrm{MET}(G) \subseteq \mathrm{MET}(G, E^-)$.

First, we prove that z^* satisfies constraints (2).

Claim 1. $z^* \in [0, 1]^E$.

Proof of claim. By contradiction, suppose there exists $e \in E^-$ such that $z^*_e < 0$. If we increase z^*_e by any $\epsilon > 0$, since $w_e < 0$, we will decrease the objective value. Our goal is to contradict the optimality of z^* by showing that for a sufficiently small $\epsilon > 0$ this perturbation of z^* yields a feasible solution to (MET$^-$). To this purpose, we show that for each $C \in \mathcal{C}_e$ the corresponding cycle inequality (4) is *not* tight at z^*:

$$z^*(C^- \setminus \{e\}) + z^*_e - z^*(C^+) < |C^-| - 1, \tag{7}$$

since $z^*_h \leq 1$ for all $h \in C^-$, $z^*_e < 0$ and $z^*_h \geq 0$ for all $h \in C^+$.

Similarly, if there exists $e \in E^+$ such that $z^*_e > 1$, we can decrease z^*_e by ϵ without violating any cycle constraint (4), which contradicts the optimality of z^*. ⋄

Next, we show that z^* satisfies all inequalities (1). Recall that each inequality (1) is associated to a pair (C, D), where C is an odd cycle of G and D is an odd cardinality subset of C. We will proceed by induction on $|C^+ \cap D| + |C^- \setminus D|$. In the base case, we prove that z^* satisfies every inequality (1) such that $|C^+ \cap D| + |C^- \setminus D| = 0$. In this case, $D = C^-$ and, since z^* is feasible for (MET$^-$), the inequality (1) corresponding to (C, D) is satisfied by z^*. Let ℓ be a nonnegative integer, and suppose that z^* satisfies every inequality (1) corresponding to a cycle C and odd cardinality subset D of C such that $|C^+ \cap D| + |C^- \setminus D| \leq \ell$. Consider a cycle C_0 of G and an odd cardinality subset D_0 of C_0, such that $|C_0^+ \cap D_0| + |C_0^- \setminus D_0| = \ell + 1$. Our goal is to prove that z^* satisfies the inequality (1) corresponding to (C_0, D_0).

Let $\bar{D}_0 = C_0 \setminus D_0$. By contradiction we suppose that

$$z^*(D_0) - z^*(\bar{D}_0) \geq |D_0| - 1 + \delta, \tag{8}$$

where $\delta > 0$. We define the sets:

$$D_0^+ = C_0^+ \cap D_0 \qquad \bar{D}_0^+ = C_0^+ \setminus D_0$$
$$D_0^- = C_0^- \cap D_0 \qquad \bar{D}_0^- = C_0^- \setminus D_0.$$

Since $|D_0^+ \cup \bar{D}_0^-| \geq 1$, there is an edge $e \in D_0^+ \cup \bar{D}_0^-$.

Claim 2. *There exists a cycle $C \in \mathcal{C}_e$ that is tight at z^*.*

Proof of claim. In order for (8) to hold, it should be $z^*_e > 0$ for all $e \in D_0$ and $z^*_e < 1$ for all $e \in \bar{D}_0$. Thus $z^*_e > 0$ if $e \in D_0^+$ and $z^*_e < 1$ if $e \in \bar{D}_0^-$. By contradiction suppose that, for all $C \in \mathcal{C}_e$, $z^*(C^-) - z^*(C^+) \leq |C^-| - 1 - \epsilon$, with $\epsilon > 0$. Define vector $\bar{z}$ as follows

$$\bar{z}_h = \begin{cases} z^*_h & \text{if } h \neq e \\ z^*_h - \epsilon' & \text{if } h = e \text{ and } e \in D_0^+ \\ z^*_h + \epsilon' & \text{if } h = e \text{ and } e \in \bar{D}_0^-, \end{cases}$$

where $\epsilon' = \min\{\epsilon, z^*_e\}$ if $e \in D_0^+$ and $\epsilon' = \min\{\epsilon, 1 - z^*_e\}$ if $e \in \bar{D}_0^-$. By construction, $\bar{z}$ is feasible for (MET$^-$). Moreover, the value of the objective function of (MET$^-$) in $\bar{z}$ satisfies

$$w^T \bar{z} = \sum_{h \neq e} w_h z^*_h + w_e(z^*_e \pm \epsilon') = w^T z^* - |w_e|\epsilon' < w^T z^*,$$

since $\epsilon' > 0$. This contradicts optimality of z^*. ◇

Let C_1 be an odd cycle containing e that is tight at z^*. We have

$$z^*(C_1^-) - z^*(C_1^+) = |C_1^-| - 1. \tag{9}$$

Summing up (8) and (9) we obtain

$$z^*(D_0) + z^*(C_1^-) - z^*(\bar{D}_0) - z^*(C_1^+) \geq |D_0| + |C_1^-| + \delta - 2.$$

This inequality can be rewritten as follows, see Fig. 3:

$$\begin{aligned} z^*(D_0 \setminus C_1) + z^*(C_1^- \setminus C_0) - z^*(\bar{D}_0 \setminus C_1) - z^*(C_1^+ \setminus C_0) \geq |D_0 \setminus C_1| + |C_1^- \setminus C_0| + \delta \\ +|D_0 \cap C_1| + |C_1^- \cap C_0| - 2 \\ -2z^*(D_0 \cap C_1^-) + 2z^*(\bar{D}_0 \cap C_1^+). \end{aligned} \tag{10}$$

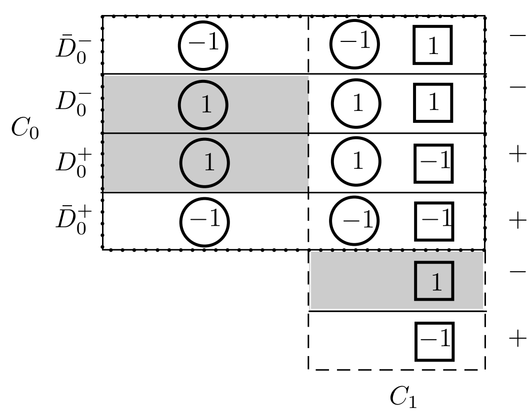


Figure 3: Pictorial representation of the edge subsets C_0 (beaded rectangle) and C_1 (dashed rectangle), and of their intersection. On the right the "−" and "+" labels indicate the negative and positive edges, respectively. The partition of C_0 into D_0 and $\bar{D}_0$ is indicated on the left. The numbers enclosed in circles are the coefficients of (8) and the numbers enclosed in squares are the coefficients of (9) (the zero coefficients are omitted for ease of representation). The greyed cells represent S_0.

Note that $|D_0 \cap C_1| + |C_1^- \cap C_0| = |\bar{D}_0 \cap C_1^-| + |D_0 \cap C_1^+| + 2|D_0 \cap C_1^-|$, thus we can rewrite the right-hand-side of (10) as

$$(|D_0 \setminus C_1| + |C_1^- \setminus C_0| + \delta) + (|\bar{D}_0 \cap C_1^-| + |D_0 \cap C_1^+| - 2)$$

$$+(2|D_0 \cap C_1^-| - 2z^*(D_0 \cap C_1^-)) + 2z^*(\bar{D}_0 \cap C_1^+) \geq$$
$$(|D_0 \setminus C_1| + |C_1^- \setminus C_0| + \delta) + (|\bar{D}_0 \cap C_1^-| + |D_0 \cap C_1^+| - 2), \tag{11}$$

where (11) is implied by $z^* \in [0,1]^E$.

Consider the cycle given by the symmetric difference of C_0 and C_1, i.e., $S = C_0 \Delta C_1 = (C_0 \setminus C_1) \cup (C_1 \setminus C_0)$. Let $S_0 = (D_0 \setminus C_1) \cup (C_1^- \setminus C_0) \subseteq S$, see Fig. 3. We also define a parameter $\sigma(S_0)$, that is 0 if $|S_0|$ is even, and 1 if $|S_0|$ is odd. Our next goal is to prove the following claim.

Claim 3. $|\bar{D}_0 \cap C_1^-| + |D_0 \cap C_1^+| - 2 \geq -\sigma(S_0)$

Proof of claim. First, since $e \in \bar{D}_0^- \cup D_0^+$ and $e \in C_1$, we have $|\bar{D}_0 \cap C_1^-| + |D_0 \cap C_1^+| \geq 1$. Thus the claim holds if $|S_0|$ is odd. We now consider the case where $|S_0|$ is even and we prove that $|\bar{D}_0 \cap C_1^-| + |D_0 \cap C_1^+| \geq 2$.

Recall that $|D_0|$ is odd and that C_1 is an odd cycle:

$$|D_0| = |D_0 \cap C_1| + |D_0 \setminus C_1| \equiv_2 1, \tag{12}$$
$$|C_1^-| = |C_0 \cap C_1^-| + |C_1^- \setminus C_0| \equiv_2 1. \tag{13}$$

Since $C_0 \cap C_1^- = (D_0 \cap C_1^-) \cup (\bar{D}_0 \cap C_1^-)$ (see Fig. 3), we rewrite (13) as

$$|D_0 \cap C_1^-| + |\bar{D}_0 \cap C_1^-| + |C_1^- \setminus C_0| \equiv_2 1. \tag{14}$$

By contradiction, suppose $|\bar{D}_0 \cap C_1^-| + |D_0 \cap C_1^+| = 1$.
Case 1. If $|\bar{D}_0 \cap C_1^-| = 1$ and $|D_0 \cap C_1^+| = 0$, then $D_0 \cap C_1 = D_0 \cap C_1^-$. From (14) we obtain:

$$|D_0 \cap C_1^-| + |\bar{D}_0 \cap C_1^-| + |C_1^- \setminus C_0| = |D_0 \cap C_1^-| + 1 + |C_1^- \setminus C_0| \equiv_2 1$$
$$\Rightarrow |D_0 \cap C_1^-| \equiv_2 |C_1^- \setminus C_0|. \tag{15}$$

Since $D_0 \cap C_1 = D_0 \cap C_1^-$, from (12) we obtain:

$$|D_0 \cap C_1^-| + |D_0 \setminus C_1| \equiv_2 1.$$

This, together with (15), implies $|C_1^- \setminus C_0| + |D_0 \setminus C_1| = |S_0|$ is odd, a contradiction.
Case 2. If $|\bar{D}_0 \cap C_1^-| = 0$ and $|D_0 \cap C_1^+| = 1$, then from (14) we obtain:

$$|D_0 \cap C_1^-| + |\bar{D}_0 \cap C_1^-| + |C_1^- \setminus C_0| = |D_0 \cap C_1^-| + |C_1^- \setminus C_0| \equiv_2 1,$$
$$\Rightarrow |D_0 \cap C_1^-| + 1 \equiv_2 |C_1^- \setminus C_0|. \tag{16}$$

Since $D_0 \cap C_1 = (D_0 \cap C_1^-) \cup (D_0 \cap C_1^+)$ and $|D_0 \cap C_1^+| = 1$, from (12) we obtain:

$$|D_0 \cap C_1^-| + 1 + |D_0 \setminus C_1| \equiv_2 1.$$

This, together with (16), implies $|C_1^- \setminus C_0| + |D_0 \setminus C_1| = |S_0|$ is odd, a contradiction.

$\diamond$

From (10), (11) and Claim 3 we obtain

$$z^*(D_0 \setminus C_1) + z^*(C_1^- \setminus C_0) - z^*(\bar{D}_0 \setminus C_1) - z^*(C_1^+ \setminus C_0) \geq$$
$$|D_0 \setminus C_1| + |C_1^- \setminus C_0| + \delta - \sigma(S_0). \tag{17}$$

By Claim 1, since $z^* \in [0,1]^E$, we also have:

$$z^*(D_0 \setminus C_1) + z^*(C_1^- \setminus C_0) - z^*(\bar{D}_0 \setminus C_1) - z^*(C_1^+ \setminus C_0) \leq |D_0 \setminus C_1| + |C_1^- \setminus C_0|,$$

which immediately contradicts (17) if $|S_0|$ is even. Now consider the case where $|S_0|$ is odd. Recall that $S_0 = (D_0 \setminus C_1) \cup (C_1^- \setminus C_0)$, see the greyed cells of Fig. 3. We have $S^- \setminus S_0 = \bar{D}_0^- \setminus C_1 \subseteq \bar{D}_0^- \setminus \{e\}$ and

$S^+ \cap S_0 = D_0^+ \setminus C_1 \subseteq D_0^+ \setminus \{e\}$, see Fig. 3. Thus $|S^+ \cap S_0| + |(S^- \setminus S_0)| \le \ell$ and by our inductive hypothesis we obtain

$$z^*(S_0) - z^*(S \setminus S_0) \le |S_0| - 1$$

This last inequality can be rewritten as follows, see Fig. 3:

$$z^*(D_0 \setminus C_1) + z^*(C_1^- \setminus C_0) - z^*(\bar{D}_0 \setminus C_1) - z^*(C_1^+ \setminus C_0) \le |D_0 \setminus C_1| + |C_1^- \setminus C_0| - 1,$$

which contradicts (17).

We finally prove the "only if" direction, i.e., we now assume that z^* is an optimum of (MET) and we prove that z^* is an optimum of (MET$^-$). If not, there is an optimum of (MET$^-$) with $w^T\bar{z} < w^T z^*$. Since any optimum of (MET$^-$) is also optimal for (MET), we must have $w^T\bar{z} = w^T z^*$, a contradiction. □

We remark that Theorem 2 implies that a vector $z^* \in \mathbb{R}^E$ is the unique optimum of (MET) if and only if it is the unique optimum of (MET$^-$). We now use Theorem 2 to prove Proposition 2. First, we recall some basic definitions. Let $P \subseteq R^E$ be a polyhedron. The *normal cone* of P at $\bar{z} \in P$ is the set $N(P, \bar{z}) = \{\bar{w} \in \mathbb{R}^E : \bar{w}^T(z - \bar{z}) \le 0 \ \forall z \in P\}$. Recall that for problem $\min\{\bar{w}^T z : z \in P\}$, we have that a point $\bar{z} \in P$ is an optimum if and only if $-\bar{w} \in N(P, \bar{z})$. Moreover a vertex $\bar{z}$ is a unique optimum if and only if $-\bar{w}$ is in the relative interior of $N(P, \bar{z})$.

Proof of Proposition 2. We denote by vert(MET(G, E^-)) the vertices of MET(G, E^-). We first show that z is a vertex of MET(G, E^-) if and only if it is a vertex of MET(G) whose normal cone contains in its relative interior a vector in $\mathcal{R}(E^-)$. Let $z \in$ vert(MET(G, E^-)) and let $\bar{w}$ be a weight vector such that z is a unique minimizer of $\bar{w}^T z$ over MET(G, E^-). Note that $\bar{w} \in \mathcal{R}(E^+)$, since otherwise this problem would be unbounded. Thus $E^-(\bar{w}) = E^-(w) = E^-$. By Theorem 2, we have that z is a unique optimum of (MET) with weight vector $\bar{w}$, thus z is a vertex of MET(G) and $-\bar{w} \in \mathcal{R}(E^-)$ belongs to the relative interior of the normal cone of MET(G) at z. Conversely, let z be a vertex of MET(G) whose normal cone contains a vector $-\bar{w} \in \mathcal{R}(E^-)$ in its relative interior. Thus z is the unique optimum of (MET) with weight vector $\bar{w} \in \mathcal{R}(E^+)$. By Theorem 2 we have that z is the unique optimum of (MET$^-$) with weight vector $\bar{w} \in \mathcal{R}(E^+)$. This implies that z is a vertex of MET(G, E^-).

Now, we prove MET(G, E^-) = MET$(G) + \mathcal{R}(E^+)$. First, we show that MET$(G, E^-) \supseteq$ MET$(G) + \mathcal{R}(E^+)$. Let $z = x + y$, where $x \in$ MET(G) and $y \in \mathcal{R}(E^+)$. Since MET$(G) \subseteq$ MET(G, E^-), we have $x \in$ MET(G, E^-). It can be easily verified that $x + y$ satisfies (4), (5) and (6), thus $x + y \in$ MET(G, E^-). Next, we show that MET$(G, E^-) \subseteq$ MET$(G) + \mathcal{R}(E^+)$. Let $z \in$ MET(G, E^-). By the Minkowski-Weyl Theorem [23, 28] we have

$$\mathrm{MET}(G, E^-) = \mathrm{conv}(\mathrm{vert}(\mathrm{MET}(G, E^-))) + \mathcal{R}(E^+), \tag{18}$$

since the recession cone of MET(G, E^-) is $\mathcal{R}(E^+)$. Thus $z = x + y$, where $x \in$ conv(vert(MET(G, E^-))) and $y \in \mathcal{R}(E^+)$. From the characterization of the vertices of MET(G, E^-) proved above, we have that vert(MET(G, E^-)) $\subseteq$ vert(MET(G)). This implies that $x \in$ conv(vert(MET(G)), and conv(vert(MET(G)) = MET(G) because MET(G) is bounded. □

We now prove Theorem 1 and Proposition 1. Define a bijective affine map $\theta : \mathbb{R}^E \to \mathbb{R}^E$ that complements the variables indexed by the negative edges of (G, E^-), i.e. $\theta_e(z) = 1 - z_e$ for $e \in E^-$ and $\theta_e(z) = z_e$ for $e \in E^+$. By setting $\zeta = \theta(z)$ we can rewrite (MET$^-$) as

$$\begin{array}{lll} \min & \sum_{e \in E^+} w_e \zeta_e + \sum_{e \in E^-} w_e(1 - \zeta_e) & \\ \text{s.t.} & \sum_{e \in C^-}(1 - \zeta_e) - \sum_{e \in C^+} \zeta_e \le |C^-| - 1 & C \in \mathcal{C}(G, E^-) \\ & \zeta_e \ge 0 & e \in E^+, \\ & 1 - \zeta_e \le 1 & e \in E^-. \end{array}$$

By removing the constant in the objective function we obtain:

$$\begin{array}{llll} \min & |w|^T \zeta & & \\ \text{s.t.} & \zeta(C) & \ge 1 & C \in \mathcal{C}(G, E^-) \\ & \zeta_e \ge 0 & & e \in E, \end{array} \tag{19}$$

where $|w|$ is the vector of absolute values of w. We will exploit a powerful result due to Guenin [15], characterizing weakly bipartite graphs.

Theorem 4 (Guenin [15])**.** *Let* (G, Σ) *be a signed graph, where* $G = (V, E)$*. The polyhedron* $\{\zeta \in \mathbb{R}_+^E : \zeta(C) \geq 1 \;\; \forall\, C$ *odd cycle of* $(G, \Sigma)\}$ *is integral if and only if* (G, Σ) *has no odd-*K_5 *minor.*

Proposition 4. *We have that* z^* *is an optimal vertex of* $\mathrm{MET}(G)$ *in* (MET) *if and only if* $\theta(z^*)$ *is an optimal vertex solution to* (19)*.*

Proof. First, by Theorem 2, the optima of (MET) coincide with the optima of (MET$^-$), thus the optimal face of $\mathrm{MET}(G)$ in (MET) coincides with the optimal face of $\mathrm{MET}(G, E^-)$ in (MET$^-$). As a consequence, z^* is an optimal vertex of $\mathrm{MET}(G)$ in (MET) if and only if it is an optimal vertex of $\mathrm{MET}(G, E^-)$ in (MET$^-$). Moreover, z^* is an optimum of (MET$^-$) if and only if $\theta(z^*)$ is an optimum of (19). Finally, since θ is an affine bijective map, it preserves the combinatorial structure of $\mathrm{MET}(G, E^-)$, thus z^* is a vertex of $\mathrm{MET}(G, E^-)$ if and only if $\theta(z^*)$ is a vertex of the polyhedron defined by (19). □

Proof of Theorem 1. Suppose that the signed graph (G, E^-) does not contain an odd-K_5 minor. Then, by Theorem 4, the polyhedron $\{\zeta \in \mathbb{R}_+^E : \zeta(C) \geq 1 \; \forall\, C$ odd cycle of $(G, E^-)\}$ is integral. As a consequence, all optimal vertices of (19) are integral. By Proposition 4, any optimal vertex z^* of (MET) is such that $\theta(z^*)$ is integral. Since θ^{-1} is an affine map that preserves integrality, we conclude that z^* is integral. □

Proof of Proposition 1. Suppose that (G, E^-) contains an odd-K_5 minor. By Theorem 4, the polyhedron

$$\{\zeta \in \mathbb{R}_+^E : \zeta(C) \geq 1 \; \forall\, C \text{ odd cycle of } (G, E^-)\}$$

has a fractional vertex ζ^*. Let $\omega \in \mathbb{R}^E$ be such that ζ^* is the unique minimizer of $\omega^T \zeta$. First, $\omega > 0$: if there exists $e \in E$ with $\omega_e \leq 0$, increasing ζ_e^* by $\epsilon > 0$ would give an alternate optimum, contradicting the fact that ζ^* is the unique minimizer of $\omega^T \zeta$. Define $w \in \mathbb{R}^E$ as $w_e = -\omega_e$ for $e \in E^-$ and $w_e = \omega_e$ for $e \in E^+$. By Proposition 4, z^* is an optimal vertex of (MET) if and only if $\theta(z^*)$ is an optimal vertex of (19). Moreover, since θ is a bijective map, z^* is the unique optimum to (MET). As $\omega = |w|$, we have $\zeta^* = \theta(z^*)$. Since θ^{-1} is an affine map that preserves integrality, we conclude that z^* is the unique optimum to (MET), and it is a fractional vertex of $\mathrm{MET}(G)$. □

5 Optimizing over the cycle relaxation of the boolean quadric polytope

To prove Theorem 3 and Proposition 3, we exploit the connection between the boolean quadric polytope and the cut polytope, independently discovered by several authors [1, 16, 7], see also [8]. Let $G + r$ denote the graph obtained from G by adding a new node r and joining r to all nodes of G, and define $E' = \{ri : i \in V\}$. The *covariance mapping* $\pi : \mathbb{R}^{V \cup E} \to \mathbb{R}^{E' \cup E}$ maps any $(x, y) \in \mathbb{R}^{V \cup E}$ to $\pi(x, y) = z \in \mathbb{R}^{E' \cup E}$ defined as

$$z_{ri} = x_i \qquad ri \in E' \tag{20}$$

$$z_{ij} = x_i + x_j - 2y_{ij} \qquad ij \in E. \tag{21}$$

The boolean quadric polytope of G is in one-to-one correspondence with the cut polytope of $G + r$ via the covariance mapping [7]. Moreover, the image of $\mathrm{BQP}_3(G)$ through the covariance mapping is exactly $\mathrm{MET}(G + r)$, see also [8]. This is easy to check, since Barahona and Mahjoub [2] proved that (1) are facet-defining only for chordless cycles, while (2) are facet-defining only for the edges that do not belong to any triangle. Thus, $\mathrm{CUT}(G + r) = \pi(\mathrm{BQP}(G))$ and $\mathrm{MET}(G + r) = \pi(\mathrm{BQP}_3(G))$.

The proof of Theorem 3 relies on Theorem 1 and on the following property of the covariance mapping.

Proposition 5. *Let* $(x, y) \in \mathrm{BQP}_3(G)$ *and* $z = \pi(x, y)$*. Then* (x, y) *is integral if and only if* z *is integral.*

Proof. From the definition of the covariance mapping π in (20) and (21) it is clear that if (x, y) is integral, then $z = \pi(x, y)$ is integral. To prove the converse, suppose that z is integral and that (x, y) is fractional. Since $z \in \mathrm{MET}(G + r)$, z is binary. As $x_i = z_{ri}$ for all $i \in V$, it follows that x is binary. Then there exists an edge $ij \in E$ such that $y_{ij} = \frac{1}{2}(z_{ri} + z_{rj} - z_{ij})$ is fractional. This happens if $z_{ri} + z_{rj} - z_{ij} = 1$. If $z_{ri} = z_{ij} = 0$ and $z_{rj} = 1$, then the triangle inequality $z_{rj} - z_{ri} - z_{ij} \leq 0$ of $\mathrm{MET}(G + r)$ is violated. Similarly, if $z_{rj} = z_{ij} = 0$ and $z_{ri} = 1$, then the triangle inequality $z_{ri} - z_{rj} - z_{ij} \leq 0$ of $\mathrm{MET}(G + r)$ is violated. Finally, if $z_{rj} = z_{ri} = z_{ij} = 1$, then the triangle inequality $z_{ri} + z_{rj} + z_{ij} \leq 2$ of $\mathrm{MET}(G + r)$ is violated. □

Proof of Theorem 3. Suppose that (G, E^-) does not contain an odd-K_4 minor. First, $\pi(\text{BQP}_3(G)) = \text{MET}(G+r)$. Recall that, since π is a linear bijective map, it preserves the combinatorial structure of $\text{BQP}_3(G)$, thus vertices of $\text{BQP}_3(G)$ are mapped onto vertices of $\text{MET}(G+r)$.

Let $E' = \{ri : i \in V\}$ and define $w \in \mathbb{R}^{E \cup E'}$ as the vector of edge weights of $G+r$, such that

$$\begin{array}{lll} w_{ij} & = \frac{q_{ij}}{2} & ij \in E \\ w_{ri} & = -c_i - \frac{1}{2}\sum_{ij \in E} q_{ij} & ri \in E'. \end{array}$$

Then (BQP3) is equivalent to $\min\{w^T z : z \in \text{MET}(G+r)\}$, since for each $(x,y) \in \text{BQP}_3(G)$, $c^T x + q^T y = -w^T \pi(x,y)$. Note that q and w have the same sign pattern on E. Let $H = G+r$ and

$$F^- = \{e \in E \cup E' : w_e < 0\} = E^- \cup \{e \in E' : w_e < 0\}.$$

Then, the signed graph (H, F^-) does not contain an odd-K_5 minor. This immediately implies, by Theorem 1, that any optimal vertex of $\text{MET}(G+r)$ is integral. By Proposition 5, we have that any optimal vertex of $\text{BQP}_3(G)$ is integral. □

Similarly, the proof of Proposition 3 relies on Proposition 1 and on Proposition 5.

Proof of Proposition 3. Suppose that (G, Σ) has an odd-K_4 minor and let $H = G+r$ and $E' = \{ri : i \in V\}$. Let Σ' be such that $\Sigma' \supseteq \Sigma$ and (H, Σ') has an odd-K_5 minor. Then, by Proposition 1, there exists $w \in \mathbb{R}^{E \cup E'}$ such that $\Sigma' = \{ij \in E \cup E' : w_{ij} < 0\}$ and (MET) has a unique optimum z^* that is a fractional vertex of $\text{MET}(G)$. Define $c \in \mathbb{R}^V$ and $q \in \mathbb{R}^E$ such that:

$$\begin{array}{lll} q_{ij} & = 2w_{ij} & ij \in E \\ c_j & = -w_{rj} - \sum_{ij \in E} w_{ij} & i \in V. \end{array}$$

Then, $\pi(\text{BQP}_3(G)) = \text{MET}(G+r)$ and (BQP3) is equivalent to $\min\{w^T z : z \in \text{MET}(G+r)\}$. This implies that there exists a unique optimum (x^*, y^*) of (BQP3) such that $\pi(x^*, y^*) = z^*$. Since z^* is fractional, by Proposition 5, we conclude that (x^*, y^*) is fractional.

□

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